Research on the Algorithm of Adjacent Vertex Reducible Total Labeling for Graphs

Jiang Wang, Jingwen Li, Xin Gao, Cong Huang

***Abstract*—For an undirected connected graph $G(V,E)$, if there exists a unique mapping $f: V(G) \cup E(G) \to \{1,2,\cdots,|V|+|E|\}$, such that for any edge $uv \in E(G)$ and $d(u)=d(v)$, it holds that $S(u)=S(v)$, where $S(u)=f(u)+\sum_{uz\in E(G)} f(uz)$, and $d(u)$ represents the degree of vertex u, then f is termed as an Adjacent Vertex Reducible Total Labeling (AVRTL) of G. Building upon existing research on total labeling algorithms, a heuristic search algorithm is devised by combining principles from genetic algorithm and particle swarm optimization. By studying the labeling patterns within finite-point graphs and extending them to describe the labeling patterns of infinite-point analogous graphs, it is discovered that some composite graphs of AVRTL are derived from several special subgraphs that are also AVRTL through a graph operation. Several theorems summarizing the labeling characteristics of these composite graphs are formulated and proven, along with defining the graph operation. Finally, a conjecture is proposed: if subgraphs G_1 and G_2 are AVRTL graphs, then their composite graph $G_1 \uparrow_{ab} G_2$ is also an AVRTL graph, where $\uparrow_{ab}$ denotes the graph operation.**

***Index Terms*—Adjacent Vertex Reducible Total Labeling, AVRTL graph, heuristic search algorithm, joint-graph, graph operation**

I. Introduction

GRAPH theory, as a significant branch of mathematics, owes its significant position in various fields such as computer science, social network analysis, transportation planning, communication network design[1], bioinformatics, logistics, and transportation planning to its concise representation and rich applications. The origin of graph theory can be traced back to the 18th century with Euler laying its foundation by solving the Seven Bridges of Königsberg problem. Since then, graph theory has gradually evolved into an independent and prolific research tool, finding extensive applications in practical problems such as coding theory, X-ray crystallography, radar, astronomy, circuit design, secret sharing schemes, and cryptography.

Graph labeling, as an important research focus in graph theory, saw the inception of the Graceful Tree Conjecture in Rosa's paper in 1967, proposing that all trees are graceful. In 1970, Anton et al[2].introduced the concept of edge-magic and conjectured that every tree has an edge-magic total labeling. Further, in 1998, Enomoto et al. extended this by proposing that all tree graphs have super edge-magic total labeling advanced the proposition that all trees have super edge magic total labeling [3]. Subsequently, in 1999 [4], MacDougall et al. introduced the concepts of vertex-magic labeling and super vertex-magic labeling. In 2020, literature presented vertex magic total labeling and related patterns for general graphs. Other labeling schemes are referenced in [5-10], while summarizes the current status of labeling research.

This paper addresses the point-edge partitioning strategy problem in logistics and supply chain management, social network analysis, energy network optimization, transportation planning, and power distribution networks. This algorithm synthesizes the principles of genetic algorithms and particle swarm optimization. By integrating principles from genetic algorithm and particle swarm optimization, a heuristic search algorithm is designed. Through analysis of experimental results, it is observed that some composite graphs of AVRTL are derived from several special subgraphs that are also AVRTL through a graph operation. Several theorems summarizing the labeling characteristics of these composite graphs are formulated and proven, along with defining the graph operation. Finally, a conjecture is proposed: If subgraphs G_1 and G_2 are AVRTL graphs, then their composite graph $G_1 \uparrow_{ab} G_2$ is also an AVRTL graph, where $\uparrow_{ab}$ denotes the graph operation.

II. Preliminary knowledge

Definition 1: Let $G(V,E)$ be an undirected connected graph , if there exists a unique mapping $f: V(G) \cup E(G) \to \{1,2,\cdots,|V|+|E|\}$, such that $uv \in E(G)$ and $d(u)=d(v)$, resulting in $S(u)=S(v)$, where $S(u)=\sum_{uz\in E(G)} f(uz)+f(u)$, $d(u)$ represents the degree of vertex u, then f is designated as an AVRTL of G. G is called an AVRTL graph if it satisfies this condition, otherwise it is termed an N-AVRTL graph.

Definition 2: Let graph $G_1 \uparrow_{ab} G_2$ be a composite graph, where G_1 and G_2 can be one of the following: a path graph (P_n), a cycle graph (C_n), a star graph (S_n), a fan graph (F_n), or a wheel graph (W_n). Here, a represents the central node for star, fan, and wheel graphs, a 1-degree node for the path

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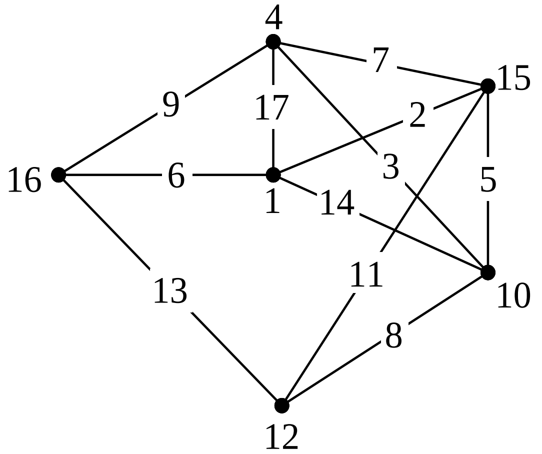
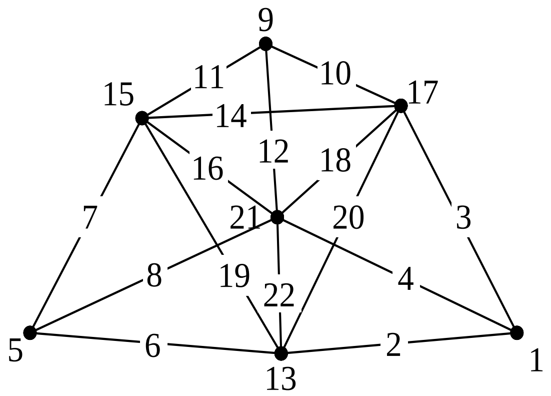
graph, and any node for the cycle graph. b represents a non-central node for star and wheel graphs, a 2-degree node for the fan graph, and a 2-degree node for the path graph. Graph $G_1 \uparrow_{ab} G_2$ is obtained by connecting node a from G_1 to node b from G_2.

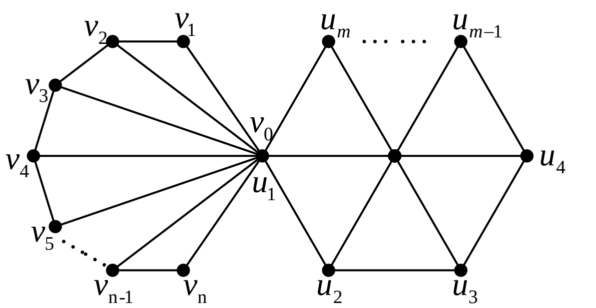


(1) $G(6,11)$

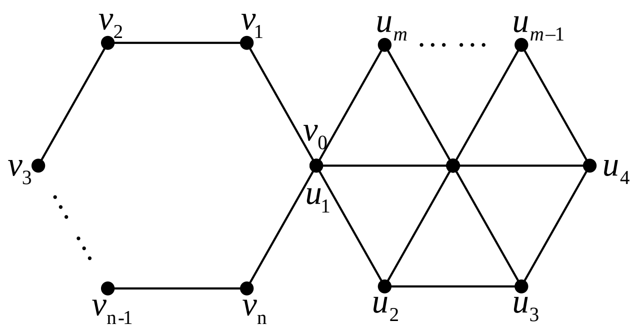


(2) $G(7,15)$

Fig. 1. Example of AVRTL graph.



(1) $C_n \uparrow_{ab} W_m$



(2) $F_n \uparrow_{ab} W_m$

Fig. 2. Example of joint graph $G_1 \uparrow_{ab} G_2$.

III. AVRTL ALGORITHM

A. Basic Principles of AVRTL Algorithm

The concept of the AVRTL algorithm integrates genetic algorithm and particle swarm algorithm. Initially, an initial labeling matrix is obtained based on the adjacency matrix of graph G. A candidate solution population is initialized, and a heuristic search algorithm is employed for exploration. During the search process, if a matrix satisfying the convergence criteria is found, it is considered as the final matrix and outputted. If, after convergence, the solution for graph G still does not satisfy the convergence criteria function, it indicates that graph G is not an AVRTL graph, and the algorithm terminates.

(1) Preprocessing function: The graph set is outputted to a file in the form of an initialized adjacency matrix. This function also calculates information such as the number of vertices and edges of graph G, the degree of each vertex, degree sequences, and sets of adjacent vertices with the same degree.

