# Jordan $f$-biderivations of Triangular Algebras

Jinhong Zhuang, Yanping Chen

***Abstract*—This research examines the Jordan $f$-biderivations on triangular algebras by utilizing a faithful bimodule structure. Let $\Omega$ be a triangular algebra and $f$ be an automorphism of $\Omega$. We demonstrate that any Jordan $f$-biderivation $\Delta$ of $\Omega$ may be expressed as $\Delta = D + \delta + \tau$, in which $D : \Omega \times \Omega \to \Omega$ is a $f$-biderivation, $\delta : \Omega \times \Omega \to f(q) \cdot l\Omega l$ is a map with $\delta(l\Omega l + q\Omega q, l\Omega l + q\Omega q) = \{0\}$ and $\tau : \Omega \times \Omega \to Z_f(\Omega)$ is a $f$-central map, $l$ is a nontrivial idempotent in $\Omega$ such that $q\Omega l = \{0\}$, $q = 1 - l$. As an application, we prove that each Jordan biderivation of triangular algebras is indeed a biderivation.**

***Index Terms*—triangular algebra, derivation, $f$-biderivation, Jordan $f$-biderivation.**

## I. Introduction

Let $\mathcal{U}$ be an unital algebra, $f$ be an automorphism of $\mathcal{U}$. A linear mapping $d$ of $\mathcal{U}$ is said to be a $f$-derivation [1], if

$$d(uv) = d(u)v + f(u)d(v)$$

for any $u, v \in \mathcal{U}$. Set $[u, v] = uv - vu$, $[u, v]_f = f(u)v - vu$. For any given $u_0 \in \mathcal{U}$, the map $d_{u_0}(u) = [u, u_0]_f$ is a $f$-derivation. We called it an inner $f$-derivation. A linear mapping $g$ of $\mathcal{U}$ is said to be a Jordan $f$-derivation, if

$$g(u^2) = g(u)u + f(u)g(u)$$

for any $u \in \mathcal{U}$. It is clear that each $f$-derivation is a Jordan $f$-derivation. Specifically, when $f$ is an identity mapping, the (Jordan) $f$-derivation is the well-known (Jordan) derivation.

A bilinear mapping $D : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ is said to be a $f$-biderivation [2], if

$$D(uw, v) = D(u, v)w + f(u)D(w, v)$$

and

$$D(u, vw) = D(u, v)w + f(v)D(u, w)$$

for any $u, v, w \in \mathcal{U}$. Clearly, any $1_{\mathcal{U}}$-biderivation is a biderivation. For any automorphism $f$ of $\mathcal{U}$, the set $Z_f(\mathcal{U}) = \{\lambda \in \mathcal{U} | f(u)\lambda = \lambda u, \forall u \in \mathcal{U}\}$ is called the $f$-center of $\mathcal{U}$. For any given $u_0 \in \mathcal{U}$, if $u_0 \notin Z_f(\mathcal{U})$ and $[[u, v], u_0]_f = 0$ for every $u, v \in \mathcal{U}$, then the bilinear map $D : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ given by $D(u, v) = [u, [v, u_0]_f]_f$ is a $f$-biderivation. We called it an extremal $f$-biderivation (see [2, Proposition 5.13]). A bilinear mapping $\Delta : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ is said to be a Jordan $f$-biderivation, if

$$\Delta(u^2, v) = \Delta(u, v)u + f(u)\Delta(u, v)$$

and

$$\Delta(u, v^2) = \Delta(u, v)v + f(v)\Delta(u, v)$$

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for all $u, v \in \mathcal{U}$. Clearly, any Jordan $1_{\mathcal{U}}$-biderivation is a Jordan biderivation and each $f$-biderivation is a Jordan $f$-biderivation.

The derivation, Jordan derivation and biderivation are significant mappings in algebras or rings, and they have a crucial function in the analysis of the structure of algebras or rings. Extensive research has been conducted on various rings and algebras to solve the problem of characterizing the structure of derivations(see [3, 9-14]). In 2006, Zhang and Yu [3] demonstrated that each Jordan derivation of triangular algebras is a derivation. In 2011, Han and Wei [4] established that if $l\Omega l$ and $q\Omega q$ have only trivial idempotents, then each Jordan $f$-derivation of a triangular algebra is also a $f$-derivation. In 2016, Benkovič [1] disproved the strong assumption and demonstrated that each Jordan $f$-derivation of triangular algebras may be broken down into the combination of a $f$-derivation and a special Jordan $f$-derivation. As a generalization of derivations, there have been more and more studies on biderivations in recent years. Benkovič [5] focused on biderivations within a certain category of triangular algebras and showed that each bilinear biderivation of this kind of triangular algebra may be expressed as the combination of an extremal biderivation and an inner biderivation. In 2017, González [2] described $f$-biderivations on triangular algebras, where $f$ is a partible automorphism. In 2020, Yuan and Li [6] investigated $f$-biderivations of certain triangular rings and proved that any $f$-biderivation of these rings can be expressed as the combination of an inner $f$-biderivation and an extremal $f$-biderivation.

