From Continuous Time Random Walks to Multidimensional Conformable Diffusion Equation

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***Abstract*—This paper discusses how to derive multidimensional conformable diffusion equation from continuous time random walks process by employing fractional Laplace transform and the conformable fractional derivative defined by Khalil et al in [10]. The results show that there exists intimate relationship between the stretched exponential function and conformable diffusion equation involving the conformable fractional derivative. The relationship is analogous to the relationship between the exponential function and usual diffusion equation involving usual derivative and the relationship between the Mittag-Leffler function and subdiffusion or slow diffusion equation involving Caputo fractional derivative. Multidimensional conformable Fokker-Planck equation is also derived here. The equation describes diffusion process influenced by external force fields. Conformable semigroup associated with the solution to the conformable diffusion equation has similar properties to the properties of the semigroup associated with the solution to usual diffusion except the semigroup property. For sufficiently large time t, the graphs of the solution to the conformable diffusion equation and the subdiffusion equation involving Caputo fractional derivative are sufficiently close. Especially, for $1/2 < \alpha < 1$, the graphs are sufficiently close for $t > 0$. In presence of nonzero constant external force fields, the particle involved in the diffusion process moves faster toward the directions of the external force fields.**

***Index Terms*—Continuous time random walks, conformable fractional derivative, fractional Laplace Transform, multidimensional conformable diffusion equation, conformable semigroup, conformable Fokker-Planck equation.**

I. Introduction

Conformable fractional derivative was introduced by Khalil et al in [10]. The use of the conformable fractional derivative has some advantages. The first advantage is that the definition of the conformable fractional derivative is simpler than those of the other types of fractional derivative such as Riemann-Liouville and Caputo fractional derivatives. The second advantage is that the conformable fractional derivative satisfies some properties of usual derivative which are not satisfied by Riemann-Liouville and Caputo fractional derivatives such as the derivations of the product and quotient of two functions, chain rule, mean value theorem, and Rolle theorem. The third advantage is that some real phenomena can be better described by using the conformable fractional derivative. Some researches showed them. Bohner and Hatipoglu in [2] used the conformable fractional derivative for dynamic cobweb models. Chung in [3] employed the conformable fractional derivative to study the fractional Newtonian Mechanics. Ekici et al in [4] applied the conformable fractional derivative to study optical solitons in presence of Hamiltonian perturbation terms. Lazzo and Torres in [12] used the conformable fractional derivative to formulate an action principle for particle influenced by frictional forces. Martinez et al in [15] described electrical circuits by employing the conformable fractional derivative. Tong et al [20] used the conformable fractional derivative in a grey model to predict network public opinion. Wu et al in [21] employed the conformable fractional derivative for studying multivariate grey system model. Yang et al in [22] replaced usual time derivative in usual advection-diffusion equation by the conformable fractional time derivative and then used the advection-diffusion equation with the conformable fractional time derivative to model the transport process of CO_2-CH_4 in Estaillades carbonate care and sandpacks. Zhou et al in [23] also replaced usual time derivative in usual diffusion equation by the conformable fractional time derivative and then used the diffusion equation with the conformable fractional time derivative to model the transport process of chloride ions in three mix types of non-saturated concrete.

In theory of fractional ordinary differential equations with the conformable fractional derivative, Feng in [5,6] obtained several oscillatory results for the fractional ordinary

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differential equations. In [7], Feng improved (G'/G} method for seeking exact solutions to the fractional ordinary differential equations. Also in theory of fractional partial differential equations with the conformable fractional derivative, Feng in [8,9] introduced some approaches for solving the fractional partial differential equations. Furthermore, Martinez et al [11] extended the conformable fractional derivative to the context of the complex-valued function of a real variable.

In this paper, we derive multidimensional diffusion equation with the conformable fractional derivative. The equation is called multidimensional conformable diffusion equation. The equation is obtained from continuous time random walks process undergone by a particle which considers stochastic aspect i.e. the waiting time of the particle to move to certain direction. In the derivation of the equation, we find that there exists intimate relationship between the stretched exponential function and the conformable diffusion equation. The relationship is analogous to the relationship between the exponential function and usual diffusion equation involving usual derivative and the relationship between the Mittag-Leffler function and the subdiffusion equation involving Caputo fractional derivative. The three special functions determine the three waiting time probability density functions involved in the three continuous time random walks processes, respectively. Then, via the three continuous time random walks processes, the three types of the diffusion equations are obtained, respectively. As far as we know, there have not been researches studying how to derive diffusion equation involving the conformable fractional derivative. Especially, we have not found the equation's derivation which is carried out from continuous time random walks process considering the stochastic aspect. Via the continuous time random walks process, we may also obtain various types of diffusion equations depending on waiting time probability density functions, as stochastic aspects, involved in the diffusion processes. It is an advantage of the derivation of diffusion equations via the continuous time random walks process. Furthermore, we here also derive an equation called multidimensional conformable fractional Fokker-Planck equation. The equation describes the diffusion process influenced by external force fields. Also, there have not been researches studying the derivation of fractional Fokker Planck equation involving the conformable fractional derivative from continuous time random walks process.

In this paper, we also find that the solution operator associated with the solution to the conformable diffusion equation, called conformable semigroup, has similar properties to the properties of semigroup associated with the solution to usual diffusion equation except the semigroup property. Furthermore, we get the fact that, under certain conditions, the conformable diffusion equation involving the conformable fractional derivative can be used as an alternative model to describe subdiffusion or slow diffusion phenomenon besides the subdiffusion equation involving Caputo fractional derivative. Of course, the use of the conformable fractional derivative is more advantageous than that of Caputo fractional derivative in describing such processes because of the advantages of the use of the conformable fractional derivative as mentioned before.

This paper is composed of four sections. In section I, the motivation and novelty of our research are explained. In section II, some theories associated with our research are provided. The main results are presented in section III. Finally, in section IV, conclusion concerning our research is given.

II. Preliminaries

Definition 2.1. [10] The conformable fractional derivative of $f:[0,\infty)\to\mathbb{R}$ of order $\alpha\in(0,1)$ is defined by

$$T_t^\alpha f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}$$

for $t>0$. If the limit exists then f is said to be α-differentiable at t. If f is α-differentiable in $(0,a)$ with $a>0$ and $\lim_{t\to 0^+}T_t^\alpha f(t)$ exists then the conformable fractional derivative of f of order α at $t=0$ is defined by

$$T_t^\alpha f(0)=\lim_{t\to 0^+}T_t^\alpha f(t).$$

Theorem 2.1. [10] If $f:[0,\infty)\to\mathbb{R}$ is a differentiable function at $t>0$ then, for $\alpha\in(0,1]$,

$$T_t^\alpha f(t)=t^{1-\alpha}\frac{d}{dt}f(t).$$

If $\alpha=1$ then the conformable fractional derivative is usual derivative

Definition 2.2. [1] Given a function $f:[0,\infty)\to\mathbb{R}$ and $\alpha\in(0,1]$. The fractional Laplace transform of order α of f is defined by

$$\mathcal{L}_\alpha\{f(t)\}(s)=\tilde{f}_\alpha(s)=\int_0^\infty e^{-s\frac{t^\alpha}{\alpha}}t^{\alpha-1}f(t)\,dt.$$

If $\alpha=1$ then the fractional Laplace transform is the Laplace transform, that is

$$\mathcal{L}_1\{f(t)\}(s)=\mathcal{L}\{f(t)\}(s)=\tilde{f}(s)=\int_0^\infty e^{-st}f(t)\,dt.$$

Teorema 2.2. [1] If $f:[0,1)\to\mathbb{R}$ is a differentiable function and $\alpha\in(0,1]$ then

$$\mathcal{L}_\alpha\{T_t^\alpha f(s)\}(s)=s\tilde{f}_\alpha(s)-f(0).$$

Example 2.1. By using Definition 2.2, we have the fractional Laplace transform of some functions :

i. $\mathcal{L}_\alpha\{1\}(s)=\frac{1}{s}, s>0,$
ii. $\mathcal{L}_\alpha\{t^p\}(s)=\alpha^{\frac{p}{\alpha}}\Gamma\left(1+\frac{1}{\alpha}\right)s^{-1-\frac{p}{\alpha}}, s>0,$
iii. $\mathcal{L}_\alpha\left\{e^{\lambda\frac{t^\alpha}{\alpha}}\right\}(s)=\frac{1}{s-\lambda}.$

Next, it is not difficult to show the following theorem.

Theorem 2.3. If f and g are piecewise continuous functions on $[0,\infty)$ then the fractional Laplace transform of the convolution $f*g$ of f and g is given by

$$\mathcal{L}_\alpha\{(f*g)(t)\}(s)=\tilde{f}_\alpha(s)\tilde{g}_\alpha(s).$$

The following theorem provides a solution to an initial value problem of an ordinary differential equation involving the conformable fractional derivative.

Theorema 2.4. [10] The solution to the initial value problem

$$T_t^\alpha y(t) = \lambda y(t),\ \ y(0) = y_0,\ \ t > 0$$

is

$$y(t) = y_0 e^{\lambda \frac{t^\alpha}{\alpha}}.$$

The Fourier transform and its invers are defined as follows.

Definition 2.3. [11] Given a function g on $\mathbb{R}^n$. The Fourier transform of g is defined by

$$\mathcal{F}\{g(x)\}(k) = \hat{g}(k) = \int_{\mathbb{R}^n} e^{ik\cdot x}\, g(x)dx$$

and the invers of the Fourier transform of $\hat{g}$ on $\mathbb{R}^n$ is defined by

$$\mathcal{F}\{\hat{g}(k)\}(x) = g(x) = \frac{1}{2\pi}\int_{\mathbb{R}^n} e^{-ik\cdot x}\, \hat{g}(k)dk.$$

III. Main results

A. Multidimensional Conformable Diffusion Equation

We assume that a particle moving in a continuous time random walks process undergoes a sequence of jumps on $\mathbb{R}^n$ with a constant jump length Δx. We next suppose that $\phi(t)$ is the probability density function of the particle to jump after waiting time $t > 0$ and $T(x;\omega)$ is the probability density function of the particle to jump from position $x \in \mathbb{R}^n$ in direction $\omega \in S^{n-1} = \{\omega \in \mathbb{R}^n : |\omega| = 1\}$. Both the probability density functions satisfy

$$\int_0^\infty \phi(t)\, dt = 1, \quad \int_{S^{n-1}} T(x;\omega)\, d\omega = 1.$$

Following Othmer et al [11], if $Q_k(x,t)$ denotes the conditional probability of the particle to reach position x at time t after k jumps then

$$Q_k(x,t) = \int_0^t \int_{S^1} \phi(t-\tau)T(x-\omega\Delta x;\omega) \times Q_{k-1}(x-\omega\Delta x,\tau)d\omega d\tau.$$

The particle reaches x at t with the probability

$$Q(x,t) = \sum_{k=0}^\infty Q_k(x,t) = Q_0(x,t) + \int_0^t \int_{S^{n-1}} \phi(t-\tau)T(x-\omega\Delta x;\omega)Q(x-\omega\Delta x,\tau)d\omega d\tau.$$

