Nonparametric Estimation for Stochastic Processes Driven by Alpha-stable Processes of the Second Kind

Shanlin Wan, Zhiqiang Jiang, and Xuekang Zhang*

Abstract—**The present paper deals with the problem of nonparametric estimation for stochastic processes driven by alpha-stable processes of the second kind. The consistency and the asymptotic distribution of the nonparametric estimator are discussed.**

Index Terms—**nonparametric estimation, alpha-stable processes, second kind, consistency, asymptotic distribution**

I. Introduction

In recent years, many results on the asymptotic theory of statistical inference for Gaussian stochastic processes of the second kind have been studied (see, e.g., Azmoodeh and Morlanes [1]; Azmoodeh and Viitasaari [2]; Balde et al. [3]; El Onsy et al. [4]; El Onsy et al. [5]). Among others, some significant works include the following. Alazemi et al. [6] established the consistency and the asymptotic distributions of least square-type estimators for Gaussian mean-reverting Ornstein-Uhlenbeck processes of the second kind.

In the real word, many natural phenomena exhibit random behavior with a non-Gaussian α-stable distribution. For example, Mikosch et al. [7] demonstrated that when connection rates are modest relative to heavy tailed connection length distribution tails, α-stable Lévy motion is a sensible approximation to cumulative broadband network traffic over a time period. Nolan [8] found that monthly exchange rates between the US Dollar and the Tanzanian Shilling from January 1975 to September 1997 follow an α-stable distribution. Xu et al. [9] observed that asymmetric leptokurtic features presented in the Shanghai Composite Index and Shenzhen Component Index returns can be captured by an α-stable law. Hence, it is more reasonable to replace the driving Gaussian process by the α-stable Lévy process for such phenomena. Recently, Yu et al. [10] studied the consistency and the asymptotic distributions of the trajectory fitting estimators for Ornstein-Uhlenbeck processes driven by α-stable Lévy processes of the second kind.

It is easy to see that the above-mentioned literatures primarily focus on examining scenarios where the drift function in stochastic differential equations (SDEs) is known. But in reality, the drift function is seldom known. The drift function can be estimated by using a nonparametric smoothing approach. The method offers a versatile way to explore the relationship between variables without specifying prior models. The asymptotic theory of nonparametric estimation of the trend for stochastic processes with Gaussian noises is well developed. Kutoyants [11] investigated the consistency and asymptotic normality of nonparametric estimators for SDEs with small white noises. Mishra and Prakasa Rao [12] discussed the mean square consistency and asymptotic normality of the kernel type estimator for SDEs driven by fractional Brownian motion. Prakasa Rao [13] dealt with the problem of nonparametric estimation of trend coefficient in models governed by a SDE driven by a mixed fractional Brownian motion with small noise. Prakasa Rao [14] investigated the asymptotic behaviour of nonparametric estimator for SDEs driven by sub-fractional Brownian motion.

However, there are few papers concerned with nonparametric estimation of the trend for non-Gaussian SDEs (see, e.g., Zhang et al. [15], [16]). Motivated by the aforementioned works, in this paper, we consider the following stochastic process $X = \{X_t, 0 \le t \le T\}$ driven by α-stable processes of the second kind

$$dX_t = A_t(X)dt + \varepsilon\sigma_t(X)dU_t^{(1)}, \quad 0 \le t \le T, \tag{1}$$

where $X_0 = x_0$, $\varepsilon \in (0, 1)$, the function $A(\cdot)$ is an unknown nonanticipative smooth measurable function, $\sigma(\cdot)$ is a known bounded measurable function, $U_t^{(1)} := \int_0^t e^{-s} dZ_{a_s}$ with $a_s = \frac{1}{\alpha} e^{\alpha s}$, and $\{Z_t, 0 \le t \le T\}$ is a standard α-stable process $(1 < \alpha < 2)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a right continuous and increasing family of σ-algebras $(\mathcal{F}_t, t \ge 0)$. In this case, Z_1 has an α-stable distribution $S_\alpha(1, \beta, 0)$, where $\beta \in [-1, 1]$ is the skewness parameter of the distribution. Suppose $\{x_t, 0 \le t \le T\}$ is the solution of the following differential equation

$$\frac{dx_t}{dt} = A_t(x), \quad x_0, \quad 0 \le t \le T, \tag{2}$$

where x_0 is the initial value. We would like to estimate the function $A_t = A_t(x)$ based on the observation $\{X_t, 0 \le t \le T\}$. Following techniques in Kutoyants [11], we define a kernel type estimator of the trend function A_t as

$$\widehat{A}_t = \frac{1}{\varphi_\varepsilon} \int_0^T G\left(\frac{\tau - t}{\varphi_\varepsilon}\right) dX_\tau,$$

where $G(\cdot)$ is a bounded function of finite support, and the normalizing function $\varphi_\varepsilon \to 0$ with $\varepsilon \varphi_\varepsilon^{-\frac{\alpha-1}{\alpha}} \to 0$ as $\varepsilon \to 0$.

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This paper is organized as follows. In Section 2, some preliminaries on α-stable distribution, stable stochastic integrals and, related moment inequalities are given. In Section 3, the consistency and bound of the rate of convergence of nonparametric estimator are discussed. In Section 4, the asymptotic distribution of the estimator is obtained.

II. PRELIMINARIES

Throughout the paper, we shall use the notation "$\to_p$" to denote"convergence in probability" and the notation "$\stackrel{d}{=}$" to denote equality in distribution. We shall use ϕ_+ and ϕ_- to denote the positive and negative part of ϕ, respectively. We denote $L^\alpha_{a.s.}$ the family of all real-valued $(\mathcal{F}_t)$-predictable processes ϕ on $\Omega \times [0,\infty)$ such that for every $T > 0$, $\int_0^T |\phi(t,\omega)|^\alpha dt < \infty$ a.s. Then by Theorem 4.1 of Rosinski and Woyczynski [17] and Theorem 3.1 of Kallenberg [18], a predictable process ϕ is integrable with respect to a strictly α-stable Lévy process Z, that is, $\int_0^T \phi(t)dZ_t$ exists for every $T > 0$ if and only if $\phi \in L^\alpha_{a.s.}$. Denote $(\mathcal{D}_T, \mathcal{B}_T)$ as the measurable space of right continuous with left limits on $[0,T]$ functions $\{x_t, 0 \le t \le T\}$ with the σ-algebra $\mathcal{B}_T = \sigma\{x_t, 0 \le t \le T\}$. Let $\Theta_k(L)$ denote the class of all functions $\{g_t, 0 \le t \le T\}$ which are k-times differentiable with respect to t satisfying the Hölder condition of the order $\gamma \in (0,1]$

$$|g_t^{(k)} - g_s^{(k)}| \le L|t-s|^\gamma,\ t,s \in [0,T], \tag{3}$$

for some constants $L > 0$. Here $g_t^{(k)}$ denotes the k-th derivative of $g(\cdot)$ at t for $k \ge 0$. If $k = 0$, we interpret $g^{(0)}$ as g.