(2) Optimal Solution Search Function: According to the principles of particle swarm algorithm, each candidate solution updates its velocity and position based on individual best positions and global best positions. Subsequently, combined with operations such as information update, selection, crossover, and mutation, the function searches for the optimal solution.

(3) Convergence Judgment Function: Criterion1; Determine whether the adjacent vertex reducible total labeling constraint is satisfied during the search process. If a solution satisfying the constraint is found, it is stored in the matrix StorageMatrix. Criterion2: $|V|+|E|-1$.

(4) Output Function: If there are graph sets satisfying the constraints of the adjacent vertex reducible total labeling, the final results are outputted in the form of a labeling matrix. If, after convergence judgment, the solution for graph G still does not satisfy the convergence judgment function, then graph G is outputted as a non-AVRTL graph.

B. Pseudocode of the AVRTL Algorithm

Input	The adjacency matrix of the graph $G(p,q)$
Output	Matrix satisfying AVRTL graph or NAVRTL
1	Read the adjacency matrix of the graph to get the initial convergence matrix AS0
2	Calculate the number of points, the number of edges, the degree sequence DegreeList, the set of adjacent points with the same degree SameList and other information of the graph G.
3	get G-AdjustMatrix, StorageMatrix ← AS0
4	isContinue = true, isSuccess = false
5	while (isContinue)
6	Permutation (p,q)
7	G-AdjustMatrix.UXOSearch # Optimal solution search initialization
8	G-AdjustMatrix. particle # Update the individual best position and fitness of each candidate solution
9	G-AdjustMatrix. Individual # Perform operations such as information update, crossover, mutation, etc.
10	if(G-AdjustMatrix.ASjudgment) # Convergence judgment
11	StorageMatrix← G-AdjustMatrix
12	isSuccess = true
13	break

14	endif
15	endwhile
16	if(!G-AdjustMatrix.equalWith(AS0)&&isSuccess)
17	Output G is not a AVRTL Graph
18	endif
19	else
20	Output G is a AVRTL Graph
21	Output StorageMatrix
22	endelse
23	end

C. Results of the Adjacent Vertex Reducible Total Labeling Algorithm

Table 1 presents the number of AVRTL graphs among the total number of graphs with different numbers of edges ranging from 3 to 6 vertices. It can be observed from Table 1 that all graphs with 3 to 6 vertices are AVRTL graphs.

Fig 3 illustrates the number and proportion of AVRTL graphs among the total number of graphs with different numbers of edges ranging from 7 to 11 vertices.

Fig 4 provides examples of AVRTL labeling for some graphs.

TABLE I
STATISTICALS OF AVRTL FOR GRAPHS WITH 3 TO 6 VERTICES

(p,q)	Total number of pictures/ piece	AVRTL picture number/ piece	(p,q)	Total number of pictures/ piece	AVRTL picture number/ piece
(3,2)	1	1	(5,10)	1	1
(3,3)	1	1	(6,5)	5	5
(4,3)	2	2	(6,6)	13	13
(4,4)	2	2	(6,7)	19	19
(4,5)	1	1	(6,8)	22	22
(4,6)	1	1	(6,2)	20	20
(5,4)	3	3	(6,9)	14	14
(5,5)	5	5	(6,10)	9	9
(5,6)	5	5	(6,11)	5	5
(5,7)	4	4	(6,12)	2	2
(5,8)	2	2	(6,13)	1	1
(5,9)	1	1	(6,14)	1	1

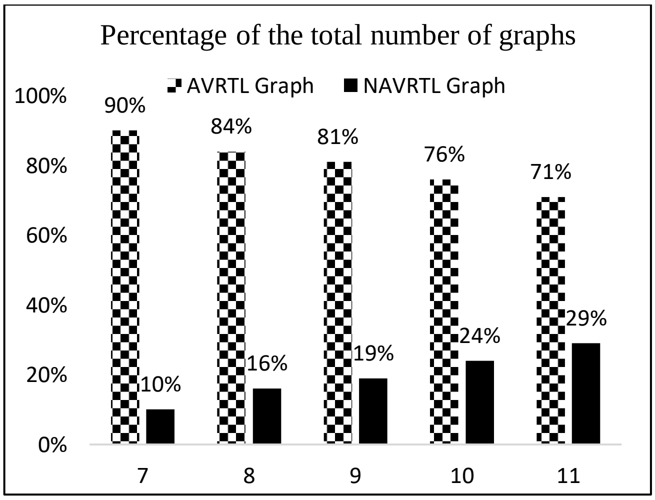


Fig. 3. Variation of the percentage of AVRTL and N-AVRTL graphs among the total number of graphs with a finite number of vertices.

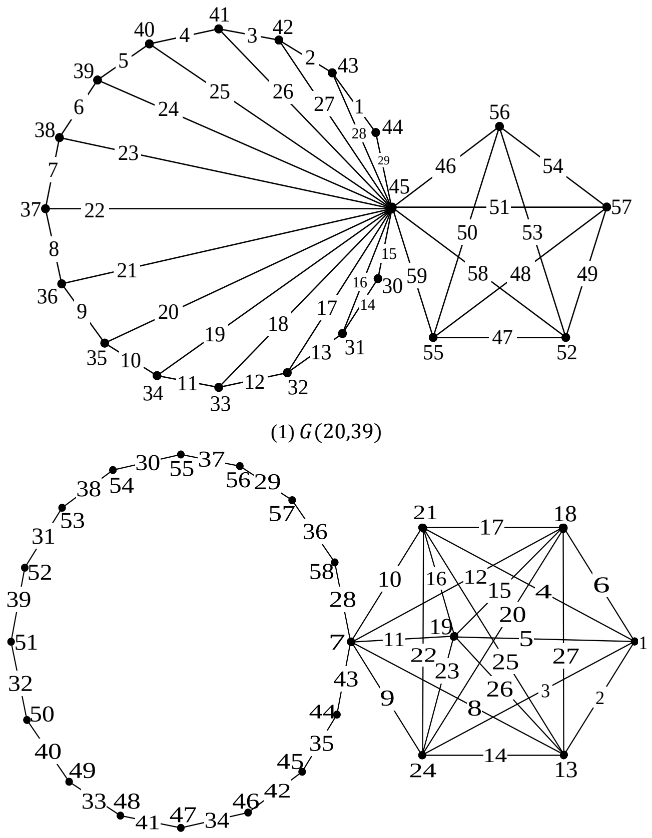


Fig. 4. Examples of AVRTL.

IV. THEOREMS AND PROOFS

For special subgraphs, paths, cycles, stars, fans, and wheels are all AVRTL graphs, and the conclusion is obviously true.

Theorem 1: For a joint graph $W_n \uparrow_{ab} S_m$, when $n \geq 3$, $m > 2$, and $m \neq n+1$, it is an AVRTL graph.

Proof: Let the vertex set of the joint graph $W_n \uparrow_{ab} S_m$, be denoted as $V(W_n \uparrow_{ab} S_m) = \{v_0, v_1, \cdots, v_n, u_0, u_1, \cdots, u_m\}$, and the edge set as $E(W_n \uparrow_{ab} S_m) = \{v_0 v_i | 1 \leq i \leq n\} \cup \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{u_0 u_i | 1 \leq i \leq m\}$.

When $n \geq 3$, $m > 2$, and $m \neq n+1$, the AVRTL of $W_n \uparrow_{ab} S_m$ is:

$f(v_i) = 3n + 1 - i, 1 \leq i \leq n$
$f(v_i v_{i+1}) = i, 1 \leq i < n$
$f(v_n v_1) = n$
$f(v_0 v_1) = n + 1$
$f(v_0 v_i) = 2n + 2 - i, 2 \leq i \leq n$
$f(u_0) = 3n + 1$
$f(u_i) = 3n + 2m + 2 - i, 1 \leq i \leq m$
$f(u_0 u_i) = 3n + 1 + i, 1 \leq i \leq m$

In this case, the vertex labeling set of $W_n \uparrow_{ab} S_m$ is:

$$f(V) = \{f(v_i) | 0 \leq i \leq n\} \cup \{f(u_j) | 0 \leq j \leq m\}$$
$$= \{2n+1, 2n+2, \cdots, 3n+1\}$$
$$\cup \{3n+m+2, 3n+m+3, \cdots, 3n+2m+1\}$$

The edge labeling set is:

$$f(E) = \{f(v_i v_{i+1}) | 1 \leq i \leq n-1\} \cup \{f(v_0 v_i) | 1 \leq i \leq n\}$$
$$\cup f(v_n v_1) \cup \{f(u_0 u_j) | 1 \leq j \leq m\}$$
$$= \{1, 2, \cdots, 2n\} \cup \{3n+2, 3n+3, \cdots, 3n+m+1\}$$

According to the definition of AVRTL, $f(V) \cup f(E) \rightarrow [1, 3n+2m+1]$, and the joint graph $W_n \uparrow_{ab} S_m$ has a

1-degree vertices $u_2, u_3, \cdots, u_m$, 3-degree vertices $v_1, v_2 \cdots, v_n$, $n+1$-degree vertices v_0/u_1, and an m-degree vertex u_0. It is only necessary to ensure that the labeling of the adjacent 3-degree vertices of the joint graph $W_n \uparrow_{ab} S_m$ are equal.