Here, $Q_0(x,t)$ is Dirac delta function, that is $Q_0(x,t) = \delta(x)\delta(t)$. Then

$$Q(x,t) = \delta(x)\delta(t) + \int_0^t \int_{S^{n-1}} \phi(t-\tau)T(x-\omega\Delta x;\omega)Q(x-\omega\Delta x,\tau)d\omega d\tau. \quad (1)$$

If $q(x,t)$ stands for the probability of the particle to be at x at t with the initial position $x = 0$ and time $t = 0$ then

$$q(x,t) = \int_0^t \Phi(t,\tau;x)Q(x,\tau)d\tau \quad (2)$$

where $\Phi(t,\tau;x)$ is the probability of the particle to reach x at $\tau < t$ and does not jump during the time interval $t-\tau$. We next assume that

$$\Phi(t,\tau;x) = \Phi(t-\tau)$$

with

$$\Phi(t) = \int_t^\infty \phi(r)dr = 1 - \int_0^t \phi(r)dr \quad (3)$$

which means that the particle does not jump during the time interval (0,*t*). Consequently, based on the equation (1), (2), and (3),

$$q(x,t) = \left(1 - \int_0^t \phi(r)dr\right)\delta(x) + \int_0^t \int_{S^{n-1}} \phi(t-\tau)T(x-\omega\Delta x,\omega)q(x-\omega\Delta x,\tau)\, d\omega d\tau. \quad (4)$$

By applying the fractional Laplace and Fourier transforms to the equation (4), we have

$$\tilde{q}_\alpha(x,s) = \delta(x)\frac{1-\tilde{\phi}_\alpha(s)}{s} + \tilde{\phi}_\alpha(s)\int_{S^{n-1}} \tilde{q}_\alpha(x-\omega\Delta x, s)\, T(x-\omega\Delta x;\omega)d\omega. \quad (5)$$

If both the sides of the equation (5) are divided by $1-\tilde{\phi}_\alpha(s)$ with $\tilde{\phi}_\alpha(s) \neq 1$ and then subtracted by $\tilde{q}_\alpha(x,s)\tilde{\phi}_\alpha(s)\left(1-\tilde{\phi}_\alpha(s)\right)^{-1}$ then we get

$$s\tilde{q}_\alpha(x,s) - \delta(x) = s\widetilde{H}_\alpha(s) \times \left(\int_{S^{n-1}} \tilde{q}_\alpha(x-\omega\Delta x, s)\, T(x-\omega\Delta x;\omega)d\omega - \tilde{q}_\alpha(x,s)\right) \quad (6)$$

with

$$\widetilde{H}_\alpha(s) = \frac{\tilde{\phi}_\alpha(s)}{1-\tilde{\phi}_\alpha(s)} = \frac{1-s\widetilde{\Phi}_\alpha(s)}{s\widetilde{\Phi}_\alpha(s)}. \quad (7)$$

If the waiting time density probability $\phi(t)$ is

$$\phi(t) = \frac{t^{\alpha-1}}{\lambda^\alpha}E_\alpha\left(-\left(\frac{t}{\lambda}\right)^\alpha\right), 0 < \alpha < 1, \lambda > 0, \quad (8)$$

where E_α is the Mittag-Leffler function

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n+1)}, \alpha > 0, z \in \mathbb{C},$$

then the survival function $\Phi(t)$ associated with $\phi(t)$ expressed by the equation (8) is

$$\Phi(t) = E_\alpha\left(-\left(\frac{t}{\lambda}\right)^\alpha\right), 0 < \alpha < 1, \lambda > 0. \quad (9)$$

The Laplace transform of the waiting time density probability (8) is ([18])

$$\tilde{\phi}(s) = \frac{1}{1+\lambda^\alpha s^\alpha}.$$

Consequently,

$$\widetilde{H}(s) = \frac{\tilde{\phi}(s)}{1-\tilde{\phi}(s)} = \frac{1}{\lambda^\alpha s^\alpha}. \quad (10)$$

It implies that

$$H(t) = \frac{1}{\lambda^\alpha}\cdot\frac{t^{\alpha-1}}{\Gamma(\alpha)}. \quad (11)$$

If $\alpha = 1$ then the equation (6) is reduced to

$$s\tilde{q}(x,s) - \delta(x) = s\widetilde{H}(s) \times \left(\int_{S^{n-1}} \tilde{q}(x-\omega\Delta x, s)\, T(x-\omega\Delta x;\omega)d\omega - \tilde{q}(x,s)\right). \quad (12)$$

Then by dividing both the sides of the equation (12) by s and using the equation (10), we have

$$\tilde{q}(x,s)-\frac{\delta(x)}{s}=\frac{1}{\lambda^{\alpha}s^{\alpha}}$$
$$\times\left(\int_{S^{n-1}}\tilde{q}(x-\omega\Delta x,s)\,T(x-\omega\Delta x;\omega)d\omega-\tilde{q}(x,s)\right). \tag{13}$$

By applying the invers of the Laplace transform to the equation (13) and using the equation (11), we obtain

$$q(x,t)-q(x,0)=\frac{1}{\lambda^{\alpha}}\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$$
$$\times\int_{S^{n-1}}T(x-\omega\Delta x;\omega)[q(x-\omega\Delta x,\tau)-q(x,\tau)]\,d\omega d\tau$$
$$=\frac{1}{\lambda^{\alpha}}I_t^{\alpha}\int_{S^{n-1}}T(x-\omega\Delta x;\omega)[q(x-\omega\Delta x,t)-q(x,t)]\,d\omega \tag{14}$$

where I_t^{α} is the Riemann-Liouville fractional integral operator defined by ([6,12])

$$I_t^{\alpha}f(t)=\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau)d\tau.$$

By applying the Caputo fractional derivative operator d^{α}/dt^{α} defined by ([11,18])

$$\frac{d^{\alpha}}{dt^{\alpha}}f(t)=\int_0^t\frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\cdot\frac{d}{d\tau}f(\tau)d\tau,$$

to the equation (14) and since ([6,12])

$$\frac{d^{\alpha}}{dt^{\alpha}}I_t^{\alpha}f(t)=f(t)$$

we have

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}q(x,t)=\frac{1}{\lambda^{\alpha}}\int_{S^{n-1}}T(x-\omega\Delta x;\omega)$$
$$\times[q(x-\omega\Delta x,t)-q(x,t)]d\omega. \tag{15}$$

We now assume that $T(x;\omega)$ is a constant which means that the particle in the random walks process moves without any external force field. Since

$$\int_{S^{n-1}}T(x;\omega)d\omega=1$$

and

$$\int_{S^{n-1}}d\omega=|S^{n-1}|=\frac{2^{n/2}}{\Gamma(n/2)}$$

then

$$T(x;\omega)=\frac{1}{|S^{n-1}|}=\frac{\Gamma(n/2)}{2^{n/2}}$$

where Γ is Gamma function defined by

$$\Gamma(z)=\int_0^{\infty}r^{z-1}e^{-z}\,dz,\ \ \mathrm{Re}(z)>0.$$

Then equation (15) is reduced to

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}q(x,t)=\frac{1}{\lambda^{\alpha}|S^{n-1}|}\int_{S^{n-1}}[q(x-\omega\Delta x,t)-q(x,t)]\,d\omega. \tag{16}$$

We next consider the following lemma.

Lemma 3.1. [19] If $f=f(x)$ is a continuously twice differentiable function then

$$\int_{S^{n-1}}[f(x+\omega\Delta x)-f(x)]d\omega$$
$$=\frac{1}{2n}|S^{n-1}|(\Delta x)^2\Delta f(x)+o((\Delta x)^2)$$

as $\Delta x\to 0$.

By Lemma 3.1, the equation (16) becomes

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}q(x,t)=\frac{(\Delta x)^2}{2n\lambda^{\alpha}}\Delta q(x,t)+o((\Delta x)^2) \tag{17}$$

as $\Delta x\to 0$. If $\Delta x\to 0$, $\lambda\to 0$, and $(\Delta\mathbf{x})^2/\lambda^{\alpha}$ is kept finite, the equation (17) is reduced to

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}q(x,t)=K_{\alpha}\Delta q(x,t) \tag{18}$$

with

$$K_{\alpha}=\frac{(\Delta x)^2}{2n\lambda^{\alpha}}.$$

The equation (18) is called subdiffusion equation involving Caputo fractional derivative. The subdiffusion equation (18) is associated with the survival function $\Phi(t)$ expressed by the equation (9).

We now turn back to the equation (6) and consider again the survival function $\Phi(t)$ expressed by the equation (9). If we take only the first two terms of (9), that is

$$E_{\alpha}\left(-\left(\frac{t}{\lambda}\right)^{\alpha}\right)\approx 1-\frac{t^{\alpha}}{\lambda^{\alpha}\Gamma(1+\alpha)}\approx e^{-\frac{t^{\alpha}}{\lambda^{\alpha}\Gamma(1+\alpha)}},$$

Then we have which is called stretched exponential function. If we use the stretched exponential function as a survival function, that is

$$\Phi(t)=e^{-\frac{t^{\alpha}}{\lambda^{\alpha}\Gamma(1+\alpha)}} \tag{19}$$

then, by Example 2.1(iii), the fractional Laplace transform of the survival function $\Phi(t)$ expressed by equation (19) is

$$\tilde{\Phi}_{\alpha}(s)=\frac{\lambda^{\alpha}\Gamma(\alpha)}{\lambda^{\alpha}\Gamma(\alpha)s+1}. \tag{20}$$

By the equation (7) and (20), it follows that

$$\tilde{H}_{\alpha}(s)=\frac{1}{s\lambda^{\alpha}\Gamma(\alpha)}. \tag{21}$$

By substituting the equation (21) into the equation (6), we have

$$s\tilde{q}_{\alpha}(x,s)-\delta(x)=\frac{1}{\lambda^{\alpha}\Gamma(\alpha)}$$
$$\times\left(\int_{S^{n-1}}T(x-\omega\Delta x;\omega)\tilde{q}_{\alpha}(x-\omega\Delta x,s)\,d\omega-\tilde{q}_{\alpha}(x,s)\right)$$
$$=\frac{1}{\lambda^{\alpha}\Gamma(\alpha)}\int_{S^{n-1}}T(x-\omega\Delta x;\omega)$$
$$\times\big(\tilde{q}_{\alpha}(x-\omega\Delta x,s)-\tilde{q}_{\alpha}(x,s)\big)d\omega. \tag{22}$$

If T is a constant then the equation (22) is reduced to

$$s\tilde{q}_{\alpha}(x,s)-\delta(x)=\frac{1}{\lambda^{\alpha}\Gamma(\alpha)|S^{n-1}|}$$
$$\times\int_{S^{n-1}}[\tilde{q}_{\alpha}(x-\omega\Delta x,s)-\tilde{q}_{\alpha}(x,s)]d\omega. \tag{23}$$

By using Lemma 3.1, the equation (23) is reduced to

$$s\tilde{q}_{\alpha}(x,s)-\delta(x)$$
$$=\frac{1}{\lambda^{\alpha}\Gamma(\alpha)}\frac{(\Delta x)^2}{2n}\Delta\tilde{q}_{\alpha}(x,s)+o((\Delta x)^2) \tag{24}$$

as $\Delta\mathbf{x}\to 0$. If $\Delta x\to 0$, $\lambda\to 0$, and $(\Delta\mathbf{x})^2/\lambda^{\alpha}$ is kept finite then the equation (24) becomes

$$s\tilde{q}_\alpha(x,s) - \delta(x) = \frac{(\Delta x)^2}{2n\lambda^\alpha \Gamma(\alpha)} \Delta \tilde{q}_\alpha(x,s). \qquad (25)$$

By applying the invers of the fractional Laplace transform to the equation (25) and using Theorem 2.2, we obtain

$$T_t^\alpha q(x,t) = D_\alpha \Delta q(x,t) \qquad (26)$$

with

$$D_\alpha = \frac{(\Delta x)^2}{2n\lambda^\alpha \Gamma(\alpha)}.$$

We call the equation (26) multidimensional conformable diffusion equation.

