Weakly Regular Ordered Semigroups Characterized in Terms of Generalized Interval Valued Bipolar Fuzzy Ideals

Pannawit Khamrot, Aiyared Iampan, Thiti Gaketem

Abstract—**In this article, we give a definition of types generalized interval valued bipolar fuzzy ideal in ordered semigroups. We study some interesting properties of generalized interval valued bipolar fuzzy ideal by using weakly regular ordered semigroups. In the goal results, we characterize weakly regular ordered semigroups by using generalized interval valued bipolar fuzzy ideals.**

Index Terms—**Ordered semigroup, weakly regular, IVBF sets, BF-sets**

I. Introduction

TOOLS USED deal world of uncertainty is the theory fuzzy set was established with the work by L. A. Zadeh in 1965 [1]. In 1979, Kuroki [2] used knowledge of fuzzy set in theory semigroups. Later the theory of fuzzy sets extened by interval valued fuzzy sets was introduced by L. A. Zadeh in 1975 [3]. Interval valued fuzzy sets have various applications in several areas like medical science [4], image processing [5], decision making [6], etc. In 2006, A. L. Narayanan and T. Manikantan [7] used knowledge the theory of interval valued fuzzy sets in algebraic system. In 1994, W. R. Zhang [8] introduced the notion of bipolar fuzzy sets with the extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$, and used them for modeling and decision analysis. In 2000, K. Lee [9] used the term bipolar valued fuzzy sets and applied it to algebraic structures. Moreover research, in types bipolar fuzzy ideals such as M. K. Kang and J. G. Kang [10] studied bioplar fuzzy subsemigroups in semigroups. V. Chinnadurau and K. Arulmozhi [11] discussed the bipolar fuzzy ideal in orederd Γ-semigroups, P. Khamrot and M. Siripitukdet [12] explained generalized bipolar fuzzy subsemigroups in semigroups. T. Gaketem and P. Khamrot [13] studied bipolar weakly interior ideals in semigorups. T. Gaketem et al. [14] expand cubic bipoar fuzzy subsemigroups and ideals in semigroups. In 2024 T. Gaketem and T. Prommai [15] studied properties of bipolar fuzzy bi-interior ideals in semigroups. In 2019, K. Arulmozhi et al. studied interval valued bipolar fuzzy set in algebra structure. M. Shabir and A. Khan [16] characterize of types ordered semigroups in terms of fuzzy ideals. In 2021, S. Lekkoksung [17] deveoped interval valued bipolar fuzzy ideal in ordered semigroup and characterized regular ordered semigroup in terms generalized interval valued bipolar fuzzy ideal and bi-ideal.

In this paper, we establish definition types of generalized interval valued bipolar fuzzy ideals in ordered semigroups. The remainder of this paper is organized in the following In Section 3, prove properties of generalized interval valued bipolar fuzzy ideal by using weakly regular ordered semigroups. In Section 4, characterize weakly regular ordered semigroups by using generalized interval valued bipolar fuzzy ideals. The conclusion are presented in Section 5.

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P. Khamrot is a lecturer at the Department of Mathematics, Faculty of Science and Agricultural Technology, Rajamangala University Technology Lanna Phitsanulok, Phitsanulok, Thailand. (e-mail: pk_g@rmutl.ac.th).

A. Iampan is a lecturer at the Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, University of Phayao, Phayao, Thailand. (e-mail: aiyared.ia@up.ac.th).

T. Gaketem is a lecturer at the Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, University of Phayao, Phayao, Thailand. (corresponding author to provide email: thiti.ga@up.ac.th).

II. Preliminaries

In this section, we give some definition and theory helpful in later sections.

An ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation. A non-empty subset $\mathfrak{L}$ of an ordered semigroup $\mathfrak{G}$ is called

(1) a *subsemigroup* of $\mathfrak{G}$ if $\mathfrak{L}^2 \subseteq \mathfrak{L}$,
(2) a *left* (right) ideal of $\mathfrak{G}$ if $\mathfrak{G}\mathfrak{L} \subseteq \mathfrak{L}$ ($\mathfrak{L}\mathfrak{G} \subseteq \mathfrak{L}$),
(3) a *genralized bi-ideal* of $\mathfrak{G}$ if $\mathfrak{L}\mathfrak{G}\mathfrak{L} \subseteq \mathfrak{L}$,
(4) a *quasi-ideal* of $\mathfrak{G}$ if $\mathfrak{G}\mathfrak{L} \cap \mathfrak{L}\mathfrak{G} \subseteq \mathfrak{L}$,
(5) a *bi-ideal* of $\mathfrak{G}$ if $\mathfrak{L}$ is a subsemigroup and $\mathfrak{L}\mathfrak{G}\mathfrak{L} \subseteq \mathfrak{L}$,
(6) an *interior ideal* of $\mathfrak{G}$ if $\mathfrak{L}$ is a subsemigroup and $\mathfrak{G}\mathfrak{L}\mathfrak{G} \subseteq \mathfrak{L}$.

For any $\mathfrak{m}_i \in [0, 1]$, where $i \in \mathcal{A}$, define

$$\underset{i\in\mathcal{A}}{\vee} \mathfrak{m}_i := \sup_{i\in\mathcal{A}}\{\mathfrak{m}_i\} \quad \text{and} \quad \underset{i\in\mathcal{A}}{\wedge} \mathfrak{m}_i := \inf_{i\in\mathcal{A}}\{\mathfrak{m}_i\}.$$

We see that for any $\mathfrak{m}, \mathfrak{q} \in [0, 1]$, we have

$$\mathfrak{m} \vee \mathfrak{q} = \max\{\mathfrak{m}, \mathfrak{q}\} \quad \text{and} \quad \mathfrak{m} \wedge \mathfrak{q} = \min\{\mathfrak{m}, \mathfrak{q}\}.$$

A *fuzzy set* of a non-empty set $\mathfrak{T}$ is a function $\Upsilon : \mathfrak{L} \to [0, 1]$.

Let $\mathrm{CS}[0, 1]$ be the set of all closed subintervals of $[0, 1]$, i.e.,

$$\mathrm{CS}[0, 1] = \{\overline{\Upsilon} = [\Upsilon^-, \Upsilon^+] \mid 0 \leq \Upsilon^- \leq \Upsilon^+ \leq 1\}.$$

We note that $[\Upsilon, \Upsilon] = \{\Upsilon\}$ for all $\Upsilon \in [0, 1]$. For $\Upsilon = 0$ or 1 we shall denote $[0, 0]$ by $\overline{0}$ and $[1, 1]$ by $\overline{1}$.

Let $\overline{\Upsilon} = [\Upsilon^-, \Upsilon^+]$ and $\overline{\rho} = [\rho^-, \rho^+] \in \mathrm{CS}[0, 1]$. Define the operations $\preceq$, $=$, $\curlywedge$ and $\curlyvee$ as follows:

(1) $\overline{\Upsilon} \preceq \overline{\rho}$ if and only if $\Upsilon^- \leq \rho^-$ and $\Upsilon^+ \leq \rho^+$
(2) $\overline{\Upsilon} = \overline{\rho}$ if and only if $\Upsilon^- = \rho^-$ and $\Upsilon^+ = \rho^+$

(3) $\overline{\Upsilon} \curlywedge \overline{\rho} = [(\Upsilon^- \wedge \rho^-), (\Upsilon^+ \wedge \rho^+)]$
(4) $\overline{\Upsilon} \curlyvee \overline{\rho} = [(\Upsilon^- \vee \rho^-), (\Upsilon^+ \vee \rho^+)]$.
If $\overline{\Upsilon} \succeq \overline{\rho}$, we mean $\overline{\rho} \preceq \overline{\Upsilon}$.

For each interval $\overline{\Upsilon}_i = [\Upsilon_i^-, \Upsilon_i^+] \in \mathrm{CS}[0,1]$, $i \in \mathcal{A}$ where $\mathcal{A}$ is an index set, we define

$$\curlywedge_{i\in\mathcal{A}} \overline{\Upsilon}_i = [\wedge_{i\in\mathcal{A}} \Upsilon_i^-, \wedge_{i\in\mathcal{A}} \Upsilon_i^+] \quad \text{and} \quad \curlyvee_{i\in\mathcal{A}} \overline{\Upsilon}_i = [\vee_{i\in\mathcal{A}} \Upsilon_i^-, \vee_{i\in\mathcal{A}} \Upsilon_i^+].$$

Definition 2.1. *[7] Let $\mathfrak{T}$ be a non-empty set. Then the function $\overline{\Upsilon} : \mathfrak{T} \to CS[0,1]$ is called* interval valued fuzzy set *(shortly, IVF set) of $\mathfrak{T}$.*

Definition 2.2. *[7] Let $\mathfrak{L}$ be a subset of a non-empty set $\mathfrak{G}$. An* interval valued characteristic function *of $\mathfrak{L}$ is defined to be a function $\overline{\chi}_{\mathfrak{L}} : \mathfrak{G} \to CS[0,1]$ by*

$$\overline{\chi}_{\mathfrak{L}}(\mathfrak{e}) = \begin{cases} \overline{1} & \text{if} \quad \mathfrak{e} \in \mathfrak{L}, \\ \overline{0} & \text{if} \quad \mathfrak{e} \notin \mathfrak{L} \end{cases}$$

for all $\mathfrak{e} \in \mathfrak{G}$.

Now, we reivew definition of bipolar valued fuzzy set and basic properties used in next section.

Definition 2.3. *[9] Let $\mathfrak{T}$ be a non-empty set. A* ***bipolar fuzzy set*** *(BF set) Υ on $\mathfrak{T}$ is an object having the form*

$$\Upsilon := \{(\mathfrak{e}, \Upsilon^p(\mathfrak{e}), \Upsilon^n(\mathfrak{e})) \mid \mathfrak{e} \in \mathfrak{T}\},$$

where $\Upsilon^p : \mathfrak{T} \to [0,1]$ and $\Upsilon^n : \mathfrak{T} \to [-1,0]$.

Remark 2.4. *For the sake of simplicity we shall use the symbol $\Upsilon = (\mathfrak{T}; \Upsilon^p, \Upsilon^n)$ for the BF set $\Upsilon = \{(\mathfrak{e}, \Upsilon^p(\mathfrak{e}), \Upsilon^n(\mathfrak{e})) \mid \mathfrak{k} \in \mathfrak{T}\}$.*

The following example of a BF set.

