A New Hyperchaotic System with Exponential Function Non Linearity: Dynamical Properties, Control Hyperchaos and Complete Synchronization Study

Rayene Abir Meskine, Smail Kaouache, and Orhan Ozgur Aybar

***Abstract*—In this paper, we construct a new $4-D$ hyperchaotic system with a nonlinear term in the form of an exponential function. The system is derived from a modified $3-D$ Lü system. Firstly, we discuss the qualitative properties of the proposed system, using tools such as hyperchaotic attractors, symmetry, dissipation, equilibrium points, and Lyapunov spectrum for analysis. Next, we present an adaptive controller for stability analysis of the system. An active controller is designed to achieve complete synchronization between two identical systems. All stability results are established using Lyapunov stability theory. Finally, numerical examples and computer simulations are provided to illustrate the main results.**

***Index Terms*—Hyperchaotic system; adaptive control; active control; Lyapunov function; sunchronization.**

I. Introduction

CHAOS, a fascinating nonlinear phenomenon, has been the hotbed of groundbreaking developments over the last thirty years. Chaotic systems are inherently nonlinear and exhibit intense sensitivity to their initial conditions, as well as dense periodic orbits[1]. The remarkable sensitivity of chaotic systems is truly impressive, particularly when we observe a positive Lyapunov exponent, which quantifies the dynamic nature of the system. It is mesmerising that even small changes can lead to significantly different outcomes, demonstrating the intricate elegance of chaos theory.

A hyperchaotic system is a truly remarkable entity, typically characterised as a chaotic system with at least two positive Lyapunov exponents. This unique feature endows hyperchaotic systems with unparalleled capabilities in terms of capacity, security, and efficiency. As a result of these exceptional attributes, hyperchaotic systems have become prominent contenders for critical applications in various fields, including cryptosystems [2], secure communications [3] and encryption [4].

Since the first hyperchaos was reported by Rossler in 1979 [5], many $4-D$ hyperchaotic systems have been reported in the literature such as hyperchaotic Lorenz system [6], hyperchaotic Lü system [7], hyperchaotic Chen system [8] and hyperchaotic Wang system [9], etc.

Recently, various feedback control methods have been applied to stabilize or regulate the outputs of chaotic systems. The prevalence of chaotic systems in various industries and fields drives the need for effective control methods. These control methods play a crucial role in maintaining stability and predictability in dynamic and unpredictable environments. Examples include active control [10], adaptive control [11], modified adaptive control [12], observe-rbased control [13], impulsive control [14], H_∞ control [15], quantized H_∞ control [16] and finite-time control [17], etc.

In the field of control theory, adaptive control stands out as a widely employed technique for stabilizing systems in situations where the system parameters are not known [18], [19], [20]. On the other hand, active control methods, is applied when the parameters are accessible and measurable [21], [22], [23].

Since the revolutionary work of Pecora and Carroll [24], the concept of chaos synchronization in chaotic systems has garnered significant attention within the academic and scientific communities. This intriguing phenomenon has captivated researchers across diverse fields such as cryptosystems [25], encryption [26] and secure communication protocols [27], [28], etc.

The realm of chaos literature explores a diverse range of methodologies for synchronizing chaotic systems. These approaches aim to establish coherence and coordination among inherently unpredictable systems, facilitating a deeper understanding of complex dynamical behaviors, such as complete synchronization [29], anti-synchronization [30], generalized synchronization [31], $Q-S$ synchronization [32], projective synchronization [33], generalized projective synchronization [34], modified projective synchronization [35], hybrid projective synchronization [36], inverse matrix projective synchronization [37], Finite-time synchronization [38], and combination synchronization [39], [40], [41] etc.

This work presents a novel $4-D$ hyperchaotic system derived from the modified $3-D$ Lü system[7]. The system exhibits exponential function non-linearity. We conduct a comprehensive analysis of the system's qualitative properties, stability, and feasibility for complete synchronization. This in-depth exploration sheds light on the intricate characteristics and dynamics of this complex system. The manuscript is organized as follows: Section II introduces a novel hyperchaotic system with exponential function non-linearity and

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analyzes its basic dynamic properties. Section III presents an adaptive controller designed to stabilize the proposed system with unknown parameters. Section IV develops an active controller for synchronizing two identical hyperchaotic systems. The feasibility and effectiveness of both control schemes are verified by numerical simulations. Finally, Section V concludes the research.