For the 3-degree vertices $v_1, v_2, \cdots, v_n$ adjacent to the joint graph $W_n \uparrow_{ab} S_m$, the sum of the labeling of each vertex is:

$$Sum(v_i|1 \le i \le n)$$
$$= f(v_i) + \sum_{ve \in E(v_i)} f(ve)|1 \le i \le n$$
$$= f(v_i) + f(v_{i-1}v_i) + f(v_i v_{i+1}) f(v_0 v_i)|2 \le i \le n-1||$$
$$f(v_1) + f(v_n v_1) + f(v_1 v_2) + f(v_0 v_1)|f(v_n) + f(v_{n-1}v_n)$$
$$+ f(v_n v_1) + f(v_0 v_n)$$
$$= 5n + 2$$

Thus, it is proven that when $n \ge 3$, $m > 2$, and $m \ne n+1$, $W_n \uparrow_{ab} S_m$ is an AVRTL graph.

The labeling results for some joint graphs $W_n \uparrow_{ab} S_m$ are shown in Fig 5.

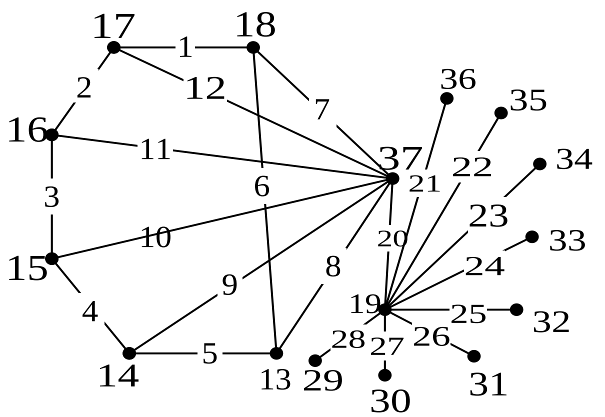


Fig. 5. Labeling Results for $W_6 \uparrow_{ab} S_9$

Theorem 2: For a joint graph $P_n \uparrow_{ab} W_m$, when $n \ge 3$, $m > 3$, and $m \ne 4$, it is an AVRTL graph.

Proof: Let the vertex set of the joint graph $P_n \uparrow_{ab} W_m$, be denoted as $V(P_n \uparrow_{ab} W_m) = \{v_1, v_2, \cdots, v_n, u_0, u_1, \cdots, u_m\}$, and the edge set as $E(P_n \uparrow_{ab} W_m) = \{v_i v_{i+1}|1 \le i \le n-1\} \cup \{u_0 u_i|1 \le i \le m\} \cup \{u_i u_{i+1}|1 \le i \le m-1\} \cup \{u_m u_1\}$.

Case 1: When $n \ge 3$, $m > 3$, $m \ne 4$, and $n \equiv 0(mod 2)$, the AVRTL of $P_n \uparrow_{ab} W_m$ is:

$$f(v_i) = 2n - i, 2 \le i \le n$$
$$f(v_i v_{i+1}) = \begin{cases} \frac{n}{2} + \frac{i}{2}, i \equiv 0(mod 2) \; and \; 2 \le i < n \\ i - \left\lfloor \frac{i}{2} \right\rfloor, i \equiv 1(mod 2) \; and \; 1 \le i < n \end{cases}$$
$$f(u_0) = 2n + 3m - 1$$
$$f(u_i) = 2n + 3m - 1 - i, 1 \le i \le m$$
$$f(u_i u_{i+1}) = 2n - 2 + i, 1 \le i < m$$
$$f(u_m u_1) = 2n + m - 2$$
$$f(u_0 u_1) = 2n + m - 1$$
$$f(u_0 u_i) = 2(n+m) - i, 2 \le i \le m$$

In this case, the vertex labeling set of $P_n \uparrow_{ab} W_n$ is:

$$f(V) = \{f(v_i)|1 \le i \le n\} \cup \{f(u_j)|0 \le j \le m\}$$
$$= \{n, n+1, \cdots, 2n-2\}$$
$$\cup \{2n+2m-1, 2n+2m, \cdots, 2n+3m-1\}$$

The edge labeling set is:

$$f(E) = \{f(v_i v_{i+1})|1 \le i \le n-1\} \cup \{f(u_0 u_j)|1 \le j \le m\}$$
$$\cup \{f(u_j u_{j+1})|1 \le j \le m-1\} \cup f(u_m u_1)$$
$$= \{1, 2, \cdots, n-1\} \cup \{2n-1, 2n, \cdots, 2n+2m-2\}$$

According to the definition of AVRTL, $f(V) \cup f(E) \to [1, 2n+3m-1]$, and the joint graph $P_n \uparrow_{ab} W_m$ has a 1-degree vertex v_n, 2-degree vertices $v_2, v_3, \cdots, v_{n-1}$, 3-degree vertices $u_2, u_3, \cdots, u_m$, 4-degree vertices v_1/u_1, and an m-degree vertex u_0. It is only necessary to ensure that the labeling of the adjacent 2-degree vertices and 3-degree vertices of the joint graph $P_n \uparrow_{ab} W_m$ are equal.

For the 2-degree vertices $v_2, v_3, \cdots, v_{n-1}$ adjacent to the joint graph $P_n \uparrow_{ab} W_m$, the sum of their vertex labels is:

$$Sum(v_i|2 \le i \le n-1)$$
$$= f(v_i) + \sum_{ve \in E(v_i)} f(ve)|2 \le i \le n-1$$
$$= f(v_i) + f(v_{i-1}v_i) + f(v_i v_{i+1})|2 \le i \le n-1$$
$$= 3n - \left\lfloor \frac{n-1}{2} \right\rfloor - 1$$

For the 3-degree vertices $u_2, u_3, \cdots, u_m$ adjacent to the joint graph $P_n \uparrow_{ab} W_m$, the sum of their vertex labels is:

$$Sum(u_j|2 \le j \le m)$$
$$= f(u_j) + \sum_{ve \in E(u_j)} f(ue)|2 \le j \le m$$
$$= f(u_j) + f(u_{j-1}u_j) + f(u_j u_{j+1}) + f(u_0 u_j)|2 \le j \le m$$
$$|| f(u_m) + f(u_{m-1}u_m) + f(u_m u_1) + f(u_0 u_m)$$
$$= 8n + 5m - 6$$

Thus, it is proven that when $n \ge 3$, $m > 3$, $m \ne 4$, and $n \equiv 0(mod 2)$, $P_n \uparrow_{ab} W_m$ is an AVRTL graph.

Case 2: When $n \ge 3$, $m > 3$, $m \ne 4$, and $n \equiv 1(mod 2)$, the AVRTL of $P_n \uparrow_{ab} W_m$ is:

$$f(v_i) = 2n - i, 2 \le i \le n$$
$$f(v_i v_{i+1}) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor + \frac{i}{2}, i \equiv 0(mod 2) \; and \; 2 \le i < n \\ i - \left\lfloor \frac{i}{2} \right\rfloor, i \equiv 1(mod 2) \; and \; 1 \le i < n \end{cases}$$
$$f(u_0) = 2n + 3m - 1$$
$$f(u_i) = 2n + 3m - 1 - i, 1 \le i \le m$$
$$f(u_i u_{i+1}) = 2n - 2 + i, 1 \le i < m$$
$$f(u_m u_1) = 2n + m - 2$$
$$f(u_0 u_1) = 2n + m - 1$$
$$f(u_0 u_i) = 2(n+m) - i, 2 \le i \le m$$

In this case, the vertex labeling set of $P_n \uparrow_{ab} W_m$ is:

$$f(V) = \{f(v_i)|1 \le i \le n\} \cup \{f(u_j)|0 \le j \le m\}$$
$$= \{n, n+1, \cdots, 2n-2\}$$
$$\cup \{2n+2m-1, 2n+2m, \cdots, 2n+3m-1\}$$

The edge labeling set is:

$$f(E) = \{f(v_i v_{i+1})|1 \le i \le n-1\} \cup \{f(u_0 u_j)|1 \le j \le m\}$$
$$\cup \{f(u_j u_{j+1})|1 \le j \le m-1\} \cup f(u_m u_1)$$
$$= \{1, 2, \cdots, n-1\} \cup \{2n-1, 2n, \cdots, 2n+2m-2\}$$

According to the AVRTL definition, $f(V) \cup f(E) \to [1, 2n+3m-1]$, and the joint graph $P_n \uparrow_{ab} W_m$ has a 1-degree vertex v_n, 2-degree vertices $v_2, v_3, \cdots, v_{n-1}$, 3-degree vertices $u_2, u_3, \cdots, u_m$, 4-degree vertices v_1/u_1, and an m-degree vertex u_0. It is only necessary to ensure that the labeling of the adjacent 2-degree and 3-degree vertices of the joint graph $P_n \uparrow_{ab} W_m$ are equal.

