Enhanced Turan-type Inequalities for Polynomials

Reingachan N., Robinson Soraisam, Khangembam Babina Devi, M. Singhajit Singh, Barchand Chanam

***Abstract*—If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, Jain [Bull. Math. Soc. Sci. Math. Roumania Tome, 59(2016), 339-347] proved**

$$\max_{|z|=1} |p'(z)| \ge nA_k \max_{|z|=1} |p(z)|,$$

where

$$A_k = \left(\frac{|a_0| + |a_n| k^{n+1}}{|a_o| (1 + k^{n+1}) + |a_n| (k^{n+1} + k^{2n})} \right).$$

In this paper, we initially derive a generalized form that not only encompasses but also enhances the aforementioned inequality. Additionally, we extend this formulation to a more comprehensive result, thereby producing an improved outcome for certain established inequalities as a specific instance.

***Index Terms*—polynomial, zeros, derivatives, Turán-type inequality, maximum modulus, Schwarz lemma.**

I. Introduction

Consider a polynomial $p(z)$ of degree n. Turán [16] proved his calibrated result that if $p(z)$ has all its zeros in $|z| \le 1$, then

$$\max_{|z|=1} |p'(z)| \ge \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1}$$

Inequality (1) is sharp and equality holds for $p(z) = \alpha z^n + \beta$, where $|\alpha| = |\beta|$.
Inequality (1) was refined by Aziz and Dawood [1] in the form

$$\max_{|z|=1} |p'(z)| \ge \frac{n}{2} \left\{ \max_{|z|=1} |p(z)| + \min_{|z|=1} |p(z)| \right\}. \tag{2}$$

Inequality (1) of Turán [16] has been of considerable interest and applications and it would be of interest to seek its generalization for polynomials having all their zeros in $|z| \le k, k > 0$. The case when $0 < k \le 1$ was for the first time settled by Malik [8] and proved

$$\max_{|z|=1} |p'(z)| \ge \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{3}$$

While for the case $k \ge 1$, Govil [4] proved

Manuscript received December 2, 2023; revised April 25, 2024. This work was financially supported in part by University Grant Commission with reference no. 1118/(CSIR-UGC NET JUNE 2017).

Reingachan N. is a PhD candidate of the Department of Mathematics, National Institute of Technology Manipur, Langol-795004, India (Corresponding author, phone: +917085223940; e-mail: reinga14@gmail.com).
Robinson Soraisam is a PhD candidate of the Department of Mathematics, National Institute of Technology Manipur, Langol-795004, India (e-mail: soraisam.robinson@gmail.com).
Khangembam Babina Devi is a PhD candidate of the Department of Mathematics, National Institute of Technology Manipur, Langol-795004, India (e-mail: Khangembambabina@gmail.com).
M. Singhajit Singh is a PhD candidate of the Department of Mathematics, National Institute of Technology Manipur, Langol-795004, India (e-mail: msinghasingho@gmail.com).
Barchand Chanam is Professor of the Department of Mathematics, National Institute of Technology Manipur, Langol-795004, India (e-mail: barchand_2004@yahoo.co.in).

$$\max_{|z|=1} |p'(z)| \ge \frac{n}{1+k^n} \max_{|z|=1} |p(z)|. \tag{4}$$

Equality in (4) is satisfied for $p(z) = z^n + k^n, k \ge 1$.
Under the same hypothesis, it was Govil [5] who improved upon (4) by proving

$$\max_{|z|=1} |p'(z)| \ge \frac{n}{1+k^n} \left\{ \max_{|z|=1} |p(z)| + \min_{|z|=k} |p(z)| \right\}. \tag{5}$$

Equality in (5) holds for $p(z) = z^n + k^n, k \ge 1$.
For a better insight into the recent works on polynomial inequalities one can see [2], [9], [11], [12], [13], [14].

Recently, Jain [7] proved an improvement of inequality (4), incorporating the leading coefficient and the constant term of the polynomial by using the generalized form of the classical Schwarz's lemma.

Theorem 1. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then*

$$\max_{|z|=1} |p'(z)| \ge n \max_{|z|=1} |p(z)|, \tag{6}$$

where

$$A_k = \left(\frac{|a_0| + |a_n| k^{n+1}}{|a_o| (1 + k^{n+1}) + |a_n| (k^{n+1} + k^{2n})} \right)$$

II. Lemmas

To establish the proposed theorems, we will rely on the following lemmas. Consider a polynomial $p(z)$ of degree n, and let $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$. The first lemma is credited to Frappier et al. [3].