In the recent paper [7], Ren and Liang investigated Jordan biderivations on triangular algebras and showed that any Jordan biderivation can be expressed as the combination of an inner biderivation and an extremal biderivation under certain moderate conditions. Building upon the findings of references [1] and [7], our study focuses on analyzing the composition of Jordan $f$-biderivations within triangular algebras.

## II. Preliminaries

This section will provide some definitions and preliminary informations.

An unital algebra $\Omega$ with an idempotent $l \neq 0, 1$ is a triangular algebra [1], if $l\Omega q$ is a $(l\Omega l, q\Omega q)$-bimodule and $q\Omega l = \{0\}$, in which $q = 1 - l$. Each triangular algebra $\Omega$ has the form $\Omega = l\Omega l \oplus l\Omega q \oplus q\Omega q$. It is clear that $l\Omega l$ and $q\Omega q$ are subalgebra of $\Omega$ with unitary elements $l$ and $q$. Furthermore, we assume that $(l\Omega l, q\Omega q)$-bimodule $l\Omega q$ is faithful. Recall that the $(l\Omega l, q\Omega q)$-bimodule $l\Omega q$ is faithful if $ltl \cdot l\Omega q = \{0\}$ implies $ltl = 0$ and $l\Omega q \cdot qtq = \{0\}$ implies $qtq = 0$.

Suppose $f : \Omega \times \Omega \to \Omega$ is an automorphism, $l$ and $q$ are orthogonal idempotents in $\Omega$, then $L = f(l)$ and $Q = f(q)$ are likewise orthogonal idempotents in $\Omega$. Since $Q\Omega L =$

$f(q)f(\Omega)f(l) = f(q\Omega l) = \{0\}$, then $\Omega = L\Omega L \oplus L\Omega Q \oplus Q\Omega Q$ is also a triangular algebra. Furthermore, $L\Omega L$ is a subalgebra of $\Omega$ isomorphic to $l\Omega l$, $Q\Omega Q$ is a subalgebra of $\Omega$ isomorphic to $q\Omega q$, $L\Omega Q$ is a $(L\Omega L, Q\Omega Q)$-bimodule isomorphic to the $(l\Omega l, q\Omega q)$-bimodule $l\Omega q$. Each $t \in \Omega$ may be expressed in a unique way as

$$t = (L+Q)t(l+q) = Ltl + Ltq + Qtl + Qtq.$$

Thus, we can rewrite $\Omega = L\Omega l \oplus L\Omega q \oplus Q\Omega l \oplus Q\Omega q$.

**Proposition 2.1.**[8] Let $\Omega$ be a triangular algebra and $f$ be an automorphism of $\Omega$. Then

$$Z_f(\Omega) = \{r + t \in L\Omega l + Q\Omega q | rs = f(s)t, \forall s \in l\Omega q\}.$$

Specifically, a linear map $\tau : \Omega \times \Omega \to Z_f(\Omega)$ will be referred to as a $f$-central.

Following the same logic as the Jordan derivation(see [1, section 2]), we obtain

**Lemma 2.2.** Let $\Omega$ be a 2-torsion free algebra (meaning that if $2t = 0$, then $t = 0$ for all $t \in \Omega$). Set $\Delta : \Omega \times \Omega \to \Omega$ be a Jordan $f$-biderivation. Then for every $x, y, z, r, s, t \in \Omega$,

$$\begin{aligned}
\text{(i)}\Delta(xy + yx, s) &= \Delta(x, s)y + f(x)\Delta(y, s) \\
&\quad + \Delta(y, s)x + f(y)\Delta(x, s), \\
\Delta(y, st + ts) &= \Delta(y, s)t + f(s)\Delta(y, t) \\
&\quad + \Delta(y, t)s + f(t)\Delta(y, s); \\
\text{(ii)}\Delta(xyx, s) &= \Delta(x, s)yx + f(x)\Delta(y, s)x \\
&\quad + f(x)f(y)\Delta(x, s), \\
\Delta(y, sts) &= \Delta(y, s)ts + f(s)\Delta(y, t)s \\
&\quad + f(s)f(t)\Delta(y, s); \\
\text{(iii)}\Delta(xyz + zyx, s) &= \Delta(x, s)yz + f(x)\Delta(y, s)z \\
&\quad + f(x)f(y)\Delta(z, s) + \Delta(z, s)yx \\
&\quad + f(z)\Delta(y, s)x + f(z)f(y)\Delta(x, s), \\
\Delta(y, rst + tsr) &= \Delta(y, r)st + f(r)\Delta(y, s)t \\
&\quad + f(r)f(s)\Delta(y, t) + \Delta(y, t)sr \\
&\quad + f(t)\Delta(y, s)r + f(t)f(s)\Delta(y, r).
\end{aligned}$$

## III. MAIN RESULTS

This section will focus on the examination of Jordan $f$-biderivations on triangular algebras.

**Lemma 3.1.** Let $\Omega$ be an algebra and $\Delta : \Omega \times \Omega \to \Omega$ be a Jordan $f$-biderivation. Then $[[s, t], \Delta(s, t)]_f = 0$ for any $s, t \in \Omega$.