For $\alpha = 1$, the equation (18) or (26) is reduced to be usual diffusion equation

$$\frac{\partial}{\partial t} q(x,t) = D\Delta q(x,t) \qquad (27)$$

which is associated with the survival function $\Phi(t)$ of the form exponential function

$$\Phi(t) = e^{-\frac{t}{\lambda}} \qquad (28)$$

and

$$D = \frac{(\Delta x)^2}{2n\lambda}.$$

B. Fundamental Solution to Multidimensional Conformable Diffusion Equation

The fundamental solution to the subdiffusion equation (18) involving Caputo fractional derivative for $x \in \mathbb{R}$ and $t > 0$ is ([13,16,18])

$$q(x,t) = \frac{1}{\sqrt{4K_\alpha t^\alpha}} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma\left(-\frac{1}{2}\alpha n + 1 - \frac{1}{2}\alpha\right)} \left(\frac{x^2}{K_\alpha t^\alpha}\right)^{\frac{j}{2}}. \qquad (29)$$

We now find the fundamental solution to the multidimensional conformable diffusion equation (26)

$$T_t^\alpha q(x,t) = D_\alpha \Delta q(x,t), x \in \mathbb{R}^n, t > 0,$$
$$q(x,0) = \delta(x).$$

By applying Fourier transform to equation (26), we have

$$T_t^\alpha \hat{q}(k,t) = -D_\alpha |k|^2\, \hat{q}(k,t),$$
$$\hat{q}(k,0) = 1.$$

By Theorem 2.4, we obtain

$$\hat{q}(k,t) = e^{-\mathrm{D}_\alpha |k|^2 \frac{t^\alpha}{\alpha}}.$$

Since for $x, k \in \mathbb{R}$,

$$\mathcal{F}\left\{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{|x|^2}{2\sigma^2}}\right\}(k) = e^{-\frac{\sigma^2 |k|^2}{2}},$$

we obtain the fundamental solution to the multidimensional conformable diffusion equation (26), that is

$$q(x,t) = \left(\frac{\alpha}{4\pi D_\alpha t^\alpha}\right)^{n/2} e^{-\frac{\alpha}{4D_\alpha t^\alpha}|x|^2}. \qquad (30)$$

Then the mean square displacement (MSD) of the particle moving in the process is

$$\langle x^2(t)\rangle = \int_{\mathbb{R}^n} |x|^2 \left(\frac{\alpha}{4\pi D_\alpha t^\alpha}\right)^{n/2} e^{-\frac{\alpha}{4D_\alpha t^\alpha}|x|^2} dx$$
$$= \frac{2n\mathrm{D}_\alpha t^\alpha}{\alpha}. \qquad (31)$$

It is similar to the MSD of the particle moving in the diffusion process described by the subdiffusion equation (18) with the survival function $\Phi(t)$ expressed by equation (9) ([16]). For $\alpha = 1$, the fundamental solution (29) and (30) are the fundamental solution to usual diffusion equation (27) with MSD which is linearly proportional to t.

C. Multidimensional Conformable Fokker-Planck Equation

In this section, we assume that $T(x;\omega)$ is not a constant by assuming

$$T(x;\omega) - T(x;-\omega) = F(x;\omega) \qquad (32)$$
$$T(x;\omega) + T(x;-\omega) = G(x;\omega) \qquad (33)$$

where

$$F(x;\omega) = \frac{\Delta x}{|S^{n-1}|} \omega \cdot f(x)$$

with $f(x) = \big(f_1(x), \dots, f_n(x)\big) \in \mathbb{R}^n$ is an external force field influencing the particle's jumps at position x, $f_i\colon \mathbb{R} \to \mathbb{R}$, $i = 1,2, \dots n$, and $G(x\colon\omega)$ is a constant. Then, since

$$\int_{S^{n-1}} T(x;\omega) d\omega = 1,$$

we have

$$G(x\colon\omega) = \frac{2}{|S^{n-1}|}.$$

Next, by employing Taylor Series expansion, we get

$$T(x+\omega\Delta x;-\omega)\tilde{q}_\alpha(x+\omega\Delta x,s)$$
$$= T(x;-\omega)\tilde{q}_\alpha(x,s) + \Delta x\{\omega\cdot\nabla\}T(x;-\omega)\tilde{q}_\alpha(x,s)$$
$$+\frac{1}{2}(\Delta x)^2\{\omega\cdot\nabla\}^2 T(x;-\omega)\tilde{q}_\alpha(x,s) + o((\Delta x)^2) \qquad (34)$$

and

$$T(x-\omega\Delta x;\omega)\tilde{q}_\alpha(x-\omega\Delta x,s)$$
$$= T(x;\omega)\tilde{q}_\alpha(x,s) - \Delta x\{\omega\cdot\nabla\}T(x;\omega)\tilde{q}_\alpha(x,s)$$
$$+\frac{1}{2}(\Delta x)^2\{\omega\cdot\nabla\}^2 T(x;\omega)\tilde{q}_\alpha(x,s) + o((\Delta x)^2). \qquad (35)$$

By the equation (32), (33), (34), and (35), we obtain

$$T(x+\omega\Delta x;-\omega)\tilde{q}_\alpha(x+\omega\Delta x,s)$$
$$+ T(x-\omega\Delta x;\omega)\tilde{q}_\alpha(x-\omega\Delta x,s)$$
$$= G(x;\omega)\tilde{q}_\alpha(x,s) - \Delta x\{\omega\cdot\nabla\}F(x;\omega)\tilde{q}_\alpha(x,s)$$
$$+\frac{1}{2}(\Delta x)^2\{\omega\cdot\nabla\}^2 G(x;\omega)\tilde{q}_\alpha(x,s) + o((\Delta x)^2) \qquad (36)$$

Since

$$\int_{S^{n-1}} \tilde{q}_\alpha(x-\omega\Delta x,s)\, T(x-\omega\Delta x;\omega) d\omega$$
$$= \frac{1}{2}\int_{S^{n-1}} [T(x+\omega\Delta x;-\omega)\tilde{q}_\alpha(x+\omega\Delta x,s)$$
$$+T(x-\omega\Delta x;\omega)\tilde{q}_\alpha(x-\omega\Delta x,s)]d\omega,$$

and

$$\int_{S^{n-1}} \omega_i d\omega = 0, \;\; i = 1,2,\dots,n,$$

$$\int_{S^{n-1}} \omega_i\omega_j d\omega = \delta_{ij}\frac{|S^{n-1}|}{n}, \;\; i = 1,2,\dots,n$$

where δ_{ij} is the kronecker delta function, then by the equation (36) we have

$$\int_{S^{n-1}} \tilde{q}_\alpha(x-\omega\Delta x,s)\, T(x-\omega\Delta x;\omega) d\omega - \tilde{q}_\alpha(x,s)$$
$$= \frac{(\Delta x)^2}{2n}[\Delta\tilde{q}_\alpha(x,s) - \nabla\cdot f(x)\tilde{q}_\alpha(x,s)] + o((\Delta x)^2). \qquad (37)$$

By the equation (37), the equation (22) is reduced to

$$s\tilde{q}_\alpha(x,s) - \delta(x)$$
$$= \frac{(\Delta x)^2}{2n\lambda^\alpha\Gamma(\alpha)}[\Delta\tilde{q}_\alpha(x,s) - \nabla\cdot f(x)\tilde{q}_\alpha(x,s)] + o((\Delta x)^2). \qquad (38)$$

By applying the invers of fractional Laplace transform to the equation (38), we get

$$T_t^\alpha q(x,t) = \frac{(\Delta x)^2}{2n\lambda^\alpha \Gamma(\alpha)}[\Delta q(x,t) - \nabla \cdot f(x) q(x,t)] + o((\Delta x)^2). \quad (39)$$

If $\Delta x \to 0$, $\lambda \to 0$, and $(\Delta x)^2/\lambda^\alpha$ is kept finite then the equation (39) is reduced to

$$T_t^\alpha q(x,t) = D_\alpha[\Delta q(x,t) - \nabla \cdot f(x) q(x,t)]. \quad (40)$$

The equation (40) is called multidimensional conformable Fokker-Planck equation.

If $f(x) = 0$ or there are no external force fields involved in the process then we have the conformable diffusion equation (26). If $f(x) = v$ where $v = (v_1, \dots, v_n)$ with v_i is a constant for $i = 1,2,\dots,n$ then the equation (40) becomes

$$T_t^\alpha q(x,t) = D_\alpha[\Delta q(x,t) - v \cdot \nabla q(x,t)]. \quad (41)$$

The equation (41) is called multidimensional conformable advection-diffusion equation. In [22], the equation (41) was used to study fluid flow and salute transport in porous media.

