Parameter Estimation for Ornstein-Uhlenbeck Process Driven by Liu Process

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***Abstract*—Statistical inference is a critical issue in the applications of uncertain differential equations. In this paper, such a parameter estimation problem is formulated for Ornstein-Uhlenbeck process driven by Liu process from discrete observation. The contrast function is given to obtain the least squares estimators. The consistency and asymptotic distribution of two estimators are derived. Some numerical simulations and an empirical analysis on the loan interest rates of RMB under the real data are provided to verify the effectiveness of the estimation methods.**

***Index Terms*—Least squares estimation; Ornstein-Uhlenbeck process; Liu process; consistency; asymptotic distribution**

I. Introduction

Almost all systems are affected by noise and exhibit certain random characteristics ([9], [10]). When modeling or optimizing a stochastic system, due to the complexity of the internal structure and the uncertainty of the external environment, parameters of the system are unknown. In the past few decades, many authors studied the parameter estimation problem for stochastic differential equations. For example, Prakasa Rao ([15]) discussed the asymptotic properties of the maximum likelihood estimator and Bayes estimator for linear stochastic differential equations driven by a mixed fractional Brownian motion. Ginovyan ([4]) studied parameter estimation for Lévy-driven continuous-time linear models with tapered data. When the system is observed discretely, Hu et al. ([5]) derived the strong consistency of the least squares estimator for the fractional stochastic differential system. At the same year, Hu et al. ([6]) studied the parameter estimation for fractional Ornstein-Uhlenbeck processes with general Hurst parameter. Wei ([17]) analyzed the estimation for Cox-Ingersoll-Ross model driven by small symmetrical stable noises. Kaino and Uchida ([8]) considered a linear parabolic stochastic partial differential equation with one space dimension. When the system is observed partially, Xiao et al. ([20]) provided least squares estimators for Vasicek processes, derived the strong consistency and asymptotic distribution of estimators. Wei ([19]) analyzed state and parameter estimation for nonlinear stochastic systems by extended Kalman filtering. Botha et al. ([1]) investigated particle methods for stochastic differential equation mixed effects models.

In practical problems, it is difficult to apply the general theory to build models because of some emergencies. Liu ([12]) created the uncertainty theory to address this uncertainty. Then, Liu ([13]) perfected the uncertainty theory by establishing four axioms and proposed the Liu process. Liu process is the uncertain process for dealing with dynamic systems in uncertain environments. In recent years, parameter estimation for stochastic differential equations driven by Liu process has been discussed in some literature. For instance, Yao and Liu ([22]) used the method of moments to estimate the parameters in uncertain differential equations. Sheng et al. ([16]) employed least squares estimation for uncertain differential equations and proposed a principle of minimum noise. Yang et al. ([21]) used α-path approach to estimate the parameter of uncertain differential equations from discretely sampled data. Lio and Liu ([11]) applied the method of moments to estimate the time-varying parameters in uncertain differential equations. Liu and Liu ([14]) provided a new method in uncertain differential equation based on uncertain maximum likelihood estimation.

The Ornstein-Uhlenbeck process is extensively used in finance during the past few decades as the one-factor short-term interest rate model. Therefore, statistical inference for Ornstein-Uhlenbeck processes has been studied by many authors. For example, Chen et al. ([2]) showed the Berry-Esseen bound of the least squares estimator for fractional Ornstein-Uhlenbeck processes based on continuous-time observation. Chen and Zhou ([3]) considered an inference problem for an Ornstein-Uhlenbeck process driven by a general one-dimensional centered Gaussian process. Zhang et al. ([23]) discussed the parameter estimation for Ornstein-Uhlenbeck driven by Ornstein-Uhlenbeck processes with small Lévy noises. Hu and Xi ([7]) proposed generalized moment estimators to estimate the parameters and proved the strong consistency and asymptotic normality. Wei and Xu ([18]) studied least squares estimation for Ornstein-Uhlenbeck process driven by sub-fractional Brownian processes from discrete observations. Different from above literature, we considered the parameter estimation for Ornstein-Uhlenbeck process driven by Liu process. In this paper, the contrast function is introduced to obtain the least squares estimators. The consistency and asymptotic distribution of the estimators are derived by Markov inequality, Cauchy-Schwarz inequality and Gronwall's inequality. The rest of this paper is organized as follows. In Section 2, we give the contrast function to obtain the least squares estimators. In Section 3, we obtain the consistency and asymptotic distribution of the estimators. In Section 4, some numerical simulations are provided. An empirical analysis on the loan interest rates of RMB under the real data is provided. The conclusion is given in Section 5.