**Proof.** Set $\Delta$ be a Jordan $f$-biderivation of $\Omega$. For any $s, t \in \Omega$,

$$\begin{aligned}
\Delta(s^2, t^2) &= \Delta(s, t^2)s + f(s)\Delta(s, t^2) \\
&= (\Delta(s, t)t + f(t)\Delta(s, t))s \\
&\quad + f(s)(\Delta(s, t)t + f(t)\Delta(s, t)) \qquad (1) \\
&= \Delta(s, t)ts + f(t)\Delta(s, t)s \\
&\quad + f(s)\Delta(s, t)t + f(s)f(t)\Delta(s, t))
\end{aligned}$$

and

$$\begin{aligned}
\Delta(s^2, t^2) &= \Delta(s^2, t)t + f(t)\Delta(s^2, t) \\
&= (\Delta(s, t)s + f(s)\Delta(s, t))t \\
&\quad + f(t)(\Delta(s, t)s + f(s)\Delta(s, t)) \qquad (2) \\
&= \Delta(s, t)st + f(s)\Delta(s, t)t \\
&\quad + f(t)\Delta(s, t)s + f(t)f(s)\Delta(s, t)).
\end{aligned}$$

Comparing (1) and (2), one can obtain $[[s, t], \Delta(s, t)]_f = 0$.□

**Lemma 3.2.** Let $\Omega$ be a triangular algebra and $\Delta : \Omega \times \Omega \to \Omega$ be a Jordan $f$-biderivation. Then
(i) $\Delta(0, t) = 0, \Delta(t, 0) = 0$ for all $t \in \Omega$;
(ii) $\Delta(1, t) = 0, \Delta(t, 1) = 0$ for all $t \in \Omega$;
(iii) $\Delta(l, l) = -\Delta(l, q) = -\Delta(q, l) = \Delta(q, q)$.

**Proof.** (i) Since $\Delta$ is a Jordan $f$-biderivation, we have

$$\Delta(0, t) = \Delta(0^2, t) = \Delta(0, t) \cdot 0 + f(0) \cdot \Delta(0, t) = 0$$

for all $t \in \Omega$. Similarly, we can prove $\Delta(t, 0) = 0$.
(ii) According to the definition of Jordan $f$-biderivation, we obtain

$$\Delta(1, t) = \Delta(1^2, t) = \Delta(1, t) \cdot 1 + f(1) \cdot \Delta(1, t).$$

This implies that $\Delta(1, t) = 0$. Similarly, one can prove $\Delta(t, 1) = 0$.
(iii) Using (ii) and the relation $l + q = 1$, we have

$$\Delta(l, l) = \Delta(l, 1 - q) = \Delta(l, 1) - \Delta(l, q) = -\Delta(l, q),$$

$$\Delta(l, l) = \Delta(1 - q, l) = \Delta(1, l) - \Delta(q, l) = -\Delta(q, l)$$

and

$$\Delta(q, q) = \Delta(q, 1 - l) = \Delta(q, 1) - \Delta(q, l) = -\Delta(q, l).$$

Thus, $\Delta(l, l) = -\Delta(l, q) = -\Delta(q, l) = \Delta(q, q)$. □

**Proposition 3.3.** Let $\Omega$ be a triangular algebra and $\Delta : \Omega \times \Omega \to \Omega$ be a Jordan $f$-biderivation. If $\Delta(l, l) \neq 0$, then $\Delta = \Delta_1 + \Delta_2$, where $\Delta_1(s, t) = [s, [t, \Delta(l, l)]_f]_f$ is an extremal biderivation, $\Delta_2$ is a Jordan $f$-biderivation satisfing $\Delta_2(l, l) = 0$.

**Proof.** Set $\Delta$ be a Jordan $f$-biderivation with $\Delta(l, l) \neq 0$. Using the definition of Jordan $f$-biderivation, we get

$$\Delta(l, l) = \Delta(l^2, l) = \Delta(l, l)l + L\Delta(l, l),$$

where $L = f(l)$. This implies that $L\Delta(l, l)l = 0$ and $Q\Delta(l, l)q = 0$. Thus

$$\Delta(l, l) = L\Delta(l, l)q + Q\Delta(l, l)l \notin Z_f(\Omega).$$

For any $s, t \in \Omega$, by Lemma 3.1, one can obtain

$$[[s, t], \Delta(s, t)]_f = 0.$$

Replacing $t$ by $t + l$ , we get

$$[[s, t], \Delta(s, l)]_f + [[s, l], \Delta(s, t)]_f = 0. \qquad (3)$$

Set $s = l$ in (3), then we obtain $[[l, t], \Delta(l, l)]_f = 0$. Since $[l, t] = ltq$, we have $[ltq, \Delta(l, l)]_f = 0$.
Replacing $s$ by $s + l$ in (3) and summarizing the above conclusions, we obtain

$$[[s, t], \Delta(l, l)]_f + [[s, l], \Delta(l, t)]_f + [[l, t], \Delta(s, l)]_f = 0.$$

This implies that

$$[[lsl, ltl], \Delta(l, l)]_f = 0$$

and

$$[[qsq, qtq], \Delta(l, l)]_f = 0.$$

Hence,

$$\begin{aligned}
&[[s, t], \Delta(l, l)]_f \\
&= [l[s, t]l + l[s, t]q + q[s, t]q, \Delta(l, l)]_f \\
&= [l[s, t]l + q[s, t]q, \Delta(l, l)]_f \\
&= [[lsl, ltl] + [qsq, qtq], \Delta(l, l)]_f = 0.
\end{aligned}$$