Lemma 2. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n, then for $R \ge 1$*

$$\max_{|z|=R} |p(z)| \le R^n \max_{|z|=1} |p(z)| - \left(R^n - R^{n-2}\right) |p(0)|, \tag{7}$$

if $n \ge 2$ and

$$\max_{|z|=R} |p(z)| \le R \max_{|z|=1} |p(z)| - (R-1) |p(0)|, \quad \text{if } n = 1. \tag{8}$$

Lemma 3. *Let $f(z)$ be analytic in $|z| < 1$, with $f(0) = a$ and $|f(z)| \le M$, $|z| < 1$. Then*

$$|f(z)| \le M \frac{M|z| + |a|}{|a||z| + M}, \quad |z| < 1. \tag{9}$$

Lemma 3 is a well-known generalization of Schwarz's lemma [15, p.212].

Lemma 4. *Let $f(z)$ be analytic in $|z| \le 1$, with $f(0) = a$ and $|f(z)| \le M$, $|z| \le 1$. Then*

$$|f(z)| \le M \frac{M|z| + |a|}{|a||z| + M}, \quad |z| \le 1. \tag{10}$$

Proof: It follows easily from Lemma 3. ∎

Lemma 5. *If* $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ *is a polynomial of degree* n *having all its zeros in* $|z|\le 1$, *then*

$$|q'(z)|\le |p'(z)| \quad on \quad |z|=1. \tag{11}$$

The above lemma is due to Jain [7].

Lemma 6. *If* $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ *is a polynomial of degree* n, *then*

$$\max_{|z|=1}|p'(z)|+\max_{|z|=1}|q'(z)|\ge n\max_{|z|=1}|p(z)|. \tag{12}$$

The result is due to Govil et al. [6].

Lemma 7. *If* $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ *is a polynomial of degree* n *having all its zeros in* $|z|\le k$, $k>0$, *then for any real or complex number* λ *with* $|\lambda|<1$ *and* $m=\min_{|z|=k}|p(z)|$

$$k^n|a_n|\ge|\lambda|m+|a_0|. \tag{13}$$

Proof: By hypothesis, $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z|\le k$, $k>0$. Then, the polynomial $P(z)=e^{-i\arg a_0}p(z)$ has the same zeros as $p(z)$. Now,

$$\begin{aligned}&P(z)=e^{-i\arg a_0}\\&\times\left\{|a_0|e^{i\arg a_0}+a_1z+\cdots+a_{n-1}z^{n-1}+a_nz^n\right\}\\&=|a_0|+e^{-i\arg a_0}\\&\times\left\{a_1z+\cdots+a_{n-1}z^{n-1}+a_nz^n\right\}.\end{aligned} \tag{14}$$

Now, on $|z|=k$ for any real or complex number λ with $|\lambda|<1$ and $m=\min_{|z|=k}|p(z)|\ne 0$, we have

$$|\lambda|m<m\le|P(z)|.$$

Then by Rouche's theorem, $R(z)=P(z)+|\lambda|m$ has all its zeros in $|z|<k$ and in case $m=0$, $R(z)=P(z)$. Thus, in any case $R(z)$ has all its zeros in $|z|\le k$. Now, applying Vieta's formula to $R(z)$, we get

$$\frac{|a_0|+|\lambda|m}{|a_n|}\le k^n, \tag{15}$$

i.e.

$$k^n|a_n|\ge|\lambda|m+|a_0|. \tag{16}$$

∎

III. Main Result

In this paper, our focus is on the category of polynomials with degree $n\ge 2$ having zero of order s at the origin, where $0\le s\le n-2$ only. For a polynomial of degree 1, expressed as $p(z)=a_0+a_1z$, the evaluation of $\max_{|z|=1}|p(z)|$ becomes straightforward as $|a_0|+|a_1|$, and $\max_{|z|=1}|p'(z)|$ is simply $|a_1|$. In the case where $s=n-1$, the polynomial takes the form $p(z)=a_{n-1}z^{n-1}+a_nz^n$, and consequently, we trivially obtain $\max_{|z|=1}|p(z)|=|a_{n-1}|+|a_n|$ and $\max_{|z|=1}|p'(z)|=(n-1)|a_{n-1}|+n|a_n|$. In both the instances, precise values are known, which eliminates the need for estimations. Our primary objective is to establish the following generalization and enhancement of Theorem 1. More precisely, we prove