We next find a solution to the equation (41)

$$T_t^\alpha q(x,t) = D_\alpha[\Delta q(x,t) - v \cdot \nabla q(x,t)]$$
$$q(x,0) = \delta(x).$$

By applying Fourier transform to equation (26), we have

$$T_t^\alpha \hat{q}(k,t) = -D_\alpha(|k|^2 - iv \cdot k)\, \hat{q}(k,t),$$
$$\hat{q}(k,0) = 1.$$

By Theorem 2.4, we obtain

$$\hat{q}(k,t) = e^{-D_\alpha(|k|^2 - iv\cdot k)\frac{t^\alpha}{\alpha}}.$$

Since for $x, k \in \mathbb{R}$

$$\mathcal{F}\left\{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{|x-\mu|^2}{2\sigma^2}}\right\}(k) = e^{-\frac{\sigma^2|k|^2}{2} + i\mu k},$$

then

$$q(x,t) = \left(\frac{\alpha}{4\pi D_\alpha t^\alpha}\right)^{\frac{n}{2}} e^{-\frac{\alpha}{4D_\alpha t^\alpha}\left|x - D_\alpha \frac{t^\alpha}{\alpha} v\right|^2}. \quad (42)$$

D. Conformable Semigroup

We now consider the multidimensional conformable diffusion equation

$$T_t^\alpha q(x,t) = \Delta q(x,t), x \in \mathbb{R}^n, t > 0, \quad (43)$$
$$q(x,0) = q_0(x), x \in \mathbb{R}^n.$$

By applying the Fourier transform and its invers to the equation (43), we obtain the solution to the equation (42), that is

$${}^\alpha q(x,t) = \int_{\mathbb{R}^n} G_\alpha(x-y,t) q_0(y)\, dy \quad (44)$$

with

$$G_\alpha(x,t) = \left(\frac{\alpha}{4\pi t^\alpha}\right)^{n/2} e^{-\frac{\alpha}{4t^\alpha}|x|^2}. \quad (45)$$

Note that $G_\alpha(x,t)$ is the fundamental solution to the conformable diffusion equation (43). If $\alpha = 1$, $G_\alpha(x,t)$ is the fundamental solution to usual diffusion equation

$$\frac{\partial}{\partial t} q(x,t) = \Delta q(x,t), x \in \mathbb{R}^n, t > 0, \quad (46)$$
$$q(x,0) = q_0(x), x \in \mathbb{R}^n$$

with

$$D = \frac{(\Delta x)^2}{2n\lambda}.$$

The solution $q(x,t)$ to the diffusion equation (46) is

$$q(x,t) = \int_{\mathbb{R}^n} G(x-y,t) q_0(y)\, dy \quad (47)$$

with

$$G(x,t) = \left(\frac{1}{4\pi t}\right)^{n/2} e^{-\frac{1}{4t}|x|^2}. \quad (48)$$

Observe that $G(x,t) = G_1(x,t) > 0$ for $x \in \mathbb{R}^n, t > 0$. Regarding $G(x,t)$ called the n-dimensional Gaussian Kernel and $q(x,t)$, we have the following theorem.

Theorem 2.5. [19] Let $G(x,t)$ be the Gaussian kernel expressed by (48) and $q(x,t)$ be the solution to the equation (46) expressed by the equation (47). The following statements hold.

(a) If $q_0(x) \geq 0$ then $q(x,t) \geq 0$ for $x \in \mathbb{R}^n, t > 0$;

(b) If $q_0(x) > 0$ then $q(x,t) > 0$ for $x \in \mathbb{R}^n, t > 0$;

(c) If $q_{01}(x) \geq q_{02}(x)$ then $q_1(x,t) \geq q_2(x,t)$ for $x \in \mathbb{R}^n, t > 0$ where $q_i(x,t)$, $i = 1,2$, is the solution to the equation (46) with the initial value $q_{0i}(x)$, $i = 1,2$;

(d) If $q_{01}(x) \geq q_{02}(x)$ and $q_{01}(x) \neq q_{02}(x)$ then $q_1(x,t) > q_2(x,t)$ for $x \in \mathbb{R}^n, t > 0$.

Consider that $q(x,t) = (G(\cdot,t) * q_0)(x)$. Then we have the following theorem.

Theorem 2.6. [19] Let $q_0 \in L^p(\mathbb{R}^n)$. Then :

(a) $\|q(\cdot,t)\|_p \leq \|q_0\|_p$, $1 \leq p < \infty$;

(b) If $q(\cdot,t) = S(t)q_0$ then $\|S(t)\|_{L^p(\mathbb{R}^n), L^p(\mathbb{R}^n)} \leq 1$;

(c) $S(t)S(s) = S(t+s)$, $t, s \geq 0$.

The operator $S(t)$ is the semigroup of Laplacian operator Δ involved in the equation (46). The statement in Theorem 2.6(b) means that the mapping

$$S(t): L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

is a contraction ([19]). The property in Theorem 2.6(c) is called the semigroup property. We also have the behavior of $S(t)$ at $t \to 0^+$ as stated in the following theorem.

Theorem 2.7. [19] If $1 \leq p < \infty$ and $q_0 \in L^p(\mathbb{R}^n)$ then

$$\lim_{t \to 0^+} \|S(t)q_0 - q_0\| = 0.$$

We now consider $G_\alpha(x,t)$ and ${}^\alpha q(x,t)$ which correspond to the conformable diffusion equation (42). We call $G_\alpha(x,t)$ the n-dimensional conformable Gaussian kernel. Observe that $G_\alpha(x,t) = G(x, t^\alpha/\alpha)$, ${}^\alpha q(x,t) = q(x, t^\alpha/\alpha)$, and ${}^\alpha q(x,t) = (G_\alpha(\cdot,t) * q_0)(x)$. Based on Theorem 2.5, it is not difficult to show that the properties of ${}^\alpha q(x,t)$ is similar to those of $q(x,t)$ as written in the following theorem.

Theorem 2.8. Let $G_\alpha(x,t)$ be the multidimensional conformable Gaussian kernel expressed by the equation (45) and ${}^\alpha q(x,t)$ be the solution to the diffusion equation (43) expressed by the equation (44). The following statements hold.

(a) If $q_0(x) \geq 0$ then ${}^\alpha q(x,t) \geq 0$ for $x \in \mathbb{R}^n, t > 0$;

(b) If $q_0(x) > 0$ then ${}^\alpha q(x,t) > 0$ for $x \in \mathbb{R}^n, t > 0$;

(c) If $q_{01}(x) \geq q_{02}(x)$ then ${}^\alpha q_1(x,t) \geq {}^\alpha q_2(x,t)$ for $x \in \mathbb{R}^n, t > 0$ where ${}^\alpha q_i(x,t)$, $i = 1,2$, is the solution to the conformable diffusion equation (43) with the initial value $q_{0i}(x)$, $i = 1,2$;

(d) If $q_{01}(x) \geq q_{02}(x)$ and $q_{01}(x) \neq q_{02}(x)$ then ${}^{\alpha}q_1(x,t) > {}^{\alpha}q_2(x,t)$ for $x \in \mathbb{R}^n, t > 0$.

If ${}^{\alpha}q(\cdot,t) = S_\alpha(t)q_o$ then $S_\alpha(t) = S(t^\alpha/\alpha)$. Based on Theorem 2.6(c), for $t, s \geq 0$,

$$\begin{aligned} S_\alpha(t)S_\alpha(s) &= S\left(\frac{t^\alpha}{\alpha}\right)S\left(\frac{s^\alpha}{\alpha}\right) \\ &= S\left(\frac{t^\alpha}{\alpha}+\frac{s^\alpha}{\alpha}\right) \\ &= S_\alpha\left((t^\alpha+s^\alpha)^{1/\alpha}\right) \end{aligned}$$

or

$$S_\alpha\left(t^{1/\alpha}\right)S_\alpha\left(s^{1/\alpha}\right) = S_\alpha\left((t+s)^{1/\alpha}\right). \qquad (49)$$

We call $S_\alpha(t)$ the conformable semigroup of Laplacian operator Δ involved in the conformable diffusion equation (43). Based on the equation (48), the conformable semigroup $S_\alpha(t)$ does not satisfy the semigroup property as stated in Theorem 2.6(c). We call the equation (49) the conformable semigroup property. Furthermore, it is also not difficult to show that ${}^{\alpha}q(x,t)$ and $S_\alpha(t)$ have similar properties to those mentioned in Theorem 2.6(a,b). Thus we obtain the following theorem.

Theorem 2.9. Let $q_0 \in L^p(\mathbb{R}^n)$. Then :

(a) $\left\| {}^{\alpha}q(\cdot,t)\right\|_p \leq \|q_0\|_p$, $1 \leq p < \infty$;

(b) If ${}^{\alpha}q(\cdot,t) = S_\alpha(t)q_0$ then $\|S_\alpha(t)\|_{L^p(\mathbb{R}^n),L^p(\mathbb{R}^n)} \leq 1$;

(c) $S_\alpha\left(t^{1/\alpha}\right)S_\alpha\left(s^{1/\alpha}\right) = S_\alpha\left((t+s)^{1/\alpha}\right)$, $t, s \geq 0$.

Since $S_\alpha(t) = S(t^\alpha/\alpha)$, based on Theorem 2.6(b) , we also have that the mapping

$$S_\alpha(t): L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

is a contraction. We next get that, by Theorem 2.7,

$$\begin{aligned} \lim_{t\to 0^+}\|S_\alpha(t)q_0 - q_0\| &= \lim_{t\to 0^+}\left\|S\left(\frac{t^\alpha}{\alpha}\right)q_0 - q_0\right\| \\ &= \lim_{\frac{t^\alpha}{\alpha}\to 0^+}\left\|S\left(\frac{t^\alpha}{\alpha}\right)q_0 - q_0\right\| \\ &= 0. \end{aligned}$$

Thus, we have the following theorem which gives the behavior of $S_\alpha(t)$ at $t \to 0^+$.

Theorem 2.10. If $1 \leq p < \infty$ and $q_0 \in L^p(\mathbb{R}^n)$ then

$$\lim_{t\to 0^+}\|S_\alpha(t)q_0 - q_0\| = 0.$$

By similar way, we can derive the solution to the conformable advection-diffusion equation

$$\begin{aligned} T_t^\alpha q(x,t) &= \Delta q(x,t) - v\cdot\nabla q(x,t) \qquad (50) \\ q(x,0) &= q_0(x). \end{aligned}$$

The solution to the equation (50) is

$${}^{\alpha}q^*(x,t) = \int_{\mathbb{R}^n} G_\alpha(x-y,t)q_0(y)\,dy$$

with

$$G_\alpha^*(x,t) = \left(\frac{\alpha}{4\pi t^\alpha}\right)^{n/2} e^{-\frac{\alpha}{4t^\alpha}\left|x-\frac{t^\alpha}{\alpha}v\right|^2}.$$

Note that $G_\alpha^*(x,t) = G_\alpha\left(x - \frac{t^\alpha}{\alpha}v, t\right)$ and ${}^{\alpha}q^*(x,t) = {}^{\alpha}q\left(x - \frac{t^\alpha}{\alpha}v, t\right)$. If ${}^{\alpha}q^*(\cdot,t) = S_\alpha^*(t)q_0$ then ${}^{\alpha}q^*(\cdot,t)$ and $S_\alpha^*(t)$ satisfy all properties stated in Theorem 2.8-2.10. The operator $S_\alpha^*(t)$ is the conformable semigroup of the linear operator $\Delta - v\cdot\nabla$ involved in the equation (50).

E. Graphs of The Fundamental Solution to Multidimensional Conformable Diffusion Equation

We here compare the graph of the fundamental solution (29) to the subdiffusion equation (18) colored by red and the graph of the fundamental solution (30) to the conformable diffusion equation (26) colored by blue. We first describe them for one-dimensional case with taking $(\Delta x)^2/(2\lambda^\alpha) = 1$.

In Fig. 1-3, we compare the one-dimensional graphs of both the fundamental solutions for $\alpha = 1/4$ and $t = 1/4, 4, 10^6$. We observe that for sufficiently large t, both the graphs are sufficiently close.