Thus, the map $\Delta_1(s,t) = [s,[t,\Delta(l,l)]_f]_f$ is an extremal biderivation of $\Omega$. Note that

$$\begin{aligned}\Delta_1(l,l) &= [l,[l,\Delta(l,l)]_f]_f\\ &= [l,[l,L\Delta(l,l)q + Q\Delta(l,l)l]_f]_f\\ &= L\Delta(l,l)q + Q\Delta(l,l)l\\ &= \Delta(l,l),\end{aligned}$$

then $\Delta_2 = \Delta - \Delta_1$ is a Jordan $f$-biderivation that satisfies $\Delta_2(l,l) = 0$. □

The following lemma outlines the structure of a Jordan $f$-biderivation of $\Omega$ which satisfies $\Delta(l,l) = 0$.

**Lemma 3.4.** Let $\Omega$ be a 2-torsion free triangular algebra and $\Delta : \Omega \times \Omega \to \Omega$ be a Jordan $f$-biderivation satisfying $\Delta(l,l) = 0$, then for all $a, a_1 \in l\Omega l$, $m, m_1 \in l\Omega q$ and $b, b_1 \in q\Omega q$,

(i)$\Delta(a,a_1) = L\Delta(a,a_1)l, \Delta(b,b_1) = Q\Delta(b,b_1)q,$
$\Delta(a,b) = 0, \Delta(b,a) = 0;$

(ii)$\Delta(a,m_1), \Delta(m,a_1) \in L\Omega q + Q\Omega l,$
$\Delta(a,m_1) = f(a)\Delta(l,m_1) + \Delta(l,m_1)a,$
$\Delta(m,a_1) = f(a_1)\Delta(m,l) + \Delta(m,l)a_1;$

(iii)$\Delta(b,m_1), \Delta(m,b_1) \in L\Omega q + Q\Omega l,$
$\Delta(b,m_1) = -f(b)\Delta(l,m_1) - \Delta(l,m_1)b,$
$\Delta(m,b_1) = -f(b_1)\Delta(m,l) - \Delta(m,l)b_1;$

(iv)$\Delta(m,m_1) = -Lf(m)Q \cdot Q\Delta(l,m_1)l + L\Delta(m,m_1)q + Q\Delta(m,m_1)l + Q\Delta(l,m_1)l \cdot lmq.$

**Proof.** (i) From the assumption $\Delta(l,l) = 0$, we get

$$\Delta(l,q) = \Delta(q,l) = \Delta(q,q) = 0.$$

According to Lemma 2.2 (ii), we have

$$\begin{aligned}\Delta(l,a) &= \Delta(l,lal)\\ &= \Delta(l,l)al + L\Delta(l,a)l + Lf(a)\Delta(l,l)\\ &= L\Delta(l,a)l.\end{aligned}$$

Using the definition of Jordan $f$-biderivation, we get

$$\Delta(l,a) = \Delta(l^2,a) = \Delta(l,a)l + L\Delta(l,a).$$

Multiplying the above equation by $L$ from the left and by $l$ from the right, one can obtain

$$L\Delta(l,a)l = 0.$$

This implies that $\Delta(l,a) = 0$ for all $a \in l\Omega l$. From Lemma 3.2 (ii), it is true that $\Delta(q,a) = 0$ for all $a \in l\Omega l$.

Similarly, we can prove $\Delta(b,l) = 0$, $\Delta(b,q) = 0$, $\Delta(a,l) = 0$, $\Delta(a,q) = 0$, $\Delta(l,b) = 0$, $\Delta(q,b) = 0$ for all $a \in l\Omega l$ and $b \in q\Omega q$.

From Lemma 2.2 (ii) and $\Delta(l,a_1) = 0$, we have

$$\begin{aligned}\Delta(a,a_1) &= \Delta(lal,a_1)\\ &= \Delta(l,a_1)al + L\Delta(a,a_1)l + Lf(a)\Delta(l,a_1)\\ &= L\Delta(a,a_1)l\end{aligned}$$

for all $a, a_1 \in l\Omega l$. Furthermore, it can be proved that $\Delta(b,b_1) = Q\Delta(b,b_1)q$ for all $b, b_1 \in q\Omega q$.

From Lemma 2.2 (ii), $\Delta(l,b) = 0$ and $\Delta(a,q) = 0$, we get

$$\begin{aligned}\Delta(a,b) &= \Delta(lal,b)\\ &= \Delta(l,b)al + L\Delta(a,b)l + Lf(a)\Delta(l,b)\\ &= L\Delta(a,b)l\end{aligned}$$

and

$$\begin{aligned}\Delta(a,b) &= \Delta(a,qbq)\\ &= \Delta(a,q)bq + Q\Delta(a,b)q\\ &\quad + Qf(b)\Delta(a,q)\\ &= Q\Delta(a,b)q,\end{aligned}$$

then $\Delta(a,b) = 0$. Similarly, we can show that $\Delta(b,a) = 0$ for all $a \in l\Omega l$ and $b \in q\Omega q$.