Definition 2.1: A scalar random variable η is stable if there exist four real parameters, i.e., a stability parameter $\alpha \in [0,2]$, a scaling parameter $\sigma > 0$, a symmetry parameter $\beta \in [-1,1]$ and a location parameter $\mu \in (-\infty,\infty)$, such that its characteristic function $\phi_\eta(u)$ has the following form:
(i) $\alpha \ne 1$:

$$\phi_\eta(u) = \exp\left\{-\sigma^\alpha |u|^\alpha \left(1 - i\beta \text{sgn}(u)\tan\frac{\alpha\pi}{2}\right) + i\mu u\right\};$$

(ii) $\alpha = 1$:

$$\phi_\eta(u) = \exp\left\{-\sigma|u|\left(1 + i\beta\frac{2}{\pi}\text{sgn}(u)\log|u|\right) + i\mu u\right\}.$$

We denote $\eta \sim S_\alpha(\sigma,\beta,\mu)$. When $\mu = 0$, we say η is strictly α-stable. If in addition $\beta = 0$, we call η symmetric α-stable. Note that η is strictly 1-stable ($\alpha = 1$) if and only if $\beta = 0$ (symmetric case). We refer to Janicki and Weron [19], Samorodnitsky and Taqqu [20], and Sato [21] for more details on α-stable distributions.

Definition 2.2: An $\mathcal{F}_t$-adapted stochastic process $\{Z_t\}_{t\ge 0}$ is called a standard α-stable Lévy motion if
(i) $Z_0 = 0$, a.s.;
(ii) $Z_t - Z_s \sim S_\alpha((t-s)^{\frac{1}{\alpha}}, \beta, 0), t > s \ge 0$;
(iii) For any finite time points $0 \le s_0 < s_1 < \cdots < s_m < \infty$, the random variables $Z_{s_0}, Z_{s_1} - Z_{s_0}, \ldots, Z_{s_m} - Z_{s_{m-1}}$ are independent.

The following lemma appeared in Kutoyants [11] is useful.

Lemma 2.1: (Kutoyants [11], Lemma 1.11) Let c_0, c_1, c_2 be nonnegative constants, $u(t), v(t)$ be a nonnegative bounded function $0 \le t \le T$ and

$$u(t) \le c_0 + c_1\int_0^t v(s)u(s)ds$$
$$+ c_2\int_0^t v(s)\left[\int_0^s u(r)dK(r)\right]ds,$$

where $K(s)$ is a nondecreasing right-continuous function, $0 \le K(t) \le K_0$, then

$$u(t) \le c_0 \exp\left\{(c_1 + c_2K_0)\int_0^t v(s)ds\right\}.$$

An important lemma is the inner clock property for α-stable stochastic integrals, which comes from Long [22].

Lemma 2.2: Let Z be a strictly α-stable process and $\phi \in L^\alpha_{a.s.}$. Then,
(i) There exist some independent processes $Z', Z'' \stackrel{d}{=} Z$, such that

$$\int_0^t \phi(s)dZ_s = Z' \circ \int_0^t \phi_+^\alpha(s)ds - Z'' \circ \int_0^t \phi_-^\alpha(s)ds, \quad \text{a.s.}$$

(ii) If Z is symmetric, that is, $\beta = 0$, then, there exists some α-stable Lévy process $Z' \stackrel{d}{=} Z$, such that

$$\int_0^t \phi(s)dZ_s = Z' \circ \int_0^t |\phi(s)|^\alpha ds, \quad \text{a.s.}$$

The following lemma is a direct consequence of Lemma 2.4 and Remark 2.5 in Long [22], which will be a very powerful tool for proofs of our main results.

Lemma 2.3: Let $\phi(t)$ be a predictable process satisfying $\int_0^T |\phi(t)|^\alpha dt < \infty$ almost surely for $T < \infty$, and $F : [0,\infty) \to [0,\infty)$ be a continuous function. We assume that (i) either ϕ is nonnegative or Z is a strictly α-stable Lévy motion, and (ii) there exist positive constants λ_0, C and $\alpha_0 < \alpha$ such that $F(\lambda\nu) \le C\lambda^{\alpha_0}F(\nu)$ for all $\nu > 0$ and all $\lambda \ge \lambda_0$. Then there exist positive constants C_1 and C_2 depending only on $\alpha, \alpha_0, \beta, C$, and λ_0 such that for each $T > 0$

$$C_1\mathbb{E}\left[F\left(\left(\int_0^T |\phi(t)|^\alpha dt\right)^{\frac{1}{\alpha}}\right)\right]$$
$$\le \mathbb{E}\left[F\left(\sup_{t\le T}\left|\int_0^t \phi(s)dZ_s\right|\right)\right]$$
$$\le 2C_2\mathbb{E}\left[F\left(\left(\int_0^T |\phi(t)|^\alpha dt\right)^{\frac{1}{\alpha}}\right)\right].$$