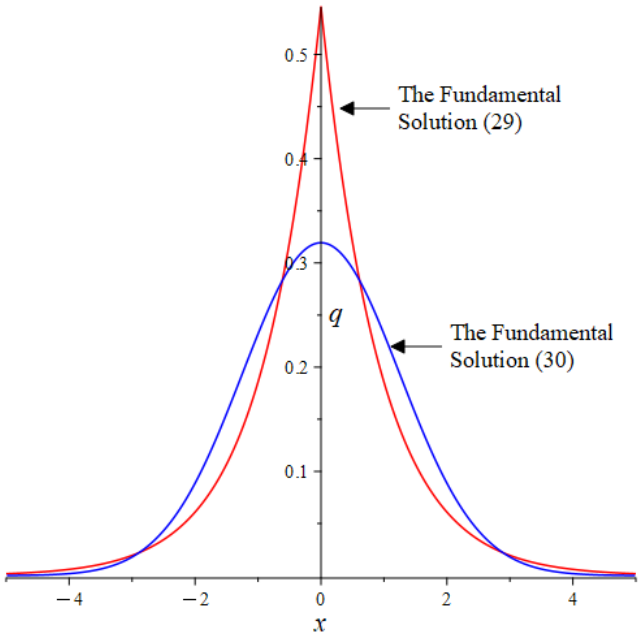

Fig. 1. The one-dimensional graphs of both the fundamental solutions for $\alpha = 1/4$, $t = 1/4$.

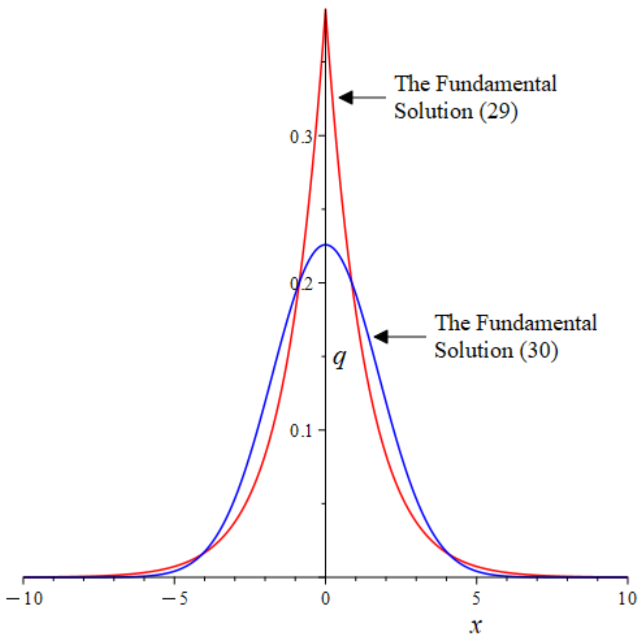

Fig. 2. The one-dimensional graphs of both the fundamental solutions for $\alpha = 1/4$, $t = 4$.

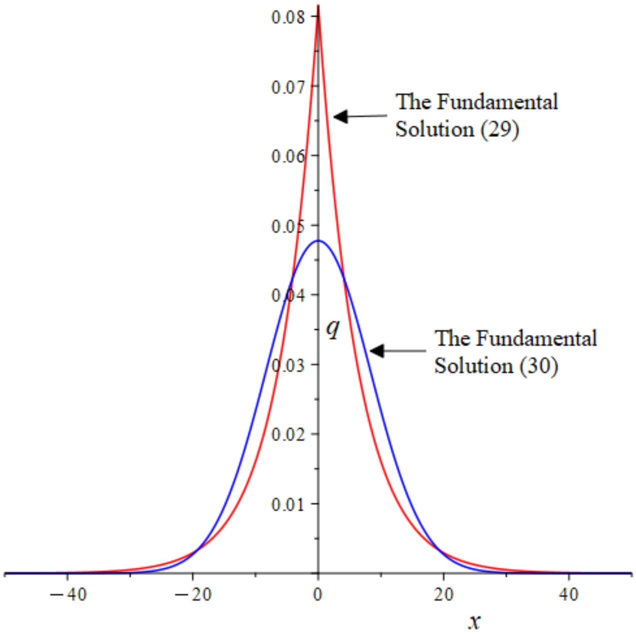

Fig. 3. The one-dimensional graphs of both the fundamental solutions for $\alpha = 1/4, t = 10^6$.

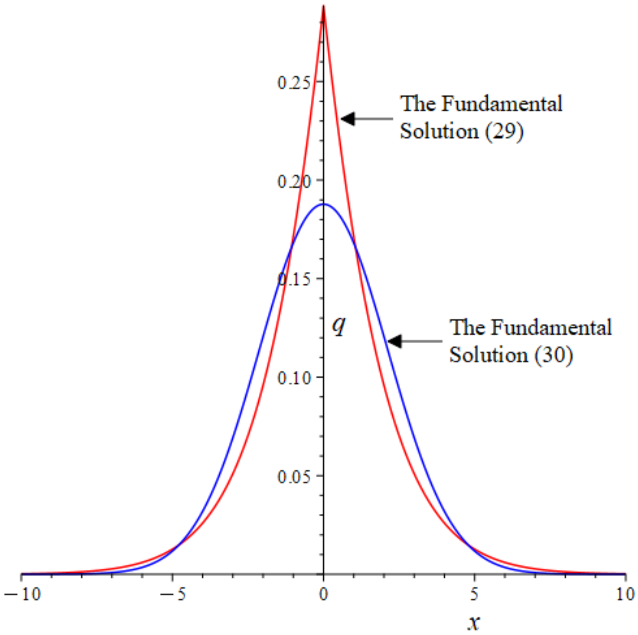

Fig. 5. The one-dimensional graphs of both the fundamental solutions for $\alpha = 1/2, t = 4$

In Fig. 4-6, we compare the one-dimensional graphs of both the fundamental solutions for $\alpha = 1/2$ and $t = 1/4, 4, 10^6$. We observe that for sufficiently large t, both the graphs are sufficiently close.

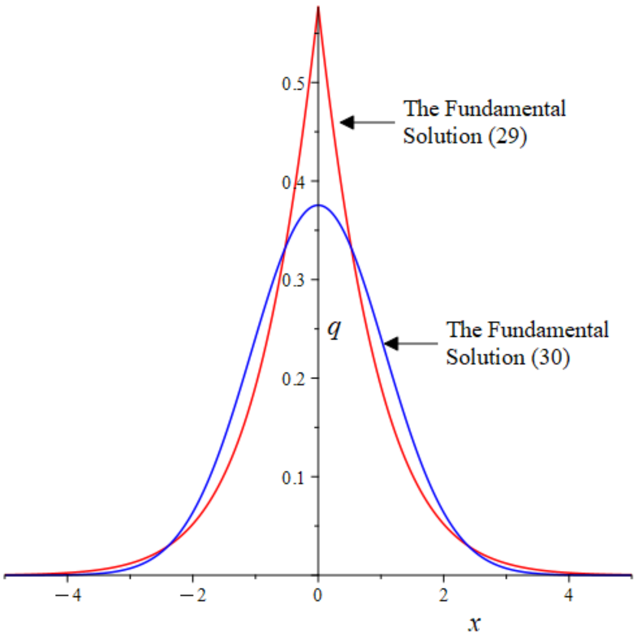

Fig. 4. The one-dimensional graphs of both the fundamental solutions for $\alpha = 1/2, t = 1/4$.

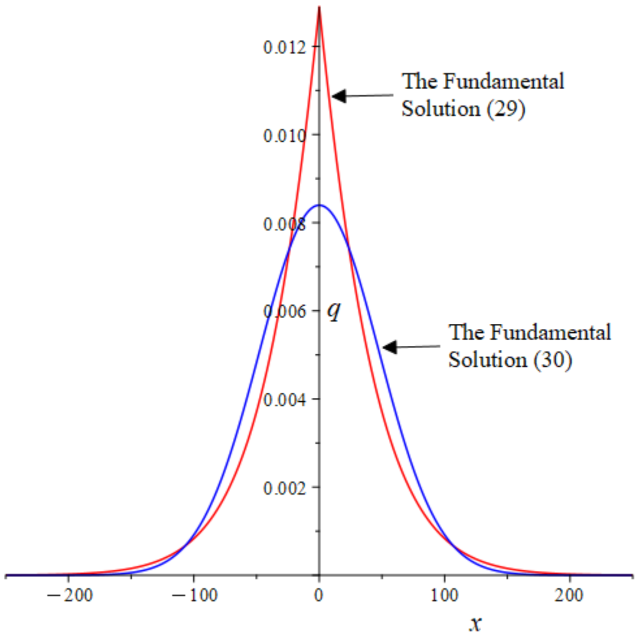

Fig. 6. The one-dimensional graphs of both the fundamental solutions for $\alpha = 1/2, t = 10^6$.

In Fig. 7-9, we compare the one-dimensional graphs of both the fundamental solutions for $\alpha = 3/4$ and $t = 1/4, 4, 10^6$. For $\alpha = 3/4$ or $1/2 < \alpha < 1$, we observe that both the graphs are sufficiently close for $t > 0$

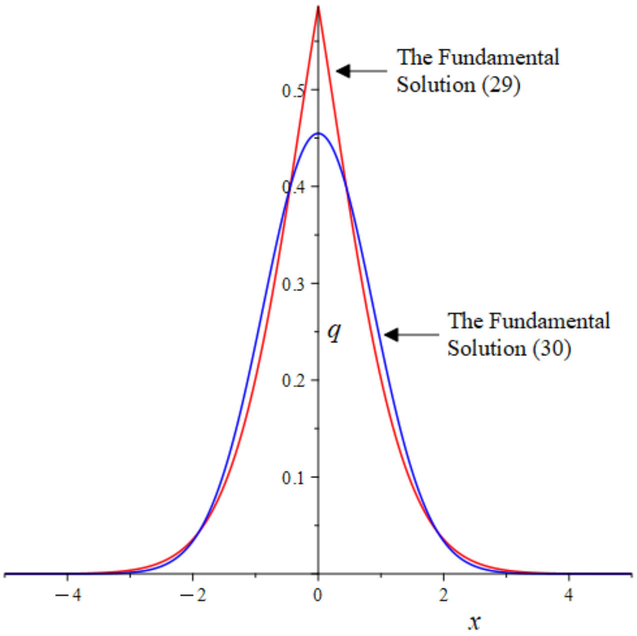

Fig. 7. The one-dimensional graphs of both the fundamental solutions for $\alpha = 3/4$, $t = 1/4$.

