# Bounds, Decay, and Integrals of Non-Increasing Functions: A Comprehensive Analysis

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***Abstract*—This scientific article investigates non-increasing functions and their integral properties, shedding light on their behavior. By examining $\int_0^\infty h(s)ds = 1 - l > 0$. Furthermore, examining $h(t) = \int_t^\infty E^{\alpha+1}(s)ds$ about $E(t)$ and $T > 0$, we establish $\int_t^\infty E^{\alpha+1}(s)ds \le TE^\alpha(0)E(t)$, showing how $E(t)$ influences $h(t)$. For $t \ge T$, $E^{\alpha+1}(t)$ follows a power-law decay. Our study unveils a set of inequalities providing insights into these relationships, offering potential applications in control theory, differential equations, and dynamical systems.**

***Index Terms*—Wave equation, Viscoelasticity, Memory term, Stabilization, Frictional damping.**

## I. Introduction

The exploration of non-increasing functions and their integral characteristics forms the cornerstone of this comprehensive analysis, titled "Bounds, Decay, and Integrals of Non-Increasing Functions." In this scientific investigation, we delve into the intricate behaviors of such functions, aiming to unravel their underlying patterns and implications. Our inquiry commences by scrutinizing the integral representation $\int_0^\infty h(s)ds = 1 - l > 0$, shedding light on its significance and offering insights into the nature of these functions.

Additionally, our study investigates the function $h(t) = \int_t^\infty E^{\alpha+1}(s)ds$ in relation to $E(t)$ and a constant $T > 0$, establishing a fundamental relationship $\int_t^\infty E^{\alpha+1}(s)ds \le TE^\alpha(0)E(t)$. This inequality not only highlights the influence of $E(t)$ on $h(t)$ but also reveals a power-law decay for $t \ge T$ in $E^{\alpha+1}(t)$.

Furthermore, this paper aims to delineate the requisite conditions for the function $h(t)$ to ensure the uniform exponential decay of the function $E$. In simpler terms, our objective is to identify specific constants $C > 1$ such that the following inequality holds for all $t \ge 0$:

$$\frac{E(t)}{E(0)} \le F\left(Ce^{\zeta\Lambda(t)}, T, \alpha\right).$$

In the course of our analysis, we derive three key lemmas that provide crucial insights into the behavior and properties of non-increasing functions. These lemmas serve as foundational building blocks for our subsequent investigations.

Additionally, we establish two significant theorems that contribute to our understanding of the bounds, decay characteristics, and integral properties of non-increasing functions. These theorems provide formal statements of our findings and their implications, further solidifying our understanding of these functions.

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Through rigorous analysis, our investigation unveils a series of inequalities that provide deeper insights into the interrelations between non-increasing functions, their bounds, decay characteristics, and integral properties. These findings hold promise for a broad spectrum of applications, including but not limited to control theory, differential equations, and dynamical systems.

The field of viscoelasticity has garnered substantial attention (see [1],[13]). Initial inquiries revolved around kernels expressed in the form of $h(t) = \exp(-\beta t)$, where $\beta > 0$. Subsequent investigations explored kernels that satisfied the inequalities $-\lambda_1 h(t) \le h'(t) \le -\lambda_2 h(t)$ for all $t \ge 0$, with $\lambda_1$ and $\lambda_2$ denoting positive constants, accompanied by additional conditions concerning the second derivative.

Over time, some researchers relaxed these constraints, considering only $h'(t) \le -\lambda_2 h(t)$ or $h'(t) \le -\lambda_2 h^\varsigma(t)$ for all $t \ge 0$, where $\lambda > 0$. Eventually, these conditions evolved further to $h'(t) \le -\theta(t)h(t)$, where $\theta(t)$ is a positive function. This expansion allowed for the derivation of decay rates extending beyond exponential or polynomial forms.

It is pertinent to mention that other studies have achieved exponential decay results under the conditions of $h'(t) \le 0$ and $\exp(\alpha t)h(t) \in L^1(0,\infty)$ for some $\alpha > 0$.

The structure of this paper is as follows: we commence by introducing several pivotal lemmas that underpin our analytical framework. Subsequently, we incorporate these lemmas into the proofs of various theorems, collectively contributing to the establishment of our primary result concerning the decay phenomenon.