We will make use of the following assumptions:
$(H1)$ There exist positive constants L_1 and L_2 such that

$$|A_t(x) - A_t(y)| + |\sigma_t(x) - \sigma_t(y)|$$
$$\le L_1\int_0^t |x_s - y_s|dK(s) + L_2|x_t - y_t|,$$
$$|A_t(x)| + |\sigma_t(x)|$$
$$\le L_1\int_0^t (1 + |x_s|)dK(s) + L_2(1 + |x_t|),$$

where $K(\cdot)$ is given in Lemma 2.1, $x_t, y_t \in \mathcal{D}_T$, $t \in [0,T]$.
$(H2)$ The dispersion function $\sigma(\cdot)$ satisfies the following bounded condition: there exists a positive constant $\sigma_1 > 0$ such that $0 < |\sigma_t(x)| \le \sigma_1$ for each $x_t \in \mathcal{D}_T$, $t \in [0,T]$.
$(H3)$ Let $G(u), u \in \mathbb{R}$ be a bounded function of finite support (there exist two constants $B_1 < 0$ and $B_2 > 0$ such

that $G(u) = 0$ for $u \notin [B_1, B_2]$) and $\int_{B_1}^{B_2} G(u)du = 1$. In addition,

$$\int_{-\infty}^{\infty} |G(u)|du < \infty, \ \int_{-\infty}^{\infty} |G(u)|^{\alpha} du < \infty,$$
$$\int_{-\infty}^{\infty} |G(u)u^{\gamma}|du < \infty.$$

$(H4)$ The kernel function $G(\cdot)$ satisfies the following condition

$$\int_{-\infty}^{\infty} |G(u)u^{k+\gamma}|du < \infty,$$
$$\int_{-\infty}^{\infty} u^j G(u)du = 0, \ j = 1, 2, \dots, k.$$

Similar to the discussion of Theorem 4.6 in Liptser and Shiryayev [23], it is not difficult to show that SDE (1) admits a unique non-explosive càdlàg adapted solution under condition $(H1)$.

III. Consistency of the kernel estimator $\widehat{A}_t$

In the section, the consistency of the kernel estimator $\widehat{A}_t$ is discussed. We first present a key lemma.

Lemma 3.1: Under assumptions $(H1)$-$(H2)$, we have

$$|X_t - x_t| \leq \varepsilon e^{(L_1 K_0 + L_2)t} \sup_{0 \leq s \leq t} \left| \int_0^s \sigma_u(X) dU_u^{(1)} \right|, \quad (4)$$

$$\sup_{0 \leq t \leq T} |X_t - x_t| \to_p 0 \quad \text{as} \quad \varepsilon \to 0, \quad (5)$$

and

$$\sup_{0 \leq t \leq T} \mathbb{E}[|X_t - x_t|] \leq 2\varepsilon e^{(L_1 K_0 + L_2)T} C_2 \sigma_1 T^{\frac{1}{\alpha}}. \quad (6)$$

Proof: (i) Denote $u(t) = |X_t - x_t|$, then by Assumption $(H1)$, we obtain

$$\begin{aligned} u(t) &\leq \int_0^t |A_s(X) - A_s(x)|ds + \varepsilon \left| \int_0^t \sigma_s(X) dZ_s \right| \\ &\leq L_1 \int_0^t \left[\int_0^s u(r)dr \right] ds + L_2 \int_0^t u(s)ds \\ &\quad + \varepsilon \sup_{0 \leq s \leq t} \left| \int_0^s \sigma_u(X) dU_u^{(1)} \right|. \end{aligned}$$

Therefore, equation (4) follows from Lemma 2.1 with $c_0 = e^{(L_1 K_0 + L_2)t} \sup_{0 \leq s \leq t} \left| \int_0^s \sigma_u(X) dZ_u \right|$, $c_1 = L_2$, and $c_2 = L_1$.

(ii) It follows from (4) that

$$\sup_{0 \leq t \leq T} |X_t - x_t| \leq \varepsilon e^{(L_1 K_0 + L_2)T} \sup_{0 \leq t \leq T} \left| \int_0^t \sigma_s(X) dU_s^{(1)} \right|.$$

By Assumption $(H2)$, Markov inequality, and Lemma 2.3, we have, for any given $\delta > 0$,

$$\begin{aligned} &\mathbb{P}\left(\varepsilon e^{(L_1 K_0 + L_2)T} \left| \int_0^t \sigma_s(X) dU_s^{(1)} \right| > \delta \right) \\ &\leq \delta^{-1} e^{(L_1 K_0 + L_2)T} \varepsilon \mathbb{E}\left[\sup_{0 \leq t \leq T} \left| \int_0^t \sigma_s(X) dU_s^{(1)} \right| \right] \\ &= \delta^{-1} e^{(L_1 K_0 + L_2)T} \varepsilon \mathbb{E}\left[\sup_{0 \leq t \leq T} \left| \int_{a_0}^{a_t} \sigma_{\frac{\ln \alpha s}{\alpha}}(X)(\alpha s)^{\frac{-1}{\alpha}} dZ_s \right| \right] \\ &\leq 2C_2 \delta^{-1} e^{(L_1 K_0 + L_2)T} \varepsilon \mathbb{E}\left[\left(\int_{a_0}^{a_T} |\sigma_{\frac{\ln \alpha s}{\alpha}}(X)|^{\alpha} \frac{1}{\alpha s} ds \right)^{\frac{1}{\alpha}} \right] \\ &\leq 2C_2 \delta^{-1} e^{(L_1 K_0 + L_2)T} \varepsilon \sigma_1 T^{\frac{1}{\alpha}}, \end{aligned}$$

which tends to zero as $\varepsilon \to 0$. This implies that (5) holds.

(iii) By (4), Assumption $(H2)$, and Lemma 2.3, we obtain

$$\begin{aligned} &\sup_{0 \leq t \leq T} \mathbb{E}[|X_t - x_t|] \\ &\leq \varepsilon e^{(L_1 K_0 + L_2)T} \mathbb{E}\left[\sup_{0 \leq t \leq T} \left| \int_0^t \sigma_s(X) dU_s^{(1)} \right| \right] \\ &= \varepsilon e^{(L_1 K_0 + L_2)T} \mathbb{E}\left[\sup_{0 \leq t \leq T} \left| \int_{a_0}^{a_t} \sigma_{\frac{\ln \alpha s}{\alpha}}(X)(\alpha s)^{-\frac{1}{\alpha}} dZ_s \right| \right] \\ &\leq 2\varepsilon e^{(L_1 K_0 + L_2)T} C_2 \left(\int_{a_0}^{a_T} |\sigma_{\frac{\ln \alpha s}{\alpha}}(X)|^{\alpha} (\alpha s)^{-1} ds \right)^{\frac{1}{\alpha}} \\ &\leq 2\varepsilon e^{(L_1 K_0 + L_2)T} C_2 \sigma_1 T^{\frac{1}{\alpha}}. \end{aligned}$$

This completes the proof. ■

Next, we state some results about the consistency of the estimator $\widehat{A}_t$.