Example 2.5. *Let $\mathfrak{T} = \{21, 22, 23...\}$.*
Define $\Upsilon^p : \mathfrak{T} \to [0,1]$ is a function

$$\Upsilon^p(\mathfrak{e}) = \begin{cases} 0 & \text{if } \mathfrak{e} \text{ is old number} \\ 1 & \text{if } \mathfrak{e} \text{ is even number} \end{cases}$$

and $\Upsilon^n : \mathfrak{T} \to [-1,0]$ is a function

$$\Upsilon^n(\mathfrak{e}) = \begin{cases} -1 & \text{if } \mathfrak{e} \text{ is old number} \\ 0 & \text{if } \mathfrak{e} \text{ is even number.} \end{cases}$$

Then $\Upsilon = (\mathfrak{T}; \Upsilon^p, \Upsilon^n)$ is a BF set.

For $\mathfrak{e} \in \mathfrak{T}$, define $F_{\mathfrak{e}} = \{(\mathfrak{y}, \mathfrak{z}) \in \mathfrak{T} \times \mathfrak{T} \mid \mathfrak{e} = \mathfrak{y}\mathfrak{z}\}$.

Definition 2.6. *[18] Let $\mathfrak{I}$ be a non-empty subset of a semigroup $\mathfrak{T}$. A* ***positive characteristic function and a negative characteristic function*** *are respectively defined by*

$$\chi_{\mathfrak{I}}^p : \mathfrak{T} \to [0,1], \mathfrak{e} \mapsto \lambda_{\mathfrak{I}}^p(\mathfrak{e}) := \begin{cases} 1 & \mathfrak{e} \in \mathfrak{I}, \\ 0 & \mathfrak{e} \notin \mathfrak{I}, \end{cases}$$

and

$$\chi_{\mathfrak{I}}^n : \mathfrak{T} \to [-1,0], k \mapsto \lambda_I^n(e) := \begin{cases} -1 & \mathfrak{e} \in \mathfrak{I}, \\ 0 & \mathfrak{e} \notin \mathfrak{I}. \end{cases}$$

Remark 2.7. *For the sake of simplicity we shall use the symbol $\chi_{\mathfrak{I}} = (\mathfrak{T}; \chi_{\mathfrak{I}}^p, \chi_{\mathfrak{I}}^n)$ for the BF set $\chi_{\mathfrak{I}} := \{(\mathfrak{e}, \chi_{\mathfrak{I}}^p(\mathfrak{e}), \chi_{\mathfrak{I}}^n(\mathfrak{e})) \mid \mathfrak{k} \in \mathfrak{I}\}$.*

Now, we review definition of an interval valued bipolar fuzzy set and basic properties used in next section.

Definition 2.8. *[17] An* interval valued bipolar fuzzy set *(shortly, IVBF subset) $\overline{\mathcal{T}}$ on an ordered semigroup $\mathfrak{G}$ is form*

$$\overline{\mathcal{T}} := \{\mathfrak{e}, \overline{\Upsilon}^p(\mathfrak{e}), \overline{\Upsilon}^n(\mathfrak{e}) \mid \mathfrak{e} \in \mathfrak{G}\},$$

where $\overline{\Upsilon}^p : \mathfrak{G} \to \Upsilon[0,1]$ and $\overline{\Upsilon}^n : \mathfrak{G} \to \Upsilon[-1,0]$.

In this page we shall use the symbol $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$ instead of the IVBF set $\overline{\mathcal{T}} := \{\mathfrak{e}, \overline{\Upsilon}^p(\mathfrak{e}), \overline{\Upsilon}^n(\mathfrak{e}) \mid \mathfrak{e} \in \mathfrak{G}\}$.

For two IVBF sets $\overline{\mathcal{T}}_1 = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$ and $\overline{\mathcal{T}}_2 = (\overline{\rho}^p, \overline{\rho}^n)$ of an ordered semigroup $\mathfrak{G}$, define

(1) $\overline{\mathcal{T}}_1 \sqsubseteq \overline{\mathcal{T}}_2$ if and only if $\overline{\Upsilon}^p(\mathfrak{e}) \leq \overline{\rho}^p(\mathfrak{e})$ and $\overline{\Upsilon}^n(\mathfrak{e}) \leq \overline{\rho}^n(\mathfrak{e})$ for all $\mathfrak{e} \in \mathfrak{G}$,
(2) $\overline{\mathcal{T}}_1 = \overline{\mathcal{T}}_1$ if and only if $\overline{\mathcal{T}}_1 \sqsubseteq \overline{\mathcal{T}}_2$ and $\overline{\mathcal{T}}_2 \sqsubseteq \overline{\mathcal{T}}_1$,
(3) $\overline{\mathcal{T}}_1 \sqcup \overline{\mathcal{T}}_2$ if and only if $\overline{\Upsilon} \cup \overline{\rho}$ where $(\overline{\Upsilon}^p \cup \overline{\rho}^p)(\mathfrak{e}) = \overline{\Upsilon}^p(\mathfrak{e}) \vee \overline{\rho}^p(\mathfrak{e})$ and $(\overline{\Upsilon}^n \cup \overline{\rho}^n)(\mathfrak{e}) = \overline{\Upsilon}^n(\mathfrak{e}) \wedge \overline{\rho}^n(\mathfrak{e})$ for all $\mathfrak{e} \in \mathfrak{G}$,
(4) $\overline{\mathcal{T}}_1 \sqcap \overline{\mathcal{T}}_2$ if and only if $\overline{\Upsilon} \cap \overline{\rho}$ where $(\overline{\Upsilon}^p \cap \overline{\rho}^p)(\mathfrak{e}) = \overline{\Upsilon}^p(\mathfrak{e}) \wedge \overline{\rho}^p(\mathfrak{e})$ and $(\overline{\Upsilon}^n \cap \overline{\rho}^n)(\mathfrak{e}) = \overline{\Upsilon}^n(\mathfrak{e}) \vee \overline{\rho}^n(\mathfrak{e})$for all $\mathfrak{e} \in \mathfrak{G}$,
(5) $\overline{\mathcal{T}}_1 \overline{\circ} \overline{\mathcal{T}}_2$ if and on if $\overline{\Upsilon} \circ \overline{\rho}$ where

$$(\overline{\Upsilon}^p \circ \overline{\rho}^p)(\mathfrak{e}) = \begin{cases} \bigvee_{(\mathfrak{t},\mathfrak{h})\in F_{\mathfrak{e}}} \{\overline{\Upsilon}^p(\mathfrak{t}) \wedge \overline{\rho}^p(\mathfrak{h})\} & \text{if } F_{\mathfrak{e}} \neq \emptyset, \\ \overline{0} & \text{if } F_{\mathfrak{e}} = \emptyset, \end{cases}$$

and

$$(\overline{\Upsilon}^n \overline{\circ} \overline{\rho}^n)(\mathfrak{e}) = \begin{cases} \bigwedge_{(t,h)\in F_{\mathfrak{e}}} \{\overline{\Upsilon}^n(\mathfrak{t}) \vee \overline{\rho}^n(\mathfrak{h})\} & \text{if } F_{\mathfrak{e}} \neq \emptyset, \\ \overline{0} & \text{if } F_{\mathfrak{e}} = \emptyset, \end{cases}$$

where $F_{\mathfrak{e}} := \{(\mathfrak{t}, \mathfrak{h}) \in \mathfrak{G} \times \mathfrak{G} \mid \mathfrak{e} \leq \mathfrak{t}\mathfrak{h}\}$ for all $\mathfrak{e} \in \mathfrak{G}$.

Definition 2.9. *[17] Let $\mathfrak{I}$ be a non-empty set of an ordered semigroup $\mathfrak{G}$. An* ***interval valued bipolar characteristic function*** *are respectively defined by*

$$\overline{\chi}_{\mathfrak{I}}^p : \mathfrak{G} \to CS[0,1], \mathfrak{e} \mapsto \overline{\chi}_{\mathfrak{I}}^p(\mathfrak{e}) := \begin{cases} \overline{1} & \mathfrak{e} \in \mathfrak{I}, \\ \overline{0} & \mathfrak{e} \notin \mathfrak{I}, \end{cases}$$

and

$$\overline{\chi}_{\mathfrak{I}}^n : \mathfrak{T} \to CS[-1,0], \mathfrak{e} \mapsto \overline{\chi}_I^n(e) := \begin{cases} -\overline{1} & \mathfrak{e} \in \mathfrak{I}, \\ \overline{0} & \mathfrak{e} \notin \mathfrak{I}. \end{cases}$$

Remark 2.10. *For the sake of simplicity we shall use the symbol $\overline{\chi}_{\mathfrak{I}} = (\mathfrak{G}; \overline{\chi}_{\mathfrak{I}}^p, \overline{\chi}_{\mathfrak{I}}^n)$ for the IVBF set $\overline{\chi}_{\mathfrak{I}} := \{(\mathfrak{e}, \overline{\chi}_{\mathfrak{I}}^p(\mathfrak{e}), \overline{\chi}_{\mathfrak{I}}^n(\mathfrak{e})) \mid \mathfrak{e} \in \mathfrak{I}\}$.*

Now, we let $\overline{\lambda}^p$, $\overline{\delta}^p \in \mathrm{CS}[0,1]$ be such that $\overline{0} \leq \overline{\lambda}^p < \overline{\delta}^p \leq \overline{1}$ and $\overline{\lambda}^n$, $\overline{\delta}^n \in \mathrm{CS}[-1,0]$ be such that $-\overline{1} \leq \overline{\delta}^n < \overline{\lambda}^n \leq \overline{1}$. Both $\overline{\lambda}, \overline{\delta}$ are arbitrary but fixed.

Definition 2.11. *[17] Let $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$ be an IVBF set of ordered semigroup $\mathfrak{G}$ is called an $(\overline{\lambda}, \overline{\delta})$-IVBF subsemigroup of $\mathfrak{G}$ if*
(1) $\overline{\Upsilon}^p(\mathfrak{e}_1\mathfrak{e}_2) \vee \overline{\lambda}^p \geq \overline{\Upsilon}^p(\mathfrak{e}_1) \wedge \overline{\Upsilon}^p(\mathfrak{e}_2) \wedge \overline{\delta}^p$.
(2) *If* $\mathfrak{e}_1 \geq \mathfrak{e}_2$, *then* $\overline{\Upsilon}^p(\mathfrak{e}_1) \vee \overline{\lambda}^p \geq \overline{\Upsilon}^p(\mathfrak{e}_2) \wedge \overline{\delta}^p$.
(3) $\overline{\Upsilon}^n(\mathfrak{e}_1\mathfrak{e}_2) \wedge \overline{\lambda}^n \leq \overline{\Upsilon}^n(\mathfrak{e}_1) \vee \overline{\Upsilon}^n(\mathfrak{e}_2) \curlywedge \overline{\delta}^n$.
(4) *If* $\mathfrak{e}_1 \geq \mathfrak{e}_2$, *then* $\overline{\Upsilon}^n(\mathfrak{e}_1) \wedge \overline{\lambda}^n \leq \overline{\Upsilon}^n(\mathfrak{e}_2) \vee \overline{\delta}^n$,
for all $\mathfrak{e}_1, \mathfrak{e}_2 \in \mathfrak{G}$.