Theorem 8. *If* $p(z)=\sum_{\nu=s}^{n}a_\nu z^\nu$, $0\le s\le n-2$, *is a polynomial of degree* $n\ge 2$ *having all its zeros in* $|z|\le k$, $k\ge 1$, *then*

$$\begin{aligned}&\max_{|z|=1}|zp'(z)-sp(z)|\ge(n-s)\\&\times\left(\frac{|a_s|+|a_n|k^{n-s+1}}{|a_s|\left(1+k^{n-s+1}\right)+|a_n|\left(k^{n-s+1}+k^{2n-2s}\right)}\right)\\&\times\max_{|z|=1}|p(z)|\\&+\frac{|a_s|k+|a_n|k^{n-s}}{|a_s|\left(1+k^{n-s+1}\right)+|a_n|\left(k^{n-s+1}+k^{2n-2s}\right)}\\&\times k^{n-s-4}\left(k^4-1\right)|a_{s+1}|,\quad \text{for } s\le n-3\end{aligned} \tag{17}$$

and

$$\begin{aligned}&\max_{|z|=1}|zp'(z)-sp(z)|\ge(n-s)\\&\times\left(\frac{|a_s|+|a_n|k^3}{|a_s|\left(1+k^3\right)+|a_n|\left(k^4+k^3\right)}\right)\max_{|z|=1}|p(z)|\\&+\frac{|a_s|k+|a_n|k^2}{|a_s|\left(1+k^3\right)+|a_n|\left(k^4+k^3\right)}\\&\times(k^2-1)|a_{s+1}|,\quad \text{for } s=n-2.\end{aligned} \tag{18}$$

Remark 9. *Setting* $s=0$ *in Theorem 8, we get the following improvement of Theorem 1 recently proved by Jain [7] as well as inequality* (4), *for polynomials of degree* $n\ge 2$.

Corollary 10. *If* $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ *is a polynomial of degree* $n\ge 2$ *having all its zeros in* $|z|\le k$, $k\ge 1$, *then*

$$\begin{aligned}&\max_{|z|=1}|p'(z)|\ge n\left(\frac{|a_0|+|a_n|k^{n+1}}{|a_0|\left(1+k^{n+1}\right)+|a_n|\left(k^{n+1}+k^{2n}\right)}\right)\\&\times\max_{|z|=1}|p(z)|+\frac{|a_0|k+|a_n|k^n}{|a_0|\left(1+k^{n+1}\right)+|a_n|\left(k^{n+1}+k^{2n}\right)}\\&\times k^{n-4}\left(k^4-1\right)|a_1|,\quad \text{for } n\ge 3\end{aligned} \tag{19}$$

and

$$\begin{aligned}&\max_{|z|=1}|p'(z)|\ge n\left(\frac{|a_0|+|a_2|k^3}{|a_0|\left(1+k^3\right)+|a_2|\left(k^4+k^3\right)}\right)\\&\times\max_{|z|=1}|p(z)|+\left(\frac{|a_0|k+|a_2|k^2}{|a_0|\left(1+k^3\right)+|a_2|\left(k^4+k^3\right)}\right)\\&\times(k^2-1)|a_1|,\quad \text{for } n=2.\end{aligned} \tag{20}$$

Remark 11. *Since* $k\ge 1$, *it immediately follows that Corollary 10 provides an enhanced bound compared to Theorem 1 when* $a_1\ne 0$. *To demonstrate that the bounds of Corollary 10 represent an improvement over* (4), *it suffices to establish that*

$$\frac{|a_0|+|a_n|k^{n+1}}{|a_0|\left(1+k^{n+1}\right)+|a_n|\left(k^{n+1}+k^{2n}\right)}\ge\frac{1}{1+k^n},$$

which is equivalent to

$$|a_n|\left(k^{2n+1}-k^{2n}\right)\ge|a_0|\left(k^{n+1}-k^n\right),$$

that is

$$k^n|a_n|\ge|a_0|,$$

which clearly holds by Lemma 7 with $\lambda = 0$.

Remark 12. *In some cases, the enhancement can be quite significant, as is illustrated through the following examples.*

Example 13. *Consider $p(z) = z^3 + 3z^2 + \frac{11}{4}z + \frac{3}{4}$. Clearly $p(z)$ is a polynomial of degree 3 having all its zeros in $|z| \le \frac{3}{2}$. We take $k = 2$ and find that*

$$\max_{|z|=1} |p(z)| = 7.5.$$

$$\min_{|z|=2} |p(z)| = 0.75.$$

$$\max_{|z|=1} |p'(z)| \ge 2.5, \quad \text{(by (4))}. \tag{21}$$

$$\max_{|z|=1} |p'(z)| \ge 4.06, \quad \text{(by Theorem 1)}. \tag{22}$$

$$\max_{|z|=1} |p'(z)| \ge 6.17, \quad \text{(by (19) of Corollary 10)}. \tag{23}$$

Example 14. *Consider $p(z) = z^2 + 2z + \frac{3}{4}$. Clearly $p(z)$ is a polynomial of degree 2 having all its zeros in $|z| \le \frac{3}{2}$. We take $k = 2$ and find that*

$$\max_{|z|=1} |p(z)| = 3.75.$$

$$\min_{|z|=3} |p(z)| = 0.75.$$

$$\max_{|z|=1} |p'(z)| \ge 1.5, \quad \text{(by (4))}. \tag{24}$$

$$\max_{|z|=1} |p'(z)| \ge 2.13, \quad \text{(by Theorem 1)}. \tag{25}$$

$$\max_{|z|=1} |p'(z)| \ge 3.2, \quad \text{(by (20) of Corollary 10)}. \tag{26}$$

Further, we not only extend Theorem 8 to a more generalized result but also yields an improved result of some known inequalities as particular cases.