(ii) By Lemma 2.2 (i), we have

$$\begin{aligned}\Delta(a,m_1) &= \Delta(a,lm_1 + m_1 l)\\ &= \Delta(a,l)m_1 + L\Delta(a,m_1)\\ &\quad + \Delta(a,m_1)l + f(m_1)\Delta(a,l)\\ &= L\Delta(a,m_1) + \Delta(a,m_1)l.\end{aligned}$$

This implies that

$$L\Delta(a,m_1)l = 0, Q\Delta(a,m_1)q = 0.$$

So $\Delta(a,m_1) \in L\Omega q + Q\Omega l$ for all $a \in l\Omega l$ and $m_1 \in l\Omega q$. From Lemma 2.2 (ii), we have

$$\begin{aligned}\Delta(a,m_1) &= \Delta(lal,m_1)\\ &= \Delta(l,m_1)al + L\Delta(a,m_1)l + Lf(a)\Delta(l,m_1)\\ &= f(a)\Delta(l,m_1) + \Delta(l,m_1)a.\end{aligned}$$

Similarly, we can prove that $\Delta(m,a_1) \in L\Omega q + Q\Omega l$ and $\Delta(m,a_1) = f(a_1)\Delta(m,l) + \Delta(m,l)a_1$ for any $a_1 \in l\Omega l$ and $m \in l\Omega q$.

(iii) Set $a = l$ in (ii), we obtain $\Delta(l,m_1) \in L\Omega q + Q\Omega l$, then

$$\Delta(l,m_1) = L\Delta(l,m_1)q + Q\Delta(l,m_1)l.$$

By applying Lemma 3.2(ii) and utilizing the relationship $l + q = 1$, we have $\Delta(q,m_1) = -\Delta(l,m_1)$. By Lemma 2.2 (i), one can obtain

$$\begin{aligned}\Delta(b,m_1) &= \Delta(b,qm_1 + m_1 q)\\ &= \Delta(b,q)m_1 + Q\Delta(b,m_1)\\ &\quad + \Delta(b,m_1)q + f(m_1)\Delta(b,q)\\ &= Q\Delta(b,m_1) + \Delta(b,m_1)q\end{aligned}$$

for all $m_1 \in l\Omega q$. This implies that

$$L\Delta(b,m_1)l = 0, Q\Delta(b,m_1)q = 0,$$

So $\Delta(b,m_1) \in L\Omega q + Q\Omega l$. From Lemma 2.2 (ii), one can obtain

$$\begin{aligned}\Delta(b,m_1) &= \Delta(qbq,m_1)\\ &= \Delta(q,m_1)bq + Q\Delta(b,m_1)q + Qf(b)\Delta(q,m_1)\\ &= -f(b)\Delta(l,m_1) - \Delta(l,m_1)b.\end{aligned}$$

Similarly, we can prove that $\Delta(m,b_1) \in L\Omega q + Q\Omega l$ and $\Delta(m,b_1) = -f(b_1)\Delta(m,l) - \Delta(m,l)b_1$ for any $m \in l\Omega q$

and $b_1 \in q\Omega q$.

(iv) From Lemma 2.2 (iii), we obtain

$$\begin{aligned}&\Delta(m, m_1)\\&= \Delta(qml + lmq, m_1)\\&= \Delta(q, m_1)ml + Q\Delta(m, m_1)l + Qf(m)\Delta(l, m_1)\\&\quad + \Delta(l, m_1)mq + L\Delta(m, m_1)q + Lf(m)\Delta(q, m_1)\\&= -Lf(m)Q \cdot Q\Delta(l, m_1)l + L\Delta(m, m_1)q\\&\quad + Q\Delta(m, m_1)l + Q\Delta(l, m_1)l \cdot lmq\end{aligned}$$

for all $m, m_1 \in l\Omega q$. □

**Lemma 3.5.** Assume that $\Omega$ be a 2-torsion free triangular algebra, $\Delta : \Omega\times\Omega \to \Omega$ be a Jordan $f$-biderivation satisfying $\Delta(l, l) = 0$. Set $D : \Omega \times \Omega \to \Omega$ be a map. If for any $a, a_1 \in l\Omega l$, $m, m_1 \in l\Omega q$ and $b, b_1 \in q\Omega q$,

(i) $D(a, a_1) = L\Delta(a, a_1)l, D(b, b_1) = Q\Delta(b, b_1)q,$
$D(a, b) = 0, D(b, a) = 0;$

(ii) $D(a, m_1) = L\Delta(a, m_1)q = f(a)\Delta(l, m_1),$
$D(m, a_1) = L\Delta(m, a_1)q = f(a_1)\Delta(m, l);$

(iii) $D(b, m_1) = L\Delta(b, m_1)q = -\Delta(l, m_1)b,$
$D(m, b_1) = L\Delta(m, b_1)q = -\Delta(m, l)b_1;$

(iv) $D(m, m_1) = L\Delta(m, m_1)q,$

then $D$ is a $f$-biderivation.

**Proof.** At first, we verify that $D(x, a_1)$ and $D(x, b_1)$ are $f$-derivations of $\Omega$ for any given $a_1 \in l\Omega l$ and $b_1 \in q\Omega q$.