For the 2-degree vertices $v_2, v_3, \cdots, v_{n-1}$ adjacent to the

joint graph $P_n \uparrow_{ab} W_m$, the sum of their vertex labels is:
$Sum(v_i|2 \le i \le n-1)$
$= f(v_i) + \sum_{ve\in E(v_i)} f(ve)|2 \le i \le n-1$
$= f(v_i) + f(v_{i-1}v_i) + f(v_iv_{i+1})|2 \le i \le n-1$
$= 2n + \frac{n-1}{2}$

For the 3-degree vertices $u_2, u_3, \cdots, u_m$ adjacent to the joint graph $P_n \uparrow_{ab} W_m$, the sum of their vertex labels is:
$Sum(u_j|2 \le j \le m)$
$= f(u_j) + \sum_{ve\in E(u_j)} f(ue)|2 \le j \le m$
$= f(u_j) + f(u_{j-1}u_j) + f(u_ju_{j+1}) + f(u_0u_j)|2 \le j \le m$
$||f(u_m) + f(u_{m-1}u_m) + f(u_mu_1) + f(u_0u_m)$
$= 8n + 5m - 6$

Thus, it is proven that when $n \ge 3$, $m > 3$, $m \ne 4$, and $n \equiv 1(mod 2)$, $P_n \uparrow_{ab} W_m$ is an AVRTL graph.

In summary, for the joint graph $P_n \uparrow_{ab} W_m$, when $n \ge 3$, $m > 3$, and $m \ne 4$, it is an AVRTL graph.

Partial graph labeling results for $P_n \uparrow_{ab} W_m$ are shown in Fig 6.

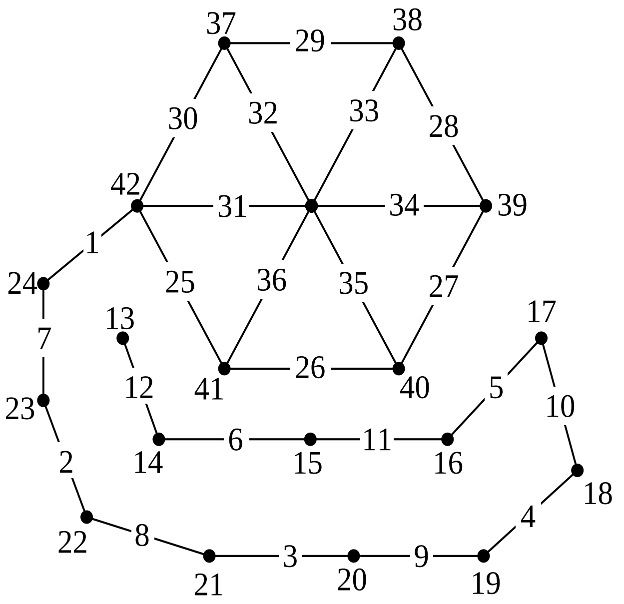


Fig. 6. Labeling Results for $P_{13} \uparrow_{ab} W_6$.

Theorem 3: For a joint graph $F_n \uparrow_{ab} S_m$, when $n > 3$, $m > 2$, and $m \ne n+1$, it is an AVRTL graph.

Proof: Let the vertex set of the joint graph $F_n \uparrow_{ab} S_m$ be denoted as $V(F_n \uparrow_{ab} S_m) = \{v_0, v_1, \cdots, v_n, u_0, u_1, \cdots, u_m\}$, and the edge set as $E(F_n \uparrow_{ab} S_m) = \{v_0v_i|1 \le i \le n\} \cup \{v_iv_{i+1}|1 \le i \le n-1\} \cup \{u_0u_i|1 \le i \le m\}$.

When $n > 3$, $m > 2$, and $m \ne n+1$, the ARVRL of $F_n \uparrow_{ab} S_m$ is:
$f(v_i) = 3n - i, 1 \le i \le n$
$f(v_iv_{i+1}) = i, 1 \le i < n$
$f(v_0v_i) = 2n - i, 1 \le i \le n$
$f(u_0) = 3n$
$f(u_i) = 3n + 2m + 1 - i, 1 \le i \le m$
$f(u_0u_i) = 3n + i, 1 \le i \le m$

In this case, the vertex labeling set of $F_n \uparrow_{ab} S_m$ is:
$f(V) = \{f(v_i)|0 \le i \le n\} \cup \{f(u_j)|0 \le j \le m\}$
$= \{2n, 2n+1, \cdots, 3n\}$
$\cup \{3n+m+1, 3n+m+2, \cdots, 3n+2m\}$

The edge labeling set is:
$f(E) = \{f(v_iv_{i+1})|1 \le i \le n-1\} \cup \{f(v_0v_i)|1 \le i \le n\}$
$\cup \{f(u_0u_j)|1 \le j \le m\}$
$= \{1, 2, \cdots, 2n-1\} \cup \{3n+1, 3n+2, \cdots, 3n+m\}$

According to the definition of AVRTL, $f(V) \cup f(E) \to [1, 3n+2m]$, and the joint graph $W_n \uparrow_{ab} S_m$ has a 1-degree vertices $u_2, u_3, \cdots, u_m$, 2-degree vertices v_1/v_n, 3-degree vertices $v_2, v_3, \cdots, v_{n-1}$, $n+1$-degree vertices v_0/u_1, and an m-degree vertex u_0. It is only necessary to ensure that the labeling of the adjacent 3-degree vertices of the joint graph $F_n \uparrow_{ab} S_m$ are equal.

For the 3-degree vertices $v_2, v_3, \cdots, v_{n-1}$ adjacent to the joint graph $F_n \uparrow_{ab} S_m$, the sum of the labeling of each vertex is:
$Sum(v_i|2 \le i \le n-1)$
$= f(v_i) + \sum_{ve\in E(v_i)} f(ve)|2 \le i \le n-1$
$= f(v_i) + f(v_{i-1}v_i) + f(v_iv_{i+1}) f(v_0v_i)|2 \le i \le n-1$
$= 5n - 1$

Thus, it is proven that when $n > 3$, $m > 2$, and $m \ne n+1$, $F_n \uparrow_{ab} S_m$ is an AVRTL graph.

The labeling results for some joint graphs $F_n \uparrow_{ab} S_m$ are shown in Fig 7.

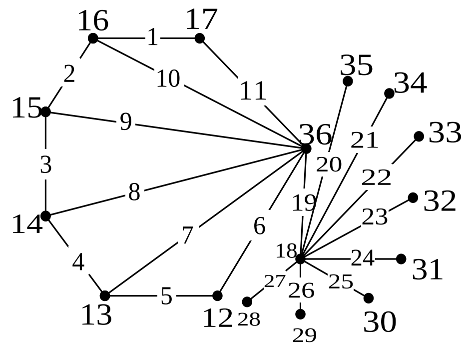


Fig. 7. Labeling Results for $F_6 \uparrow_{ab} S_9$

Theorem 4: For a joint graph $C_n \uparrow_{ab} W_m$, when $n > 3$, $m > 3$, and $m \ne 5$, it is an AVRTL graph.

Proof: Let the vertex set of the joint graph $C_n \uparrow_{ab} W_m$ be denoted as $V(C_n \uparrow_{ab} W_m) = \{v_1, v_2, \cdots, v_n, u_0, u_1, \cdots, u_m\}$, and the edge set as $E(C_n \uparrow_{ab} W_m) = \{v_nv_1\} \cup \{u_mu_1\} \cup \{v_iv_{i+1}|1 \le i \le n-1\} \cup \{u_0u_i|1 \le i \le m\} \cup \{u_iu_{i+1}|1 \le i \le m-1\}$.