Theorem 15. *If $p(z) = \sum_{\nu=s}^{n} a_\nu z^\nu$, $0 \le s \le n-2$, is a polynomial of degree $n \ge 2$ having all its zeros in $|z| \le k$, $k \ge 1$, then for $0 \le l < 1$ and $m = \min_{|z|=k} |p(z)|$*

$$\begin{aligned}
&\max_{|z|=1} |zp'(z) - sp(z)| \ge (n-s) \\
&\times \frac{|a_s|k^s + lm + |a_n|k^{n+1}}{(|a_s|k^s + lm)(1 + k^{n-s+1}) + |a_n|(k^{n+1} + k^{2n-s})} \\
&\times \left\{ \max_{|z|=1} |p(z)| + \frac{lm}{k^s} \right\} \\
&+ \frac{(|a_s|k^s + lm)k + |a_n|k^n}{(|a_s|k^s + lm)(1 + k^{n-s+1}) + |a_n|(k^{n+1} + k^{2n-s})} \\
&\times k^{n-s-4}\left(k^4 - 1\right)|a_{s+1}|, \quad \text{for } s \le n-3
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
&\max_{|z|=1} |zp'(z) - sp(z)| \ge (n-s) \\
&\times \frac{|a_s|k^s + lm + |a_n|k^{n+1}}{(|a_s|k^s + lm)(k^3 + 1) + |a_n|k^{n+1}(k+1)} \\
&\times \left\{ \max_{|z|=1} |p(z)| + \frac{lm}{k^s} \right\} \\
&+ \frac{(|a_s|k^s + lm)k + |a_n|k^n}{(|a_s|k^s + lm)(k^3 + 1) + |a_n|k^{n+1}(k+1)} \\
&\times (k^2 - 1)|a_{s+1}|, \quad \text{for } s = n-2.
\end{aligned} \tag{28}$$

Remark 16. *Putting $l = 0$, Theorem 15 reduces to Theorem 8.*

Remark 17. *Setting $s = 0$ in Theorem 15, we get the following generalization of Corollary 10 as well as improvement of inequality (5) and also a result recently proved by Mir [10, Theorem 2], for polynomials of degree $n \ge 2$.*

Corollary 18. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree $n \ge 2$ having all its zeros in $|z| \le k$, $k \le 1$, then for $0 \le l < 1$ and $m = \min_{|z|=k} |p(z)|$*

$$\begin{aligned}
&\max_{|z|=1} |p'(z)| \ge n \\
&\times \frac{|a_0| + lm + |a_n|k^{n+1}}{(|a_0| + lm)(1 + k^{n+1}) + |a_n|(k^{n+1} + k^{2n})} \\
&\times \left\{ \max_{|z|=1} |p(z)| + lm \right\} \\
&+ \frac{(|a_0| + lm)k + |a_n|k^n}{(|a_0| + lm)(1 + k^{n+1}) + |a_n|(k^{n+1} + k^{2n})} \\
&\times k^{n-4}\left(k^4 - 1\right)|a_1|, \quad \text{for } n \ge 3
\end{aligned} \tag{29}$$

and

$$\begin{aligned}
\max_{|z|=1} |p'(z)| \ &\ge \ n \frac{|a_0| + lm + |a_2|k^3}{(|a_0| + lm)(k^3 + 1) + |a_n|k^3(k+1)} \\
&\times \ \left\{ \max_{|z|=1} |p(z)| + lm \right\} \\
&+ \ \frac{(|a_0| + lm)k + |a_2|k^2}{(|a_0| + lm)(k^3 + 1) + |a_2|k^3(k+1)} \\
&\times \ (k^2 - 1)|a_1|, \quad \text{for } n = 2.
\end{aligned} \tag{30}$$

Remark 19. *Since $k \ge 1$, to show that the inequalities of Corollary 18 are improvements of inequality (5), it is sufficient to verify that*

$$\frac{|a_0| + lm + |a_n|k^{n+1}}{|a_0|(1 + k^{n+1}) + |a_n|(k^{n+1} + k^{2n})} \ge \frac{1}{1 + k^n},$$

which is equivalent to

$$|a_n|\left(k^{2n+1} - k^{2n}\right) \ge (|a_0| + lm)\left(k^{n+1} - k^n\right),$$

that is

$$k^n |a_n| \ge |a_0| + lm,$$

which clearly holds by Lemma 7.