Finally, we conclude our analysis by synthesizing our findings and discussing their implications for future research directions. This comprehensive exploration contributes to the broader understanding of non-increasing functions and their role in various mathematical and scientific contexts.

*Remark 1:* For $\exp(\alpha t)h(t)$ to be in $L^1(0,\infty)$, it is essential that $\alpha > 0$ This condition ensures that the exponential growth of $\exp(\alpha t)$ compensates for the behavior of $\mid h(t) \mid$, making their product Lebesgue integrable over the positive real line.

## II. Bounds on the Function $h(t)$ under Specific Conditions

This section explores the function $h(t)$ within the constraints $\gamma > \gamma' + b$ and $\gamma' > 2a$, $\gamma > \gamma' + 2b$, $t > 0$, aiming to determine its range under these conditions. By delving into its behavior within this defined domain, we seek to uncover insights applicable across disciplines.

*Lemma 1:* If a $C^1-$function $h(t) > 0$ satisfies for $t \ge 0$,

$$h(t) \le ae^{-\gamma' t} + b\int_0^t e^{-\gamma(t-s)}h(s)ds, \tag{1}$$

TABLE I
DIFFERENT EXAMPLES OF $\gamma'$, $a$, $\gamma$, $b$.

| Example | $\gamma'$ | $a$ | $\gamma$ | $b$ |
|---|---|---|---|---|
| 1 | 3.8104 | 0.5864 | 6.9486 | 0.8172 |
| 2 | 7.5582 | 3.3627 | 9.6528 | 0.8127 |
| 3 | 7.8606 | 1.3123 | 9.9073 | 0.8480 |

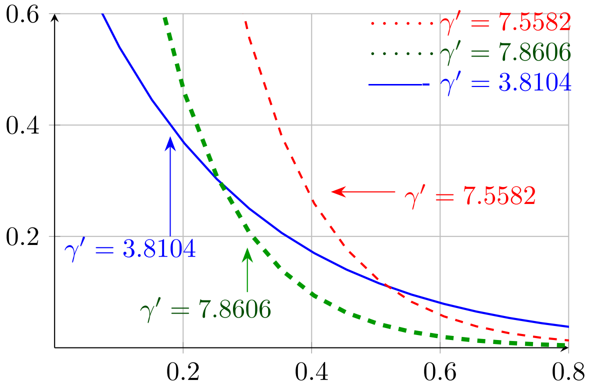


Fig. 1. Function $f(t) = a\left(\frac{\gamma-\gamma'}{\gamma-\gamma'-b}\right)e^{-\gamma' t}$ for different $\gamma'$ values

with $a, b, \gamma, \gamma' > 0$, for $\gamma > \gamma' + b$, $t > 0$,

$$\int_0^t e^{\gamma(s)} h(s)ds \leq \frac{a}{(\gamma-\gamma'-b)} e^{(\gamma-\gamma')t}.$$

then

$$h(t) \leq a\left(\frac{\gamma-\gamma'}{\gamma-\gamma'-b}\right)e^{-\gamma' t}. \tag{2}$$

and

$$h'(t) \leq -\gamma' h(t), \quad h'(t) \leq -\theta(t)h(t) \tag{3}$$

where

$$\theta(t) = \gamma' e^{-\gamma' t},$$

$$0 < \frac{\theta'(t)}{\theta(t)} < \beta.$$

*Remark 2:* Under the conditions $\gamma > \gamma' + b$ and $t > 0$, the function $h(t)$ is bounded above by a positive value. Consequently, the integral of $\mid h(t) \mid$ over the positive real line is finite, indicating that $h(t)$ lies in the space $L^1(0, \infty)$.

*Proof:*
We begin by defining the function:

$$r(t) = \int_0^t e^{\gamma(s)} h(s)ds,$$

and deduce from Equation (1) that:

$$r'(t) = e^{\gamma t} h(t) \leq ae^{(\gamma-\gamma')t} + b\int_0^t e^{\gamma(s)} h(s)ds.$$

This implies

$$r'(t) \leq b \cdot r(t) + ae^{(\gamma-\gamma')t}.$$

Next, we rewrite this as:

$$r'(t) \leq \beta(t) \cdot r(t) + \varphi(t).$$

where $\beta(t) = b$, and $\varphi(t) = ae^{(\gamma-\gamma')t}$

$$r(t) \leq r(0)e^{\int_0^t \beta(s)ds} + \int_0^t e^{\int_s^t \beta(\tau)d\tau} \varphi(s)ds.$$