II. Problem Formulation and Preliminaries

Firstly, we give some definitions about uncertain variables and Liu process.

Definition 1: ([12], [13]) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called

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an uncertain measure if it satisfies the following axioms:
Axiom 1: (Normality Axiom) $\mathcal{M}(\Gamma) = 1$ for the universal set Γ.
Axiom 2: (Duality Axiom) $\mathcal{M}(\Lambda) + \mathcal{M}(\Lambda^c) = 1$ for any event Λ.
Axiom 3: (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$,

$$\mathcal{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Axiom 4: (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\{\Pi_{k=1}^{\infty} \Lambda_k\} = \min_{k \geq 1} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$.

An uncertain variable ξ is a measurable function from the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers.

Definition 2: ([12]) For any real number x, let ξ be an uncertain variable and its uncertainty distribution is defined by

$$\Phi(x) = \mathcal{M}(\xi \leq x).$$

In particular, an uncertain variable ξ is called normal if it has an uncertainty distribution

$$\Phi(x) = (1 + \exp(\frac{\pi(\mu - x)}{\sqrt{3}\sigma}))^{-1}, x \in \Re,$$

denoted by $\mathcal{N}(\mu, \sigma)$. If $\mu = 0$, $\sigma = 1$, ξ is called a standard normal uncertain variable.

Definition 3: ([13]) An uncertain process C_t is called a Liu process if
(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous, (ii) C_t has stationary and independent increments, (iii) the increment $C_{s+t} - C_s$ has a normal uncertainty distribution

$$\Phi_t(x) = (1 + \exp(\frac{-\pi x}{\sqrt{3}t}))^{-1}, x \in \Re.$$

In this paper, we study the parametric estimation problem for the following uncertain Ornstein-Uhlenbeck process driven by Liu process:

$$\begin{cases} dX_t = (\alpha - \beta X_t)dt + \varepsilon dC_t, \quad t \geq 0, \\ X_0 = x_0, \end{cases} \tag{1}$$

where α and β are unknown parameters, $\varepsilon \in (0, 1]$, C_t is Liu process. It is assumed that $\{X_t, t \geq 0\}$ is observed at n regular time intervals $\{t_i = \frac{i}{n}, i = 1, 2, \cdots, n\}$.

Consider the following contrast function

$$\rho_{n,\varepsilon}(\alpha, \beta) = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}} - (\alpha - \beta X_{t_{i-1}})\Delta t_{i-1}|^2, \tag{2}$$

where $\Delta t_{i-1} = t_i - t_{i-1} = \frac{1}{n}$.

It is easy to obtain the least square estimators

$$\begin{cases} \widehat{\alpha}_{n,\varepsilon} = \frac{n \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}}) X_{t_{i-1}} \sum_{i=1}^{n} X_{t_{i-1}}}{(\sum_{i=1}^{n} X_{t_{i-1}})^2 - n \sum_{i=1}^{n} X_{t_{i-1}}^2} \\ \qquad - \frac{n \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}}) \sum_{i=1}^{n} X_{t_{i-1}}^2}{(\sum_{i=1}^{n} X_{t_{i-1}})^2 - n \sum_{i=1}^{n} X_{t_{i-1}}^2} \\ \widehat{\beta}_{n,\varepsilon} = \frac{n^2 \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}}) X_{t_{i-1}}}{(\sum_{i=1}^{n} X_{t_{i-1}})^2 - n \sum_{i=1}^{n} X_{t_{i-1}}^2} \\ \qquad - \frac{n \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}}) \sum_{i=1}^{n} X_{t_{i-1}}}{(\sum_{i=1}^{n} X_{t_{i-1}})^2 - n \sum_{i=1}^{n} X_{t_{i-1}}^2}. \end{cases} \tag{3}$$

III. Main Results and Proofs

Let $X^0 = (X_t^0, t \geq 0)$ be the solution to the following ordinary differential equation:

$$dX_t^0 = (\alpha_0 - \beta_0 X_t^0)dt, \quad X_0^0 = x_0, \tag{4}$$

where α_0 and β_0 are true values of the parameter.