II. A New Hyperchaotic Lü System with Exponential Function Non Linearity

The $3-D$ Lü system has been introduced in [7] and mathematically modeled by the following three dimensional differential system

$$\begin{cases} \dot{x_1} = a(x_2 - x_1), \\ \dot{x_2} = -x_1x_3 + bx_2, \\ \dot{x_3} = x_1x_2 - cx_3, \end{cases} \tag{1}$$

where x_1, x_2, x_3 are the states and $a; b; c$ assumed to be positive constant parameters.

When $a = 36$, $b = 3$ and $c = 20$, the Lü system has a typically critical chaotic attractor [7].

It is well known that to generate hyperchaos from dissipatively autonomous systems, the state equation must satisfy the following conditions:

(1) The dimension of the state equation is at least 4.

(2) The system has at least two positive Lyapunov exponents and the sum of all Lyapunov exponents is less than zero.

Based on above two basic conditions, our new hyperchaotic system generated from the modified Lü system [7] is given by

$$\begin{cases} \dot{x_1} = a(x_2 - x_1) + x_4, \\ \dot{x_2} = -x_1x_3 + bx_2, \\ \dot{x_3} = -1 + \exp(x_1x_2) - cx_3, \\ \dot{x_4} = dx_1, \end{cases} \tag{2}$$

where x_1, x_2, x_3, x_4 are the state variables, a, b, c and d are positive real. In addition, we demonstrate that the system (2) exhibits hyperchaotic behavior when the parameters are chosen as

$$(a,\ b,\ d) = (25,\ 20,\ 8) \text{ and } c \in [0.5, 2.5], \tag{3}$$

For the numerical simulations, we use the fourth order Runge-Kutta algorithm. In particular, when the initial values of the hyperchaotic system (2) are selected as

$$x_1(0) = x_2(0) = x_3(0) = x_4(0) = 0.001 \tag{4}$$

and the parameter values are taken as

$$(a, b, c, d) = (25,\ 20, 1.78,\ 8), \tag{5}$$

the hyperchaotic attractors of the system (2) are represented in Fig. 1 in various coordinate planes.

The system under consideration demonstrates the following dynamical properties:

A. Symmetry

The system (2) remains invariant under the function: $(x_1,\ x_2,\ x_3,\ x_4) \ \to (-x_1,\ -x_2,\ x_3,\ -x_4)$. In addition, this system is symmetric with respect to the x_3 - axis.

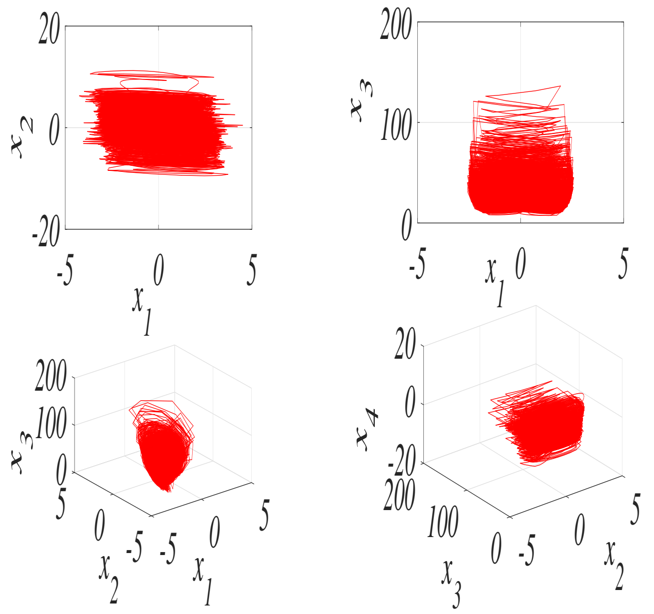

Fig. 1. Different attractors of the hyperchaotic system (2).