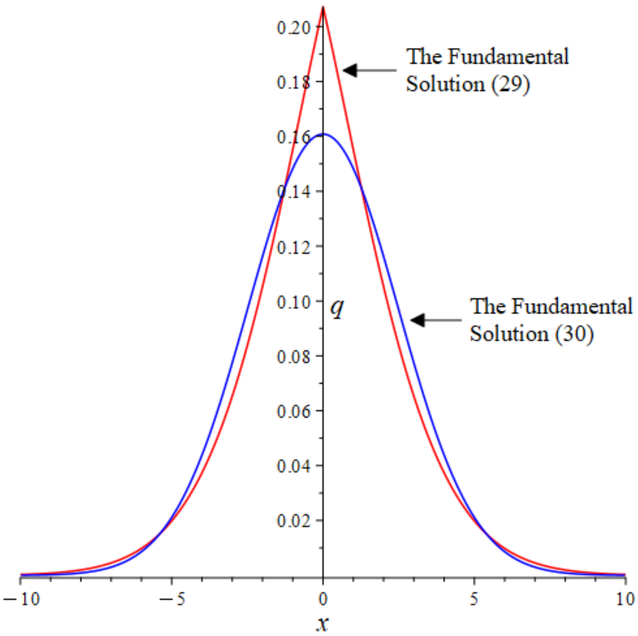

Fig. 8. The one-dimensional graphs of both the fundamental solutions for $\alpha = 3/4$, $t = 4$.

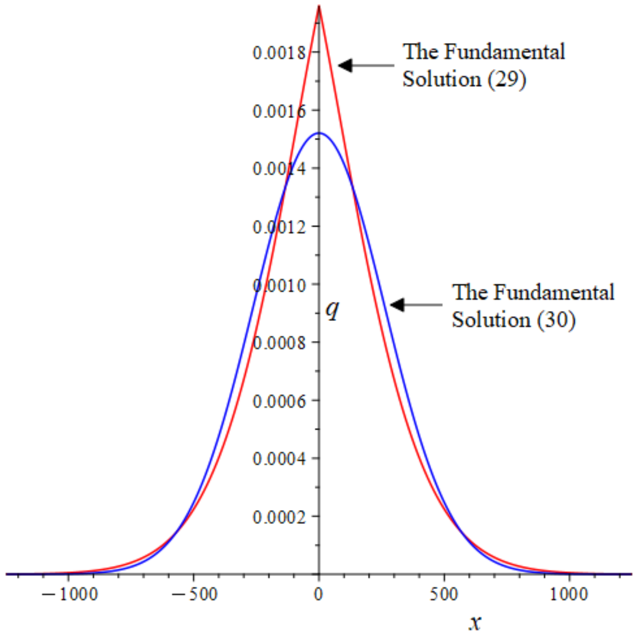

Fig. 9. The one-dimensional graphs of both the fundamental solutions for $\alpha = 3/4$, $t = 10^6$.

F. Graphs of The Solution to Multidimensional Conformable Advection-Diffusion Equation

In this section, we figure the graphs of the solution (42) to the equation (41). We first figure them for one-dimensional case. The solution is

$$q(x,t) = \left(\frac{\alpha}{4\pi D_\alpha t^\alpha}\right)^{\frac{1}{2}} e^{-\frac{\alpha}{4D_\alpha t^\alpha}\left|x - D_\alpha \frac{t^\alpha}{\alpha} v\right|^2}$$

with $x \in \mathbb{R}$ and a constant $v \in \mathbb{R}$. We describe them with taking $(\Delta x)^2/(2\lambda^\alpha) = 1$

In Fig. 10, the one-dimensional graphs of the solution (42) to the equation (41) for $v = 0,1,2,3$ are colored by red, blue, green, and orange, respectively with $\alpha = 1/4$ and $t = 4$. Based on Fig. 10, we observe that, for $v \geq 0$, the greater the value of v, the faster the particle movement in the direction of the positive x-axis.

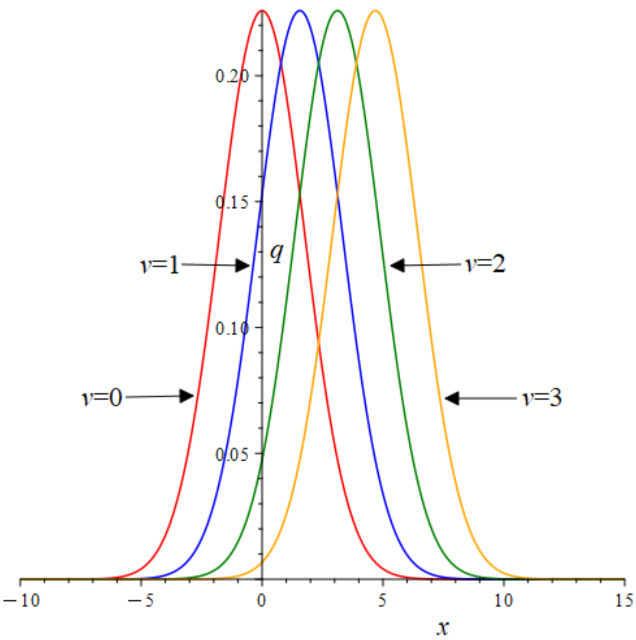

Fig. 10. The one-dimensional graphs of the solution (42) to the equation (41) for $v = 0,1,2,3$ with $\alpha = 1/4$, $t = 4$.

In Fig. 11, the one-dimensional graphs of the solution (42) to the equation (41) for $v = -3, -2, -1, 0$ are colored by orange, green, blue, and red, respectively with $\alpha = 1/4$ and $t = 4$. Based on Fig. 11, we observe that, for $v \leq 0$, the greater the value of $|v|$, the faster the particle movement in the direction of the negative x-axis.

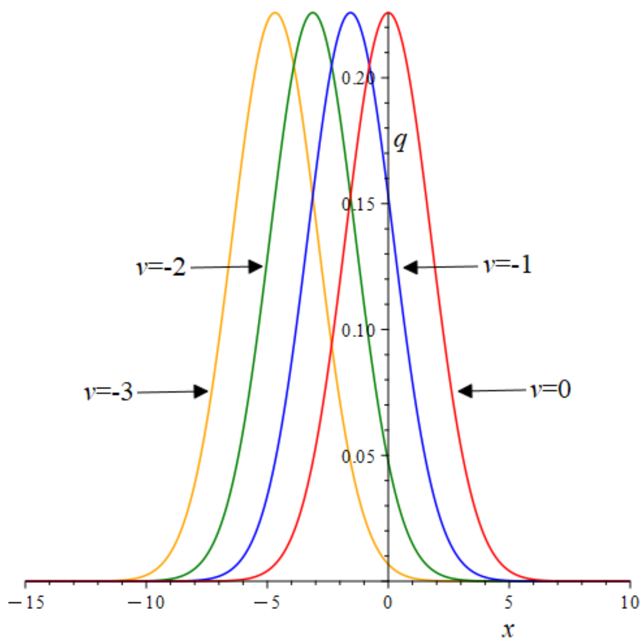

Fig. 11. The one-dimensional graphs of the solution (42) to the equation (41) for $v = -3, -2, -1, 0$ with $\alpha = 1/4$, $t = 4$.

In Fig. 12 and Fig. 13, the one-dimensional graphs of the solution (42) to the equation (41) for $\alpha = 1/4, 1/2, 3/4, 1$ are colored by orange, green, blue, and red, respectively with $v = 2$. In Fig. 12, $t = 1/4$, while in Fig. 13, $t = 100$. We observe that the greater the value of α, the faster the particle movement in the direction of the positive x-axis.

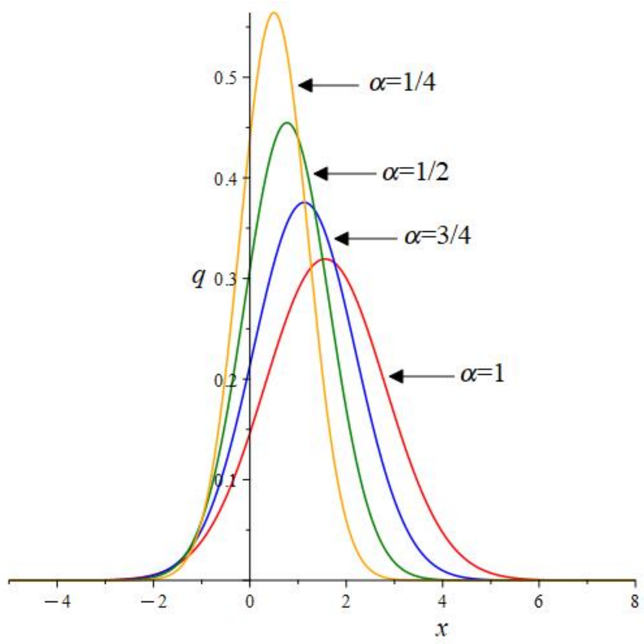

Fig. 12. The one-dimensional graphs of the solution (42) to the equation (41) for $\alpha = 1/4, 1/2, 3/4, 1$ with $t = 1/4$, $v = 2$.

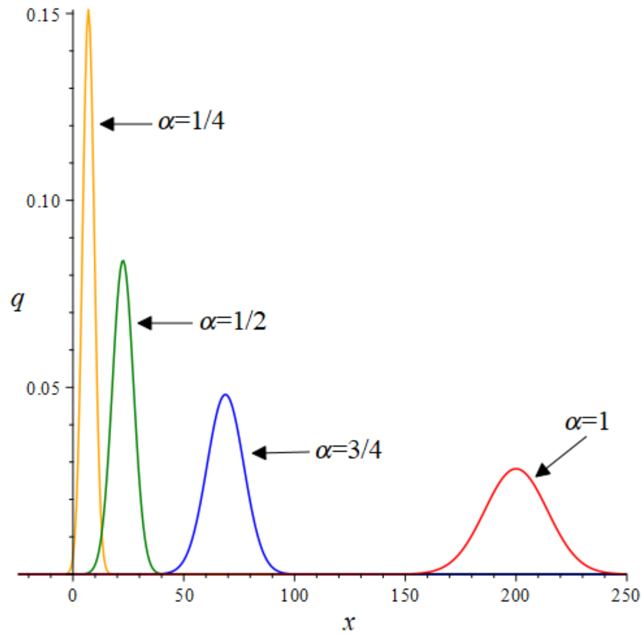

Fig. 13. The one-dimensional graphs of the solution (42) to the equation (41) for $\alpha = 1/4, 1/2, 3/4, 1$ with $t = 100$, $v = 2$.

In Fig. 14 and Fig. 15, the one-dimensional graphs of the solution (42) to the equation (41) for $\alpha = 1/4, 1/2, 3/4, 1$ are colored by orange, green, blue, and red, respectively with $v = -2$. In Fig. 14, $t = 1/4$, while in Fig. 15, $t = 100$. We observe that the greater the value of $|\alpha|$, the faster the particle movement in the direction of the negative x-axis.