Because $ma = 0$, we have

$$\begin{aligned}\Delta(am, a_1) &= \Delta(am + ma, a_1)\\&= \Delta(a, a_1)m + f(a)\Delta(m, a_1)\\&\quad + \Delta(m, a_1)a + f(m)\Delta(a, a_1)\end{aligned} \tag{4}$$

for all $a \in l\Omega l$ and $m \in l\Omega q$. Since $f(m) = Lf(m)Q$ and $\Delta(a, a_1) = L\Delta(a, a_1)l$, we see that $f(m)\Delta(a, a_1) = 0$. By (4), we get

$$\Delta(am, a_1) = \Delta(a, a_1)m + f(a)\Delta(m, a_1) + \Delta(m, a_1)a.$$

By multiplying the above equation with $L$ on the left and $q$ on the right, we have

$$\begin{aligned}L\Delta(am, a_1)q &= L\Delta(a, a_1)mq + Lf(a)\Delta(m, a_1)q\\&= L\Delta(a, a_1)l \cdot m + f(a) \cdot L\Delta(m, a_1)q.\end{aligned}$$

According to the conditions (i) and (ii), we get

$$D(am, a_1) = D(a, a_1)m + f(a)D(m, a_1)$$

for any $a \in l\Omega l$ and $m \in l\Omega q$.

For any $a, a_2 \in l\Omega l$, we have

$$D((aa_2)m, a_1) = D(aa_2, a_1)m + f(aa_2)D(m, a_1) \tag{5}$$

and

$$\begin{aligned}D(a(a_2m), a_1) &= D(a, a_1)a_2m + f(a)D(a_2m, a_1)\\&= D(a, a_1)a_2m + f(a)D(a_2, a_1)m\\&\quad + f(a)f(a_2)D(m, a_1)\end{aligned} \tag{6}$$

for all $m \in l\Omega q$.

Taking in account that $f(aa_2) = f(a)f(a_2)$, then the comparison of (5) and (6) give

$$(D(aa_2, a_1) - D(a, a_1)a_2 - f(a)D(a_2, a_1)) \cdot l\Omega q = \{0\}.$$

Since $D(aa_2, a_1) - D(a, a_1)a_2 - f(a)D(a_2, a_1) \in l\Omega l$, by the faithfulness of $l\Omega q$, we obtain

$$D(aa_2, a_1) = D(a, a_1)a_2 + f(a)D(a_2, a_1)$$

for any $a, a_2 \in l\Omega l$.

Since $bm = 0$ and $\Delta(b, a_1) = 0$, we have

$$\begin{aligned}\Delta(mb, a_1) &= \Delta(bm + mb, a_1)\\&= \Delta(b, a_1)m + f(b)\Delta(m, a_1)\\&\quad + \Delta(m, a_1)b + f(m)\Delta(b, a_1)\\&= f(b)\Delta(m, a_1) + \Delta(m, a_1)b.\end{aligned} \tag{7}$$

Since $f(b) = Qf(b)Q$ and $LQ = 0$, by (7), we have

$$D(mb, a_1) = L\Delta(mb, a_1)q = \Delta(m, a_1)b = D(m, a_1)b$$

for all $m \in l\Omega q$ and $b \in q\Omega q$.

Now, let us prove that $D(x, a_1)$ is a $f$-derivation of $\Omega$ for any given $a_1 \in l\Omega l$.

Since $\Delta$ is bilinear, then $D$ is bilinear. Let $x = a+m+b$, $y = a_2 + m_2 + b_2$ where $a, a_2 \in l\Omega l$, $m, m_2 \in l\Omega q$ and $b, b_2 \in q\Omega q$. Then

$$\begin{aligned}D(xy, a_1) &= D(aa_2 + am_2 + mb_2 + bb_2, a_1)\\&= D(a, a_1)a_2 + f(a)D(a_2, a_1) + D(a, a_1)m_2\\&\quad + f(a)D(m_2, a_1) + D(m, a_1)b_2\\&= D(a + m + b, a_1)(a_2 + m_2 + b_2)\\&\quad + f(a + m + b)D(a_2 + m_2 + b_2, a_1)\\&= D(x, a_1)y + f(x)D(y, a_1).\end{aligned}$$

Therefore, $D(x, a_1)$ is a $f$-derivations of $\Omega$.

Similarly, it can be proven that $D(x, b_1)$ is a $f$-derivation of $\Omega$ for any $b_1 \in q\Omega q$.

Next, we show that $D(x, m_1)$ is a $f$-derivation of $\Omega$ for any $m_1 \in l\Omega q$.

For any $a, a_2 \in l\Omega l$ and $b, b_2 \in q\Omega q$, we have

$$\begin{aligned}D(aa_2, m_1) &= f(aa_2)\Delta(l, m_1) = f(a)f(a_2)\Delta(l, m_1)\\&= f(a)D(a_2, m_1)\end{aligned}$$

and

$$D(bb_2, m_1) = -\Delta(l, m_1)bb_2 = D(b, m_1)b_2.$$

Since $ma = 0$ for all $a \in l\Omega l$, $m \in l\Omega q$, we have

$$\begin{aligned}\Delta(am, m_1) &= \Delta(am + ma, m_1)\\&= \Delta(a, m_1)m + f(a)\Delta(m, m_1)\\&\quad + \Delta(m, m_1)a + f(m)\Delta(a, m_1).\end{aligned}$$

By multiplying the above equation with $L$ on the left and $q$ on the right, using the facts that $f(m) = Lf(m)Q$ and $\Delta(a, m_1) \in L\Omega q + Q\Omega l$, we obtain

$$L\Delta(am, m_1)q = Lf(a)L \cdot L\Delta(m, m_1)q.$$

Hence, $D(am, m_1) = f(a)D(m, m_1)$ for all $a \in l\Omega l$, $m \in l\Omega q$.