Definition 2.12. *[17] Let $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$ be an IVBF set of ordered semigroup $\mathfrak{G}$ is called an $(\overline{\lambda}, \overline{\delta})$-IVBF left ideal of $\mathfrak{G}$ if*
(1) $\overline{\Upsilon}^p(\mathfrak{e}_1\mathfrak{e}_2) \vee \overline{\lambda}^p \geq \overline{\Upsilon}^p(\mathfrak{e}_2) \curlywedge \overline{\delta}^p$.
(2) *If* $\mathfrak{e}_1 \geq \mathfrak{e}_2$, *then* $\overline{\Upsilon}^p(\mathfrak{e}_1) \vee \overline{\lambda}^p \geq \overline{\Upsilon}^p(\mathfrak{e}_2) \wedge \overline{\delta}^p$.
(3) $\overline{\Upsilon}^n(\mathfrak{e}_1\mathfrak{e}_2) \wedge \overline{\lambda}^n \leq \overline{\Upsilon}^n(\mathfrak{e}_2) \vee \overline{\delta}^n$.

(4) *If* $\mathfrak{e}\mathfrak{e}_1 \geq \mathfrak{e}_2$, *then* $\overline{\Upsilon}^n(\mathfrak{e}_1) \wedge \overline{\lambda}^n \leq \overline{\Upsilon}^n(\mathfrak{e}_2) \vee \overline{\delta}^n$,

for all $\mathfrak{e}_1, \mathfrak{e}_2 \in \mathfrak{G}$.

Definition 2.13. *[17] Let* $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$ *be an IVBF set of ordered semigroup* $\mathfrak{G}$ *is called an* $(\overline{\lambda}, \overline{\delta})$*-IVBF right ideal of* $\mathfrak{G}$ *if*

(1) $\overline{\Upsilon}^p(\mathfrak{e}_1\mathfrak{e}_2) \vee \overline{\lambda}^p \geq \overline{\Upsilon}^p(\mathfrak{e}_1) \wedge \overline{\delta}^p$.
(2) *If* $\mathfrak{e}_1 \geq \mathfrak{e}_2$, *then* $\overline{\Upsilon}^p(\mathfrak{e}_1) \vee \overline{\lambda}^p \geq \overline{\Upsilon}^p(\mathfrak{e}_2) \wedge \overline{\delta}^p$.
(3) $\overline{\Upsilon}^n(\mathfrak{e}_1\mathfrak{e}_2) \wedge \overline{\lambda}^n \geq \overline{\Upsilon}^n(\mathfrak{e}_1) \vee \overline{\delta}^n$.
(4) *If* $\mathfrak{e}_1 \geq \mathfrak{e}_2$, *then* $\overline{\Upsilon}^n(e_1) \wedge \overline{\lambda}^n \leq \overline{\Upsilon}^n(\mathfrak{e}_2) \vee \overline{\delta}^n$,

for all $\mathfrak{e}_1, \mathfrak{e}_2 \in \mathfrak{G}$.

An IVBF set $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$ of an ordered semigroup $\mathfrak{G}$ is called $(\overline{\lambda}, \overline{\delta})$-IVBF ideal of $\mathfrak{G}$ if it is both $(\overline{\lambda}, \overline{\delta})$-IVBF left ideal and $(\overline{\lambda}, \overline{\delta})$-IVBF right ideal of $\mathfrak{G}$.

Definition 2.14. *[17] Let* $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$ *be an IVBF set of ordered semigroup* $\mathfrak{G}$ *is called an* $(\overline{\lambda}, \overline{\delta})$*-IVBF bi-ideal of* $\mathfrak{G}$ *if*

(1) $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$ *is an* $(\overline{\lambda}, \overline{\delta})$*-IVBF subsemigroup of* $\mathfrak{G}$
(2) $\overline{\Upsilon}^p(\mathfrak{e}_1\mathfrak{e}_2\mathfrak{e}_3) \vee \overline{\lambda}^p \geq \overline{\Upsilon}^p(\mathfrak{e}_1) \wedge \overline{\Upsilon}^p(\mathfrak{e}_3) \wedge \overline{\delta}^p$
(3) $\overline{\Upsilon}^n(\mathfrak{e}_1\mathfrak{e}_2\mathfrak{e}_3) \wedge \overline{\lambda}^n \leq \overline{\Upsilon}^n(e_1) \vee \overline{\Upsilon}^n(\mathfrak{e}_3) \vee \overline{\delta}^n$,

for all $\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3 \in \mathfrak{G}$.

III. Relation of Types Generalized Interval Valued Bipolar Fuzzy Ideals

In this section, we give some definiton of a generalized interval valued bipolar fuzzy ideal and study relations of types generalized interval valued bipolar fuzzy ideal in ordered semigroups.

Definition 3.1. *[17] Let* $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$ *be an IVBF set of ordered semigroup* $\mathfrak{G}$ *is called an* $(\overline{\lambda}, \overline{\delta})$*-IVBF generalized bi-ideal of* $\mathfrak{G}$ *if*

(1) $\overline{\Upsilon}^p(\mathfrak{e}_1\mathfrak{e}_2\mathfrak{e}_3) \vee \overline{\lambda}^p \geq \overline{\Upsilon}^p(\mathfrak{e}_1) \wedge \overline{\Upsilon}^p(\mathfrak{e}_3) \wedge \overline{\delta}^p$
(2) $\overline{\Upsilon}^n(\mathfrak{e}_1\mathfrak{e}_2\mathfrak{e}_3) \wedge \overline{\lambda}^n \leq \overline{\Upsilon}^n(e_1) \vee \overline{\Upsilon}^n(\mathfrak{e}_3) \vee \overline{\delta}^n$,

for all $\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3 \in \mathfrak{G}$.

Definition 3.2. *Let* $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$ *be an IVBF set of ordered semigroup* $\mathfrak{G}$ *is called an* $(\overline{\lambda}, \overline{\delta})$*-IVBF interior ideal of* $\mathfrak{G}$ *if*

(1) $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$ *is an* $(\overline{\lambda}, \overline{\delta})$*-IVBF subsemigroup of* $\mathfrak{G}$
(2) $\overline{\Upsilon}^p(\mathfrak{e}_1\mathfrak{e}_2\mathfrak{e}_3) \vee \overline{\lambda}^p \geq \overline{\Upsilon}^p(\mathfrak{e}_2) \wedge \overline{\delta}^p$
(3) $\overline{\Upsilon}^n(\mathfrak{e}_1\mathfrak{e}_2\mathfrak{e}_3) \wedge \overline{\lambda}^n \leq \overline{\Upsilon}^n(\mathfrak{e}_2) \vee \overline{\delta}^n$

for all $\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3 \in \mathfrak{G}$.

Definition 3.3. *Let* $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$ *be an IVBF set of ordered semigroup* $\mathfrak{G}$ *is called an* $(\overline{\lambda}, \overline{\delta})$*-IVBF quasi-ideal of* $\mathfrak{G}$ *if*

(1) $(\overline{\mathfrak{G}} \circ \overline{\mathcal{T}})^{\overline{\lambda}}_{\overline{\delta}} \sqcap (\overline{\mathcal{T}} \circ \overline{\mathfrak{G}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq \overline{\mathcal{T}}^{\overline{\lambda}}_{\overline{\delta}}$.
(2) *If* $\mathfrak{e}_1 \geq \mathfrak{e}_2$, *then* $\overline{\omega}^p(\mathfrak{G}_1) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{e}_2) \wedge \overline{\delta}^p$ *and* $\overline{\omega}^n(\mathfrak{e}_1) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(\mathfrak{e}_2) \vee \overline{\delta}^n$,

for all $\mathfrak{e}_1, \mathfrak{e}_2 \in \mathfrak{G}$.

From studying the above definitions of types we see that every $(\overline{\lambda}, \overline{\delta})$-IVBF ideal is an $(\overline{\lambda}, \overline{\delta})$-IVBF interior ideal, every $(\overline{\lambda}, \overline{\delta})$-IVBF ideal is an $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal, every $(\overline{\lambda}, \overline{\delta})$-IVBF qausi-ideal is an $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal and every $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal is an $(\overline{\lambda}, \overline{\delta})$-IVBF generalized bi-ideal.

The following example is an $(\overline{\lambda}, \overline{\delta})$-IVBF interior ideal of a semigroup.

Example 3.4. *Let us consider an ordered semigroup* $(\mathfrak{G}, \cdot)$ *defined by the following table:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	c	c
d	a	c	c	c

An IVBF set $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ *in* $\mathfrak{G}$ *as follows:*
$\overline{\omega}^p = \{((a), [0.7, 0.8]), ((b), [0.4, 0.6]), ((c), [0.6, 0.7]), ((d), [0.3, 0.5])\}$ *and* $\overline{\omega}^n = \{((a), [-0.6, -0.7]), ((b), [-0.5, -0.6]), ((c), [-0.5, -0.6]), ((d), [-0.3, -0.5])\}$ *and define a partial order raltion* $\leq$ *on* G *as follows:* $\leq$: $\{(a,b), (a,c), (a,d), (b,c), (b,d), (d,c)\} \cup \triangle_{\mathfrak{G}}$, *where* $\triangle_{\mathfrak{G}}$ *is an equality relation on* $\mathfrak{G}$. *By routine calculation,* $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ *is an* $([0.3, 0.3], [0.5, 0.5])$*-IVBF interior ideal of* $\mathfrak{G}$.

Remark 3.5. *In example 3.4 we can show that the converse of the above theorem is not true in general. Consider* $\overline{\Upsilon}^p(bd) \vee \overline{\lambda}^p = [0.3, 0.5] \ngeq [0.4, 0.5] = \overline{\Upsilon}^p(b) \wedge \overline{\delta}^p$. *Thus* $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$ *is an* $(\overline{\lambda}, \overline{\delta})$*-IVBF ideal of* G.