B. Dissipativity

The system (2) can be written in its vector form as

$$\dot{x} = f(x) = \begin{pmatrix} f_1(x_1, x_2, x_3, x_4) \\ f_2(x_1, x_2, x_3, x_4) \\ f_3(x_1, x_2, x_3, x_4) \\ f_4(x_1, x_2, x_3, x_4)) \end{pmatrix},$$

where $x(t) = (x_1,\ x_2,\ x_3,\ x_4)$ and

$$\begin{cases} f_1(x_1,\ x_2,\ x_3,\ x_4) = a(x_2 - x_1) + x_4, \\ f_2(x_1,\ x_2,\ x_3,\ x_4) = -x_1x_3 + bx_2, \\ f_3(x_1,\ x_2,\ x_3,\ x_4) = -1 + \exp(x_1x_2) - cx_3, \\ f_4(x_1,\ x_2,\ x_3,\ x_4) = dx_1. \end{cases}$$

The divergence of the $4-D$ system (2) is given by

$$\begin{aligned} \nabla f &= \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} + \frac{\partial f_4}{\partial x_4} \\ &= -a + b - c. \end{aligned} \tag{6}$$

By using Liouville's theorem, we obtain

$$V(t) = V_0 \exp((-a + b + c)t) \tag{7}$$

If $\nabla f < 0$, the system (2) will be dissipative. In this case, all the trajectories of the system tend to an attractor, when $t \to +\infty$.

C. Equilibrium Points and Stability

The equilibrium points of the system (2) are obtained by solving the following system of equations

$$\begin{cases} a(x_2 - x_1) + x_4 = 0 \\ -x_1x_3 + bx_2 = 0 \\ -1 + \exp(x_1x_2) - cx_3 = 0 \\ dx_1 = 0, \end{cases}$$

It is easy to see that the system (2) has only one equilibrium point $P(0, 0, 0, 0)$.

The Jacobian matrix at this point is

$$\begin{pmatrix} -a & a & 0 & 1 \\ 0 & b & 0 & 0 \\ 0 & 0 & -c & 0 \\ d & 0 & 0 & 0 \end{pmatrix}, \tag{8}$$

and its characteristic polynomial is given by

$$P(\lambda) = (\lambda - b)(\lambda + c)\left(\lambda^2 + a\lambda - d\right). \tag{9}$$

The eigenvalues corresponding to the equilibrium point P are

$$\begin{aligned}\lambda_1 &= -c,\ \lambda_2 = -\frac{a}{2} - \frac{1}{2}\sqrt{4d + a^2},\ \lambda_3 = b,\\ \text{and } \lambda_4 &= -\frac{a}{2} + \frac{1}{2}\sqrt{4d + a^2}. \end{aligned}\tag{10}$$

Here, λ_1 and λ_2 are negative real numbers, λ_3 and λ_4 are positive real numbers.
Hence, the equilibrium point P is an unstable saddle point.

D. Lyapunov Exponents

In this section, we assume that the parameters a, b, d remain constant and c is varied in $[0.5, 4.5]$. By using Wolf algorithm [42], the Lyapunov exponents spectrum of system (2) with $a = 25$, $b = 20$ and $d = 8$ is represented in Fig. 2.

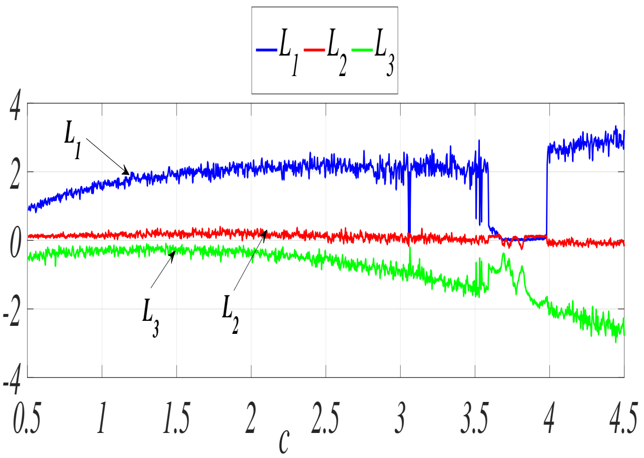

Fig. 2. The three largest lyapunov exponents of the system (2) versus c.

In particular, for the parameter values are taken as in case (5), the values of Lyapunov exponents of non-linear system (2) are given by

$$\begin{cases} L_1 = 1.90,\\ L_2 = 0.39,\\ L_3 = 0,\\ L_4 = -9.312. \end{cases}\tag{11}$$

Since $L_1 + L_2 + L_3 + L_4 < 0$ and $L_1 > 0$, $L_1 > 0$, then the proposed system is dissipative and hyperchaotic.

The Kaplan-Yorke dimension of the hyperchaotic system is obtained as:

$$\begin{aligned} D_{KY} &= 3 + \frac{L_1 + L_2 + L_3}{|L_4|}\\ &= 3.25, \end{aligned}$$

which is a fractal dimension.