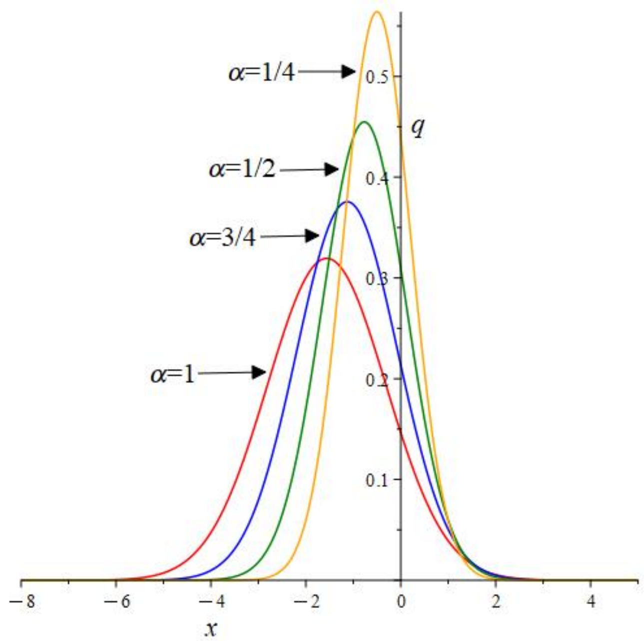

Fig. 14. The one-dimensional graphs of the solution (42) to the equation (41) for $\alpha = 1/4, 1/2, 3/4, 1$ with $t = 1/4$, $v = -2$.

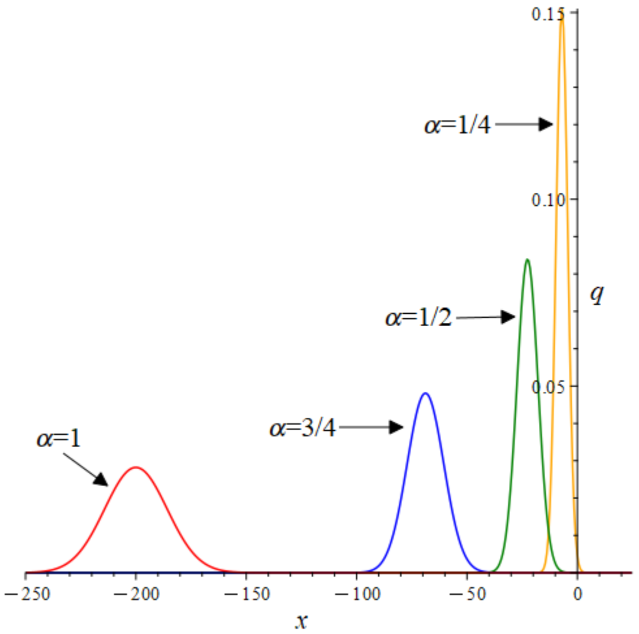

Fig. 15. The one-dimensional graphs of the solution (42) to the equation (41) for $\alpha = 1/4, 1/2, 3/4, 1$ with $t = 100$, $v = -2$.

We next figure the graphs of the solution (42) to the equation (41) for two-dimensional case. The solution is

$$q(x,t) = \frac{\alpha}{4\pi D_\alpha t^\alpha} \cdot e^{-\frac{\alpha}{4D_\alpha t^\alpha}\left\{\left(x_1 - D_\alpha \frac{t^\alpha}{\alpha} v_1\right)^2 + \left(x_2 - D_\alpha \frac{t^\alpha}{\alpha} v_2\right)^2\right\}}$$

with $x = (x_1, x_2) \in \mathbb{R}^2$ and a constant vector $v = (v_1, v_2) \in \mathbb{R}^2$. We describe them with taking $(\Delta x)^2/(4\lambda^\alpha) = 1$

In Fig. 16, the two-dimensional graphs of the solution (42) to the equation (41) for $v = (0,0), (1,1), (2,2), (3,3)$ are colored by red, blue, green, and orange, respectively with $\alpha = 1/4$ and $t = 4$. Fig. 17 shows the contours of the two-dimensional graphs figured in Fig. 16. Fig 16 and Fig. 17 show us that the greater the value of $\|v\|$, the faster the particle movement in the direction of the vector v.

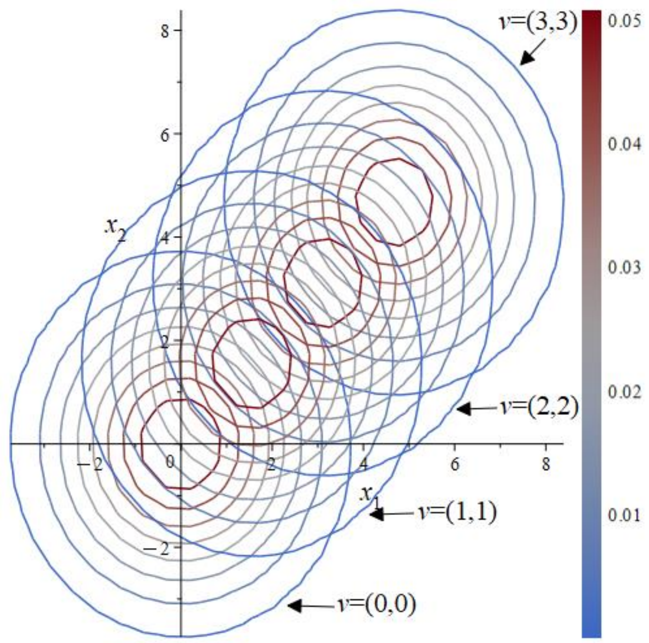

Fig. 17. The contours of the surfaces figured in Fig. 16.

In Fig. 18, the two-dimensional graphs of the solution (42) to the equation (41) for $v = (-3,-3), (-2,-2), (-1,-1), (0,0)$ are colored by orange, green, blue, and red, respectively with $\alpha = 1/4$ and $t = 4$. Fig. 19 shows the contours of the two-dimensional graphs figured in Fig. 18. Fig 18 and Fig. 19 also show us that the greater the value of $\|v\|$, the faster the particle movement in the direction of the vector v.

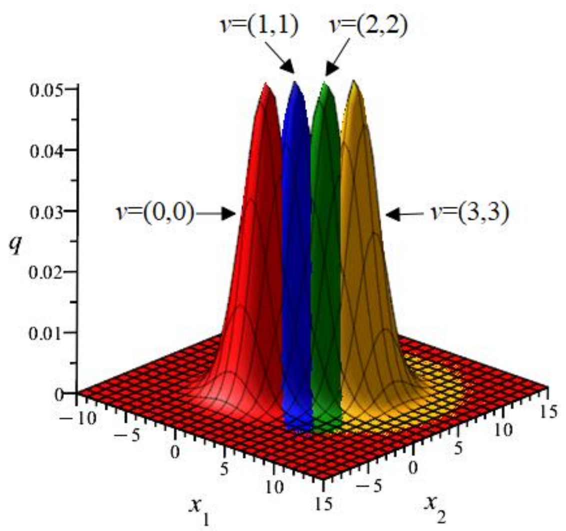

Fig. 16. The two-dimensional graphs of the solution (42) to the equation (41) for $v = (0,0), (1,1), (2,2), (3,3)$ colored by red, blue, green, and orange, respectively with $\alpha = 1/4$, $t = 4$.

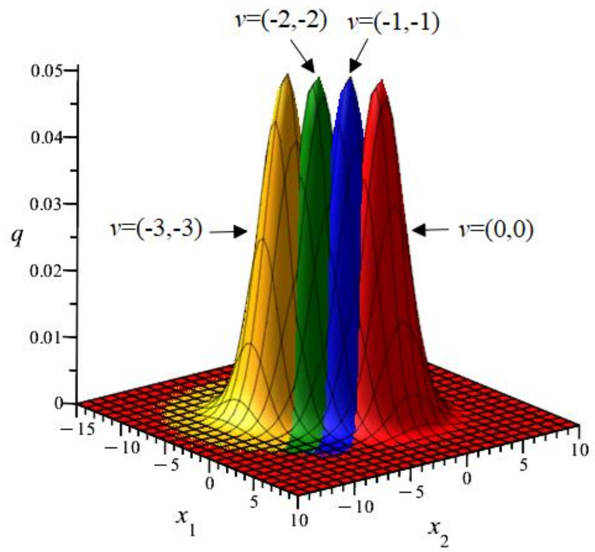

Fig. 18. The two-dimensional graphs of the solution (42) to the equation (41) for $v = (-3,-3), (-2,-2), (-1,-1), (0,0)$ colored by orange, green, blue, and red, respectively with $\alpha = 1/4$, $t = 4$.

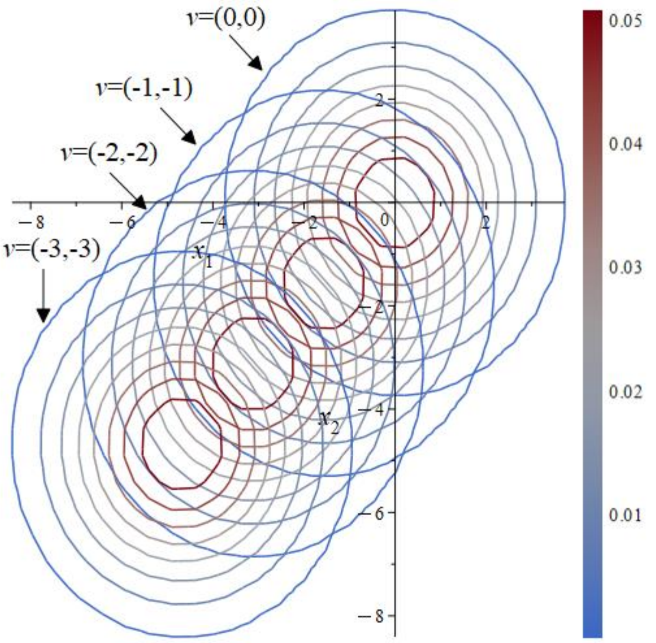

Fig. 19. The contours of the surfaces figured in Fig. 18.

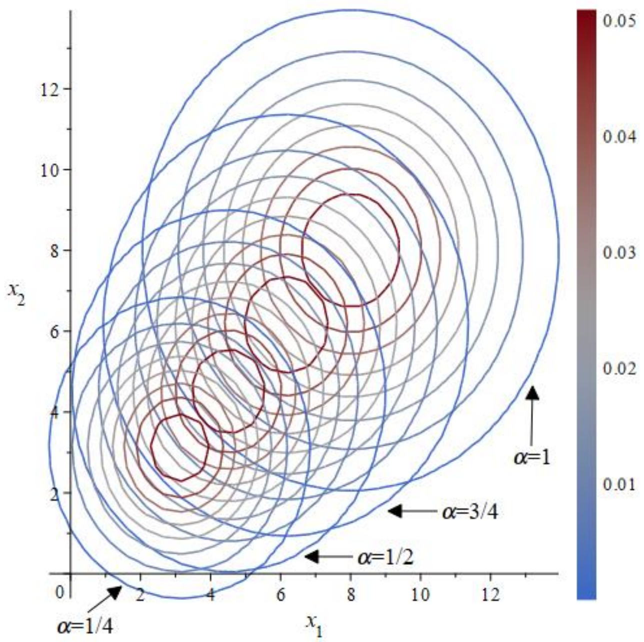

Fig. 21. The contours of the surfaces figured in Fig. 20.