Remark 20. *Similarly, in certain instances, the enhancement is noteworthy, and we demonstrate this with the aid of the previous examples 13 and 14.*

Example 21. *For Example 13 and $k = 2$, we have*

$$\max_{|z|=1} |p'(z)| \ge 2.75, \quad \text{(by (5))}, \tag{31}$$

whereas

$$\max_{|z|=1} |p'(z)| \ge 6.65, \quad \text{(by (29) of Corollary 18 for } l = 1\text{)}. \tag{32}$$

Example 22. *While for Example 14 and $k = 2$, we have*

$$\max_{|z|=1} |p'(z)| \ge 1.8, \quad \text{(by (5))}, \tag{33}$$

whereas

$$\max_{|z|=1}|p'(z)| \geq 3.4, \quad \text{(by (30) of Corollary 18 for } l=1). \tag{34}$$

IV. Proofs of the Theorems

We first prove Theorem 15.

Proof of Theorem 15: Firstly, we shall prove inequality (27).
Since in inequality (27) $s \leq n-3$, our polynomial $p(z) = \sum_{\nu=s}^{n} a_\nu z^\nu$ must be of degree $n \geq 3$. By hypothesis, $p(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$. Now,

$$p(z) = z^s j(z), \tag{35}$$

where

$$j(z) = a_s + a_{s+1}z + \cdots + a_n z^{n-s}, \tag{36}$$

is a polynomial of degree $n-s \geq 3$. Consider a polynomial

$$R(z) = p(z) + \frac{m}{k^s}\lambda z^s, \tag{37}$$

where λ is any real or complex number with $|\lambda| < 1$ and $m = \min_{|z|=k}|p(z)|$.
Suppose $m \neq 0$, then for $|z| = k$

$$|\frac{m}{k^s}\lambda z^s| < m \leq |p(z)|.$$

Then by Rouche's theorem, it follows that $R(z)$ has all its zeros in $|z| < k$ and in case $m = 0$, $R(z) = p(z)$. Thus in any case, $R(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$.
Now,

$$\begin{aligned} R(z) &= \frac{\lambda m}{k^s}z^s + a_s z^s + a_{s+1}z^{s+1} + \cdots + a_n z^n \\ &= z^s h(z), \end{aligned} \tag{38}$$

where

$$h(z) = \frac{\lambda m}{k^s} + a_s + a_{s+1}z + \cdots + a_n z^{n-s}, \tag{39}$$

and

$$g(z) = z^{n-s}\overline{h\left(\frac{1}{\overline{z}}\right)}. \tag{40}$$

From (36) and (39), we have

$$j'(z) = h'(z). \tag{41}$$

We observe that

$$H(z) = h(kz), \tag{42}$$

is a polynomial of degree $n-s \geq 3$ having all its zeros in $|z| \leq 1$ and

$$\begin{aligned} G(z) &= z^{n-s}\overline{H\left(\frac{1}{\overline{z}}\right)} \\ &= k^{n-s}\left(\frac{z}{k}\right)^{n-s}\overline{h\left(\frac{k}{\overline{z}}\right)} \\ &= k^{n-s}g\left(\frac{z}{k}\right), \quad \text{(by (40))}. \end{aligned} \tag{43}$$

By Lemma 5, we have

$$G'(z) \leq H'(z), \quad |z| = 1. \tag{44}$$

Using (44) we can say that a zero z_j, with $|z_j| = 1$ and multiplicity m_j, of $H'(z)$ will also be a zero, with multiplicity ($\geq m_j$), of $G'(z)$, thereby helping us to write

$$H'(z) = \phi(z)H_1(z), \tag{45}$$

$$G'(z) = \phi(z)G_1(z), \tag{46}$$

where

$$\phi(z) = \begin{cases} 1, \\ \prod_{j=1}^{p}(z-z_j)^{m_j}; |z_j| = 1 \ \forall \ j, \end{cases} \tag{47}$$

for $H'(z) \neq 0$ on $|z| = 1$ and $H'(z)$ has certain number of zeros on $|z| = 1$ respectively.
Now,

$$H_1(z) \neq 0, \quad |z| = 1. \tag{48}$$

By (44), (45) and (46), we have

$$G_1(z) \leq H_1(z), \quad |z| = 1. \tag{49}$$

Now as $H(z)$ has all its zeros in $|z| \leq 1$, by Gauss-Lucas theorem, $H'(z)$ will also have all its zeros in $|z| \leq 1$. Therefore by (45), (47) and (48), we can conclude that

$$\psi(z) = \frac{G_1(z)}{H_1(z)} \tag{50}$$

is analytic in $|z| > r$, for certain r, with $0 < r < 1$, including ∞ and accordingly