III. Control Hyperchaos

In this section, we construct an adaptive control law in the objective to stabilize the proposedsystem with unknown system parameters. The main adaptive control result is established via Lyapunov stability theory.

Consider the controlled hyperchaotic system as follows

$$\begin{cases} \dot{x_1} = a(x_2 - x_1) + x_4 + u_1,\\ \dot{x_2} = -x_1 x_3 + b x_2 + u_2,\\ \dot{x_3} = -1 + \exp(x_1 x_2) - c x_3 + u_3,\\ \dot{x_4} = d x_1 + u_4, \end{cases}\tag{12}$$

where x_1, x_2, x_3, x_4 are the state variables, a, b, c, d are unknown constant parameters and u_i, $i = 1, 2, 3, 4$ is adaptive control law to be determined using estimates $\hat{a}(t)$, $\hat{b}(t)$, $\hat{c}(t)$ and $\hat{d}(t)$ of the unknown parameters a, b, c and d, respectively.

Hypothesis 1: Consider the following conditions
1. Assume that the adaptive controllers u_i, $i = 1, 2, 3, 4$ are taken by

$$\begin{cases} u_1 = -\hat{a}(x_2 - x_1) - x_4 - k_1 x_1,\\ u_2 = x_2^2\\ u_3 = 1 - \exp(x_1 x_2) + \hat{c} x_3 - k_3 x_3,\\ u_4 = -\hat{d} x_1 - k_4 x_4, \end{cases}\tag{13}$$

where k_1, k_2, k_3, k_4 are positive gain constants, and $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ are estimates of the unknown parameters a, b, c, d, respectively.
2. Assume that the parameter update law are chosen as

$$\begin{cases} \dot{\hat{a}} = x_1 x_2 - x_1^2,\\ \dot{\hat{b}} = x_2^2,\\ \dot{\hat{c}} = -x_3^2,\\ \dot{\hat{d}} = x_1 x_4, \end{cases}\tag{14}$$

then, we have the following result.

Theorem 1: The controlled hyperchaotic system (12) is globally and exponentially stabilized under the adaptive control law (13) and the parameter update law (14).

Proof: Substituting (13) into (12), we obtain

$$\begin{cases} \dot{x_1} = e_a(x_2 - x_1) - k_1 x_1,\\ \dot{x_2} = e_b x_2 - k_2 x_2,\\ \dot{x_3} = -e_c x_3 - k_3 x_3,\\ \dot{x_4} = e_d x_1 - k_4 x_4, \end{cases}\tag{15}$$

where e_a, e_b, e_c, e_d are the parameter estimation errors defined by

$$\begin{cases} e_a = a - \hat{a},\\ e_b = b - \hat{b},\\ e_c = c - \hat{c},\\ e_d = d - \hat{d}. \end{cases}\tag{16}$$

Differentiating (16) with respect to t, we obtain

$$\begin{cases} \dot{e}_a = -\dot{\hat{a}},\\ \dot{e}_b = -\dot{\hat{b}},\\ \dot{e}_c = -\dot{\hat{c}},\\ \dot{e}_d = -\dot{\hat{d}}. \end{cases}$$

Consider the quadratic Lyapunov function defined by

$$V = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2 + x_4^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2\right), \tag{17}$$

which is positive definite on $\mathbb{R}^8$.
Differentiating (17) along the trajectories of (15) and (16), we get

$$\begin{aligned} \dot{V} &= -k_1 x_1^2 - k_2 x_2^2 - k_3 x_3^2 - k_4 x_4^2\\ &\quad + e_a\left(x_1 x_2 - x_1^2 - \dot{a}\right) + e_b\left(x_2^2 - \dot{b}\right)\\ &\quad + e_c\left(-x_3^2 - c\right) + e_d\left(x_1 x_4 - \dot{d}\right). \end{aligned}\tag{18}$$

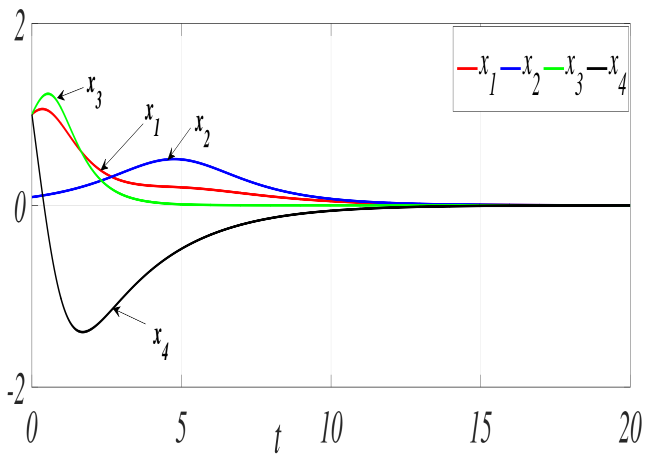

Fig. 3. Time-history of the controlled hyperchaotic system (12).