In Fig. 20 and Fig. 22, the two-dimensional graphs of the solution (42) to the equation (41) for $\alpha = 1/4, 1/2, 3/4, 1$ are colored by orange, green, blue, and red, respectively with $v = (2,2)$. In Fig. 20, $t = 4$, while in Fig. 22, $t = 100$. Fig. 21 and Fig. 23 show the contours of two-dimensional graphs figured in Fig. 20 and Fig. 22, respectively. Based on Fig. 20-23, we observe that the greater the value of α, the faster the particle movement in the direction of the vector v.

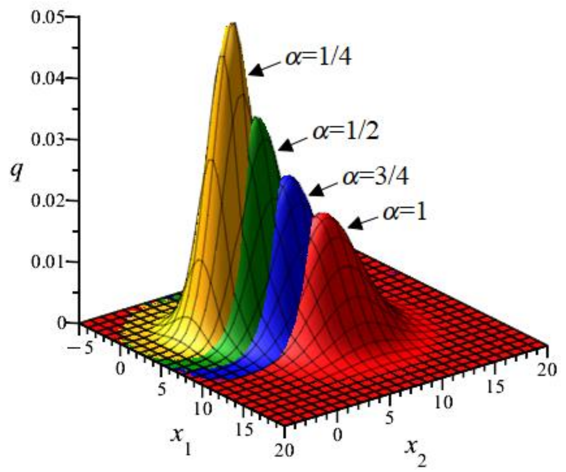

Fig. 20. The two-dimensional graphs of the solution (42) to the equation (41) for $\alpha = 1/4, 1/2, 3/4, 1$ colored orange, green, blue, and red, respectively with $t = 4$, $v = (2,2)$.

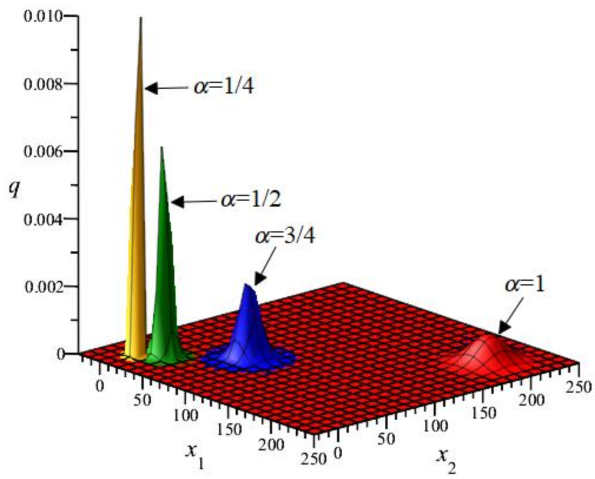

Fig. 22. The two-dimensional graphs of the solution (42) to the equation (41) for $\alpha = 1/4, 1/2, 3/4, 1$ colored by orange, green, blue, and red, respectively with $t = 100$, $v = (2,2)$.

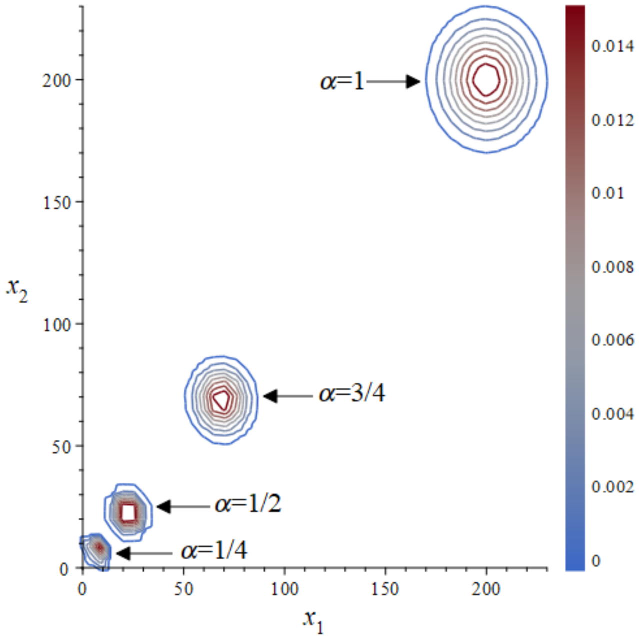

Fig. 23. The contours of the surfaces figured in Fig. 22.

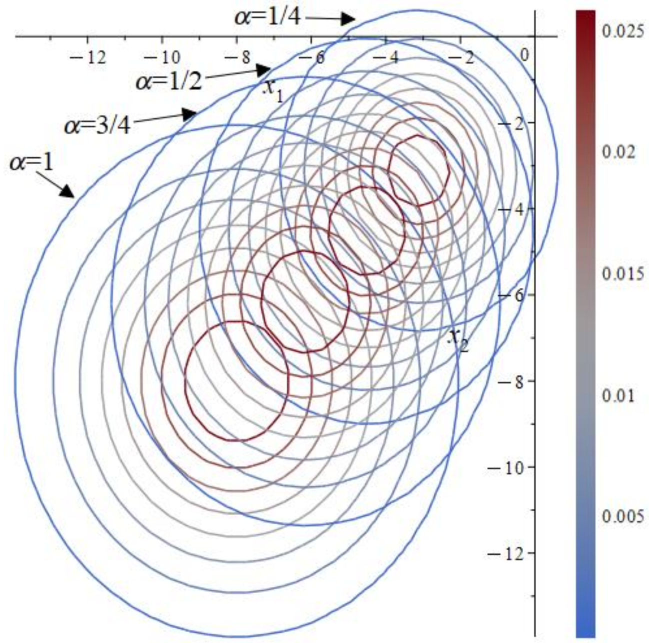

Fig. 25. The contours of the surfaces figured in Fig. 24.

In Fig. 24 and Fig. 26, the two-dimensional graphs of the solution (42) to the equation (41) for $\alpha = 1/4, 1/2, 3/4, 1$ are colored by orange, green, blue, and red, respectively with $v = (-2, -2)$. In Fig. 24, $t = 4$, while in Fig. 26, $t = 100$. Fig. 25 and Fig. 27 show the contours of the two-dimensional graphs figured in Fig. 24 and Fig. 26, respectively. Based on Fig. 24-27, we observe that the greater the value of α, the faster the particle movement in the direction of the vector v.

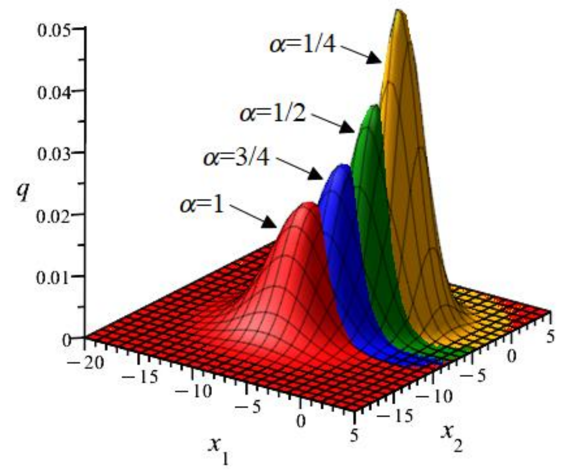

Fig. 24. The two-dimensional graphs of the solution (42) to the equation (41) for $\alpha = 1/4, 1/2, 3/4, 1$ colored by orange, green, blue, and red, respectively with $t = 4$, $v = (-2, -2)$.

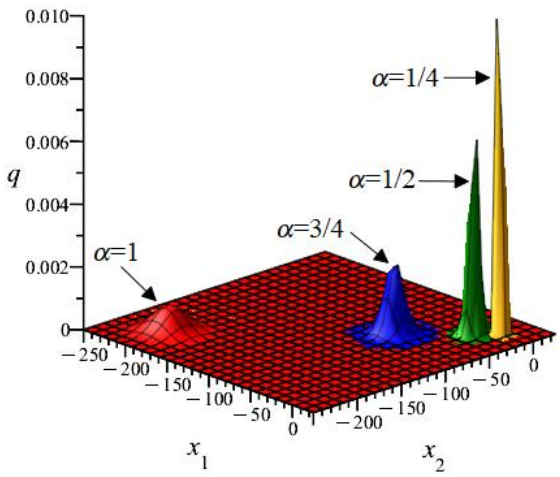

Fig. 26. The two-dimensional graphs of the solution (42) to the equation (41) for $\alpha = 1/4, 1/2, 3/4, 1$ colored by orange, green, blue, and red, respectively with $t = 100$, $v = (-2, -2)$.

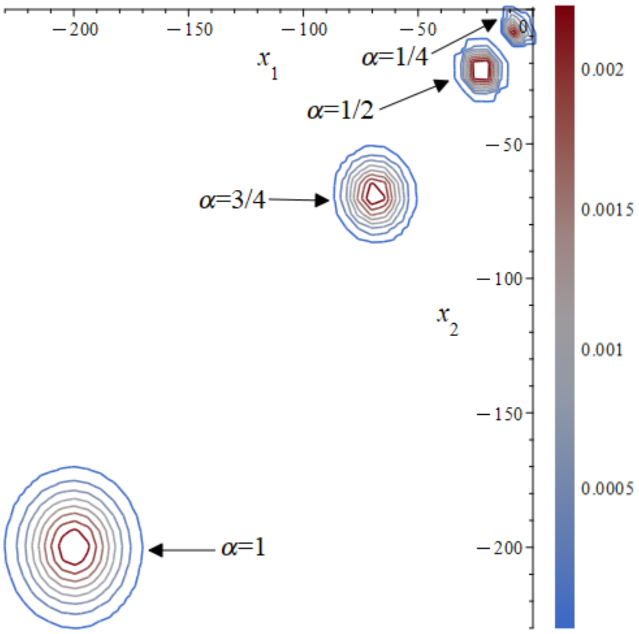

Fig. 27. The contours of the surfaces figured in Fig. 26.

IV. Conclusion

The use of the stretched exponential function as a survival function in the continuous time random walks process yields the diffusion equation involving the conformable fractional derivative. The equation is called the conformable diffusion equation. The conformable diffusion equation can be used as an alternative equation, besides the subdiffusion equation involving Caputo fractional derivative, to describe subdiffusion or slow diffusion phenomenon since MSD of the particle moving in the process is proportional to t^α for $t > 0$ with $0 < \alpha < 1$ as expressed by the equation (31). The fact is also supported by the comparison of the graphs of the fundamental solutions to the subdiffusion equation (18) and the conformable diffusion equation (26) as shown in section III. It is the other advantage of the use of the conformable fractional derivative in modeling subdiffusion or slow diffusion phenomenon besides the advantages as mentioned in section I. The greater the value of α, the faster the particle movement in the process. In presence of the constant external force field $f(x) = v \neq 0$, the particle involved in the diffusion process moves faster toward the direction of the external force field v. The greater the value of $\|v\|$, the faster the particle movement in the process.

The conformable semigroup associated with the solution to the conformable diffusion equation (43) and the semigroup associated with the solution to usual diffusion equation (46) have similar properties except the semigroup property as stated in Theorem 2.6(c).

For further researches, the topic on the conformable diffusion equation with the other reaction terms such as chemotaxis is very interesting to investigate.

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