Since

$$X_{t_i} - X_{t_{i-1}} = \frac{1}{n}\alpha_0 - \beta_0 \int_{t_{i-1}}^{t_i} X_s ds + \varepsilon \int_{t_{i-1}}^{t_i} dC_s. \tag{5}$$

Then, $\widehat{\alpha}_{n,\varepsilon}$ and $\widehat{\beta}_{n,\varepsilon}$ can be rewritten as follows:

$$\begin{aligned} \widehat{\alpha}_{n,\varepsilon} &= \alpha_0 + \frac{n\beta_0 \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} X_s ds \sum_{i=1}^{n} X_{t_{i-1}}^2}{(\sum_{i=1}^{n} X_{t_{i-1}})^2 - n \sum_{i=1}^{n} X_{t_{i-1}}^2} \\ &\quad - \frac{n\beta_0 \sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds \sum_{i=1}^{n} X_{t_{i-1}}}{(\sum_{i=1}^{n} X_{t_{i-1}})^2 - n \sum_{i=1}^{n} X_{t_{i-1}}^2} \\ &\quad + \frac{n\varepsilon \sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dC_s \sum_{i=1}^{n} X_{t_{i-1}}}{(\sum_{i=1}^{n} X_{t_{i-1}})^2 - n \sum_{i=1}^{n} X_{t_{i-1}}^2} \\ &\quad - \frac{n\varepsilon \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} dC_s \sum_{i=1}^{n} X_{t_{i-1}}^2}{(\sum_{i=1}^{n} X_{t_{i-1}})^2 - n \sum_{i=1}^{n} X_{t_{i-1}}^2} \\ &= \alpha_0 + \frac{\beta_0 \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}^2}{(\frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}^2} \\ &\quad - \frac{\beta_0 \sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}}{(\frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}^2} \\ &\quad + \frac{\varepsilon \sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dC_s \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}}{(\frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}^2} \\ &\quad - \frac{\varepsilon \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} dC_s \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}^2}{(\frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}^2}. \end{aligned}$$

$$\begin{aligned} \widehat{\beta}_{n,\varepsilon} &= \frac{n\beta_0 \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} X_s ds \sum_{i=1}^{n} X_{t_{i-1}}}{(\sum_{i=1}^{n} X_{t_{i-1}})^2 - n \sum_{i=1}^{n} X_{t_{i-1}}^2} \\ &\quad - \frac{n^2\beta_0 \sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds}{(\sum_{i=1}^{n} X_{t_{i-1}})^2 - n \sum_{i=1}^{n} X_{t_{i-1}}^2} \\ &\quad + \frac{n^2\varepsilon \sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dC_s}{(\sum_{i=1}^{n} X_{t_{i-1}})^2 - n \sum_{i=1}^{n} X_{t_{i-1}}^2} \\ &\quad - \frac{n\varepsilon \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} dC_s \sum_{i=1}^{n} X_{t_{i-1}}}{(\sum_{i=1}^{n} X_{t_{i-1}})^2 - n \sum_{i=1}^{n} X_{t_{i-1}}^2} \\ &= \frac{\beta_0 \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}}{(\frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}^2} \\ &\quad - \frac{\beta_0 \sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds}{(\frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}^2} \\ &\quad + \frac{\varepsilon \sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dC_s}{(\frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}^2} \\ &\quad - \frac{\varepsilon \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} dC_s \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}}{(\frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}^2}. \end{aligned}$$

Next, we give some lemmas which are very important for proving the main results.