Using Gronwall's lemma and the initial condition $r(0) = 0$, we obtain :

$$r(t) \leq \int_0^t e^{\int_s^t \beta(\tau)d\tau} \varphi(s)ds.$$

Further simplification leads to:

$$r(t) \leq a\int_0^t e^{b\int_s^t d\tau} e^{(\gamma-\gamma')s} ds.$$

Which can be expressed as:

$$r(t) \leq a\int_0^t e^{b(t-s)} e^{(\gamma-\gamma')s} ds.$$

This inequality implies:

$$r(t) \leq ae^{bt}\int_0^t e^{(\gamma-\gamma'-b)s} ds.$$

Next, we arrive at:

$$r(t) \leq \frac{a}{(\gamma-\gamma'-b)} e^{bt}\left(e^{(\gamma-\gamma'-b)t} - 1\right).$$

Hence:

$$r(t) \leq \frac{a}{(\gamma-\gamma'-b)}\left(e^{(\gamma-\gamma')t} - e^{bt}\right).$$

then

$$r(t) \leq \frac{a}{(\gamma-\gamma'-b)} e^{(\gamma-\gamma')t}.$$

Now, considering $r'(t) = e^{\gamma t} h(t)$ , we obtain:

$$r'(t) = e^{\gamma t} h(t) \leq ae^{(\gamma-\gamma')t} + b\frac{a}{(\gamma-\gamma'-b)} e^{(\gamma-\gamma')t}.$$

Further simplifying:

$$h(t) \leq ae^{-\gamma' t} + b\frac{a}{(\gamma-\gamma'-b)} e^{-\gamma' t}.$$

next

$$h(t) \leq a\left[1 + \frac{b}{(\gamma-\gamma'-b)}\right] e^{-\gamma' t}.$$

This yields:

$$h(t) \leq a\left[\frac{\gamma-\gamma'}{(\gamma-\gamma'-b)}\right] e^{-\gamma' t}$$

$$\leq a\left[\frac{\gamma-\gamma'}{(\gamma-\gamma'-b)}\right] = \eta.$$

Therefore:

$$h'(t) \leq -a\gamma'\left[\frac{\gamma-\gamma'}{(\gamma-\gamma'-b)}\right] e^{-\gamma' t},$$

and

$$\gamma' h(t) \leq a\gamma'\left[\frac{\gamma-\gamma'}{(\gamma-\gamma'-b)}\right] e^{-\gamma' t}.$$

Leading to:

$$h'(t) + \gamma' h(t) \leq 0.$$

However:

$$h'(t) \leq -\gamma' \eta e^{-\gamma' t}, \quad and \quad \gamma' e^{-\gamma' t} h(t) \leq \gamma' \eta e^{-\gamma' t}.$$

Resulting in:

$$h'(t) \leq -\gamma' \eta e^{-\gamma' t} = -\eta \theta(t),$$

where:

$$\theta(t) h(t) = \gamma' e^{-\gamma' t} h(t) \leq \gamma' \eta e^{-\gamma' t} = \eta \theta(t).$$

Hence:

$$h'(t) + \theta(t) h(t) \leq 0.$$

■

*Remark 3:* The inequality

$$h'(t) + \gamma' h(t) \leq 0$$

signifies that the rate of change of $h(t)$ decreases at least as fast as exponential decay with a decay constant $\gamma'$.

*Lemma 2:* If $\gamma' > 2a > 0$ and $\gamma > \gamma' + 2b > 0$ then

$$1 - \int_0^\infty h(s) ds = l > 0.$$

*Proof:* We have established that:

$$h(t) \leq a \left( \frac{\gamma - \gamma'}{\gamma - \gamma' - b} \right) e^{-\gamma' t}. \quad (4)$$

Now, we can integrate this inequality from 0 to infinity:

$$\int_0^\infty h(s) ds \leq \frac{a}{\gamma'} \left( \frac{\gamma - \gamma'}{\gamma - \gamma' - b} \right). \quad (5)$$