Case 1: When $n > 3$, $m > 3$, $m \ne 5$ and $n \equiv 0(mod 2)$, the AVRTL of $C_n \uparrow_{ab} W_m$ is:
$f(v_i) = 2n + 1 - i, 2 \le i \le n$
$f(v_nv_1) = n$

$$f(v_iv_{i+1}) = \begin{cases} \frac{n}{2} + \frac{i}{2}, i \equiv 0(mod 2)\ and\ 2 \le i < n \\ i - \left\lfloor \frac{i}{2} \right\rfloor, i \equiv 1(mod 2)\ and\ 1 \le i < n \end{cases}$$

$f(u_0) = 2n + 3m$
$f(u_i) = 2n + 3m - i, 1 \le i \le m$
$f(u_iu_{i+1}) = 2n - 1 + i, 1 \le i < m$
$f(u_mu_1) = 2n + m - 1$

$f(u_0u_1) = 2n + m$
$f(u_0u_i) = 2(n + m) + 1 - i, 2 \le i \le m$

In this case, the vertex labeling set of $C_n \uparrow_{ab} W_m$ is:

$$f(V) = \{f(v_i)|1 \le i \le n\} \cup \{f(u_j)|0 \le j \le m\}$$
$$= \{n + 1, n + 2, \cdots, 2n - 1\}$$
$$\cup \{2n + 2m, 2n + 2m + 1, \cdots, 2n + 3m\}$$

The edge labeling set is:

$$f(E) = \{f(v_iv_{i+1})|1 \le i \le n - 1\} \cup f(v_nv_1) \cup f(u_mu_1)$$
$$\cup \{f(u_ju_{j+1})|1 \le j \le m - 1\} \cup \{f(u_0u_j)|1 \le j \le m\}$$
$$= \{1,2, \cdots, n\} \cup \{2n, 2n + 1, \cdots, 2n + 2m - 1\}$$

According to the AVRTL definition, $f(V) \cup f(E) \to [1,2n + 3m]$, and the joint graph $C_n \uparrow_{ab} W_m$ has 2-degree vertices $v_2, v_3, \cdots, v_n$, 3-degree vertices $u_2, u_3, \cdots, u_m$, 5-degree vertices v_1/u_1, and an m-degree vertex u_0. It is only necessary to ensure that the labeling of the adjacent 2-degree and 3-degree vertices of the joint graph $C_n \uparrow_{ab} W_m$ are equal.

For the 2-degree vertices $v_2, v_3, \cdots, v_n$ adjacent to the joint graph $C_n \uparrow_{ab} W_m$, the sum of the labeling of each vertex is:

$$Sum(v_i|2 \le i \le n)$$
$$= f(v_i) + \sum_{ve\in E(v_i)} f(ve)|2 \le i \le n$$
$$= f(v_i) + f(v_{i-1}v_i) + f(v_iv_{i+1})|2 \le i \le n - 1$$
$$||f(v_n) + f(v_{n-1}v_n) + f(v_nv_1)$$
$$= 3n - \left\lfloor\frac{n-1}{2}\right\rfloor$$

For the 3-degree vertices $u_2, u_3, \cdots, u_m$ adjacent to the joint graph $C_n \uparrow_{ab} W_m$, the sum of the labeling of each vertex is:

$$Sum(u_j|2 \le j \le m)$$
$$= f(u_j) + \sum_{ue\in E(u_j)} f(ue)|2 \le j \le m$$
$$= f(u_j) + f(u_{j-1}u_j) + f(u_ju_{j+1}) + f(u_0u_j)|2 \le j \le m$$
$$||f(u_m) + f(u_{m-1}u_m) + f(u_mu_1) + f(u_0u_m)$$
$$= 8n + 5m - 2$$

Thus, it is proven that when $n > 3, m > 3, m \ne 5$, and $n \equiv 0(mod2)$, $C_n \uparrow_{ab} W_m$ is an AVRTL graph.

Case 2: When $n > 3, m > 3, m \ne 5$, and $n \equiv 1(mod2)$, the AVRTL of $C_n \uparrow_{ab} W_m$ is:

$f(v_i) = 2n + 1 - i, 2 \le i \le n$

$$f(v_nv_1) = n - \left\lfloor\frac{n}{2}\right\rfloor$$

$$f(v_iv_{i+1}) = \begin{cases} \left\lceil\frac{n}{2}\right\rceil + \frac{i}{2}, i \equiv 0(mod2)\ and\ 2 \le i < n \\ i - \left\lfloor\frac{i}{2}\right\rfloor, i \equiv 1(mod2)\ and\ 1 \le i < n \end{cases}$$

$f(u_0) = 2n + 3m$
$f(u_i) = 2n + 3m - i, 1 \le i \le m$
$f(u_iu_{i+1}) = 2n - 1 + i, 1 \le i < m$
$f(u_mu_1) = 2n + m - 1$
$f(u_0u_1) = 2n + m$
$f(u_0u_i) = 2(n + m) + 1 - i, 2 \le i \le m$

In this case, the vertex labeling set of $C_n \uparrow_{ab} W_m$ is:

$$f(V) = \{f(v_i)|1 \le i \le n\} \cup \{f(u_j)|0 \le j \le m\}$$
$$= \{n + 1, n + 2, \cdots, 2n - 1\}$$
$$\cup \{2n + 2m, 2n + 2m + 1, \cdots, 2n + 3m\}$$

The edge labeling set is:

$$f(E) = \{f(v_iv_{i+1})|1 \le i \le n - 1\} \cup f(v_nv_1) \cup f(u_mu_1)$$
$$\cup \{f(u_ju_{j+1})|1 \le j \le m - 1\} \cup \{f(u_0u_j)|1 \le j \le m\}$$
$$= \{1,2, \cdots, n\} \cup \{2n, 2n + 1, \cdots, 2n + 2m - 1\}$$

According to the AVRTL definition, $f(V) \cup f(E) \to [1,2n + 3m]$, and the joint graph $C_n \uparrow_{ab} W_m$ has 2-degree vertices $v_2, v_3, \cdots, v_n$, 3-degree vertices $u_2, u_3, \cdots, u_m$, 5-degree vertices v_1/u_1, and an m-degree vertex u_0. It is only necessary to ensure that the labeling of the adjacent 2-degree and 3-degree vertices of the joint graph $C_n \uparrow_{ab} W_m$ are equal.

For the 2-degree vertices $v_2, v_3, \cdots, v_n$ adjacent to the joint graph $C_n \uparrow_{ab} W_m$, the sum of the labeling of each vertex is:

$$Sum(v_i|2 \le i \le n)$$
$$= f(v_i) + \sum_{ve\in E(v_i)} f(ve)|2 \le i \le n$$
$$= f(v_i) + f(v_{i-1}v_i) + f(v_iv_{i+1})|2 \le i \le n - 1$$
$$||f(v_n) + f(v_{n-1}v_n) + f(v_nv_1)$$
$$= 2n + 1 + \left\lceil\frac{n}{2}\right\rceil$$

For the 3-degree vertices $u_2, u_3, \cdots, u_m$ adjacent to the joint graph $C_n \uparrow_{ab} W_m$, the sum of the labeling of each vertex is:

$$Sum(u_j|2 \le j \le m)$$
$$= f(u_j) + \sum_{ue\in E(u_j)} f(ue)|2 \le j \le m$$
$$= f(u_j) + f(u_{j-1}u_j) + f(u_ju_{j+1}) + f(u_0u_j)|2 \le j \le m$$
$$||f(u_m) + f(u_{m-1}u_m) + f(u_mu_1) + f(u_0u_m)$$
$$= 8n + 5m - 2$$

Thus, it is proven that when $n > 3, m > 3, m \ne 5$, and $n \equiv 1(mod2)$, $C_n \uparrow_{ab} W_m$ is an AVRTL graph.

In summary, for the joint graph $C_n \uparrow_{ab} W_m$, when $n > 3$, $m > 3$, and $m \ne 5$, it is an AVRTL graph.

Partial graph labeling results for $C_n \uparrow_{ab} W_m$ are shown in Fig 8.

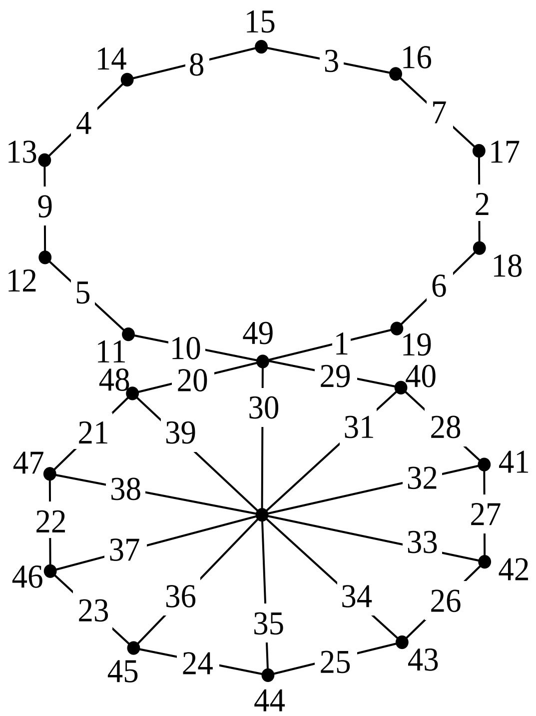


Fig. 8. Labeling Results for $C_{10} \uparrow_{ab} W_{10}$

Theorem 5: For a joint graph $S_n \uparrow_{ab} S_m$, when $n > 3$, $m > 3$, and $m \neq n + 1$, it is an AVRTL graph.