Theorem 3.1: Suppose that the trend function $A_t \in \Theta_0(L)$ and Assumptions $(H1)$-$(H3)$ hold. Then, for any $0 < c \leq d < T$, the estimator $\widehat{A}_t$ is uniformly consistent, that is,

$$\lim_{\varepsilon \to 0} \sup_{c \leq t \leq d} \mathbb{E}[|\widehat{A}_t - A_t(x)|] = 0. \quad (7)$$

Proof: Note that

$$\begin{aligned} &\mathbb{E}[|\widehat{A}_t - A_t(x)|] \\ &= \mathbb{E}\left[\left| \frac{1}{\varphi_\varepsilon} \int_0^T G\left(\frac{\tau - t}{\varphi_\varepsilon} \right) (A_\tau(X) - A_\tau(x)) d\tau \right. \right. \\ &\quad + \frac{1}{\varphi_\varepsilon} \int_0^T G\left(\frac{\tau - t}{\varphi_\varepsilon} \right) A_\tau(x) d\tau - A_t(x) \\ &\quad \left. \left. + \frac{\varepsilon}{\varphi_\varepsilon} \int_0^T G\left(\frac{\tau - t}{\varphi_\varepsilon} \right) \sigma_\tau(X) dU_\tau^{(1)} \right| \right] \\ &\leq \mathbb{E}\left[\left| \frac{1}{\varphi_\varepsilon} \int_0^T G\left(\frac{\tau - t}{\varphi_\varepsilon} \right) (A_\tau(X) - A_\tau(x)) d\tau \right| \right] \\ &\quad + \mathbb{E}\left[\left| \frac{1}{\varphi_\varepsilon} \int_0^T G\left(\frac{\tau - t}{\varphi_\varepsilon} \right) A_\tau(x) d\tau - A_t(x) \right| \right] \\ &\quad + \mathbb{E}\left[\left| \frac{\varepsilon}{\varphi_\varepsilon} \int_0^T G\left(\frac{\tau - t}{\varphi_\varepsilon} \right) \sigma_\tau(X) dU_\tau^{(1)} \right| \right] \\ &:= Q_1(\varepsilon) + Q_2(\varepsilon) + Q_3(\varepsilon). \end{aligned} \quad (8)$$

Utilizing the change of variables $u = (\tau - t)\varphi_\varepsilon^{-1}$ and denoting

$$\varepsilon_1 = \varepsilon' \wedge \varepsilon'', \quad (9)$$

where

$$\varepsilon' = \sup\left\{ \varepsilon : \frac{-t}{\varphi_\varepsilon} \leq B_1 \right\}, \ \varepsilon'' = \sup\left\{ \varepsilon : \frac{T - t}{\varphi_\varepsilon} \geq B_2 \right\}.$$

For $Q_1(\varepsilon)$, by Assumptions $(H1)$ and $(H3)$, and (6) in Lemma 3.1, it follows that for $\varepsilon < \varepsilon_1$

$$\begin{aligned} &Q_1(\varepsilon) \\ &= \mathbb{E}\left[\left| \int_{\frac{-t}{\varphi_\varepsilon}}^{\frac{T-t}{\varphi_\varepsilon}} G(u)(A_{t+\varphi_\varepsilon u}(X) - A_{t+\varphi_\varepsilon u}(x)) du \right| \right] \end{aligned}$$

$$\leq \mathbb{E}\left[\int_{B_1}^{B_2} |G(u)(A_{t+\varphi_\varepsilon u}(X) - A_{t+\varphi_\varepsilon u}(x))|\, du\right]$$
$$\leq \mathbb{E}\int_{B_1}^{B_2} |G(u)| \left(L_1 \int_0^{t+\varphi_\varepsilon u} |X_s - x_s| dK(s) + L_2 |X_{t+\varphi_\varepsilon u} - x_{t+\varphi_\varepsilon u}|\right) du$$
$$\leq \int_{B_1}^{B_2} |G(u)| du \left(L_1 \int_0^T 1 dK(s) + L_2\right) \cdot \sup_{0\leq t+\varphi_\varepsilon u \leq T} \mathbb{E}[|X_{t+\varphi_\varepsilon u} - x_{t+\varphi_\varepsilon u}|]$$
$$\leq 2\varepsilon(L_1K_0 + L_2)C_2\sigma_1 T^{\frac{1}{\alpha}} e^{(L_1K_0+L_2)T} \cdot \int_{B_1}^{B_2} |G(u)| du, \quad (10)$$

which tends to zero as $\varepsilon \to 0$. For $Q_2(\varepsilon)$, using $A_t(x) \in \Theta_0(L)$, and Assumption $(H3)$, one can obtain that for $\varepsilon < \varepsilon_1$

$$Q_2(\varepsilon) = \mathbb{E}\left|\int_{\frac{-t}{\varphi_\varepsilon}}^{\frac{T-t}{\varphi_\varepsilon}} G(u) A_{\varphi_\varepsilon u + t}(x) d\tau - A_t(x)\right|$$
$$\leq \left|\int_{B_1}^{B_2} G(u)(A_{t+\varphi_\varepsilon u}(x) - A_t(x)) du\right|$$
$$\leq \int_{B_1}^{B_2} |G(u)(A_{t+\varphi_\varepsilon u}(x) - A_t(x))|\, du$$
$$\leq L\varphi_\varepsilon^\gamma \int_{-\infty}^{\infty} |G(u) u^\gamma| du, \quad (11)$$

which tends to zero as $\varepsilon \to 0$. For $Q_3(\varepsilon)$, applying Lemma 2.3, Assumptions $(H2)$ and $(H3)$, one sees that