Given:

$$\gamma' > 2a > 0$$

and

$$\gamma > \gamma' + 2b > 0$$

We need to prove:

$$\int_0^\infty h(s)\, ds < 1$$

Since $\gamma' > 2a > 0$, we have:

$$\frac{\gamma' - a}{a} > 1$$

This implies:

$$\frac{ba}{\gamma' - a} < b$$

Also, because:

$$\gamma > \gamma' + 2b > \gamma' + b + \frac{ba}{\gamma' - a}$$

We get:

$$\gamma - \gamma' - b > \frac{ba}{\gamma' - a}$$

TABLE II
EXAMPLES OF $\int_0^\infty h(s)\, ds < 1$

| Example | Integral | $\gamma'$ | $a$ | $\gamma$ | $b$ |
|---|---|---|---|---|---|
| 1 | 0.2081 | 3.8104 | 0.5864 | 6.9486 | 0.8172 |
| 2 | 0.7270 | 7.5582 | 3.3627 | 9.6528 | 0.8127 |
| 3 | 0.2851 | 7.8606 | 1.3123 | 9.9073 | 0.8480 |

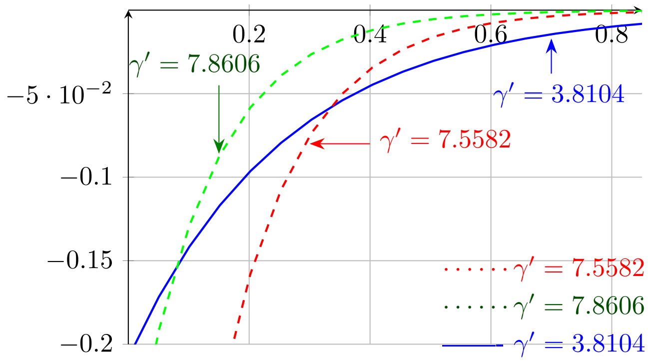


Fig. 2. Function $g(t) = -\frac{a}{\gamma'} \left( \frac{\gamma - \gamma'}{\gamma - \gamma' - b} \right) e^{-\gamma' t}$ for different $\gamma'$ values

Then:

$$\frac{1}{\gamma - \gamma' - b} < \frac{\gamma' - a}{ba}$$

Hence:

$$\frac{a}{\gamma'} \frac{b}{\gamma - \gamma' - b} < \frac{a}{\gamma'} \frac{\gamma' - a}{a}$$

This leads to:

$$\frac{a}{\gamma'} \frac{b}{\gamma - \gamma' - b} < \frac{\gamma' - a}{\gamma'}$$

Then:

$$\frac{a}{\gamma'} + \frac{a}{\gamma'} \frac{b}{\gamma - \gamma' - b} < 1$$

Finally:

$$\frac{a}{\gamma'} \left( 1 + \frac{b}{\gamma - \gamma' - b} \right) < 1$$

Which simplifies to:

$$\frac{a}{\gamma'} \left( \frac{\gamma - \gamma' - b + b}{\gamma - \gamma' - b} \right) < 1$$

Thus:

$$\int_0^\infty h(s)\, ds < 1$$

This concludes the proof. In conclusion, we have shown that the integral of $h(t)$ over the entire range from 0 to infinity is less than or equal to 1.

■

## III. PROOFS OF PROPERTIES FOR NON-INCREASING FUNCTIONS

In this section, we offer concise proofs validating key properties of non-increasing functions. These properties encompass inequalities dictating the behavior of the function under various conditions. Through rigorous demonstrations, we elucidate the relationships and constraints governing non-increasing functions, advancing our comprehension of their fundamental characteristics.

*Lemma 3:* Let $E$: $R^+ \to R^+$ be a non-increasing function.
Then

$$\frac{E(t)}{E(0)} \le Ce^{\frac{-2}{\gamma'}\theta(t)}. \tag{6}$$

*Proof:* By applying Lemma 1, we derive the following expression:

$$(h^2E)' = 2hh'E + h^2E'. \tag{7}$$

This leads to:

$$\le -2\theta(t)h^2E + h^2E'.$$

Further simplifying:

$$\le (-2\theta(t)E + E')\, h^2 \le 0.$$

And subsequently:

$$-2\theta(t)E + E' \le 0. \tag{8}$$

Then:

$$E' \le 2\theta(t)E. \tag{9}$$

Finally:
which yields the desired result:

$$\frac{E(t)}{E(0)} \le Ce^{\frac{-2}{\gamma'}\theta(t)}. \tag{10}$$

■

*Theorem 1:* Let $E$: $R^+ \to R^+$ be a non-increasing function.
If

$$h(t) = e^{\frac{t}{T}} \int_t^\infty E(s)ds, \quad t \in R^+,\ T > 0. \tag{11}$$

then

$$\int_t^\infty E(s)ds \le \frac{T}{\gamma' T + 1} E(t). \tag{12}$$

and

$$E(t) \le E(0)e^{1-\frac{t}{T}}, \quad t \ge T. \tag{13}$$

*Proof:* We begin by noting that $h$ is locally absolutely continuous and non-increasing, as established in Lemma 1:

$$h'(t) = \frac{1}{T}h(t) - e^{\frac{t}{T}}E(t), \quad t \in R^+, \quad T > 0.$$

This implies:

$$h'(t) + \gamma' h(t) = \left(\gamma' + \frac{1}{T}\right) h(t) - e^{\frac{t}{T}}E(t)$$
$$\le 0, \quad t \in R^+, \quad T > 0.$$

then

$$\left(\gamma' + \frac{1}{T}\right) h(t) \le e^{\frac{t}{T}}E(t), \quad t \in R^+, \quad T > 0.$$

Hence

$$\int_t^\infty E(s)ds \le \frac{T}{\gamma' T + 1} E(t).$$

This inequality is almost everywhere in $R^+$. Using equation (11) again, we find:

$$h(t) \le h(0) = \int_0^\infty E(s)ds \le \frac{T}{\gamma' T + 1} E(0), \quad t \in R^+.$$

In other words:

$$\int_t^\infty E(s)ds \le \frac{T}{\gamma' T + 1} E(0)e^{\frac{-t}{T}}, \quad t \in R^+. \tag{14}$$

Since $E$ is nonnegative and non-increasing, we can deduce:

$$\int_t^\infty E(s)ds \ge \int_t^{t+T} E(s)ds \ge TE(t+T).$$

Substituting this into equation (14), we obtain:

$$E(t+T) \le \frac{E(0)}{\gamma' T + 1} e^{\frac{-t}{T}}, \quad t \in R^+.$$

setting $t := t+T$ and $\gamma' = \frac{1}{T}$, we conclude equation (13):

$$E(t) \le \frac{E(0)}{\gamma' T + 1} e^{1-\frac{t}{T}} = \frac{E(0)}{2} e^{1-\frac{t}{T}} \le E(0)e^{1-\frac{t}{T}}, \quad t \ge T.$$

■

*Theorem 2:* Let $E : R^+ \to R^+$ be a non-increasing function and assume that there is a constant $\alpha > 0$.
If

$$h(t) = \int_t^\infty E^{\alpha+1}(s)ds, \quad t \in R^+, \tag{15}$$

then

$$\int_t^\infty E^{\alpha+1}(s)ds \le TE^\alpha(0)E(t), \quad t \in R^+, \quad T > 0. \tag{16}$$

and

$$E^{\alpha+1}(t) \le \left(\frac{T + \alpha t}{T + \alpha T}\right)^{\frac{-1}{\alpha}}, \quad T \le t.$$

*Proof:* We first establish that h is non-increasing and locally absolutely continuous. By differentiating and using Lemma (1), we find that:

$$h'(t) + \gamma' h(t) \tag{17}$$
$$= -E^{\alpha+1}(t) + \gamma' \int_t^\infty E^{\alpha+1}(s)ds, \quad t \in R^+, \quad T > 0.$$

This leads to

$$\gamma' \int_t^\infty E^{\alpha+1}(s)ds \le E^{\alpha+1}(t), \quad t \in R^+, \quad T > 0. \tag{18}$$

Next, we have:

$$\int_t^\infty E^{\alpha+1}(s)ds \le TE^\alpha(t)E(t), \tag{19}$$
$$t \in R^+, \quad T = \frac{1}{\gamma'} > 0.$$

Hence:

$$\int_t^\infty E^{\alpha+1}(s)ds \le TE^\alpha(0)E(t), \tag{20}$$
$$t \in R^+, \quad T = \frac{1}{\gamma'} > 0.$$