Definition 3.6. *An ordered semigroup* $\mathfrak{G}$ *is weakly regular if and only if for every* $\mathfrak{e} \in \mathfrak{G}$, *there exist* $\mathfrak{x}, \mathfrak{y} \in \mathfrak{G}$ *such that* $\mathfrak{e} \leq \mathfrak{e}\mathfrak{x}\mathfrak{e}\mathfrak{y}$.

The following theorem show that the $(\overline{\lambda}, \overline{\delta})$-IVBF interior ideals and $(\overline{\lambda}, \overline{\delta})$-IVBF ideals coincide for some types of ordered semigroups.

Theorem 3.7. *Let* $\mathfrak{G}$ *be a weakly regular ordered semigroup. Then the following statements are holds.*

(1) *Every* $(\overline{\lambda}, \overline{\delta})$*-IVBF interior ideal of* $\mathfrak{G}$ *is an* $(\overline{\lambda}, \overline{\delta})$*-IVBF ideal of* $\mathfrak{G}$.
(2) *Every* $(\overline{\lambda}, \overline{\delta})$*-IVBF generalized bi-ideal of* $\mathfrak{G}$ *is an* $(\overline{\lambda}, \overline{\delta})$*-IVBF bi-ideal of* $\mathfrak{G}$.

Proof:

(1) Suppose that $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVF interior ideal of $\mathfrak{G}$ and let $\mathfrak{e}_1, \mathfrak{e}_2 \in \mathfrak{G}$. Since $\mathfrak{G}$ is weakly regular, there exists $\mathfrak{t}, \mathfrak{n} \in \mathfrak{G}$ such that $\mathfrak{e}_1 \leq \mathfrak{e}_1\mathfrak{t}\mathfrak{e}_1\mathfrak{n}$. Thus,

$$\begin{aligned} \overline{\Upsilon}^p(\mathfrak{e}_1\mathfrak{e}_2) \vee \overline{\lambda}^p &= \overline{\Upsilon}^p((\mathfrak{e}_1\mathfrak{t}\mathfrak{e}_1\mathfrak{n})\mathfrak{e}_2) \vee \overline{\lambda}^p \\ &= \overline{\Upsilon}^p((\mathfrak{e}_1\mathfrak{t})\mathfrak{e}_1(\mathfrak{n}\mathfrak{e}_2)) \vee \overline{\lambda}^p \\ &\geq \overline{\Upsilon}^p(\mathfrak{e}_1) \wedge \overline{\delta}^p \end{aligned}$$

and

$$\begin{aligned} \overline{\Upsilon}^n(\mathfrak{e}_1\mathfrak{e}_2) \wedge \overline{\lambda}^n &= \overline{\Upsilon}^n((\mathfrak{e}_1\mathfrak{t}\mathfrak{e}_1\mathfrak{n})\mathfrak{e}_2) \wedge \overline{\lambda}^n \\ &= \overline{\Upsilon}^n((\mathfrak{e}_1\mathfrak{t})\mathfrak{e}_1(\mathfrak{n}\mathfrak{e}_2)) \wedge \overline{\lambda}^n \\ &\leq \overline{\Upsilon}^n(\mathfrak{e}_1) \vee \overline{\delta}^n. \end{aligned}$$

Hence $\overline{\mathcal{T}}$ is an $(\overline{\lambda}, \overline{\delta})$-IVF right ideal of $\mathfrak{G}$. Similarly, we can prove that $\overline{\mathcal{T}}$ is an $(\overline{\lambda}, \overline{\delta})$-IVF left ideal of $\mathfrak{G}$. Thus $\overline{\mathcal{T}}$ is an $(\overline{\lambda}, \overline{\delta})$-IVF ideal of $\mathfrak{G}$.

(2) Suppose that $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVF generalized bi-ideal of $\mathfrak{G}$ and let $\mathfrak{e}_1, \mathfrak{e}_2 \in \mathfrak{G}$. Since $\mathfrak{G}$ is weakly regular, there exists $\mathfrak{t}, \mathfrak{n} \in \mathfrak{G}$ such that $\mathfrak{e}_1 \leq \mathfrak{e}_1\mathfrak{t}\mathfrak{e}_1\mathfrak{n}$. Thus,

$$\begin{aligned} \overline{\Upsilon}^p(\mathfrak{e}_1\mathfrak{e}_2) \vee \overline{\lambda}^p &= \overline{\Upsilon}^p(\mathfrak{e}_1(\mathfrak{t}\mathfrak{e}_1\mathfrak{n})\mathfrak{e}_2) \vee \overline{\lambda}^p \\ &\geq \overline{\Upsilon}^p(\mathfrak{e}_1) \wedge \overline{\Upsilon}^p\mathfrak{e}_2 \wedge \overline{\delta}^p \end{aligned}$$

and

$$\begin{aligned} \overline{\Upsilon}^n(\mathfrak{e}_1\mathfrak{e}_2) \wedge \overline{\lambda}^n &= \overline{\Upsilon}^n(\mathfrak{e}_1(\mathfrak{t}\mathfrak{e}_1\mathfrak{n})\mathfrak{e}_2) \wedge \overline{\lambda}^n \\ &\leq \overline{\Upsilon}^n(\mathfrak{e}_1) \vee \overline{\Upsilon}^p\mathfrak{e}_2 \vee \overline{\delta}^n. \end{aligned}$$

Hence $\overline{\mathcal{T}}$ is an $(\overline{\lambda},\overline{\delta})$-IVF subsemigroup of $\mathfrak{G}$. By Definition 2.14 we have $\overline{\mathcal{T}}$ is an $(\overline{\lambda},\overline{\delta})$-IVF bi-ideal of $\mathfrak{G}$.

■

In the following theorem, we give a relationship between a subsemigroup and the intervval valued bipolar characteristic function which proved easily.

Theorem 3.8. *Let $\mathfrak{I}$ be a non-empty subset of an ordered semigroup $\mathfrak{G}$. Then $\mathfrak{I}$ is a left ideal (right ideal, generalized bi-ideal, bi-ideal, interior ideal, quasi-ideal) of $\mathfrak{G}$ with $\overline{\lambda}^p < \overline{\delta}^p$ and $\overline{\lambda}^n > \overline{\delta}^n$ if and only if $\overline{\chi}_{\mathfrak{I}} = (\mathfrak{G};\overline{\chi}^p_{\mathfrak{I}},\overline{\chi}^n_{\mathfrak{I}})$ is an $(\overline{\lambda},\overline{\delta})$-IVBF left ideal (right ideal, generalized bi-ideal, bi-ideal, interior ideal, quasi-ideal) of $\mathfrak{G}$.*

Proof: Suppose that $\mathfrak{I}$ is a left ideal of $\mathfrak{G}$ and let $\mathfrak{e}_1,\mathfrak{e}_2\in\mathfrak{G}$.

If $\mathfrak{e}_1\geq\mathfrak{e}_2$, then $\overline{\Upsilon}^p(\mathfrak{e}_1)\vee\overline{\lambda}^p\geq\overline{\Upsilon}^p(\mathfrak{e}_2)\wedge\overline{\delta}^p$.

If $\mathfrak{e}_2\in\mathfrak{I}$, then $\mathfrak{e}_1\mathfrak{e}_2\in\mathfrak{I}$. Thus, $\overline{1}=\overline{\chi}^p_{\mathfrak{I}}(\mathfrak{e}_2)=\overline{\chi}^p_{\mathfrak{I}}(\mathfrak{e}_1\mathfrak{e}_2)$ and $-\overline{1}=\overline{\chi}^n_{\mathfrak{I}}(\mathfrak{e}_2)=\overline{\chi}^n_{\mathfrak{I}}(\mathfrak{e}_1\mathfrak{e}_2)$.

Hence, $\overline{\chi}^p_{\mathfrak{I}}(\mathfrak{e}_1\mathfrak{e}_2)\vee\overline{\lambda}^p\geq\overline{\chi}^p_{\mathfrak{I}}(e_2)\wedge\overline{\delta}^p$ and $\overline{\chi}^n_{\mathfrak{I}}(\mathfrak{e}_1\mathfrak{e}_2)\wedge\overline{\lambda}^n\leq\overline{\chi}^n_{\mathfrak{I}}(\mathfrak{e}_2)\vee\overline{\delta}^n$.

If $\mathfrak{e}_2\notin\mathfrak{I}$, then $\mathfrak{e}_1\mathfrak{e}_2\in\mathfrak{I}$. Thus, $\overline{0}=\overline{\chi}^p_{\mathfrak{I}}(\mathfrak{e}_2)=\overline{\chi}^n_{\mathfrak{I}}(\mathfrak{e}_2)$, $1=\overline{\chi}^p_{\mathfrak{I}}(\mathfrak{e}_1\mathfrak{e}_2)$ and $-\overline{1}=\overline{\chi}^n_{\mathfrak{I}}(\mathfrak{e}_1\mathfrak{e}_2)$.
Hence $\overline{\chi}^p_{\mathfrak{I}}(\mathfrak{e}_1\mathfrak{e}_2)\vee\overline{\lambda}^p\geq\overline{\chi}^p_{\mathfrak{I}}(e_2)\wedge\overline{\delta}^p$ and
$\overline{\chi}^n_{\mathfrak{I}}(\mathfrak{e}_1\mathfrak{e}_2)\wedge\overline{\lambda}^n\leq\overline{\chi}^n_{\mathfrak{I}}(\mathfrak{e}_2)\vee\overline{\delta}^n$.
We conclude that, $\overline{\chi}_{\mathfrak{I}}=(\mathfrak{G};\overline{\chi}^p_{\mathfrak{I}},\overline{\chi}^n_{\mathfrak{I}})$ is an $(\overline{\lambda},\overline{\delta})$-IVBF left ideal of $\mathfrak{G}$.