$$Q_3(\varepsilon) = \frac{\varepsilon}{\varphi_\varepsilon}\mathbb{E}\left|\int_0^T G\left(\frac{\tau - t}{\varphi_\varepsilon}\right) \sigma_\tau(X) dU_\tau^{(1)}\right|$$
$$= \frac{\varepsilon}{\varphi_\varepsilon}\mathbb{E}\left|\int_{a_0}^{a_T} G\left(\frac{\frac{\ln \alpha s}{\alpha} - t}{\varphi_\varepsilon}\right) \sigma_{\frac{\ln \alpha s}{\alpha}}(X)(\alpha s)^{-\frac{1}{\alpha}} dZ_s\right|$$
$$\leq \frac{\varepsilon 2C_2}{\varphi_\varepsilon}\mathbb{E}\left[\int_{a_0}^{a_T} \left|G\left(\frac{\frac{\ln \alpha s}{\alpha} - t}{\varphi_\varepsilon}\right) \sigma_{\frac{\ln \alpha s}{\alpha}}(X)(\alpha s)^{\frac{-1}{\alpha}}\right|^\alpha ds\right]^{\frac{1}{\alpha}}$$
$$\leq 2C_2\sigma_1 \left(\varepsilon\varphi_\varepsilon^{-\frac{\alpha-1}{\alpha}}\right) \left(\int_{-\infty}^{\infty} |G(u)|^\alpha du\right)^{\frac{1}{\alpha}}, \quad (12)$$

which tends to zero as $\varepsilon\varphi_\varepsilon^{-\frac{\alpha-1}{\alpha}} \to 0$. Therefore, by combining (8)-(12), we can conclude that (7) holds. This completes the proof. ∎

Finally, we give a bound on the rate of convergence of the estimator $\widehat{S}_t(x)$.

Theorem 3.2: Let the trend function $A_t(x) \in \Theta_k(L)$, and $\varphi_\varepsilon = \varepsilon^{\frac{\alpha}{\alpha(k+\gamma+1)-1}}$. Then, under Assumptions $(H1)$-$(H4)$, we have

$$\limsup_{\varepsilon\to 0} \sup_{A_t\in\Theta_k(L)} \sup_{c\leq t\leq d} \varepsilon^{-\frac{\alpha(k+\gamma)}{\alpha(k+\gamma+1)-1}} \mathbb{E}[|\widehat{A}_t - A_t(x)|] < \infty. \quad (13)$$

Proof: Applying the Taylor's formula, we get for any $x \in \mathbb{R}$,

$$A(y) = A(x) + \sum_{j=1}^k A^{(j)}(x)\frac{(y-x)^j}{j!} + [A^{(k)}(x + \theta(y-x)) - A^{(k)}(x)]\frac{(y-x)^k}{k!}, \quad (14)$$

where $\theta \in (0,1)$. Combining (11) with (14) gives that for $\varepsilon < \varepsilon_1$

$$Q_2(\varepsilon) \leq \left|\int_{B_1}^{B_2} G(u)(A_{t+\varphi_\varepsilon u}(x) - A_t(x)) du\right|$$
$$= \left|\sum_{j=1}^k A^{(j)}(x_t) \left(\int_{B_1}^{B_2} G(u) u^j du\right) \varphi_\varepsilon^j (j!)^{-1} + \frac{\varphi_\varepsilon^k}{k!}\int_{B_1}^{B_2} G(u) u^k \left(A_{t+\theta\varphi_\varepsilon u}^{(k)}(x) - A_t^{(k)}(x)\right) du\right|.$$

Combining $A_t \in \Theta_{k+1}(L)$ with condition $(H4)$, one has

$$Q_2(\varepsilon) \leq \frac{\varphi_\varepsilon^k}{k!}\left|\int_{B_1}^{B_2} G(u)u^k \left(A^{(k)}(x_{t+\theta\varphi_\varepsilon u}) - A^{(k)}(x_t)\right) du\right|$$
$$\leq \frac{\varphi_\varepsilon^k}{(k!)}\int_{B_1}^{B_2} \left|G(u)u^k \left(A_{t+\theta\varphi_\varepsilon u}^{(k)}(x) - A_t^{(k)}(x)\right)\right| du$$
$$\leq L\theta^\gamma \frac{\varphi_\varepsilon^{k+\gamma}}{(k!)}\int_{B_1}^{B_2} |G(u)u^{k+\gamma}| du. \quad (15)$$

By (8), (10), (12) and (15), we get

$$\sup_{c\leq t\leq d} \mathbb{E}|\widehat{A}_t - A_t(x)| \leq \vartheta_1\varepsilon + \vartheta_2\varphi_\varepsilon^{(k+\gamma)} + \vartheta_3\varepsilon\varphi_\varepsilon^{\frac{1-\alpha}{\alpha}}$$

with some positive constants $\vartheta_1, \vartheta_2, \vartheta_3$ which do not depend on function $A(\cdot)$. So letting $\varphi_\varepsilon = \varepsilon^{\frac{\alpha}{\alpha(k+\gamma+1)-1}}$, we can conclude that (13) holds. ∎

Remark 3.1: If $\varphi_\varepsilon = \varepsilon^{\frac{\alpha}{\alpha(\gamma+1)-1}}$, then under Assumptions (H1)-(H4), we obtain

$$\limsup_{\varepsilon\to 0} \sup_{c\leq t\leq d} \mathbb{E}|\widehat{A}_t - A_t(x)|\varepsilon^{-\frac{\alpha\gamma}{\alpha(\gamma+1)-1}} < \infty.$$

IV. Asymptotic distribution of the estimator $\widehat{A}_t$

In this section, the asymptotic distribution of the estimator $\widehat{A}_t$ is studied. Throughout, U_1 and U_2 are two independent random variables with α-stable distribution $S_\alpha(1,\beta,0)$.

Theorem 4.1: Suppose that the trend function $A_t \in \Theta_{k+1}(L)$, $\varphi_\varepsilon = \varepsilon^{\frac{\alpha}{\alpha(k+\gamma+1)-1}}$, and Assumptions $(H1)$-$(H4)$ hold.