According to the definition of the total label reducible from adjacent vertices, Theorem 5 is evidently valid.

Theorem 6: For a joint graph $F_n \uparrow_{ab} W_m$, when $n > 3$, $m > 3$, and $m \neq n + 3$, it is an AVRTL graph.

Proof: Let the vertex set of the joint graph $F_n \uparrow_{ab} W_m$ be denoted as $V(F_n \uparrow_{ab} W_m) = \{v_0, v_1, \cdots, v_n, u_0, u_1, \cdots, u_m\}$, and the edge set as $E(F_n \uparrow_{ab} W_m) = \{v_0 v_i | 1 \leq i \leq n\} \cup \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{u_0 u_i | 1 \leq i \leq m\} \cup \{u_m u_1\} \cup \{u_i u_{i+1} | 1 \leq i \leq m-1\}$.

When $n > 3$, $m > 3$, and $m \neq n + 3$, the AVRTL of $F_n \uparrow_{ab} W_m$ is:

$f(v_i) = 3n - i, 1 \leq i \leq n$

$f(v_i v_{i+1}) = i, 1 \leq i < n$

$f(v_0 v_i) = 2n - i, 1 \leq i \leq n$

$f(u_0) = 3n + 3m$

$f(u_i) = 3n + 3m - i, 1 \leq i \leq m$

$f(u_i u_{i+1}) = 3n - 1 + i, 1 \leq i < m$

$f(u_m u_1) = 3n + m - 1$

$f(u_0 u_1) = 3n + m$

$f(u_0 u_i) = 3n + 2m + 1 - i, 2 \leq i \leq m$

In this case, the vertex labeling set of $F_n \uparrow_{ab} W_m$ is:

$$\begin{aligned} f(V) &= \{f(v_i) | 0 \leq i \leq n\} \cup \{f(u_j) | 0 \leq j \leq m\} \\ &= \{2n, 2n+1, \cdots, 3n-1\} \\ &\cup \{3n+2m, 3n+2m+1, \cdots, 3n+3m\} \end{aligned}$$

The edge labeling set is:

$$\begin{aligned} f(E) &= \{f(v_0 v_i) | 1 \leq i \leq n\} \cup \{f(v_i v_{i+1}) | 1 \leq i \leq n-1\} \\ &\cup \{f(u_j u_{j+1}) | 1 \leq j \leq m-1\} \cup \{f(u_0 u_j) | 1 \leq j \leq m\} \\ &\cup f(u_m u_1) \\ &= \{1, 2, \cdots, 2n-1\} \cup \{3n, 3n+1, \cdots, 3n+2m-1\} \end{aligned}$$

According to the AVRTL definition, $f(V) \cup f(E) \rightarrow [1, 3n + 3m]$, and the joint graph $F_n \uparrow_{ab} W_m$ has 2-degree vertices v_1, v_n, 3-degree vertices $v_2, v_3, \cdots, v_{n-1}, u_2, u_3, \cdots u_m$, $n + 3$-degree vertices v_0/u_1, and an m-degree vertex u_0. It is only necessary to ensure that the labeling of the adjacent 3-degree vertices of the joint graph $F_n \uparrow_{ab} W_m$ are equal.

For the 3-degree vertices $v_2, v_3, \cdots, v_{n-1}$ adjacent to the joint graph $F_n \uparrow_{ab} W_m$, the sum of the labeling of each vertex is:

$$\begin{aligned} & Sum(v_i | 2 \leq i \leq n-1) \\ &= f(v_i) + \sum_{ve \in E(v_i)} f(ve) | 2 \leq i \leq n-1 \\ &= f(v_i) + f(v_{i-1} v_i) + f(v_i v_{i+1}) | 2 \leq i \leq n-1 \\ &= 5n - 1 \end{aligned}$$

For the 3-degree vertices $u_2, u_3, \cdots, u_m$ adjacent to the joint graph $F_n \uparrow_{ab} W_m$, the sum of the labeling of each vertex is:

$$\begin{aligned} & Sum(u_j | 2 \leq j \leq m) \\ &= f(u_j) + \sum_{ue \in E(u_j)} f(ue) | 2 \leq j \leq m \\ &= f(u_j) + f(u_{j-1} u_j) + f(u_j u_{j+1}) + f(u_0 u_j) | 2 \leq j \leq m-1 \\ & || f(u_m) + f(u_{m-1} u_m) + f(u_m u_1) + f(u_0 u_m) \\ &= 12n + 5m - 2 \end{aligned}$$

Thus, it is proven that when $n > 3$, $m > 3$, and $m \neq n + 3$, $F_n \uparrow_{ab} W_m$ is an AVRTL graph.

Partial graph labeling results for $F_n \uparrow_{ab} W_m$ are shown in Fig 9.

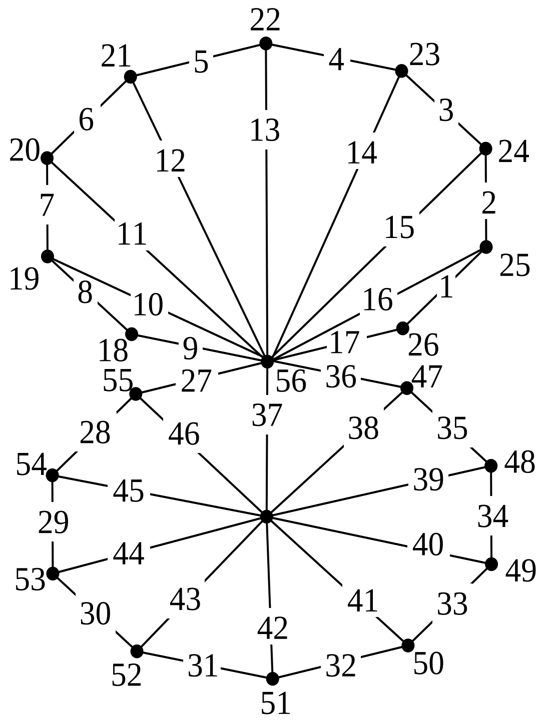


Fig. 9. Labeling Results for $F_9 \uparrow_{ab} W_{10}$.

Theorem 7: For a joint graph $P_n \uparrow_{ab} W_m$, when $n \geq 3$, $m > 3$, and $m \neq 4$, it is an AVRTL graph.

According to the definition of the total label reducible from adjacent vertices, Theorem 7 is evidently valid.

Theorem 8: For a joint graph $S_n \uparrow_{ab} W_m$, when $n > 2$, $m > 3$, and $m \neq n + 3$, it is an AVRTL graph.

Proof: Let the vertex set of the joint graph $F_n \uparrow_{ab} S_m$, be denoted as $V(S_n \uparrow_{ab} W_m) = \{v_0, v_1, \cdots, v_n, u_0, u_1, \cdots, u_m\}$, and the edge set as $E(S_n \uparrow_{ab} W_m) = \{v_0 v_i | 1 \leq i \leq n\} \cup \{u_0 u_i | 1 \leq i \leq m\} \cup \{u_i u_{i+1} | 1 \leq i \leq m-1\} \{u_m u_1\}$.

When $n > 2$, $m > 3$, and $m \neq n + 3$, the AVRTL of $S_n \uparrow_{ab} W_m$ is:

$f(v_i) = 2n + 1 - i, 1 \leq i \leq n$

$f(v_0 v_i) = i, 1 \leq i \leq n$

$f(u_0) = 2n + 3m + 1$

$f(u_i) = 2n + 3m + 1 - i, 1 \leq i \leq m$

$f(u_i u_{i+1}) = 2n + i, 1 \leq i < m$

$f(u_m u_1) = 2n + m$

$f(u_0 u_1) = 2n + m + 1$

$f(u_0 u_i) = 2(n + m + 1) - i, 2 \leq i \leq m$

In this case, the vertex labeling set of $S_n \uparrow_{ab} W_m$ is:

$$\begin{aligned} f(V) &= \{f(v_i) | 0 \leq i \leq n\} \cup \{f(u_j) | 0 \leq j \leq m\} \\ &= \{n+1, n+2, \cdots, 2n\} \\ &\cup \{2n+2m+1, 2n+2m+2, \cdots, 2n+3m+1\} \end{aligned}$$

The edge labeling set is:

$$\begin{aligned} f(E) &= \{f(v_0 v_i) | 1 \leq i \leq n\} \cup \{f(u_0 u_j) | 1 \leq j \leq m\} \\ &\cup \{f(u_j u_{j+1}) | 1 \leq j \leq m-1\} \cup f(u_m u_1) \\ &= \{1, 2, \cdots, n\} \cup \{2n+1, 2n+2, \cdots, 2n+2m\} \end{aligned}$$

According to the AVRTL definition, $f(V) \cup f(E) \rightarrow [1, 2n + 3m + 1]$, and the joint graph $S_n \uparrow_{ab} W_m$ has a 1-degree vertices $v_1, v_2, \cdots, v_n$, 3-degree vertices $u_2, u_3, \cdots$, u_m, $n + 3$-degree vertices v_0/u_1, and an m-degree vertex u_0. It is only necessary to ensure that the labeling of the adjacent

3-degree vertices of the joint graph $S_n \uparrow_{ab} W_m$ are equal.