Conversely, suppose that $\overline{\chi}_{\mathfrak{I}}=(\mathfrak{G};\overline{\chi}^p_{\mathfrak{I}},\overline{\chi}^n_{\mathfrak{I}})$ is an $(\overline{\lambda},\overline{\delta})$-IVBF left ideal of $\mathfrak{G}$ with $\overline{\lambda}^p<\overline{\delta}^p$ and $\overline{\lambda}^n>\overline{\delta}^n$ and let $\mathfrak{e}_1,\mathfrak{e}_2\in\mathfrak{G}$ with $\mathfrak{e}_2\in\mathfrak{I}$. Then $\overline{\chi}^p_{\mathfrak{I}}(\mathfrak{e}_2)=\overline{1}$ and $\overline{\chi}^n_{I}(\mathfrak{e}_2)=-\overline{1}$. By assumtion,

$$\overline{\chi}^p_{\mathfrak{I}}(\mathfrak{e}_1\mathfrak{e}_2)\vee\overline{\lambda}^p\geq\overline{\chi}^p_{\mathfrak{I}}(e_2)\wedge\overline{\delta}^p$$

and

$$\overline{\chi}^n_{\mathfrak{I}}(\mathfrak{e}_1\mathfrak{e}_2)\wedge\overline{\lambda}^n\leq\overline{\chi}^n_{\mathfrak{I}}(\mathfrak{e}_2)\vee\overline{\delta}^n.$$

If $\mathfrak{e}_1\mathfrak{e}_2\notin\mathfrak{I}$, then $\overline{\lambda}^p\geq\overline{\delta}^p$and $\overline{\lambda}^n\leq\overline{\delta}^n$. It is a contradiction. Hence $\mathfrak{e}_1\mathfrak{e}_2\in\mathfrak{I}$. Therefore $\mathfrak{I}$ is a left ideal of $\mathfrak{G}$. ■

IV. Characterize weakly ordered semigroups in terms generalized Interval valued bipolar fuzzy ideals.

In this topic, we study a characterization of weakly ordered semigroups in terms of $(\overline{\lambda},\overline{\delta})$-IVBF ideals.

First, we will defined the following symbools.

For two IVBF sets $\overline{\mathcal{T}}_1=(\overline{\Upsilon}^p,\overline{\Upsilon}^n)$ and $\overline{\mathcal{T}}_2=(\overline{\rho}^p,\overline{\rho}^n)$ of an ordered semigroup $\mathfrak{G}$, define

(1) $\overline{\mathcal{T}}^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e}):=\left((\overline{\Upsilon}^p)^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e}),(\overline{\Upsilon}^n)^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e})\right)$,

(2) $(\overline{\mathcal{T}}_1\sqcap\overline{\mathcal{T}}_2)^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e}):=\left((\overline{\Upsilon}^p\cap\overline{\rho}^p)^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e}),(\overline{\Upsilon}^n\cap\overline{\rho}^n)^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e})\right)$

(3) $(\overline{\mathcal{T}}_1\circ\overline{\mathcal{T}}_2)^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e}):=\left((\overline{\Upsilon}^p\circ\overline{\rho}^p)^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e}),(\overline{\Upsilon}^n\circ\overline{\rho}^n)^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e})\right)$.

where

$$(\overline{\Upsilon}^p\circ\overline{\rho}^p)(\mathfrak{e})=\begin{cases}\bigvee\limits_{(\mathfrak{t},\mathfrak{h})\in F_{\mathfrak{e}}}\{\overline{\Upsilon}^p(\mathfrak{t})\wedge\overline{\rho}^p(\mathfrak{h})\} & \text{if } F_{\mathfrak{e}}\neq\emptyset,\\ \overline{0} & \text{if } F_{\mathfrak{e}}=\emptyset,\end{cases}$$

and

$$(\overline{\Upsilon}^n\circ\overline{\rho}^n)(\mathfrak{e})=\begin{cases}\bigwedge\limits_{(\mathfrak{t},\mathfrak{h})\in F_{\mathfrak{e}}}\{\overline{\Upsilon}^n(\mathfrak{t})\vee\overline{\rho}^n(\mathfrak{h})\} & \text{if } F_{\mathfrak{e}}\neq\emptyset,\\ \overline{0} & \text{if } F_{\mathfrak{e}}=\emptyset,\end{cases}$$

Remark 4.1. *Since $\overline{\chi}_{\mathcal{L}}$ is an interval valued characteristic function we have*

$$(\overline{\chi}^p_{\mathcal{L}})^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e})=\begin{cases}\overline{\lambda}^p & \text{if } \mathfrak{e}\in\mathcal{L},\\ \overline{\delta}^p & \text{if } \mathfrak{e}\notin\mathcal{L}\end{cases}$$

and

$$(\overline{\chi}^n_{\mathcal{L}})^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e})=\begin{cases}\overline{\delta}^n & \text{if } \mathfrak{e}\in\mathcal{L},\\ \overline{\lambda}^n & \text{if } \mathfrak{e}\notin\mathcal{L}\end{cases}$$

This lemmas allows us to prove the Theorem 4.4.

Lemma 4.2. *[17] If $\overline{\mathcal{T}}=(\overline{\Upsilon}^p,\overline{\Upsilon}^n)$ is an $(\overline{\lambda},\overline{\delta})$-IVBF right ideal of an ordered semigroup $\mathfrak{G}$, then $(\overline{\mathcal{T}}\circ\overline{\mathfrak{G}})^{\overline{\lambda}}_{\overline{\delta}}\sqsubseteq\overline{\mathcal{T}}^{\overline{\lambda}}_{\overline{\delta}}$.*

Lemma 4.3. *[17] If $\overline{\mathcal{T}}=(\overline{\Upsilon}^p,\overline{\Upsilon}^n)$ is an $(\overline{\lambda},\overline{\delta})$-IVBF left ideal of an ordered semigroup $\mathfrak{G}$, then $(\overline{\mathfrak{G}}\circ\overline{\mathcal{T}})^{\overline{\lambda}}_{\overline{\delta}}\sqsubseteq\overline{\mathcal{T}}^{\overline{\lambda}}_{\overline{\delta}}$.*

Theorem 4.4. *If $\overline{\mathcal{T}}=(\overline{\Upsilon}^p,\overline{\Upsilon}^n)$ is an $(\overline{\lambda},\overline{\delta})$-IVBF right ideal and $\overline{\mathcal{J}}=(\overline{\rho}^p,\overline{\rho}^n)$ is an $(\overline{\lambda},\overline{\delta})$-IVBF left ideal of an ordered semigroup $\mathfrak{G}$, then $(\overline{\mathcal{T}}\circ\overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}\sqsubseteq(\overline{\mathcal{T}}\sqcap\overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$.*

Proof: Assume that $\overline{\mathcal{T}}=(\overline{\Upsilon}^p,\overline{\Upsilon}^n)$ is an $(\overline{\lambda},\overline{\delta})$-IVBF right ideal and $\overline{\mathcal{J}}=(\overline{\rho}^p,\overline{\rho}^n)$ is an $(\overline{\lambda},\overline{\delta})$-IVBF left ideal of $\mathfrak{G}$. By Lemma 4.2 and 4.3 we have Thus, $(\overline{\mathcal{T}}\circ\overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}\sqsubseteq(\overline{\mathcal{T}}\circ\overline{\mathfrak{G}})^{\overline{\lambda}}_{\overline{\delta}}\sqsubseteq\overline{\mathcal{T}}^{\overline{\lambda}}_{\overline{\delta}}$. and $(\overline{\mathcal{T}}\circ\overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}\sqsubseteq(\overline{\mathfrak{G}}\circ\overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}\sqsubseteq\overline{\mathcal{J}}^{\overline{\lambda}}_{\overline{\delta}}$. Hence, $(\overline{\mathcal{T}}\circ\overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}\sqsubseteq(\overline{\mathcal{T}}\sqcap\overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$. ■

This theorem is tool of characterizations weakly regular ordered semigroups in term $(\overline{\lambda},\overline{\delta})$-IVBF ideals.

Theorem 4.5. *[17] Let $\mathfrak{I}$ and $\mathfrak{K}$ be a non-empty subsets of $\mathfrak{G}$. Then*

(1) $(\overline{\chi}_{\mathfrak{I}}\circ\overline{\chi}_{\mathfrak{K}})^{\overline{\lambda}}_{\overline{\delta}}=(\overline{\chi}_{\mathfrak{I}\mathfrak{K}})^{\overline{\lambda}}_{\overline{\delta}}$ *i.e.* $\langle(\overline{\chi}^p_{\mathfrak{I}}\circ\overline{\chi}^p_{\mathfrak{K}})^{\overline{\delta}}_{\overline{\lambda}},(\overline{\chi}^n_{\mathfrak{I}}\circ\overline{\chi}^n_{\mathfrak{K}})^{\overline{\lambda}}_{\overline{\delta}}\rangle=\langle(\overline{\chi}^p_{\mathfrak{I}\mathfrak{K}})^{\overline{\delta}}_{\overline{\lambda}},(\overline{\chi}^n_{\mathfrak{I}\mathfrak{K}})^{\overline{\lambda}}_{\overline{\delta}}\rangle$

(2) $(\overline{\chi}_{\mathfrak{I}}\sqcap\overline{\chi}_{\mathfrak{K}})^{\overline{\lambda}}_{\overline{\delta}}=(\overline{\chi}^p_{\mathfrak{I}}\sqcap\mathfrak{K})^{\overline{\lambda}}_{\overline{\delta}}$ *i.e.* $\langle(\overline{\chi}^p_{\mathfrak{I}}\cap\overline{\chi}^p_{\mathfrak{K}})^{\overline{\delta}}_{\overline{\lambda}},(\overline{\chi}^n_{\mathfrak{I}}\cup\overline{\chi}^n_{\mathfrak{K}})^{\overline{\lambda}}_{\overline{\delta}}\rangle=\langle(\overline{\chi}^p_{\mathfrak{I}\cap\mathfrak{K}})^{\overline{\delta}}_{\overline{\lambda}},(\overline{\chi}^n_{\mathfrak{I}\cup\mathfrak{K}})^{\overline{\lambda}}_{\overline{\delta}}\rangle$, *where* $\overline{\chi}_{\mathfrak{I}}=(\mathfrak{G};\overline{\chi}^p_{\mathfrak{I}},\overline{\chi}^n_{\mathfrak{I}})$ *and* $\overline{\chi}_{\mathfrak{K}}=(\mathfrak{G};\overline{\chi}^p_{\mathfrak{K}},\overline{\chi}^n_{\mathfrak{K}})$

Now we characterize weakly regular semigroups in terms of $(\overline{\lambda},\overline{\delta})$-IVBF ideals.