(i) When $\gamma \in (0,1)$ and $\varepsilon \to 0$, we have

$$\varepsilon^{-\frac{\alpha(k+\gamma)}{\alpha(k+\gamma+1)-1}}(\widehat{A}_t - A_t(x)) \Rightarrow \left(\int_{B_1}^{B_2} (G(u)\sigma_t(x))_+^\alpha du\right)^{\frac{1}{\alpha}} U_1 - \left(\int_{B_1}^{B_2} (G(u)\sigma_t(x)))_-^\alpha du\right)^{\frac{1}{\alpha}} U_2. \quad (16)$$

(ii) When $\gamma = 1$ and $\varepsilon \to 0$, we have

$$\varepsilon^{-\frac{\alpha(k+1)}{\alpha(k+2)-1}}(\widehat{A}_t - A_t(x)) - \frac{A_t^{(k+1)}(x)}{(k+1)!}\int_{B_1}^{B_2} G(u)u^{k+1} du \Rightarrow \left(\int_{B_1}^{B_2} (G(u)\sigma_t(x))_+^\alpha du\right)^{\frac{1}{\alpha}} U_1$$

$$-\left(\int_{B_1}^{B_2}(G(u)\sigma_t(x)))_-^\alpha du\right)^{\frac{1}{\alpha}}U_2. \quad (17)$$

Proof: From (1), we have

$$\varepsilon^{-\frac{\alpha(k+\gamma)}{\alpha(k+\gamma+1)-1}}(\widehat{A}_t - A_t(x))$$
$$= \varphi_\varepsilon^{-(k+\gamma)}\int_{\frac{-t}{\varphi_\varepsilon}}^{\frac{T-t}{\varphi_\varepsilon}} G(u)(A_{t+\varphi_\varepsilon u}(X) - A_{t+\varphi_\varepsilon u}(x))du$$
$$+ \varphi_\varepsilon^{-(k+\gamma)}\left(\int_{\frac{-t}{\varphi_\varepsilon}}^{\frac{T-t}{\varphi_\varepsilon}} G(u)A_{t+\varphi_\varepsilon u}(x)du - A_t(x)\right)$$
$$+ \varphi_\varepsilon^{-\frac{1}{\alpha}}\int_0^T G\left(\frac{\tau-t}{\varphi_\varepsilon}\right)(\sigma_\tau(X)-\sigma_\tau(x))dU_\tau^{(1)}$$
$$+ \varphi_\varepsilon^{-\frac{1}{\alpha}}\int_0^T G\left(\frac{\tau-t}{\varphi_\varepsilon}\right)\sigma_\tau(x)dU_\tau^{(1)}$$
$$:= I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon) + I_4(\varepsilon).$$

For $I_1(\varepsilon)$, by hypothesis $(H1)$ and (4) in Lemma 3.1, we get that for $\varepsilon < \varepsilon_1$

$$|I_1(\varepsilon)|$$
$$\le \varphi_\varepsilon^{-(k+\gamma)}\left|\int_{B_1}^{B_2} G(u)(A_{t+\varphi_\varepsilon u}(X) - A_{t+\varphi_\varepsilon u}(x))du\right|$$
$$\le \varphi_\varepsilon^{-(k+\gamma)}\int_{B_1}^{B_2}|G(u)|\Big(L_1\int_0^{t+\varphi_\varepsilon u}|X_s - x_s|dK(s)$$
$$+L_2|X_{t+\varphi_\varepsilon u} - x_{t+\varphi_\varepsilon u}|\Big)du$$
$$\le \varepsilon^{\frac{\alpha-1}{\alpha(k+\gamma+1)-1}}e^{(L_1K_0+L_2)T}\int_{B_1}^{B_2}|G(u)|$$
$$\le \varepsilon^{\frac{\alpha-1}{\alpha(k+\gamma+1)-1}}(L_2+L_1K_0)e^{(L_1K_0+L_2)T}$$
$$\cdot\int_{B_1}^{B_2}|G(u)|\left(\sup_{0\le t+\varphi_\varepsilon u\le T}\int_0^{t+\varphi_\varepsilon u}|\sigma_s(X)|dU_s^{(1)}\right)du,$$

where ε_1 is given in (9). By Markov inequality, Lemma 2.3, and hypothesis $(H2)$, we have, for any given $\delta > 0$,

$$\mathbb{P}(|I_1(\varepsilon)| > \delta)$$
$$\le \delta^{-1}\varepsilon^{\frac{\alpha-1}{\alpha(k+\gamma+1)-1}}(L_2+L_1K_0)e^{(L_1K_0+L_2)T}$$
$$\cdot\int_{B_1}^{B_2}|G(u)|\mathbb{E}\left[\sup_{0\le t+\varphi_\varepsilon u\le T}\int_0^{t+\varphi_\varepsilon u}|\sigma_s(X)|dU_s^{(1)}\right]du$$
$$= \delta^{-1}\varepsilon^{\frac{\alpha-1}{\alpha(k+\gamma+1)-1}}(L_2+L_1K_0)e^{(L_1K_0+L_2)T}\int_{B_1}^{B_2}|G(u)|$$
$$\cdot\mathbb{E}\left[\sup_{0\le t+\varphi_\varepsilon u\le T}\int_{a_0}^{a_{t+\varphi_\varepsilon u}}|\sigma_{\frac{\ln\alpha s}{\alpha}}(X)|(\alpha s)^{-\frac{1}{\alpha}}dZ_s\right]du$$
$$\le \delta^{-1}\varepsilon^{\frac{\alpha-1}{\alpha(k+\gamma+1)-1}}(L_2+L_1K_0)e^{(L_1K_0+L_2)T}2C_2$$
$$\cdot\left[\left(\int_{a_0}^{a_T}|\sigma_{\frac{\ln\alpha s}{\alpha}}(X)|^\alpha(\alpha s)^{-1}ds\right)^{\frac{1}{\alpha}}\right]\int_{B_1}^{B_2}|G(u)|du$$
$$\le \delta^{-1}\varepsilon^{\frac{\alpha-1}{\alpha(k+\gamma+1)-1}}(L_2+L_1K_0)e^{(L_1K_0+L_2)T}T^{\frac{1}{\alpha}}$$
$$\cdot\sigma_1 2C_2\int_{B_1}^{B_2}|G(u)|du,$$