Since $bm = 0$ for all $m \in l\Omega q$, $b \in q\Omega q$, we have

$$\begin{aligned}\Delta(mb, m_1) &= \Delta(bm + mb, m_1)\\&= \Delta(b, m_1)m + f(b)\Delta(m, m_1)\\&\quad + \Delta(m, m_1)b + f(m)\Delta(b, m_1).\end{aligned}$$

By multiplying the above equation with $L$ on the left and $q$ on the right, using the facts that $f(b) = Qf(b)Q$ and $\Delta(b, m_1) \in L\Omega q + Q\Omega l$, one can obtain

$$L\Delta(mb, m_1)q = L\Delta(m, m_1)q \cdot qbq.$$

Therefore, $D(mb, m_1) = D(m, m_1)b$, for all $m \in l\Omega q$, $b \in q\Omega q$.

Now, let us show that $D(x, m_1)$ is a $f$-derivation of $\Omega$ for any given $m_1 \in l\Omega q$.

Let $x = a + m + b$, $y = a_2 + m_2 + b_2$ where $a, a_2 \in l\Omega l$, $m, m_2 \in l\Omega q$ and $b, b_2 \in q\Omega q$. Since $D$ is bilinear and

$$\begin{aligned} &D(a, m_1)b_2 + f(a)D(b_2, m_1) \\ &= (f(a)\Delta(l, m_1))b_2 + f(a)(-\Delta(l, m_1)b_2) \\ &= 0, \end{aligned}$$

it follows

$$\begin{aligned} D(xy, m_1) &= D(aa_2 + am_2 + mb_2 + bb_2, m_1) \\ &= f(a)D(a_2, m_1) + f(a)D(m_2, m_1) \\ &\quad + D(m, m_1)b_2 + D(b, m_1)b_2 \\ &= D(a + m + b, m_1)(a_2 + m_2 + b_2) \\ &\quad + f(a + m + b)D(a_2 + m_2 + b_2, m_1) \\ &= D(x, m_1)y + f(x)D(y, m_1). \end{aligned}$$

Therefore, $D(x, m_1)$ is a $f$-derivation of $\Omega$.

Let $\overline{y} = a_1 + m_1 + b_1$ be arbitrary element of $\Omega$. Since $D(x, a_1)$, $D(x, b_1)$ and $D(x, m_1)$ are $f$-derivations of $\Omega$, one can obtain

$$\begin{aligned} D(xy, \overline{y}) &= D(xy, a_1) + D(xy, m_1) + D(xy, b_1) \\ &= D(x, a_1)y + f(x)D(y, a_1) + D(x, m_1)y \\ &\quad + f(x)D(y, m_1) + D(x, b_1)y + f(x)D(y, b_1) \\ &= D(x, a_1 + m_1 + b_1)y + f(x)D(y, a_1 + m_1 + b_1) \\ &= D(x, \overline{y})y + f(x)D(y, \overline{y}). \end{aligned}$$

Therefore, $D$ is a $f$-derivation in relation to the first component.

In a similar way, we can show that $D$ is a $f$-derivation in relation to the second component. Hence, $D$ is a $f$-biderivation. □

**Theorem 3.6.** Let $\Omega$ be a 2-torsion free triangular algebra, then any Jordan $f$-biderivation $\Delta : \Omega \times \Omega \to \Omega$ may be expressed as $\Delta = D + \delta + \tau$, in which $D : \Omega \times \Omega \to \Omega$ is a $f$-biderivation, $\delta : \Omega \times \Omega \to f(q) \cdot l\Omega l$ is a map with $\delta(l\Omega l + q\Omega q, l\Omega l + q\Omega q) = \{0\}$ and $\tau : \Omega \times \Omega \to Z_f(\Omega)$ is a $f$-central map.

**Proof.** Let $\Delta : \Omega \times \Omega \to \Omega$ be a Jordan $f$-biderivation. According to Proposition 3.3, we may assume $\Delta(l, l) = 0$.

The mapping $D : \Omega \times \Omega \to \Omega$ defined in Lemma 3.5 is a $f$-biderivation.

Let $\delta : \Omega \times \Omega \to \Omega$ be a mapping defined by

$$\begin{aligned} &\delta(a, a_1) = 0, \delta(b, b_1) = 0, \delta(a, b_1) = 0, \delta(b, a_1) = 0, \\ &\delta(a, m_1) = Q\Delta(a, m_1)l = \Delta(l, m_1)a, \\ &\delta(m, a_1) = Q\Delta(m, a_1)l = \Delta(m, l)a_1, \\ &\delta(b, m_1) = Q\Delta(b, m_1)l = -f(b)\Delta(l, m_1), \\ &\delta(m, b_1) = Q\Delta(m, b_1)l = -f(b_1)\Delta(m, l), \\ &\delta(m, m_1) = Q\Delta(m, m_1)l \end{aligned}$$

for all $a, a_1 \in l\Omega l$, $m, m_1 \in l\Omega q$ and $b, b_1 \in q\Omega q$.