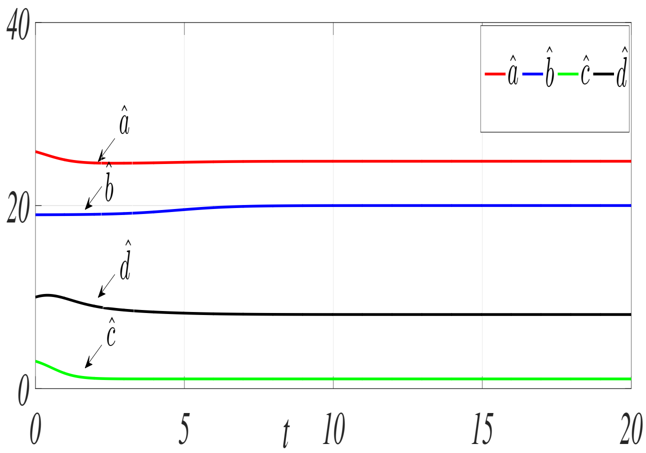

Fig. 4. Temporal evolution of the parameter estimates (14).

Substituting (14) into (18), we obtain

$$\dot{V} = -k_1 x_1^2 - k_2 x_2^2 - k_3 x_3^2 - k_4 x_4^2, \tag{19}$$

which is a negative semi-definite function on $\mathbb{R}^8$.

We define $k = min(k_1, k_2, k_3, k_4)$. Then it follows from (19) that

$$\begin{aligned} \dot{V} &= -k_1 x_1^2 - k_2 x_2^2 - k_3 x_3^2 - k_4 x_4^2 \\ &\le -k x_1^2 - k x_2^2 - k x_3^2 - k x_4^2 \\ &= -k \left\| x \right\|^2, \end{aligned} \tag{20}$$

where $x = (x_1, x_2, x_3, x_4)$.
From (20), it follows that

$$\left\| x \right\|^2 \le \dot{V}. \tag{21}$$

Integrating the inequality (21) from 0 to t, we get

$$\int_0^t \left\| x(s) \right\|^2 ds \le V(0) - V(t). \tag{22}$$

From (22), it follows that $x(t) \in L_2$, while from (15), it can be deduced that $\dot{x}(t) \in L_\infty$.

Thus, by Barbalat's lemma [43], we conclude that $x(t) \to 0$ exponentially, as $t \to \infty$, for all initial conditions $x(0) \in \mathbb{R}^4$. This completes the proof.

To validate the theoretical findings mentioned above, we implement the proposed adaptive control law (13) to stabilize the unstable equilibrium point $P(0, 0, 0, 0)$.

The system of differential equations (12) and (14), along with the control law (13), are solved numerically using a Matlab code based on the classical fourth-order Runge-Kutta method with a step size of $h = 0.02$.

For numerical simulations, the parameter values of the new hyperchaotic system are taken as in the hyperchaotic case (5). We take the gain constants as

$$k_i = 0.5 \text{ for all } i = 1, 2, 3, 4.$$

The initial conditions of the system are taken as

$$(x_1(0),\ x_2(0),\ x_3(0),\ x_4(0)) = (1,\ 0.09,\ 1,\ 1)$$

Additionally, regarding the initial conditions for the parameter estimates, we select

$$\left(\hat{a}(0),\ \hat{b}(0),\ \hat{b}(0),\ \hat{c}(0)\right) = (25.9,\ 19,\ 3,\ 10)\,.$$

Fig. 3 describes the time-history of the controlled states x_1, x_2, x_3, x_4 of the system (12), whereas Fig. 4 shows the time-history of the parameter estimates (14). ■

IV. Complete Synchronization Study

This section focuses on achieving synchronization between two hyperchaotic systems using an active control approach. The effectiveness of the control strategy is rigorously established through the application of Lyapunov stability theory.