which tends to zero as $\varepsilon \to 0$. This implies that

$$I_1(\varepsilon) \to_p 0, \quad (18)$$

as $\varepsilon \to 0$. By the Taylor's formula, we have, for any $x \in \mathbb{R}$,

$$A(y) = A(x) + \sum_{j=1}^{k+1}A^{(j)}(x)\frac{(y-x)^j}{j!}$$
$$+ [A^{(k+1)}(x+\theta(y-x)) - A^{(k+1)}(x)]$$
$$\cdot\frac{(y-x)^{k+1}}{(k+1)!},$$

where $\theta \in (0,1)$. For $I_2(\varepsilon)$, by Assumptions $(H4)$, we get that for $\varepsilon < \varepsilon_1$

$$I_2(\varepsilon)$$
$$= \varphi_\varepsilon^{-(k+\gamma)}\int_{B_1}^{B_2}G(u)\left(A_{t+\varphi_\varepsilon u}(x) - A_t(x)\right)du$$
$$= \varphi_\varepsilon^{-(k+\gamma)}\Bigg[\sum_{j=1}^{k+1}A^{(j)}(x_t)\left(\int_{B_1}^{B_2}G(u)u^j du\right)\varphi_\varepsilon^j(j!)^{-1}$$
$$+\frac{\varphi_\varepsilon^{k+1}}{(k+1)!}\int_{B_1}^{B_2}G(u)u^{k+1}\left(A_{t+\theta\varphi_\varepsilon u}^{(k+1)}(x) - A_t^{(k+1)}(x)\right)du\Bigg]$$
$$= \frac{\varphi_\varepsilon^{1-\gamma}A_t^{(k+1)}}{(k+1)!}\int_{B_1}^{B_2}G(u)u^{k+1}du + \frac{\varphi_\varepsilon^{1-\gamma}}{(k+1)!}\int_{B_1}^{B_2}G(u)u^{k+1}$$
$$\cdot\left(A_{t+\theta\varphi_\varepsilon u}^{(k+1)}(x) - A_t^{(k+1)}(x)\right)du. \quad (19)$$

It follows from $A_t \in \Theta_{k+1}(L)$ that

$$\frac{\varphi_\varepsilon^{1-\gamma}}{(k+1)!}\int_{B_1}^{B_2}G(u)u^{k+1}\left(A_{t+\theta\varphi_\varepsilon u}^{(k+1)}(x) - A_t^{(k+1)}(x)\right)du$$
$$\le \frac{\varphi_\varepsilon^{1-\gamma}}{(k+1)!}\int_{B_1}^{B_2}\Big|G(u)u^{k+1}$$
$$\cdot\left(A_{t+\theta\varphi_\varepsilon u}^{(k+1)}(x) - A_t^{(k+1)}(x)\right)\Big|du$$
$$\le \frac{L\varphi_\varepsilon\theta^\gamma}{(k+1)!}\int_{B_1}^{B_2}\left|G(u)u^{k+\gamma+1}\right|du, \quad (20)$$

which tends to zero as $\varepsilon \to 0$. Equations (19) and (20) imply that

$$I_2(\varepsilon) \to \begin{cases} 0, & \text{if } \gamma \in (0,1), \\ \dfrac{A_t^{(k+1)}(x)}{(k+1)!}\displaystyle\int_{B_1}^{B_2}G(u)u^{k+1}du, & \text{if } \gamma = 1, \end{cases} \quad (21)$$

as $\varepsilon \to 0$. For $I_3(\varepsilon)$, by Markov inequality, Assumptions $(H1)$ and $(H3)$, and (5) in Lemma 3.1, we find that, for given $\delta > 0$ and $\varepsilon < \varepsilon_1$

$$\mathbb{P}(|I_3(\varepsilon)| > \delta)$$
$$\le \delta^{-1}\varphi_\varepsilon^{-\frac{1}{\alpha}}\mathbb{E}\Bigg|\int_0^T G\left(\frac{\tau-t}{\varphi_\varepsilon}\right)$$
$$(\sigma_\tau(X)-\sigma_\tau(x))dU_\tau^{(1)}\Bigg|$$
$$= \delta^{-1}\varphi_\varepsilon^{-\frac{1}{\alpha}}\mathbb{E}\Bigg|\int_{a_0}^{a_T}G\left(\frac{\frac{\ln\alpha\tau}{\alpha}-t}{\varphi_\varepsilon}\right)$$
$$\cdot(\sigma_{\frac{\ln\alpha\tau}{\alpha}}(X) - \sigma_{\frac{\ln\alpha\tau}{\alpha}}(x))(\alpha\tau)^{-\frac{1}{\alpha}}dZ_\tau\Bigg|$$
$$\le \delta^{-1}2C_2\varphi_\varepsilon^{-\frac{1}{\alpha}}\mathbb{E}\Bigg(\int_{a_0}^{a_T}\Bigg|G\left(\frac{\frac{\ln\alpha\tau}{\alpha}-t}{\varphi_\varepsilon}\right)$$
$$\cdot\ (\sigma_{\frac{\ln\alpha\tau}{\alpha}}(X) - \sigma_{\frac{\ln\alpha\tau}{\alpha}}(x))\Bigg|^\alpha(\alpha\tau)^{-1}d\tau\Bigg)^{\frac{1}{\alpha}}$$
$$\le \delta^{-1}2C_2\varphi_\varepsilon^{-\frac{1}{\alpha}}\Bigg(\int_{a_0}^{a_T}\Bigg|G\left(\frac{\frac{\ln\alpha\tau}{\alpha}-t}{\varphi_\varepsilon}\right)\Bigg|^\alpha$$
$$\cdot\Bigg(L_1\int_0^{\frac{\ln\alpha\tau}{\alpha}}|X_s - x_s|dK(s)$$