Lemma 1: When $\varepsilon \to 0$ and $n \to \infty$, we have

$$\sup_{0\le t\le 1}|X_t - X_t^0| \xrightarrow{P} 0.$$

Proof: Note that

$$X_t - X_t^0 = -\beta_0\int_0^t (X_s - X_s^0)ds + \varepsilon\int_0^t dC_s. \quad (6)$$

Since $C_0 = 0$, by using Cauchy-Schwarz inequality, we have

$$\begin{aligned} & |X_t - X_t^0|^2 \\ \le\ & 2\beta_0^2|\int_0^t (X_s - X_s^0)ds|^2 + 2\varepsilon^2|C_t|^2 \\ \le\ & 2\beta_0^2 t^2\int_0^t |X_s - X_s^0|^2 ds + 2\varepsilon^2\sup_{0\le t\le 1}|C_t|^2. \end{aligned}$$

By applying Gronwall's inequality, we obtain

$$|X_t - X_t^0|^2 \le 2\varepsilon^2 e^{2\beta_0^2 t^2}\sup_{0\le t\le 1}|C_t|^2. \quad (7)$$

Thus, we get

$$\sup_{0\le t\le 1}|X_t - X_t^0| \le \sqrt{2}\varepsilon e^{\beta_0^2 t^2}\sup_{0\le t\le 1}|C_t|. \quad (8)$$

Therefore, when $\varepsilon \to 0$ and $n \to \infty$, we have

$$\sup_{0\le t\le 1}|X_t - X_t^0| \xrightarrow{P} 0. \quad (9)$$

The proof is complete. ■

Lemma 2: When $\varepsilon \to 0$ and $n \to \infty$, we obtain

$$\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \xrightarrow{P} \int_0^1 (X_t^0)^2 dt.$$

Proof: Note that

$$\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 = \frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}}^0)^2 + \frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}}^2 - (X_{t_{i-1}}^0)^2). \quad (10)$$

We have

$$\frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}}^0)^2 \xrightarrow{P} \int_0^1 (X_t^0)^2 dt. \quad (11)$$

According to Lemma 1, when $\varepsilon \to 0$ and $n \to \infty$, we obtain

$$\begin{aligned} & |\frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}}^2 - (X_{t_{i-1}}^0)^2)| \\ & = |\frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}} + X_{t_{i-1}}^0)(X_{t_{i-1}} - X_{t_{i-1}}^0)| \\ & \le \frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}} - X_{t_{i-1}}^0|(|X_{t_{i-1}}| + |X_{t_{i-1}}^0|) \\ & \le \frac{1}{n}\sum_{i=1}^n (|X_{t_{i-1}} - X_{t_{i-1}}^0|^2 \\ & + 2|X_{t_{i-1}}^0||X_{t_{i-1}} - X_{t_{i-1}}^0|) \\ & = \frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}} - X_{t_{i-1}}^0|^2 \\ & + 2\frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}}^0||X_{t_{i-1}} - X_{t_{i-1}}^0| \\ & \le (\sup_{0\le t\le 1}|X_t - X_t^0|)^2 \\ & + 2\sup_{0\le t\le 1}|X_t - X_t^0|\frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}}^0| \\ & \xrightarrow{P} 0. \end{aligned}$$

Therefore, we obtain

$$\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \xrightarrow{P} \int_0^1 (X_t^0)^2 dt. \quad (12)$$

The proof is complete. ■

Theorem 1: When $\varepsilon \to 0$, $n \to \infty$, the least squares estimators $\widehat{\alpha}_{n,\varepsilon}$ and $\widehat{\beta}_{n,\varepsilon}$ are consistent, namely

$$\widehat{\alpha}_{n,\varepsilon} \xrightarrow{P} \alpha_0, \quad \widehat{\beta}_{n,\varepsilon} \xrightarrow{P} \beta_0.$$

Proof: From Lemmas 1-2, we obtain

$$(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \xrightarrow{P} (\int_0^1 X_t^0 dt)^2 - \int_0^1 (X_t^0)^2 dt. \quad (13)$$