By differentiating again and using (20), we may assume that $E(0) = 1$, and we find:

$$-h'(t) = E^{\alpha+1}(t) \ge \left(T^{-1}h(t)\right)^{\alpha+1}, \quad t \in R^+, \tag{21}$$

then

$$-h'(t) \geq T^{-\alpha-1}h^{\alpha+1}(t), \quad t \in R^+, \tag{22}$$

This further leads to the following:

$$-\alpha h'(t)h^{-\alpha-1}(t) \geq \alpha T^{-\alpha-1}, \quad t \in R^+, \tag{23}$$

then

$$\left(h^{-\alpha}(t)\right)' \geq \alpha T^{-\alpha-1}, \tag{24}$$

$$t \in (0,\ B),\ \ B = \sup\{t:\ \ E(t) > 0\}.$$

Integrating in $[0, s]$, we obtain:

$$h^{-\alpha}(s) - h^{-\alpha}(0) \geq \alpha T^{-\alpha-1}s,\ \ s \in [0,\ B), \tag{25}$$

Which further leads to the following:

$$h(s) \geq \left(\alpha T^{-\alpha-1}s + h^{-\alpha}(0)\right)^{\frac{-1}{\alpha}}, \quad s \in [0,\ B), \tag{26}$$

Since $h(s) = 0$ if $s \geq B$, this inequality holds in fact for every $s \in R^+$ . Since $h(0) \leq TE^{\alpha+1}(0) = T$ by (20), the right-hand side of (26) is less than or equal to:

$$\left(\alpha T^{-\alpha-1}s + T^{-\alpha}\right)^{\frac{-1}{\alpha}} = \left(\frac{T^{\alpha+1}}{T+\alpha s}\right)^{\frac{1}{\alpha}} \tag{27}$$

On the other hand, $E$ being nonnegative and non-increasing, the left-hand side of (26) may be estimated as follows:

$$h(s) = \int_s^\infty E^{\alpha+1}(t)dt \geq \int_s^{T+(\alpha+1)s} E^{\alpha+1}(t)dt \tag{28}$$

$$\geq (T+\alpha s)E^{\alpha+1}(T+(\alpha+1)s)$$

Therefore, we deduce from (30) the estimate:

$$(T+\alpha s)E^{\alpha+1}(T+(\alpha+1)s) \leq \left(\frac{T^{\alpha+1}}{T+\alpha s}\right)^{\frac{1}{\alpha}}, \tag{29}$$

Which further leads to the following:

$$E^{\alpha+1}(T+(\alpha+1)s) \leq \left(\frac{T}{T+\alpha s}\right)^{1+\frac{1}{\alpha}} \tag{30}$$

$$\leq \left(\frac{T}{T+\alpha s}\right)^{\frac{1}{\alpha}},$$

Choosing $T \leq t \leq T+(\alpha+1)s$ , we have:

$$\frac{T+\alpha t}{T+\alpha T} = 1 + \frac{\alpha}{(\alpha+1)}\frac{(t-T)}{T} \leq 1 + \alpha\frac{s}{T},$$

then

$$\left(\frac{T+\alpha t}{T+\alpha T}\right)^{\frac{-1}{\alpha}} \geq \left(1+\alpha\frac{s}{T}\right)^{\frac{-1}{\alpha}},$$

next

$$E^{\alpha+1}(t) \leq \left(1+\alpha\frac{s}{T}\right)^{\frac{-1}{\alpha}},$$

hence

$$E^{\alpha+1}(t) \leq \left(\frac{T+\alpha t}{T+\alpha T}\right)^{\frac{-1}{\alpha}}.$$

■

## IV. CONCLUSION

In this study, we explored the behavior of non-increasing functions and their integral properties, with a particular focus on establishing conditions for the uniform exponential decay of another function. Our goal was to find constants, denoted as $C$ and $\Lambda$, that satisfy the inequality $\frac{E(t)}{E(0)} \leq F\left(Ce^{\varsigma\Lambda(t)}, T, \alpha\right)$ for all $t \geq 0$. This investigation has broad implications in the field of viscoelasticity, where understanding decay rates is crucial. Through a series of lemmas and theorems, we unveiled the intricacies of these functions and their relationships, offering valuable insights applicable to control theory, differential equations, and dynamical systems.

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