Since

$$\begin{aligned} \delta(a, m_1) &= Q\Delta(a, m_1)l = Q(l + q)\Delta(a, m_1)l \\ &= Q \cdot l\Delta(a, m_1)l \end{aligned}$$

holds for every $m_1 \in l\Omega q$, then $\delta(a, m_1) \in Q \cdot l\Omega l$. In the same way, we can prove that

$$\delta(m, a_1), \delta(b, m_1), \delta(m, b_1), \delta(m, m_1) \in Q \cdot l\Omega l.$$

Therefore, $\delta : \Omega \times \Omega \to f(q) \cdot l\Omega l$ is a map with $\delta(l\Omega l + q\Omega q, l\Omega l + q\Omega q) = \{0\}$.

Let $\tau = \Delta - D - \delta$. From Lemma 3.4, it follows that

$$\begin{aligned} &\tau(a, a_1) = 0, \tau(b, b_1) = 0, \tau(a, b_1) = 0, \tau(b, a_1) = 0, \\ &\tau(a, m_1) = 0, \tau(m, a_1) = 0, \tau(b, m_1) = 0, \tau(m, b_1) = 0, \\ &\tau(m, m_1) = -Lf(m)Q \cdot Q\Delta(l, m_1)l \\ &\qquad\qquad + Q\Delta(l, m_1)l \cdot lmq \end{aligned}$$

for all $a, a_1 \in l\Omega l$, $m, m_1 \in l\Omega q$ and $b, b_1 \in q\Omega q$.

In the following we prove that $\tau(m, m_1) \in Z_f(\Omega)$ for any $m, m_1 \in l\Omega q$.

By Lemma 2.2 (i), we have

$$\begin{aligned} 0 = \Delta(l, 0) &= \Delta(l, mm_1 + m_1m) \\ &= \Delta(l, m)m_1 + f(m)\Delta(l, m_1) \\ &\quad + \Delta(l, m_1)m + f(m_1)\Delta(l, m) \end{aligned}$$

for all $m_1 \in l\Omega q$. By multiplying the above equation with $L$ on the left and $l$ on the right, using the facts that $f(m) = Lf(m)Q$ and $\Delta(l, m), \Delta(l, m_1) \in L\Omega q + Q\Omega l$, we obtain

$$Lf(m)Q \cdot Q\Delta(l, m_1)l + Lf(m_1)Q \cdot Q\Delta(l, m)l = 0.$$

Therefore

$$Lf(m_1)Q \cdot Q\Delta(l, m)l = -Lf(m)Q \cdot Q\Delta(l, m_1)l.$$

Similarly, we can prove that

$$Q\Delta(l, m_1)l \cdot lmq = -Q\Delta(l, m)l \cdot lm_1q.$$

Then

$$\begin{aligned} &(-Lf(m)Q \cdot Q\Delta(l, m_1)l)lm_2q \\ &= -Lf(m)Q(Q\Delta(l, m_1)l \cdot lm_2q) \\ &= -Lf(m)Q(-Q\Delta(l, m_2)l \cdot lm_1q) \\ &= (Lf(m)Q \cdot Q\Delta(l, m_2)l)lm_1q \\ &= (-Lf(m_2)Q \cdot Q\Delta(l, m)l)lm_1q \\ &= -Lf(m_2)Q(Q\Delta(l, m)l \cdot lm_1q) \\ &= -Lf(m_2)Q(-Q\Delta(l, m_1)l \cdot lmq) \\ &= Lf(m_2)Q(Q\Delta(l, m_1)l \cdot lmq) \end{aligned}$$

for all $m_2 \in l\Omega q$. According to Proposition 2.1, we have

$$-Lf(m)Q \cdot Q\Delta(l, m_1)l + Q\Delta(l, m_1)l \cdot lmq \in Z_f(\Omega),$$

that is, $\tau(m, m_1) \in Z_f(\Omega)$.

Therefore, $\tau$ is a $f$-central map. □

When $f$ is an identity mapping, the (Jordan) $f$-biderivation is the (Jordan) biderivation. In this case, $Q = f(q) = q$, $Q \cdot l = 0$, $Q\Delta(l, m_1)l = 0$, then the maps $\delta$ and $\tau$ from Theorem 3.6 equal zero. One can obtain

**Corollary 3.7.** Let $\Omega$ be a 2-torsion free triangular algebra, then any Jordan biderivation of $\Omega$ is a biderivation.

## IV. Conclusion

This study aims to examine the structure of Jordan $f$-biderivations on a triangular algebra and proves that each Jordan $f$-biderivation can be decomposed into the sum of a $f$-biderivation, a central mapping and a special mapping. The results can be applied to Jordan biderivations of a triangular algebra. The generalized matrix algebra is a natural extension of the triangular algebra. Therefore, a follow-up question is proposed: how to characterize the structure of the Jordan $f$-biderivation on a generalized matrix algebra?

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