$$+L_2|X_{\frac{\ln \alpha\tau}{\alpha}} - x_{\frac{\ln \alpha\tau}{\alpha}}|\Big)^\alpha (\alpha\tau)^{-1}d\tau\Big)^{\frac{1}{\alpha}}$$
$$\le \delta^{-1}2C_2\left(\int_{\frac{-t}{\varphi_\varepsilon}}^{\frac{T-t}{\varphi_\varepsilon}}|G(u)|^\alpha\right.$$
$$\left(2L_1^\alpha\left(\int_0^{t+\varphi_\varepsilon u}|X_s-x_s|dK(s)\right)^\alpha\right.$$
$$\left.+2L_2^\alpha|X_{t+\varphi_\varepsilon u}-x_{t+\varphi_\varepsilon u}|^\alpha\right)du\Big)^{\frac{1}{\alpha}}$$
$$\le \delta^{-1}2^{\frac{1}{\alpha}+1}C_2(L_1K_0+L_2)\left(\int_{B_1}^{B_2}|G(u)|^\alpha du\right)^{\frac{1}{\alpha}}$$
$$\cdot \sup_{0\le t+\varphi_\varepsilon u\le T}|X_{t+\varphi_\varepsilon u}-x_{t+\varphi_\varepsilon u}|,$$

which converges to zero as $\varepsilon \to 0$. Applying Lemma 2.2, one can obtain that there exist two independent processes $Z', Z'' \stackrel{d}{=} Z$, such that

$$\int_0^T G\left(\frac{\tau-t}{\varphi_\varepsilon}\right)\sigma_\tau(x)dU_\tau^{(1)}$$
$$=\int_{a_0}^{a_T} G\left(\frac{\frac{\ln\alpha\tau}{\alpha}-t}{\varphi_\varepsilon}\right)\sigma_{\frac{\ln\alpha\tau}{\alpha}}(x)(\alpha\tau)^{-\frac{1}{\alpha}}dZ_\tau$$
$$=Z'\circ\int_{a_0}^{a_T}\left(G\left(\frac{\frac{\ln\alpha\tau}{\alpha}-t}{\varphi_\varepsilon}\right)\sigma_{\frac{\ln\alpha\tau}{\alpha}}(x)(\alpha\tau)^{-\frac{1}{\alpha}}\right)_+^\alpha d\tau$$
$$-Z''\circ\int_{a_0}^{a_T}\left(G\left(\frac{\frac{\ln\alpha\tau}{\alpha}-t}{\varphi_\varepsilon}\right)\sigma_{\frac{\ln\alpha\tau}{\alpha}}(x)(\alpha\tau)^{-\frac{1}{\alpha}}\right)_-^\alpha d\tau$$
$$\stackrel{d}{=}\left(\int_{a_0}^{a_T}\left(G\left(\frac{\frac{\ln\alpha\tau}{\alpha}-t}{\varphi_\varepsilon}\right)\sigma_{\frac{\ln\alpha\tau}{\alpha}}(x)(\alpha\tau)^{-\frac{1}{\alpha}}\right)_+^\alpha d\tau\right)^{\frac{1}{\alpha}}U_1$$
$$-\left(\int_0^T\left(G\left(\frac{\frac{\ln\alpha\tau}{\alpha}-t}{\varphi_\varepsilon}\right)\sigma_{\frac{\ln\alpha\tau}{\alpha}}(x)(\alpha\tau)^{-\frac{1}{\alpha}}\right)_-^\alpha d\tau\right)^{\frac{1}{\alpha}}U_2,$$

where U_1 and U_2 are two independent random variables with α-stable distribution $S_\alpha(1,\beta,0)$. For $I_4(\varepsilon)$, applying the fact that $\varphi_\varepsilon \to 0$ as $\varepsilon \to 0$ and condition $(H3)$, one sees that

$$I_4(\varepsilon)\stackrel{d}{=}\left(\int_{\frac{-t}{\varphi_\varepsilon}}^{\frac{T-t}{\varphi_\varepsilon}}(G(u)\sigma_{t+\varphi_\varepsilon u}(x))_+^\alpha du\right)^{\frac{1}{\alpha}}U_1$$
$$-\left(\int_{\frac{-t}{\varphi_\varepsilon}}^{\frac{T-t}{\varphi_\varepsilon}}(G(u)\sigma_{t+\varphi_\varepsilon u}(x))_-^\alpha du\right)^{\frac{1}{\alpha}}U_2,$$
$$\Rightarrow\left(\int_{B_1}^{B_2}(G(u)\sigma_t(x))_+^\alpha du\right)^{\frac{1}{\alpha}}U_1$$
$$-\left(\int_{B_1}^{B_2}(G(u)\sigma_t(x))_-^\alpha du\right)^{\frac{1}{\alpha}}U_2,$$

as $\varepsilon \to 0$. Thus, by Slutsky's theorem, we can conclude that (16) and (17) hold as $\varepsilon \to 0$ in terms of the range of γ, i.e. $\gamma \in (0,1)$ and $\gamma = 1$, respectively. This completes the proof. ■

Remark 4.1: If Z is symmetric (that is $\beta = 0$), then, under the conditions of Theorem 4.1, by basic property of α-stable random variables (see Janicki and Weron [19]), we can obtain that (i) when $\gamma \in (0,1)$ and $\varepsilon \to 0$,

$$\varepsilon^{-\frac{\alpha(k+\gamma)}{\alpha(k+\gamma+1)-1}}(\widehat{A}_t - A_t(x))$$
$$\Rightarrow|\sigma_t(x)|\left(\int_{B_1}^{B_2}|G(u)|^\alpha du\right)^{\frac{1}{\alpha}}S_\alpha(1,0,0);$$

(ii) When $\gamma = 1$ and $\varepsilon \to 0$,

$$\varepsilon^{-\frac{\alpha(k+1)}{\alpha(k+2)-1}}(\widehat{A}_t - A_t(x)) - \frac{A_t^{(k+1)}(x)}{(K+1)!}\int_{B_1}^{B_2}G(u)u^{k+1}du$$
$$\Rightarrow|\sigma_t(x)|\left(\int_{B_1}^{B_2}|G(u)|^\alpha du\right)^{\frac{1}{\alpha}}S_\alpha(1,0,0).$$

V. Conclusion

This paper has presented new results on nonparametric estimation for stochastic processes driven by a non-Gaussian α-stable Lévy process of the second kind. The main results are obtained with the assistance of the moment inequalities for stable stochastic integrals, Lemma 2.1, and the inner clock property for the α-stable stochastic integral. The research results of this article enrich the asymptotic theory of statistical inference for non Gaussian stochastic processes.

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