As $\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \xrightarrow{P} \int_0^1 (X_t^0)^2 dt$ and $\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}} \xrightarrow{P} \int_0^1 X_t^0 dt$, when $\varepsilon \to 0$ and $n \to \infty$, we have

$$\beta_0\sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \xrightarrow{P} \beta_0\int_0^1 X_t^0 dt\int_0^1 (X_t^0)^2 dt, \quad (14)$$

and

$$\begin{aligned} & \beta_0\sum_{i=1}^n X_{t_{i-1}}\int_{t_{i-1}}^{t_i} X_s ds\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}} \xrightarrow{P} \\ & \beta_0\int_0^1 (X_t^0)^2 dt\int_0^1 X_t^0 dt. \end{aligned}$$

Then,

$$\begin{aligned} & \beta_0\sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s ds\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \\ & -\beta_0\sum_{i=1}^n X_{t_{i-1}}\int_{t_{i-1}}^{t_i} X_s ds\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}} \xrightarrow{P} 0. \end{aligned}$$

Since

$$\sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dC_s = \sum_{i=1}^{n} X_{t_{i-1}} (C_{t_i} - C_{t_{i-1}}), \quad (15)$$

and

$$\sum_{i=1}^{n} X_{t_{i-1}} (C_{t_i} - C_{t_{i-1}}) \xrightarrow{P} \int_0^1 X_t^0 dC_t, \quad (16)$$

when $n \to \infty$.

Then, when $\varepsilon \to 0$ and $n \to \infty$, we have

$$\varepsilon \sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dC_s \xrightarrow{P} 0. \quad (17)$$

Thus,

$$\frac{\varepsilon \sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dC_s \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} \xrightarrow{P} 0. \quad (18)$$

Furthermore, when $\varepsilon \to 0$ and $n \to \infty$, it can be checked that

$$\frac{\varepsilon \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} dC_s \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} \xrightarrow{P} 0. \quad (19)$$

Therefore, when $\varepsilon \to 0$ and $n \to \infty$, we obtain

$$\widehat{\alpha}_{n,\varepsilon} \xrightarrow{P} \alpha_0.$$

Using the same methods, we have

$$\widehat{\beta}_{n,\varepsilon} \xrightarrow{P} \beta_0.$$

The proof is complete. ■

Theorem 2: When $\varepsilon \to 0$, $n \to \infty$ and $n\varepsilon \to \infty$,

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon} - \alpha_0) \xrightarrow{d} \mathcal{N}(0, \frac{(\int_0^1 X_t^0 dt)^2 + \int_0^1 (X_t^0)^2 dt}{(\int_0^1 X_t^0 dt)^2 - \int_0^1 (X_t^0)^2 dt}),$$

$$\varepsilon^{-1}(\widehat{\beta}_{n,\varepsilon} - \beta_0) \xrightarrow{d} \mathcal{N}(0, \frac{2\int_0^1 X_t^0 dt}{(\int_0^1 X_t^0 dt)^2 - \int_0^1 (X_t^0)^2 dt}).$$

Proof: According to the explicit decomposition for $\widehat{\alpha}_{n,\varepsilon}$, it is obvious that

$$\begin{aligned}
&\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon} - \alpha_0) \\
&= \frac{\varepsilon^{-1}\beta_0 \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} \\
&- \frac{\varepsilon^{-1}\beta_0 \sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} \\
&+ \frac{\sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dC_s \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} \\
&- \frac{\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} dC_s \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2}.
\end{aligned}$$

According to Lemma 1, when $\varepsilon \to 0$, $n \to \infty$ and $n\varepsilon \to \infty$, we have

$$\begin{aligned}
&|\varepsilon^{-1}\beta_0 \sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds| \\
&\le \varepsilon^{-1}\beta_0 \sum_{i=1}^{n} |X_{t_{i-1}}| |\int_{t_{i-1}}^{t_i} X_s ds| \\
&\le \varepsilon^{-1} n^{-1} \beta_0 \sum_{i=1}^{n} (|X_{t_{i-1}} - X_{t_{i-1}}^0| + |X_{t_{i-1}}^0|) \\
&\sup_{t_{i-1} \le t \le t_i} |X_t| \\
&\xrightarrow{P} 0.
\end{aligned}$$