For the 3-degree vertices $u_2, u_3, \cdots, u_m$ adjacent to the joint graph $S_n \uparrow_{ab} W_m$, the sum of the labeling of each vertex is:

$$Sum(u_j|2 \le j \le m)$$
$$= f(u_j) + \sum_{ue \in E(u_j)} f(ue)|2 \le j \le m$$
$$= f(u_j) + f(u_{j-1}u_j) + f(u_ju_{j+1}) + f(u_0u_j)|2 \le j \le m$$
$$||f(u_m) + f(u_{m-1}u_m) + f(u_mu_{m+1}) + f(u_0u_m)$$
$$= 8n + 5m + 2$$

Thus, it is proven that when $n > 2$, $m > 3$, and $m \ne n + 3$, $S_n \uparrow_{ab} W_m$ is an AVRTL graph.

Theorem 9: For a joint graph $W_n \uparrow_{ab} W_m$, when $n > 3$, and $m > 3$, it is an AVRTL graph.

Proof: Let the vertex set of the joint graph $W_n \uparrow_{ab} W_m$ be denoted as $V(W_n \uparrow_{ab} W_m) = \{v_0, v_1, \cdots, v_n, u_0, u_1, \cdots, u_m\}$, and the edge set as $E(W_n \uparrow_{ab} W_m) = \{v_0v_i|1 \le i \le n\} \cup \{v_nv_1\} \cup \{v_iv_{i+1}|1 \le i \le n-1\} \cup \{u_0u_i|1 \le i \le m\} \cup \{u_mu_1\} \cup \{u_iu_{i+1}|1 \le i \le m-1\}$.

When $n > 3$, and $m > 3$, the AVRTL of $W_n \uparrow_{ab} W_m$ is:

$f(v_i) = 3n + 1 - i, 1 \le i \le n$
$f(v_iv_{i+1}) = i, 1 \le i < n$
$f(v_nv_1) = n$
$f(v_0v_1) = n + 1$
$f(v_0v_i) = 2n + 2 - i, 1 \le i \le n$
$f(u_0) = 3n + 3m + 1$
$f(u_i) = 3n + 3m + 1 - i, 1 \le i \le m$
$f(u_iu_{i+1}) = 3n + i, 1 \le i < m$
$f(u_mu_1) = 3n + m$
$f(u_0u_1) = 3n + m + 1$
$f(u_0u_i) = 3n + 2m + 2 - i, 2 \le i \le m$

In this case, the vertex labeling set of $W_n \uparrow_{ab} W_m$ is:

$$f(V) = \{f(v_i)|0 \le i \le n\} \cup \{f(u_j)|0 \le j \le m\}$$
$$= \{2n + 1, 2n + 2, \cdots, 3n\}$$
$$\cup \{3n + 2m + 1, 3n + 2m + 2, \cdots, 3n + 3m + 1\}$$

The edge labeling set is:

$$f(E) = \{f(v_0v_i)|1 \le i \le n\} \cup \{f(v_iv_{i+1})|1 \le i \le n-1\}$$
$$\cup f(v_nv_1) \cup \{f(u_0u_j)|1 \le j \le m\}$$
$$\cup \{f(u_ju_{j+1})|1 \le j \le m-1\} \cup f(u_mu_1)$$
$$= \{1, 2, \cdots, 2n\} \cup \{3n + 1, 3n + 2, \cdots, 3n + 2m\}$$

According to the AVRTL definition, $(V) \cup f(E) \to [1, 3n + 3m + 1]$, and the joint graph $W_n \uparrow_{ab} W_m$ has 3-degree vertices $v_1, v_2, \cdots, v_n, u_2, u_3, \cdots, u_m$, $n + 3$-degree vertices v_0/u_1, and an m-degree vertex u_0. It is only necessary to ensure that the labeling of the adjacent 3-degree vertices of the joint graph $W_n \uparrow_{ab} W_m$ are equal.

For the 3-degree vertices $v_1, v_2, \cdots, v_n$ adjacent to the joint graph $W_n \uparrow_{ab} W_m$, the sum of the labeling of each vertex is:

$$Sum(v_i|1 \le i \le n)$$
$$= (v_i|1 \le i \le n)$$
$$= f(v_i) + f(v_{i-1}v_i) + f(v_iv_{i+1}) + f(v_0v_i)|2 \le i \le n-1$$
$$||f(v_1) + f(v_nv_1) + f(v_1v_2) + f(v_0v_1)$$
$$||f(v_n) + f(v_{n-1}v_n) + f(v_nv_1) + f(v_0v_n)$$
$$= 5n + 2$$

For the 3-degree vertices $u_2, u_3, \cdots, u_m$ adjacent to the joint graph$W_n \uparrow_{ab} W_m$, the sum of the labeling of each vertex is:

$$Sum(u_j|2 \le j \le m)$$
$$= f(u_j) + \sum_{ue \in E(u_j)} f(ue)|2 \le j \le m$$
$$= f(u_j) + f(u_{j-1}u_j) + f(u_ju_{j+1}) + f(u_0u_j)|2 \le j \le m-1$$
$$||f(u_m) + f(u_{m-1}u_m) + f(u_mu_1) + f(u_0u_m)$$
$$= 12n + 5m + 2$$

Thus, it is proven that when $n > 3$, and $m > 3$, $W_n \uparrow_{ab} W_m$ is an AVRTL graph.

Partial graph labeling results for $W_n \uparrow_{ab} W_m$ are shown in Fig 10.

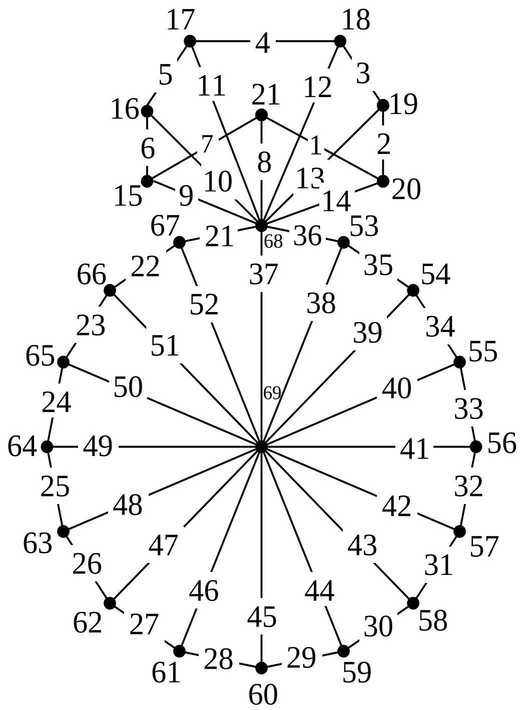


Fig. 10. Labeling Results for $W_7 \uparrow_{ab} W_{16}$.

Theorem 10: For a joint graph $C_n \uparrow_{ab} F_m$, when $n \ge 3$, $m > 4$, it is an AVRTL graph.

Proof: Let the vertex set of the joint graph $C_n \uparrow_{ab} F_m$, be denoted as $V(C_n \uparrow_{ab} F_m) = \{v_1, v_2, \cdots, v_n, u_0, u_1, \cdots, u_m\}$, and the edge set as $E(C_n \uparrow_{ab} F_m,) = \{u_0u_i|1 \le i \le m\} \cup \{v_iv_{i+1}|1 \le i \le n-1\} \cup \{u_iu_{i+1}|1 \le i \le m-1\} \cup \{v_nv_1\}$.