$$f(z) = \psi\left(\frac{1}{z}\right), \tag{51}$$

with

$$\begin{aligned} f(0) = \psi(\infty) &= \ln_{z\to\infty}\psi(z), \\ &= \ln_{z\to\infty}\frac{G'(z)}{H'(z)}, \\ &\quad \text{(by (50), (45) and (46))}, \\ &= \frac{\overline{\frac{\lambda m}{k^s} + a_s}}{a_n k^{n-s}} \end{aligned} \tag{52}$$

is analytic in $|z| < \frac{1}{r}$, $\frac{1}{r} \geq 1$. Further $|\psi(z)| \leq 1$, $|z| = 1$ by (49) and therefore

$$|f(z)| \leq 1, \quad |z| = 1, \text{ (by (51))}, \tag{53}$$

which by (52) and Lemma 4, help us to write

$$|f(z)| \leq \frac{|z| + \left|\frac{a_s + \frac{\lambda m}{k^s}}{a_n k^{n-s}}\right|}{1 + \left|\frac{a_s + \frac{\lambda m}{k^s}}{a_n k^{n-s}}\right||z|}, \quad |z| \leq 1,$$

i.e.

$$\left|f\left(re^{i\theta}\right)\right| \leq \frac{|a_n|k^n r + |a_s k^s + \lambda m|}{|a_s k^s + \lambda m|r + |a_n|k^n}, \quad r \leq 1 \text{ and } 0 \leq \theta \leq 2\pi, \tag{54}$$

i.e.

$$\left|\psi\left(\frac{1}{r}e^{-i\theta}\right)\right| \leq \frac{|a_n|k^n r + |a_s k^s + \lambda m|}{|a_s k^s + \lambda m|r + |a_n|k^n}, \quad 0 < r \leq 1 \text{ and } 0 \leq \theta \leq 2\pi, \quad \text{(by (51))},$$

i.e.

$$\left|\psi\left(Re^{-i\theta}\right)\right| \leq \frac{|a_n|k^n + |a_s k^s + \lambda m|R}{|a_s k^s + \lambda m| + |a_n|k^n R}, \quad R \geq 1 \text{ and } 0 \leq \theta \leq 2\pi,$$

i.e.

$$\left|G_1\left(Re^{-i\theta}\right)\right| \leq \frac{|a_n|k^n + |a_s k^s + \lambda m|R}{|a_s k^s + \lambda m| + |a_n|k^n R} \times \left|H_1\left(Re^{-i\theta}\right)\right|, \quad R \geq 1, \quad \text{(by (50))},$$

i.e.

$$\left|G'\left(Re^{-i\theta}\right)\right| \leq \frac{|a_n|k^n + |a_s k^s + \lambda m|R}{|a_s k^s + \lambda m| + |a_n|k^n R} \times \left|H'\left(Re^{-i\theta}\right)\right|, R \geq 1, \text{by(45)and(46)},$$

i.e.

$$|G'(z)| \leq \frac{|a_n|k^n + |a_s k^s + \lambda m||z|}{|a_s k^s + \lambda m| + |a_n|k^n|z|} \times |H'(z)|, \quad |z| \geq 1,$$

i.e.

$$k^{n-s-2}\left|g'\left(\frac{z}{k}\right)\right| \leq \frac{|a_n|k^n + |a_s k^s + \lambda m||z|}{|a_s k^s + \lambda m| + |a_n|k^n|z|} |h'(kz)|, \quad |z| \geq 1, \text{(by(42)and(43))}. \tag{55}$$

By taking $z = ke^{i\theta}$ in (55), we get

$$k^{n-s-2}\left|g'\left(e^{i\theta}\right)\right| \leq \frac{|a_n|k^n + |a_s k^s + \lambda m|k}{|a_s k^s + \lambda m| + |a_n|k^{n+1}} \left|h'\left(k^2 e^{i\theta}\right)\right|, \quad 0 \leq \theta \leq 2\pi,$$

which implies

$$k^{n-s-2}\max_{|z|=1}|g'(z)| \leq \frac{|a_n|k^n + |a_s k^s + \lambda m|k}{|a_s k^s + \lambda m| + |a_n|k^{n+1}} \times \max_{|z|=k^2}|h'(z)|. \tag{56}$$

Applying (7) of Lemma 2 to (56), we have

$$k^{n-s-2}\max_{|z|=1}|g'(z)| \leq \frac{|a_n|k^n + |a_s k^s + \lambda m|k}{|a_s k^s + \lambda m| + |a_n|k^{n+1}} \times \left\{k^{2n-2s-2}\max_{|z|=1}|h'(z)| - \left(k^{2n-2s-2} - k^{2n-2s-6}\right)|a_{s+1}|\right\}, \tag{57}$$

i.e.