Then, we obtain

$$\varepsilon^{-1}\beta_0 \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} X_s ds \xrightarrow{P} 0.$$

Thus, we have

$$\frac{\varepsilon^{-1}\beta_0 \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} \xrightarrow{P} 0, \quad (20)$$

and

$$\frac{\varepsilon^{-1}\beta_0 \sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} \xrightarrow{P} 0. \quad (21)$$

Since

$$\sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dC_s \xrightarrow{P} \int_0^1 X_t^0 dC_s,$$

$$\int_0^1 X_t^0 dt \int_0^1 X_t^0 dC_t \xrightarrow{d} \mathcal{N}(0, (\int_0^1 X_t^0 dt)^2),$$

$$C_1 \int_0^1 (X_t^0)^2 dt \xrightarrow{d} \mathcal{N}(0, \int_0^1 (X_t^0)^2 dt),$$

we obtain

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon} - \alpha_0) \xrightarrow{d} \mathcal{N}(0, \frac{(\int_0^1 X_t^0 dt)^2 + \int_0^1 (X_t^0)^2 dt}{(\int_0^1 X_t^0 dt)^2 - \int_0^1 (X_t^0)^2 dt}). \quad (22)$$

As

$$\begin{aligned}
&\varepsilon^{-1}(\widehat{\beta}_{n,\varepsilon} - \beta_0) \\
&= \frac{\varepsilon^{-1}\beta_0 \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} \\
&- \frac{\varepsilon^{-1}\beta_0 \sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} - \varepsilon^{-1}\beta_0 \\
&+ \frac{\sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dC_s}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} \\
&- \frac{\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} dC_s \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2}.
\end{aligned}$$

It is obvious that

$$\begin{aligned}
&\frac{\varepsilon^{-1}\beta_0 \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} \\
&- \frac{\varepsilon^{-1}\beta_0 \sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} X_s ds}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} \\
&- \varepsilon^{-1}\beta_0 \xrightarrow{P} 0,
\end{aligned} \quad (23)$$

TABLE I
LEAST SQUARES ESTIMATOR SIMULATION RESULTS OF α_0 AND β_0

True	Average Value			Absolute Error	
(α_0, β_0)	Size n	$\widehat{\alpha}_n$	$\widehat{\beta}_n$	$\lvert\widehat{\alpha}_n - \alpha_0\rvert$	$\lvert\widehat{\beta}_n - \beta_0\rvert$
	1000	1.0528	2.0463	0.0528	0.0463
(1,2)	2000	1.0271	2.0195	0.0271	0.0195
	5000	1.0049	2.0027	0.0049	0.0027

and

$$\frac{\sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} dC_s}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} - \frac{\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} dC_s \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2} \xrightarrow{d} \mathcal{N}(0, \frac{2\int_0^1 X_t^0 dt}{(\int_0^1 X_t^0 dt)^2 - \int_0^1 (X_t^0)^2 dt}). \quad (24)$$

Then, we have

$$\varepsilon^{-1}(\widehat{\beta}_{n,\varepsilon} - \beta_0) \xrightarrow{d} \mathcal{N}(0, \frac{2\int_0^1 X_t^0 dt}{(\int_0^1 X_t^0 dt)^2 - \int_0^1 (X_t^0)^2 dt}). \quad (25)$$

The proof is complete. ∎

IV. SIMULATION

In this experiment, we use iterative approach to generate a discrete sample $(X_{t_{i-1}})_{i=1,\dots,n}$ and compute $\widehat{\alpha}_{n,\varepsilon}$ and $\widehat{\beta}_{n,\varepsilon}$ from the sample. We let $x_0 = 0.1$, the size is increasing from 1000 to 5000. In Table 1, $\varepsilon = 0.1$. In Table 2, $\varepsilon = 0.01$. The tables list the value of least squares estimators "$\widehat{\alpha}_{n,\varepsilon}$","$\widehat{\beta}_{n,\varepsilon}$", the absolute errors "$\lvert\widehat{\alpha}_{n,\varepsilon} - \alpha_0\rvert$","$\lvert\widehat{\beta}_{n,\varepsilon} - \beta_0\rvert$" and the confidence interval.