A. General Method of Synchronization

Consider a class of master-slave hyperchaotic systems described by

$$\dot{x}(t) = Ax(t) + f(x(t)), \tag{23}$$

$$\dot{y}(t) = Ay(t) + g(y(t)) + u(t), \tag{24}$$

where x, $y \in \mathbb{R}^n$ are state variables of the master system and the slave system, respectively, $A \in \mathbb{R}^{n\times n}$, is the linear part of the proposed systems , $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are non linear functions and $u = (u_1, u_2, ..., u_n)^T \in \mathbb{R}^n$ is a control input vector.

The error between the master system (23) and the slave system (24) can be defined as

$$e(t) = y\,(t) - x\,(t)\,. \tag{25}$$

The aim in this section is that, for different initial conditions of systems (23) and (24), the two systems can be synchronized by designing a suitable active control u, such that

$$\lim_{t\to\infty} ||e(t)|| = 0. \tag{26}$$

We have the following result.

Theorem 2: The synchronization between the master system (23) and the slave system (24) can be achieved if the following conditions are satisfied

$$u(t) = \Theta e(t) - g(y(t)) + f(x(t)), \tag{27}$$

where Θ is an unknown control matrix, which is selected such that $(A + \Theta)^T + (A + \Theta)$ is a negative definite matrix.

Proof: The error dynamical system can be derived as

$$\begin{aligned} \dot{e}(t) &= \dot{y}(t) - \dot{x}(t) \\ &= Ae(t) + g(y(t)) - f(x(t)) + u(t), \end{aligned} \tag{28}$$

Substituting (27) into Eq. (28), we get

$$\dot{e}(t) = (A + \Theta)\, e(t). \tag{29}$$

Consider the quadratic Lyapunov function defined by

$$V(t) = e^T(t)e(t), \tag{30}$$

which is positive definite on $\mathbb{R}^n$.
Differentiating V along the trajectories of (23) and (24), we get

$$\begin{aligned} \dot{V}(t) &= \dot{e}^T(t)e(t) + e^T(t)\dot{e}(t) \\ &= e^T(t)(A+\Theta)^T e(t) + e^T(t)(L+\Theta)e(t) \\ &= e^T(t)\left((A+\Theta)^T + (L+\Theta)\right)e(t) \\ &< 0. \end{aligned}$$

Therefore, by stability result of Lyapunov, we can conclude that all solutions of error system (29) tend towards zero exponentially as $t \to \infty$. Hence, the complete synchronization between the identical hyperchaotic systems (23) and (24) is achieved under the condition (27). This completes the proof. ■

B. Numerical Example and Simulation Results

To verify the effectiveness and the feasibility of the presented synchronization method, we take the new hypechaotic system as a master system and its controlled system as a slave system. The master system is defined by

$$\begin{cases} \dot{x_1} = a(x_2 - x_1) + x_4, \\ \dot{x_2} = -x_1x_3 + bx_2, \\ \dot{x_3} = -1 + \exp(x_1x_2) - cx_3, \\ \dot{x_4} = dx_1, \end{cases} \tag{31}$$

The slave system is described by

$$\begin{cases} \dot{y_1} = a(y_2 - y_1) + y_4 + u_1, \\ \dot{y_2} = -y_1y_3 + by_2 + u_2, \\ \dot{y_3} = -1 + \exp(y_1y_2) - cy_3 + u_3, \\ \dot{y_4} = dy_1 + u_4. \end{cases} \tag{32}$$

where u_1, u_2, u_3, u_4 are the active control functions.
The linear part of the systems (31) and (32) is given by

$$A = \begin{pmatrix} -a & a & 0 & 1 \\ 0 & b & 0 & 0 \\ 0 & 0 & -c & 0 \\ d & 0 & 0 & 0 \end{pmatrix}.$$

According to the synchronization technique proposed in the previous section, we define the error states as

$$\begin{cases} e_1 = y_1 - x_1, \\ e_2 = y_2 - x_2, \\ e_3 = y_3 - x_3, \\ e_4 = y_4 - x_4. \end{cases}$$

To ensure the stability condition, we choose the feedback gain matrix Θ as

$$\Theta = \begin{pmatrix} 0 & -\frac{3a}{2} & 0 & -1 \\ -\frac{a}{2} & -b-a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -d & 0 & 0 & -a \end{pmatrix}. \tag{33}$$