$$\max_{|z|=1}|g'(z)| \leq \frac{|a_n|k^n + |a_s k^s + \lambda m|k}{|a_s k^s + \lambda m| + |a_n|k^{n+1}} \times \left\{k^{n-s}\max_{|z|=1}|h'(z)| - \left(k^{n-s} - k^{n-s-4}\right)|a_{s+1}|\right\}. \tag{58}$$

By Lemma 6, we have

$$\max_{|z|=1}|g'(z)| + \max_{|z|=1}|h'(z)| \geq (n-s)\max_{|z|=1}|h(z)|,$$

which on using (58), we get

$$\max_{|z|=1}|h'(z)| \geq (n-s) \times \frac{|a_s k^s + \lambda m| + |a_n|k^{n+1}}{|a_s k^s + \lambda m|\left(1 + k^{n-s+1}\right) + |a_n|\left(k^{n+1} + k^{2n-s}\right)} \times \max_{|z|=1}|h(z)| + \frac{|a_s k^s + \lambda m|k + |a_n|k^n}{|a_s k^s + \lambda m|\left(1 + k^{n-s+1}\right) + |a_n|\left(k^{n+1} + k^{2n-s}\right)} \times k^{n-s-4}\left(k^4 - 1\right)|a_{s+1}|. \tag{59}$$

By (38) and (41), we have

$$\max_{|z|=1}|j'(z)| \geq (n-s) \times \frac{|a_s k^s + \lambda m| + |a_n|k^{n+1}}{|a_s k^s + \lambda m|\left(1 + k^{n-s+1}\right) + |a_n|\left(k^{n+1} + k^{2n-s}\right)} \times \max_{|z|=1}|R(z)| + \frac{|a_s k^s + \lambda m|k + |a_n|k^n}{|a_s k^s + \lambda m|\left(1 + k^{n-s+1}\right) + |a_n|\left(k^{n+1} + k^{2n-s}\right)} \times k^{n-s-4}\left(k^4 - 1\right)|a_{s+1}|. \tag{60}$$

Again by (35) and (37), we have

$$\max_{|z|=1}|zp'(z) - sp(z)| \geq (n-s) \times \frac{|a_s k^s + \lambda m| + |a_n|k^{n+1}}{|a_s k^s + \lambda m|\left(1 + k^{n-s+1}\right) + |a_n|\left(k^{n+1} + k^{2n-s}\right)} \times \max_{|z|=1}\left|p(z) + \frac{\lambda m}{k^s}z^s\right| + \frac{|a_s k^s + \lambda m|k + |a_n|k^n}{|a_s k^s + \lambda m|\left(1 + k^{n-s+1}\right) + |a_n|\left(k^{n+1} + k^{2n-s}\right)} \times k^{n-s-4}\left(k^4 - 1\right)|a_{s+1}|. \tag{61}$$

For every real or complex number λ, we have

$$|a_s k^s + \lambda m| \leq |a_s|k^s + |\lambda|m,$$

and since both the functions $\left(\frac{x + |a_n|k^{n+1}}{x(1+k^{n-s+1}) + |a_n|(k^{n+1}+k^{2n-s})}\right)$ and $\left(\frac{xk + |a_n|k^n}{x(1+k^{n-s+1}) + |a_n|(k^{n+1}+k^{2n-s})}\right)$ are decreasing functions of x for $k \geq 1$, it follows from (61) that for every λ with $|\lambda| < 1$ and $|z| = 1$,

$$\max_{|z|=1}|zp'(z) - sp(z)| \geq (n-s) \times \frac{|a_s|k^s + |\lambda|m + |a_n|k^{n+1}}{\left(|a_s|k^s + |\lambda|m\right)\left(1 + k^{n-s+1}\right) + |a_n|\left(k^{n+1} + k^{2n-s}\right)} \times \max_{|z|=1}\left|p(z) + \frac{\lambda m}{k^s}z^s\right| + \frac{\left(|a_s|k^s + |\lambda|m\right)k + |a_n|k^n}{\left(|a_s|k^s + |\lambda|m\right)\left(1 + k^{n-s+1}\right) + |a_n|\left(k^{n+1} + k^{2n-s}\right)} \times k^{n-s-4}\left(k^4 - 1\right)|a_{s+1}|. \tag{62}$$

Suppose z_0 on $|z| = 1$ is such that

$$\max_{|z|=1}|p(z)| = |p(z_0)|. \tag{63}$$

Now,

$$\left|p(z_0) + \frac{\lambda m z_0^s}{k^s}\right| \leq \max_{|z|=1}\left|p(z) + \frac{\lambda m z^s}{k^s}\right|. \tag{64}$$