Lemma 4.6. *[16] An ordered semigroup $\mathfrak{G}$ is weakly regular if and only if $\mathfrak{I}\cap\mathfrak{L}=(\mathfrak{I}\mathfrak{L}]$ for every right ideal $\mathfrak{I}$ and every ideal $\mathfrak{L}$ of $\mathfrak{G}$.*

Theorem 4.7. *An ordered semigroup $\mathfrak{G}$ is weakly regular if and only if $(\overline{\mathcal{T}}\circ\overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}=(\overline{\mathcal{T}}\sqcap\overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$, for every $(\overline{\lambda},\overline{\delta})$-IVBF right ideal $\overline{\mathcal{T}}=(\overline{\Upsilon}^p,\overline{\Upsilon}^n)$ and every $(\overline{\lambda},\overline{\delta})$-IVBF ideal $\overline{\mathcal{J}}=(\overline{\rho}^p,\overline{\rho}^n)$ of $\mathfrak{G}$.*

Proof: Assume that $\overline{\mathcal{T}}=(\overline{\Upsilon}^p,\overline{\Upsilon}^n)$ is an $(\overline{\lambda},\overline{\delta})$-IVBF right ideal and $\overline{\mathcal{J}}=(\overline{\rho}^p,\overline{\rho}^n)$ is an $(\overline{\lambda},\overline{\delta})$-IVBF ideal of $\mathfrak{G}$. Let $\mathfrak{e}\in\mathfrak{G}$. Since $\mathfrak{G}$ is weakly regular, there exist $\mathfrak{x},\mathfrak{y}\in\mathfrak{G}$ such that $\mathfrak{e}\leq\mathfrak{e}\mathfrak{x}\mathfrak{e}\mathfrak{y}$. Thus,

$$\begin{aligned}
&(\overline{\Upsilon}^p\circ\overline{\rho}^p)^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e})=((\overline{\Upsilon}^p\circ\overline{\rho}^p)(\mathfrak{e})\wedge\overline{\delta}^p)\vee\overline{\lambda}^p\\
&=((\bigvee_{(\mathfrak{a},\mathfrak{b})\in F_{\mathfrak{e}}}\{\overline{\Upsilon}^p(\mathfrak{a})\wedge\overline{\rho}^p(\mathfrak{b})\})\wedge\overline{\delta}^p)\vee\overline{\lambda}^p\\
&=((\bigvee_{(\mathfrak{a},\mathfrak{b})\in F_{\mathfrak{e}\mathfrak{x}\mathfrak{e}\mathfrak{y}}}\{\overline{\Upsilon}^p(\mathfrak{a})\wedge\overline{\rho}^p(\mathfrak{b})\})\wedge\overline{\delta}^p)\vee\overline{\lambda}^p\\
&\geq((\overline{\Upsilon}^p(\mathfrak{e}\mathfrak{x})\wedge\overline{\rho}^p(\mathfrak{e}\mathfrak{y}))\wedge\overline{\delta}^p)\vee\overline{\lambda}^p\\
&=((\overline{\Upsilon}^p(\mathfrak{e}\mathfrak{x})\vee\overline{\lambda}^p)\wedge(\overline{\rho}^p(\mathfrak{e}\mathfrak{y})\vee\overline{\lambda}^p)\wedge\overline{\delta}^p)\vee\overline{\lambda}^p\\
&\geq((\overline{\Upsilon}^p(\mathfrak{e})\wedge\overline{\delta}^p)\wedge(\overline{\rho}^p(\mathfrak{e})\wedge\overline{\delta}^p)\wedge\overline{\delta}^p)\vee\overline{\lambda}^p\\
&=((\overline{\Upsilon}^p(\mathfrak{e})\wedge\overline{\rho}^p(\mathfrak{e})\wedge\overline{\delta}^p)\wedge\overline{\delta}^p)\vee\overline{\lambda}^p
\end{aligned}$$

$= (\overline{\Upsilon}^p(\mathfrak{e}) \wedge \overline{\rho}^p(\mathfrak{e}) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p$
$= (\overline{\Upsilon}^p \sqcap \overline{\rho}^p)^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e})$

and

$(\overline{\Upsilon}^n \circ \overline{\rho}^n)^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e}) = ((\overline{\Upsilon}^n \circ \overline{\rho}^n)(\mathfrak{e}) \wedge \overline{\delta}^n) \vee \overline{\lambda}^n$
$= (\bigwedge_{(\mathfrak{a},\mathfrak{b})\in F_{\mathfrak{e}}} \{\overline{\Upsilon}^n(\mathfrak{a}) \vee \overline{\rho}^n(\mathfrak{b})\}) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n$
$= ((\bigwedge_{(\mathfrak{a},\mathfrak{b})\in F_{\mathfrak{e}\mathfrak{x}\mathfrak{e}\mathfrak{y}}} \{\overline{\Upsilon}^n(\mathfrak{a}) \vee \overline{\rho}^n(\mathfrak{b})\}) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n$
$\leq ((\overline{\Upsilon}^n(\mathfrak{e}\mathfrak{x}) \vee \overline{\rho}^n(\mathfrak{e}\mathfrak{y})) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n$
$= ((\overline{\Upsilon}^n(\mathfrak{e}\mathfrak{x}) \wedge \overline{\lambda}^n) \wedge (\overline{\rho}^p(\mathfrak{e}\mathfrak{y}) \wedge \overline{\lambda}^n) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n$
$\leq ((\overline{\Upsilon}^n(\mathfrak{e}) \vee \overline{\delta}^n) \wedge (\overline{\rho}^n(\mathfrak{e}) \vee \overline{\delta}^n) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n$
$= ((\overline{\Upsilon}^n(\mathfrak{e}) \wedge \overline{\rho}^n(\mathfrak{e}) \vee \overline{\delta}^n) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n$
$= (\overline{\Upsilon}^n(\mathfrak{e}) \wedge \overline{\rho}^n(\mathfrak{e}) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n$
$= (\overline{\Upsilon}^n \sqcap \overline{\rho}^n)^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e})$.

Hence, $(\overline{\Upsilon}^p \circ \overline{\rho}^p)^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) \geq (\overline{\Upsilon}^p \cap \overline{\rho}^p)^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e})$ and $(\overline{\Upsilon}^n \circ \overline{\rho}^n)^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e}) \leq (\overline{\Upsilon}^n \cup \overline{\rho}^n)^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e})$.
Therefore $(\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \circ \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$. Since $\overline{\mathcal{J}} = (\overline{\rho}^p, \overline{\rho}^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF ideal of $\mathfrak{G}$, we get that $\overline{\mathcal{J}} = (\overline{\rho}^p, \overline{\rho}^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF left ideal of $\mathfrak{G}$. Thus by Theorem 4.4, $(\overline{\mathcal{T}} \circ \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$. Therefore, $(\overline{\mathcal{T}} \circ \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}} = (\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$.

Conversely, let $\mathfrak{I}$ be a right ideal and $\mathfrak{L}$ be an ideal of $\mathfrak{G}$. Then, by Theorem 3.8, $\overline{\chi}_{\mathfrak{I}} = (\mathfrak{G}; \overline{\chi}^p_{\mathfrak{I}}, \overline{\chi}^n_{\mathfrak{I}})$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF right ideal and $\overline{\chi}_{\mathfrak{L}} = (\mathfrak{G}; \overline{\chi}^p_{\mathfrak{L}}, \overline{\chi}^n_{\mathfrak{L}})$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF ideal of $\mathfrak{G}$. By supposition and Lemma 4.5, we have

$$\begin{aligned}(\overline{\chi}^p_{(\mathfrak{I}\mathfrak{L}]})^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) &= (\overline{\chi}^p_{\mathfrak{I}} \circ \overline{\chi}^p_{\mathfrak{L}})^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) = (\overline{\chi}^p_{\mathfrak{I}} \cap \overline{\chi}^p_{\mathfrak{L}})^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) \\ &= (\overline{\chi}^p_{\mathfrak{I}\cap\mathfrak{L}})^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) = \overline{\delta}^p\end{aligned}$$

and

$$\begin{aligned}(\overline{\chi}^n_{(\mathfrak{I}\mathfrak{L}]})^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e}) &= (\overline{\chi}^n_{\mathfrak{I}} \circ \overline{\chi}^n_{\mathfrak{L}})^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e}) = (\overline{\chi}^n_{\mathfrak{I}} \cup \overline{\chi}^n_{\mathfrak{L}})^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e}) \\ &= (\overline{\chi}^n_{\mathfrak{I}\cup\mathfrak{L}})^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e}) = \overline{\delta}^n.\end{aligned}$$

Thus we have $\mathfrak{e} \in (\mathfrak{I}\mathfrak{L}]$. Hence, $\mathfrak{I} \cap \mathfrak{J} = (\mathfrak{I}\mathfrak{L}]$. Consequently, $\mathfrak{G}$ is weakly regular, by Lemma 4.6. ■

The following lemma will be used in the proof of Theorem 4.9.

Lemma 4.8. *[16] Let $\mathfrak{G}$ be an ordered semigroup. Then the following statements are equivalent:*
(1) *$\mathfrak{G}$ is weakly regular.*
(2) *$\mathfrak{B} \cap \mathfrak{I} \subseteq (\mathfrak{B}\mathfrak{I}]$ for every bi-ideal $\mathfrak{B}$ and every ideal $\mathfrak{I}$ of $\mathfrak{G}$.*

Theorem 4.9. *For an ordered semigroup $\mathfrak{G}$, the following statements are equivalent.*
(1) *$\mathfrak{G}$ is weakly regular.*
(2) *$(\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \circ \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$ for every $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$ and every $(\overline{\lambda}, \overline{\delta})$-IVBF ideal $\overline{\mathcal{J}} = (\overline{\rho}^p, \overline{\rho}^n)$ of $\mathfrak{G}$.*
(3) *$(\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \circ \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$ for every $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$ and every $(\overline{s}, \overline{t})$-IVF interior ideal $\overline{\mathcal{J}} = (\overline{\rho}^p, \overline{\rho}^n)$ of $\mathfrak{G}$.*
(4) *$(\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \circ \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$ for every $(\overline{\lambda}, \overline{\delta})$-IVBF generalized bi-ideal $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$ and every $(\overline{\lambda}, \overline{\delta})$-IVBF interior ideal $\overline{\mathcal{J}} = (\overline{\rho}^p, \overline{\rho}^n)$ of $\mathfrak{G}$.*

Proof: (1) $\Rightarrow$ (4) Assume that $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF generalized bi-ideal and $\overline{\mathcal{J}} = (\overline{\rho}^p, \overline{\rho}^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF interior ideal of $\mathfrak{G}$. Let $\mathfrak{e} \in \mathfrak{G}$. Since $\mathfrak{G}$ is weakly regular, there exist $\mathfrak{x}, \mathfrak{y} \in \mathfrak{G}$ such that $\mathfrak{e} \leq \mathfrak{e}\mathfrak{x}\mathfrak{e}\mathfrak{y}$. Thus,