The tables illustrate that when n is large enough and ε is small enough, the obtained estimators are very close to the true parameter value. If we let n converge to the infinity and ε converge to zero, the estimator will converge to the true value.

TABLE II
SIMULATION RESULTS OF CONFIDENCE INTERVAL OF α_0 AND β_0

True	Average Value			0.95
(α_0, β_0)	Size n	$\widehat{\alpha}_n$	$\widehat{\beta}_n$	confidence interval of α_0
	1000	1.0364	2.0381	[0.9201,1.1965]
(1,2)	2000	1.0105	2.0113	[0.9636,1.1086]
	5000	1.0008	2.0007	[0.9920,1.0327]

We verify the results under the real data in this section. Table 3 shows the real data about loan interest rate of RMB from 4/21/1991 to 4/20/2020. The interest rate is described by uncertain Ornstein-Uhlenbeck process as Equation (1). Then, we derive the least squares estimators

$$(\widehat{\alpha}_{n,\varepsilon}, \widehat{\beta}_{n,\varepsilon}) = (9.1677, 2.3298).$$

Thus, let $\varepsilon = 0.95$, the uncertain Ornstein-Uhlenbeck process could be written as

$$dX_t = (9.1677 - 2.3298X_t)dt + 0.95dC_t.$$

Hence, the θ-path X_t^θ $(0 < \theta < 1)$ is the solution of following ordinary differential equation

$$dX_t^\theta = (9.1677 - 2.3298X_t^\theta)dt + 0.95\frac{\sqrt{3}}{\pi}\ln\frac{\theta}{1-\theta}dt.$$

TABLE III
LOAN INTEREST RATE OF RMB FROM 4/21/1991 TO 4/20/2020.

n	1	2	3	4	5	6	7	8	9	10
t_i	0	0.60	1.20	1.80	2.40	3.00	3.60	4.20	4.80	5.40
X_{t_i}	9.72	12.24	14.76	15.30	12.42	10.53	10.35	7.56	6.21	5.76
n	11	12	13	14	15	16	17	18	19	20
t_i	6.00	6.60	7.20	7.80	8.40	9.00	9.60	10.20	10.80	11.40
X_{t_i}	3.24	2.25	1.98	2.20	2.50	2.80	3.05	3.30	3.05	2.80
n	21	22	23	24	25	26	27	28	29	30
t_i	12.00	12.60	13.20	13.80	14.40	15.00	15.60	16.20	16.80	17.40
X_{t_i}	6.60	6.80	6.55	6.15	5.15	4.90	4.90	5.46	4.80	4.65

According to Figure 1, all observations fall into the area between 0.01-path $X_t^{0.01}$ and 0.99-path $X_t^{0.99}$. Therefore, the methods used in this paper are reasonable.

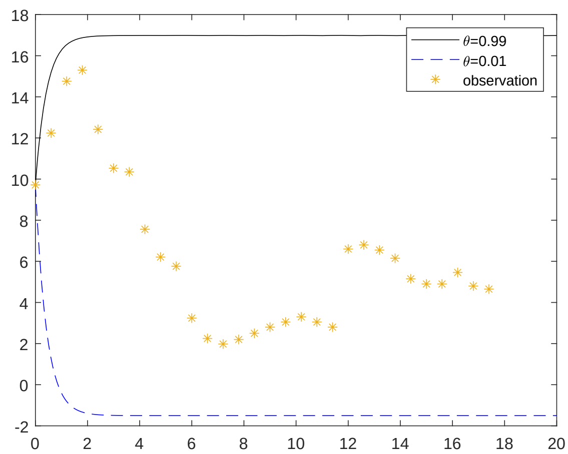

Fig. 1. Observations and θ-path of X_t

V. Conclusion

In this paper, we have studied the problem of parameter estimation for Ornstein-Uhlenbeck process driven by Liu process from discrete observations. We have derived the consistency and asymptotic distribution of the estimators. We will consider the parameter estimation for partially observed uncertain differential equations in future works.

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