From the condition (27) of the Theorem2, the vector controller $u = (u_1, u_2, u_3, u_4)$ can be constructed as follows

$$\begin{cases} u_1 = -\frac{3a}{2}e_2 - e_4, \\ u_2 = -\frac{a}{2}e_1 - (b+a)e_2 + y_1y_3 - x_1x_3, \\ u_3 = -\exp(y_1y_2) + \exp(x_1x_2), \\ u_4 = -de_1 - ae_4. \end{cases} \tag{34}$$

The resulting error dynamics can be expressed as

$$\begin{cases} \dot{e_1} = -ae_1 - \frac{a}{2}e_2, \\ \dot{e_2} = -\frac{a}{2}e_1 - ae_2, \\ \dot{e_3} = -ce_3, \\ \dot{e_4} = -ae_4. \end{cases} \tag{35}$$

The linear part of the systems (35) is given by

$$(A+\theta) = \begin{pmatrix} -a & -\frac{a}{2} & 0 & 0 \\ -\frac{a}{2} & -a & 0 & 0 \\ 0 & 0 & -c & 0 \\ 0 & 0 & 0 & -a \end{pmatrix}$$

Thus

$$(A+\theta)^T + (A+\theta) = \begin{pmatrix} -2a & -a & 0 & 0 \\ -a & -2a & 0 & 0 \\ 0 & 0 & -2c & 0 \\ 0 & 0 & 0 & -2a \end{pmatrix}$$

It is easy to see that the symetic matrix $(A+\theta)^T + (A+\theta)$ is defined negative, therefore the condition of the Theorem2 is satisfied, which ensures the stability of the error system (35).

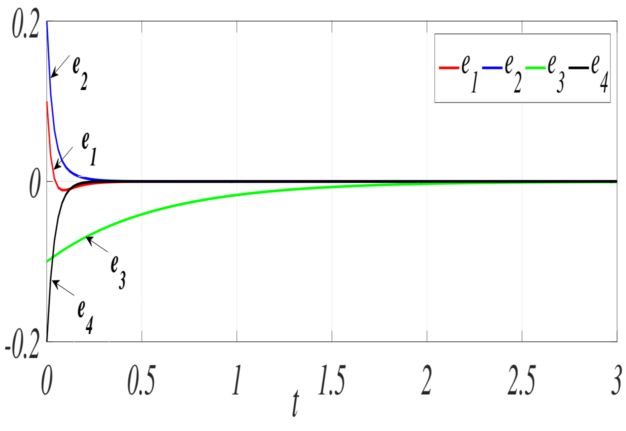

Fig. 5. Time-history of the synchronization error (35).

For the numerical simulations, the parameter values of the master and slave systems are taken as

$$(a, b, c, d) = (25,\ 20, 1.78,\ 8),$$

the initial states of the master system are taken as

$$x_1(0) = 0.1, x_2(0) = 0.1, x_3(0) = 0.1 \text{ and } x_4(0) = 0.1, \tag{36}$$

the initial states of the slave system are taken as

$$y_1(0) = 0.2, y_2(0) = 0.3, y_3(0) = 0 \text{ and } y_4(0) = -0.1, \tag{37}$$

.
thus, the error system has the initial states

$$e_1(0) = 0.1, e_2(0) = 0.2, e_3(0) = -0.1 \text{ and } e_4(0) = -0.2. \quad (38)$$

Fig. 5 describes the time-history of the synchronization error (35).
From Fig. 5, we show that the evolution of all variables of error dynamic system (35) quickly tend towards zero, which indicate that the synchronization between the hyperchaotic systems (31) and (32) is achieved.

V. Conclusion

In the chaos literature, there is a great importance in the modelling and wide applications of hyperchaotic systems. This paper introduces a new hyperchaotic system with a non-linear term in the form of an exponential function and a single unstable equilibrium point. The system's qualitative characteristics are meticulously investigated. An adaptive control algorithm is designed to regulate the hyperchaotic behavior. Furthermore, an active control approach guarantees the asymptotic stability of synchronization errors between two identical systems. This convergence implies that the follower system's trajectories asymptotically approach those of the leader system. Numerical simulations validate the effectiveness of the proposed control methods in achieving the desired outcomes for both the controlled system and its corresponding error model. Future work will explore potential applications of this novel hyperchaotic system, particularly its suitability for secure communication protocols.

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