$(\overline{\Upsilon}^p \circ \overline{\rho}^p)^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) = ((\overline{\Upsilon}^p \circ \overline{\rho}^p)(\mathfrak{e}) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p$
$= ((\bigvee_{(\mathfrak{a},\mathfrak{b})\in F_{\mathfrak{e}}} \{\overline{\Upsilon}^p(\mathfrak{a}) \wedge \overline{\rho}^p(\mathfrak{b})\}) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p$
$= ((\bigvee_{(\mathfrak{a},\mathfrak{b})\in F_{\mathfrak{e}\mathfrak{x}\mathfrak{e}\mathfrak{y}}} \{\overline{\Upsilon}^p(\mathfrak{a}) \wedge \overline{\rho}^p(\mathfrak{b})\}) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p$
$\geq ((\overline{\Upsilon}^p(\mathfrak{e}) \wedge \overline{\rho}^p(\mathfrak{x}\mathfrak{e}\mathfrak{y})) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p$
$= (\overline{\Upsilon}^p(\mathfrak{e}) \wedge (\overline{\rho}^p(\mathfrak{x}\mathfrak{e}\mathfrak{y}) \vee \overline{\lambda}^p) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p$
$\geq ((\overline{\Upsilon}^p(\mathfrak{e}) \wedge \overline{\delta}^p) \wedge (\overline{\rho}^p(\mathfrak{e}) \wedge \overline{\delta}^p) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p$
$= ((\overline{\Upsilon}^p(\mathfrak{e}) \wedge \overline{\rho}^p(\mathfrak{e}) \wedge \overline{\delta}^p) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p$
$= (\overline{\Upsilon}^p(\mathfrak{e}) \wedge \overline{\rho}^p(\mathfrak{e}) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p$
$= (\overline{\Upsilon}^p \sqcap \overline{\rho}^p)^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e})$

and

$(\overline{\Upsilon}^n \circ \overline{\rho}^n)^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e}) = ((\overline{\Upsilon}^n \circ \overline{\rho}^n)(\mathfrak{e}) \wedge \overline{\delta}^n) \vee \overline{\lambda}^n$
$= ((\bigwedge_{(\mathfrak{a},\mathfrak{b})\in F_{\mathfrak{e}}} \{\overline{\Upsilon}^n(\mathfrak{a}) \vee \overline{\rho}^n(\mathfrak{b})\}) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n$
$= ((\bigwedge_{(\mathfrak{a},\mathfrak{b})\in F_{\mathfrak{e}\mathfrak{x}\mathfrak{e}\mathfrak{y}}} \{\overline{\Upsilon}^n(\mathfrak{a}) \vee \overline{\rho}^n(\mathfrak{b})\}) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n$
$\leq ((\overline{\Upsilon}^n(\mathfrak{e}) \vee \overline{\rho}^n(\mathfrak{x}\mathfrak{e}\mathfrak{y})) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n$
$= (\overline{\Upsilon}^n(\mathfrak{e}) \wedge (\overline{\rho}^p(\mathfrak{x}\mathfrak{e}\mathfrak{y}) \wedge \overline{\lambda}^n) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n$
$\leq ((\overline{\Upsilon}^n(\mathfrak{e}) \vee \overline{\delta}^n) \wedge (\overline{\rho}^n(\mathfrak{e}) \vee \overline{\delta}^n) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n$
$= ((\overline{\Upsilon}^n(\mathfrak{e}) \wedge \overline{\rho}^n(\mathfrak{e}) \vee \overline{\delta}^n) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n$
$= (\overline{\Upsilon}^n(\mathfrak{e}) \wedge \overline{\rho}^n(\mathfrak{e}) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n$
$= (\overline{\Upsilon}^n \sqcap \overline{\rho}^n)^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e})$.

Hence, $(\overline{\Upsilon}^p \circ \overline{\rho}^p)^{\overline{\delta}}_{\overline{\lambda}}(e) \geq (\overline{\Upsilon}^p \cap \overline{\rho}^p)^{\overline{\delta}}_{\overline{\lambda}}(e)$ and $(\overline{\Upsilon}^n \circ \overline{\rho}^n)^{\overline{\lambda}}_{\overline{\delta}}(e) \leq (\overline{\Upsilon}^n \cup \overline{\rho}^n)^{\overline{\lambda}}_{\overline{\delta}}(e)$.
Therefore $(\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \circ \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2) This is obvious because every $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal is a $(\overline{\lambda}, \overline{\delta})$-IVBF generalized bi-ideal of $\mathfrak{G}$, every $(\overline{\lambda}, \overline{\delta})$-IVBF ideal is a $(\overline{\lambda}, \overline{\delta})$-IVBF interior ideal of $\mathfrak{G}$.

(2) $\Rightarrow$ (1) Let $\mathfrak{B}$ be a bi-ideal and $\mathfrak{I}$ be an ideal of $\mathfrak{G}$. Then, by Theorem 3.8, $\overline{\chi}_{\mathfrak{B}} = (\mathfrak{G}; \overline{\chi}^p_{\mathfrak{B}}, \overline{\chi}^n_{\mathfrak{B}})$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal and $\overline{\chi}_{\mathfrak{I}} = (\mathfrak{G}; \overline{\chi}^p_{\mathfrak{I}}, \overline{\chi}^n_{\mathfrak{I}})$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF ideal of $\mathfrak{G}$. By supposition and Theorem 4.5, we have

$$\begin{aligned}(\overline{\chi}^p_{\mathfrak{B}} \cap \overline{\chi}^p_{\mathfrak{I}})^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) &= (\overline{\chi}^p_{\mathfrak{B}\cap\mathfrak{I}})^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) \sqsubseteq (\overline{\chi}^p_{(\mathfrak{B}\mathfrak{I}]})^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) \\ &= (\overline{\chi}^p_{\mathfrak{B}} \circ \overline{\chi}^p_{\mathfrak{I}})^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) = \overline{\delta}^p\end{aligned}$$

and

$$\begin{aligned}(\overline{\chi}^n_{\mathfrak{B}} \cap \overline{\chi}_{\mathfrak{I}})^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) &= (\overline{\chi}^n_{\mathfrak{B}\cap\mathfrak{I}})^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) \sqsubseteq (\overline{\chi}^n_{(\mathfrak{B}\mathfrak{I}]})^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) \\ &= (\overline{\chi}^n_{\mathfrak{B}} \circ \overline{\chi}_{\mathfrak{I}}{}^n)^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) = \overline{\delta}^n.\end{aligned}$$

Thus we have $\mathfrak{e} \in (\mathfrak{B}\mathfrak{I}]$. Hence, $\mathfrak{B} \cap \mathfrak{I} \sqsubseteq (\mathfrak{B}\mathfrak{I}]$. Therefore, by Lemma 4.8, $\mathfrak{G}$ is weakly regular. ■

The following lemma allows us to prove the Theorem 4.11.

Lemma 4.10. *[16] Let $\mathfrak{G}$ be an ordered semigroup. Then the following statements are equivalent:*
(1) *$\mathfrak{G}$ is weakly regular.*
(2) *$\mathfrak{B} \cap \mathfrak{I} \cap \mathfrak{R} \subseteq (\mathfrak{B}\mathfrak{I}\mathfrak{R}]$ for every bi-ideal $\mathfrak{B}$, every ideal $\mathfrak{I}$ and every right ideal $\mathfrak{R}$ of $\mathfrak{G}$.*

Theorem 4.11. *Let $\mathfrak{G}$ be a monoid. Then the following statements are equivalent:*
(1) *$\mathfrak{G}$ is weakly regular.*
(2) *$(\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}} \sqcap \overline{\mathcal{H}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \circ \overline{\mathcal{J}} \circ \overline{\mathcal{H}})^{\overline{\lambda}}_{\overline{\delta}}$ for every $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$, every $(\overline{\lambda}, \overline{\delta})$-IVBF ideal $\overline{\mathcal{J}} = (\overline{\rho}^p, \overline{\rho}^n)$ and every $(\overline{\lambda}, \overline{\delta})$-IVBF right ideal $\overline{\mathcal{H}} = (\overline{\varrho}^p, \overline{\varrho}^n)$ of $\mathfrak{G}$.*
(3) *$(\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}} \sqcap \overline{\mathcal{H}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \circ \overline{\mathcal{J}} \circ \overline{\mathcal{H}})^{\overline{\lambda}}_{\overline{\delta}}$ for every $(\overline{\lambda}, \overline{\delta})$-IVBF generalized bi-ideal $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$, every $(\overline{\lambda}, \overline{\delta})$-IVBF*

interior ideal $\overline{\mathcal{J}} = (\overline{\rho}^p, \overline{\rho}^n)$ *and every* $(\overline{\lambda}, \overline{\delta})$*-IVBF right ideal* $\overline{\mathcal{H}} = (\overline{\varrho}^p, \overline{\varrho}^n)$ *of* $\mathfrak{G}$.

Proof: (1) $\Rightarrow$ (3) Assume that $\overline{\mathcal{T}} = (\overline{\Upsilon}^p, \overline{\Upsilon}^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF generalized bi-ideal, $\overline{\mathcal{J}} = (\overline{\rho}^p, \overline{\rho}^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF interior ideal and $\overline{\mathcal{H}} = (\overline{\varrho}^p, \overline{\varrho}^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF right ideal of S. Let $\mathfrak{e} \in \mathfrak{G}$. Then there exist $\mathfrak{x}, \mathfrak{y} \in \mathfrak{G}$ such that $\mathfrak{e} \leq \mathfrak{e}\mathfrak{x}\mathfrak{e}\mathfrak{y}$. Thus,