In the left hand side of inequality (64), for suitable choice of the argument of λ, we have

$$\left|p(z_0) + \frac{\lambda m z_0^s}{k^s}\right| = |p(z_0)| + \frac{|\lambda|m}{k^s}. \tag{65}$$

Using (63) and (65) to (64), we have

$$\max_{|z|=1}|p(z)| + \frac{|\lambda|m}{k^s} \leq \max_{|z|=1}\left|p(z) + \frac{\lambda m}{k^s}z^s\right|. \tag{66}$$

Using (66) to (62), we get

$$\max_{|z|=1}|zp'(z)-sp(z)|\geq(n-s)\times \frac{|a_s|k^s+|\lambda|m+|a_n|k^{n+1}}{(|a_s|k^s+|\lambda|m)(1+k^{n-s+1})+|a_n|(k^{n+1}+k^{2n-s})} \times\left\{\max_{|z|=1}|p(z)|+\frac{|\lambda|m}{k^s}\right\} +\frac{(|a_s|k^s+|\lambda|m)k+|a_n|k^n}{(|a_s|k^s+|\lambda|m)(1+k^{n-s+1})+|a_n|(k^{n+1}+k^{2n-s})} \times k^{n-s-4}\left(k^4-1\right)|a_{s+1}|. \tag{67}$$

Setting $|\lambda|=l$, $0\leq l<1$ in (67) gives (27).
In order to prove inequality (28) where $s=n-2$, our polynomial must be of degree $n\geq 2$ and its proof follows in a similar way as above but using inequality (8) instead of (7) of Lemma 2 to inequality (56).
This completes the proof of Theorem 15. ■

Proof of Theorem 8: The proof of Theorem 8 follows on the same lines as that of Theorem 15, simply by considering the polynomial $p(z)$ of equality (35) in place of polynomial $R(z)$ of equality (37). ■

REFERENCES

[1] A. Aziz and Q. M. Dawood, "Inequalities for a polynomial and its derivatives", *J. Approx. Theory*, vol. 54, pp. 306-313, 1988.
[2] K. B. Devi, K. Krishnadas and B. Chanam, "L^r inequalities for the derivative of a polynomial", *Note Mat.*, vol. 41(2), pp. 19-29, 2021.
[3] C. Frappier, Q. I. Rahman and St. Rucheweyh, "New Inequalities for polynomials", *Trans. Amer. Math. Soc.*, vol. 288, pp. 306-313, 1985.
[4] N. K. Govil, "On the Derivative of a Polynomial", *Proc. Amer. Math. Soc.*, vol. 41, pp. 543-546, 1973.
[5] N. K. Govil, "Some inequalities for the derivative of a polynomial", *J. Approx. Theory*, vol. 66, pp. 29-35, 1991.
[6] N. K. Govil, Q. I. Rahman and G. Schmeisser, "On the derivative of a polynomial", *IIlinois J. Math.*, vol. 23, pp. 319-329, 1979.
[7] V. K. Jain, "On the derivative of a polynomial", *Bull. Math. Soc. Sci. Math. Roumanie Tome*, vol. 59, pp. 69-99, 2016.
[8] M. A. Malik, "On the derivative of a polynomial", *J. Lond. Math. Soc.*, vol. 1(2), pp. 57-60, 1969.
[9] A. Mir, "A note on an inequality of Paul Turán concerning polynomials", *Ramanujan J.*, vol. 56, pp. 1061-1071, 2021.
[10] A. Mir, "A Turán-type inequality for polynomial", *Indian J. Pure Appl. Math.*, 2021. https://doi.org/10.1007/s13226-021-00038-3.
[11] T. B. Singh and B. Chanam, "Generalizations and sharpenings of certain Bernstein and Turán-type of inequalities for the polar derivative of a polynomial", *J. Math. Inequal.*, vol. 15(4), pp. 1663-1675, 2021.
[12] T. B. Singh, K. B. Devi and B. Chanam, "Turán-type integral mean inequalities of the polar derivative of a polynomial", *Palestine J. Math.*, vol. 12(3), pp. 85-93, 2023.
[13] T. B. Singh, R. Soraisam and B. Chanam, "Sharpening of Turán-type inequalities for polar derivative of a polynomial", *Complex Analysis synerg.*, vol. 9(1), pp. 3, 2023.
[14] R. Soraisam, B. Chanam and M. Bidkham, "Turán-type inequalities of polynomials", *Publications de l'Institut Mathematique*, vol. 114(128), pp. 91-101, 2023.
[15] E. C. Titchmarsh, "The theory of functions", *The English Language Book Society and Oxford University Press, London*, 1969.
[16] P. Turan, "Uber die Ableitung von Polynomen", *Compos. Math.*, vol. 7, pp. 89-95, 1939.