Case 1: When $n \ge 3$, $m > 4$ and $n \equiv 0(mod2)$, the AVRTL of $C_n \uparrow_{ab} F_m$ is:

$f(v_i) = 2n + 1 - i, 2 \le i \le n$
$f(v_nv_1) = n$

$$f(v_iv_{i+1}) = \begin{cases} \frac{n}{2} + \frac{i}{2}, i \equiv 0(mod2)\ and\ 2 \le i < n \\ i - \left\lfloor \frac{i}{2} \right\rfloor, i \equiv 1(mod2)\ and\ 1 \le i < n \end{cases}$$

$f(u_0) = 2n + 3m - 1$
$f(u_i) = 2n + 3m - 1 - i, 1 \le i \le m$
$f(u_iu_{i+1}) = 2n - 1 + i, 1 \le i < m$
$f(u_0u_i) = 2n + 2m - 1 - i$

In this case, the vertex labeling set of $C_n \uparrow_{ab} F_m$ is:

$$f(V) = \{f(v_i)|1 \le i \le n\} \cup \{f(u_j)|0 \le j \le m\}$$
$$= \{n + 2, n + 3, \cdots, 2n - 1\}$$
$$\cup \{2n + 2m - 1, 2n + 2m, \cdots, 2n + 3m - 1\}$$

The edge labeling set is:

$$f(E) = \{f(v_i v_{i+1}) | 1 \le i \le n-1\} \cup \{f(u_0 u_j) | 1 \le j \le m\}$$
$$\cup \{f(u_j u_{j+1}) | 1 \le j \le m-1\} \cup \{v_n v_1\}$$
$$= \{1,2,\cdots,n\} \cup \{2n, 2n+1, \cdots, 2n+2m-2\}$$

According to the AVRTL definition, $f(V) \cup f(E) \to [1, 2n+3m-1]$, and the joint graph $C_n \uparrow_{ab} F_m$ has 2-degree vertices $v_2, v_3, \cdots, v_n, u_m$, 3-degree vertices $u_2, u_3, \cdots, u_{m-1}$, 5-degree vertices v_1/u_1, and an m-degree vertex u_0. It is only necessary to ensure that the labeling of the adjacent 2-degree and 3-degree vertices of the joint graph $C_n \uparrow_{ab} F_m$ are equal.

For the 2-degree vertices $v_2, v_3, \cdots, v_n, u_m$ adjacent to the joint graph $C_n \uparrow_{ab} F_m$, the sum of the labeling of each vertex is:

$$Sum(v_i | 2 \le i \le n)$$
$$= f(v_i) + \sum_{ve \in E(v_i)} f(ve) | 2 \le i \le n$$
$$= f(v_i) + f(v_{i-1}v_i) + f(v_i v_{i+1}) | 2 \le i \le n-1$$
$$|| f(v_n) + f(v_{n-1}v_n) + f(v_n v_1)$$
$$= 3n - \left\lfloor \frac{n-1}{2} \right\rfloor$$

For the 3-degree vertices $u_2, u_3, \cdots, u_{m-1}$ adjacent to the joint graph $C_n \uparrow_{ab} F_m$, the sum of the labeling of each vertex is:

$$Sum(u_j | 2 \le j \le m-1)$$
$$= f(u_j) + \sum_{ue \in E(u_j)} f(ue) | 2 \le j \le m-1$$
$$= f(u_j) + f(u_{j-1}u_j) + f(u_j u_{j+1}) + f(u_0 u_j)$$
$$+ f(u_0 u_j) | 2 \le j \le m-1$$
$$= 8n + 5m - 5$$

Thus, it is proven that when $n \ge 3$, $m > 4$, and $n \equiv 0(mod2)$, $C_n \uparrow_{ab} F_m$ is an AVRTL graph.

Case 2: When $n \ge 3$, $m > 4$, and $n \equiv 1(mod2)$, the AVRTL of $C_n \uparrow_{ab} F_m$ is:

$$f(v_i) = 2n + 1 - i, 2 \le i \le n$$
$$f(v_n v_1) = n$$
$$f(v_i v_{i+1}) = \begin{cases} \frac{n}{2} + \frac{i}{2}, i \equiv 0(mod2) \; and \; 2 \le i < n \\ i - \left\lfloor \frac{i}{2} \right\rfloor, i \equiv 1(mod2) \; and \; 1 \le i < n \end{cases}$$
$$f(u_0) = 2n + 3m - 1$$
$$f(u_i) = 2n + 3m - 1 - i, 1 \le i \le m$$
$$f(u_i u_{i+1}) = 2n - 1 + i, 1 \le i < m$$
$$f(u_0 u_i) = 2n + 2m - 1 - i$$

In this case, the vertex labeling set of $C_n \uparrow_{ab} F_m$ is:

$$f(V) = \{f(v_i) | 1 \le i \le n\} \cup \{f(u_j) | 0 \le j \le m\}$$
$$= \{n+2, n+3, \cdots, 2n-1\}$$
$$\cup \{2n+2m-1, 2n+2m, \cdots, 2n+3m-1\}$$

The edge labeling set is:

$$f(E) = \{f(v_i v_{i+1}) | 1 \le i \le n-1\} \cup \{f(u_0 u_j) | 1 \le j \le m\}$$
$$\cup \{f(u_j u_{j+1}) | 1 \le j \le m-1\} \cup \{v_n v_1\}$$
$$= \{1,2,\cdots,n\} \cup \{2n, 2n+1, \cdots, 2n+2m-2\}$$

According to the AVRTL definition, $f(V) \cup f(E) \to [1, 2n+3m-1]$, and the joint graph $C_n \uparrow_{ab} F_m$ has 2-degree vertices $v_2, v_3, \cdots, v_n, u_m$, 3-degree vertices $u_2, u_3, \cdots, u_{m-1}$, 5-degree vertices v_1/u_1, and an m-degree vertex u_0. It is only necessary to ensure that the labeling of the adjacent 2-degree and 3-degree vertices of the joint graph $C_n \uparrow_{ab} F_m$ are equal.

For the 2-degree vertices $v_2, v_3, \cdots, v_n, u_m$ adjacent to the joint graph $C_n \uparrow_{ab} W_m$, the sum of the labeling of each vertex is:

$$Sum(v_i | 2 \le i \le n)$$
$$= f(v_i) + \sum_{ve \in E(v_i)} f(ve) | 2 \le i \le n$$
$$= f(v_i) + f(v_{i-1}v_i) + f(v_i v_{i+1}) | 2 \le i \le n-1$$
$$|| f(v_n) + f(v_{n-1}v_n) + f(v_n v_1)$$
$$= 3n - \left\lfloor \frac{n-2}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil + \frac{n-1}{2}$$

For the 3-degree vertices $u_2, u_3, \cdots, u_{m-1}$ adjacent to the joint graph $C_n \uparrow_{ab} F_m$, the sum of the labeling of each vertex is:

$$Sum(u_j | 2 \le j \le m-1)$$
$$= f(u_j) + \sum_{ue \in E(u_j)} f(ue) | 2 \le j \le m-1$$
$$= f(u_j) + f(u_{j-1}u_j) + f(u_j u_{j+1}) + f(u_0 u_j)$$
$$+ f(u_0 u_j) | 2 \le j \le m-1$$
$$= 8n + 5m - 5$$

Thus, it is proven that when $n \ge 3, m > 4$, and $n \equiv 1(mod2)$, $C_n \uparrow_{ab} F_m$ is an AVRTL graph.

In summary, for the joint graph $C_n \uparrow_{ab} F_m$, when $n \ge 3$, and $m > 4$, it is an AVRTL graph.

Partial graph labeling results for $C_n \uparrow_{ab} F_m$ are shown in Fig 11.

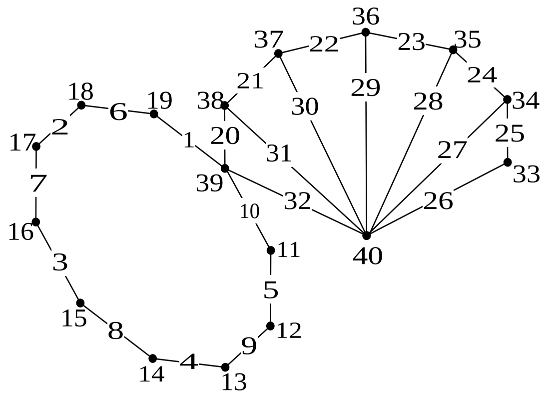


Fig. 11. Labeling Results for $C_{10} \uparrow_{ab} F_7$

Conjecture 1: If the subgraphs G_1 and G_2 are AVRTL graphs, then the joint graph $G_1 \uparrow_{ab} G_2$ is also an AVRTL graph.

V. CONCLUSION

For point-edge partitioning strategy problems such as logistics and supply chain management, social network analysis, energy network optimization, traffic planning, and power distribution networks, a heuristic search algorithm was designed by combining the algorithmic ideas of genetic algorithms and particle swarm optimization algorithms. For point-edge partitioning strategy problems such as logistics and supply chain management, social network analysis, energy network optimization, traffic planning, and power distribution networks, a heuristic search algorithm was designed by combining the algorithmic ideas of genetic algorithms and particle swarm optimization algorithms. Lastly, a conjecture is presented: If subgraphs G_1 and G_2 are AVRTL graphs, then the joint graph $G_1 \uparrow_{ab} G_2$ is also an AVRTL graph, where $\uparrow_{ab}$ is a graph operator.

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