$$\begin{aligned}
&(\overline{\Upsilon}^p \circ \overline{\rho}^p \circ \overline{\varrho}^p)^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) = ((\overline{\Upsilon}^p \circ \overline{\rho}^p \circ \overline{\varrho}^p)(\mathfrak{e}) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= ((\bigvee_{(\mathfrak{a},\mathfrak{b}) \in F_{\mathfrak{e}}} \{\overline{\Upsilon}^p(\mathfrak{a}) \wedge (\overline{\rho}^p \circ \overline{\varrho}^p)(\mathfrak{b})\}) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= ((\bigvee_{(\mathfrak{a},\mathfrak{b}) \in F_{\mathfrak{e}\mathfrak{x}\mathfrak{e}\mathfrak{y}}} \{\overline{\Upsilon}^p(\mathfrak{a}) \wedge (\overline{\rho}^p \circ \overline{\varrho}^p)(\mathfrak{b})\}) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&\geq ((\overline{\Upsilon}^p(\mathfrak{e}) \wedge (\overline{\rho}^p \circ \overline{\varrho}^p)(\mathfrak{x}\mathfrak{e}\mathfrak{y})) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= ((\overline{\Upsilon}^p(\mathfrak{e}) \wedge (\bigvee_{(\mathfrak{a},\mathfrak{b}) \in F_{\mathfrak{x}\mathfrak{e}\mathfrak{y}}} \overline{\rho}^p(\mathfrak{k}) \wedge \overline{\varrho}^p)(\mathfrak{m})) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= ((\overline{\Upsilon}^p(\mathfrak{e}) \wedge (\bigvee_{(\mathfrak{a},\mathfrak{b}) \in F_{\mathfrak{x}\mathfrak{e}\mathfrak{x}\mathfrak{e}\mathfrak{y}^2}} \overline{\rho}^p(\mathfrak{k}) \wedge \overline{\varrho}^p)(\mathfrak{m})) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= ((\overline{\Upsilon}^p(\mathfrak{e}) \wedge (\overline{\rho}^p(\mathfrak{x}\mathfrak{e}\mathfrak{x}) \wedge \overline{\varrho}^p)(\mathfrak{e}\mathfrak{y}^2)) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= (\overline{\Upsilon}^p(\mathfrak{e}) \wedge (\overline{\rho}^p(\mathfrak{x}\mathfrak{e}\mathfrak{x}) \vee \overline{\lambda}^p \wedge \overline{\varrho}^p(\mathfrak{e}\mathfrak{y}^2) \vee \overline{\lambda}^p) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&\geq (\overline{\Upsilon}^p(\mathfrak{e}) \wedge (\overline{\rho}^p(\mathfrak{e}) \wedge \overline{\delta}^p \wedge \overline{\varrho}^p(\mathfrak{e}) \wedge \overline{\delta}^p) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= (\overline{\Upsilon}^p(\mathfrak{e}) \wedge ((\overline{\rho}^p(\mathfrak{e}) \wedge \overline{\varrho}^p(\mathfrak{e})) \wedge \overline{\delta}^p) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= \overline{\Upsilon}^p(\mathfrak{e}) \wedge ((\overline{\rho}^p(\mathfrak{e}) \wedge \overline{\varrho}^p(\mathfrak{e})) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= \overline{\Upsilon}^p \sqcap (\overline{\rho}^p \sqcap \overline{\varrho}^p)^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) = (\overline{\Upsilon}^p \sqcap \overline{\rho}^p \sqcap \overline{\varrho}^p)^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e})
\end{aligned}$$

and

$$\begin{aligned}
&(\overline{\Upsilon}^n \circ \overline{\rho}^n \circ \overline{\varrho}^n)^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e}) = ((\overline{\Upsilon}^n \circ \overline{\rho}^n \circ \overline{\varrho}^n)(\mathfrak{e}) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&= ((\bigwedge_{(\mathfrak{a},\mathfrak{b}) \in F_{\mathfrak{e}}} \{\overline{\Upsilon}^n(\mathfrak{a}) \vee (\overline{\rho}^n \circ \overline{\varrho}^n)(\mathfrak{b})\}) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&= ((\bigwedge_{(\mathfrak{a},\mathfrak{b}) \in F_{\mathfrak{e}\mathfrak{x}\mathfrak{e}\mathfrak{y}}} \{\overline{\Upsilon}^n(\mathfrak{a}) \vee (\overline{\rho}^n \circ \overline{\varrho}^p)(\mathfrak{b})\}) \wedge \overline{\lambda}^p) \vee \overline{\delta}^p \\
&\leq ((\overline{\Upsilon}^n(\mathfrak{e}) \vee (\overline{\rho}^n \circ \overline{\varrho}^n)(\mathfrak{x}\mathfrak{e}\mathfrak{y})) \wedge \overline{\lambda}^p) \vee \overline{\delta}^n \\
&= ((\overline{\Upsilon}^n(\mathfrak{e}) \vee (\bigwedge_{(\mathfrak{a},\mathfrak{b}) \in F_{\mathfrak{x}\mathfrak{e}\mathfrak{y}}} \overline{\rho}^n(\mathfrak{k}) \vee \overline{\varrho}^n)(\mathfrak{m})) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&= ((\overline{\Upsilon}^n(\mathfrak{e}) \vee (\bigwedge_{(\mathfrak{a},\mathfrak{b}) \in F_{\mathfrak{x}\mathfrak{e}\mathfrak{x}\mathfrak{e}\mathfrak{y}^2}} \overline{\rho}^n(\mathfrak{k}) \vee \overline{\varrho}^n)(\mathfrak{m})) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&= ((\overline{\Upsilon}^n(\mathfrak{e}) \vee (\overline{\rho}^n(\mathfrak{x}\mathfrak{e}\mathfrak{x}) \vee \overline{\varrho}^n)(\mathfrak{e}\mathfrak{y}^2)) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&= (\overline{\Upsilon}^n(\mathfrak{e}) \vee (\overline{\rho}^n(\mathfrak{x}\mathfrak{e}\mathfrak{x}) \wedge \overline{\lambda}^n \vee \overline{\varrho}^b(\mathfrak{e}\mathfrak{y}^2) \wedge \overline{\lambda}^n) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&\leq (\overline{\Upsilon}^n(\mathfrak{e}) \vee (\overline{\rho}^n(\mathfrak{e}) \vee \overline{\delta}^n \vee \overline{\varrho}^n(\mathfrak{e}) \vee \overline{\delta}^n) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&= (\overline{\Upsilon}^n(\mathfrak{e}) \vee (\overline{\rho}^n(\mathfrak{e}) \vee \overline{\varrho}^n(\mathfrak{e})) \vee \overline{\delta}^n) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&= (\overline{\Upsilon}^n(\mathfrak{e}) \vee (\overline{\rho}^n(\mathfrak{e}) \vee \overline{\varrho}^n(\mathfrak{e})) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&= \overline{\Upsilon}^n \sqcap (\overline{\rho}^n \sqcap \overline{\varrho}^n)^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e}) = (\overline{\Upsilon}^n \sqcap \overline{\rho}^n \sqcap \overline{\varrho}^n)^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e})
\end{aligned}$$

Hence we get that $(\overline{\Upsilon}^p \sqcap \overline{\rho}^p \sqcap \overline{\varrho}^p)^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) \leq (\overline{\Upsilon}^p \circ \overline{\rho}^p \circ \overline{\varrho}^p)^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e})$ and $(\overline{\Upsilon}^n \sqcap \overline{\rho}^n \sqcap \overline{\varrho}^n)^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e}) \geq (\overline{\Upsilon}^n \circ \overline{\rho}^n \circ \overline{\varrho}^n)^{\overline{\lambda}}_{\overline{\delta}}(\mathfrak{e})$ Therefore, $(\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}} \sqcap \overline{\mathcal{H}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \circ \overline{\mathcal{J}} \circ \overline{\mathcal{H}})^{\overline{\lambda}}_{\overline{\delta}}$.

It is obvious that (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1) Let $\mathfrak{B}$ be a bi-ideal, $\mathfrak{I}$ be an ideal and $\mathfrak{R}$ be a right ideal of $\mathfrak{G}$. Then, by Theorem 3.8, $\overline{\chi}_{\mathfrak{B}} = (\mathfrak{G}; \overline{\chi}^p_{\mathfrak{B}}, \overline{\chi}^n_{\mathfrak{B}})$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal, $\overline{\chi}_{\mathfrak{I}} = (\mathfrak{G}; \overline{\chi}^p_{\mathfrak{I}}, \overline{\chi}^n_{\mathfrak{I}})$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF ideal and $\overline{\chi}_{\mathfrak{R}} = (\mathfrak{G}; \overline{\chi}^p_{\mathfrak{R}}, \overline{\chi}^n_{\mathfrak{R}})$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF right ideal of $\mathfrak{G}$. By supposition and Lemma 4.5, we have

$$\begin{aligned}
(\overline{\chi}^p_{\mathfrak{B}} \cap \overline{\chi}^p_{\mathfrak{I}} \cap \overline{\chi}^p_{\mathfrak{R}})^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) &= (\overline{\chi}^p_{\mathfrak{B}\cap\mathfrak{I}\cap\mathfrak{R}})^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) \\
&\sqsubseteq (\overline{\chi}^p_{(\mathfrak{B}\mathfrak{I}\mathfrak{R}]})^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) \\
&= (\overline{\chi}^p_{\mathfrak{B}} \circ \overline{\chi}^p_{\mathfrak{I}} \circ \overline{\chi}^p_{\mathfrak{I}})^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) \\
&= \overline{\delta}^p
\end{aligned}$$

and

$$\begin{aligned}
(\overline{\chi}^n_{\mathfrak{B}} \cap \overline{\chi}_{\mathfrak{I}} \cup \overline{\chi}^n_{\mathfrak{R}})^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) &= (\overline{\chi}^n_{\mathfrak{B}\cap\mathfrak{I}})^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) \\
&\sqsubseteq (\overline{\chi}^n_{(\mathfrak{B}\mathfrak{I}]})^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) \\
&= (\overline{\chi}^n_{\mathfrak{B}} \circ \overline{\chi}_{\mathfrak{I}}{}^n \circ \overline{\chi}_{\mathfrak{R}}{}^n)^{\overline{\delta}}_{\overline{\lambda}}(\mathfrak{e}) \\
&= \overline{\delta}^n.
\end{aligned}$$

Thus we have $\mathfrak{e} \in (\mathfrak{B}\mathfrak{I}\mathfrak{R}]$. Hence, $\mathfrak{B} \cap \mathfrak{I} \cap \mathfrak{R} \sqsubseteq (\mathfrak{B}\mathfrak{I}\mathfrak{R}]$). It thus follows, by Lemma 4.10, that $\mathfrak{G}$ is weakly regular. ■

V. Conclusion

The theory of fuzzy sets extened by interval valued fuzzy sets was introduced by L. A. Zadeh. Later, K. Arulmozhi et al. studied interval valued bipolar fuzzy set in algebra structure. In 2021, S. Lekkoksung deveoped interval valued bipolar fuzzy ideal in ordered semigroup and characterized regular ordered semigroup in terms generalized interval valued bipolar fuzzy ideal and bi-ideal. In the paper, we establish definition types of generalized interval valued bipolar fuzzy ideals. We prove properties of generalized interval valued bipolar fuzzy ideal by using weakly regular ordered semigroup. And we characterized weakly regular ordered semigroups by using generalized interval valued bipolar fuzzy ideal. We hope that the study of weakly regular ordered semigroups in terms of generalized interval valued bipolar fuzzy ideal are useful mathematical tools.

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