On Orthogonal (phi, digamma)-Contraction Type Mappings and Relevant Fixed Point Theorems with Applications

Menaha Dhanraj, Arul Joseph Gnanaprakasam*

Abstract—**In this article, we are motivated by various fixed point theorems for different types of orthogonal $(\phi, \digamma)$ contractions in orthogonal Branciari (rectangular) $\mathfrak{b}$-metric spaces. We present examples that improve upon many existing results to highlight our key findings. Additionally, we provide an application to clarify the existence and uniqueness of solutions to integral equations. Finally, we present numerical results that contrast with the analytical findings.**

Index Terms—**Orthogonal $(\phi, \digamma)$-contraction map type, fixed point, orthogonal-Cauchy sequence, orthogonal complete Branciari $\mathfrak{b}$-metric space.**

I. Introduction

THE Banach contraction principle, one of the most well-known research instruments for fixed point results, is rapidly expanding and improving the field of mathematics in many different contexts. Banach [1] proposed the notion of the famous Banach contraction principle in 1922, which is utilized in complete metric spaces. The principle of Banach contraction in fixed point theory has been further developed by many researchers. Some relevant results enhance the extension outcomes, inviting readers to explore the references for further clarification [2]- [4].

A widely recognized extension of metric spaces is found in $\mathfrak{b}$-metric spaces, which were initiated by Czerwik [5]. Numerous mathematicians have delved into the intricacies of this fascinating space; interested readers are encouraged to explore further in references [6]- [9]. More recently, in 2023, Mani et al. have proven fixed point results in various types of metric spaces with different applications (see [10]- [12]). In 2000, Branciari [13] made a new contribution by introducing generalized metric spaces, wherein the traditional triangle inequality is substituted with the quadrilateral inequality

$$\mathsf{B}_{\dot{\mathfrak{b}}}(\tilde{ß}, \propto) \leq \mathsf{B}_{\dot{\mathfrak{b}}}(\tilde{ß}, æ) + \mathsf{B}_{\dot{\mathfrak{b}}}(æ, \mathrm{q}) + \mathsf{B}_{\dot{\mathfrak{b}}}(\mathrm{q}, \propto),$$

for every pairwise different points $\tilde{ß}, \propto, æ$ and q. In 2015, George et al. [14] introduced the concept of Branciari $\mathfrak{b}$-metric space ($\mathtt{B}_b\mathtt{MS}$). Subsequently, numerous authors embarked on exploring and analyzing various fixed point theorems applicable to such spaces, as evidenced in [15] and [16].

Manuscript received March 25, 2024; revised July 22, 2024.

Menaha Dhanraj is a Research Scholar in the Department of Mathematics, College of Engineering and Technology, Faculty of Engineering and Technology, SRM Institute of Science and Technology, SRM Nagar, Kattankulathur 603203, Kanchipuram, Tamil Nadu, India. (e-mail: md4985@srmist.edu.in)

Arul Joseph Gnanaprakasam is an Assistant Professor in the Department of Mathematics, College of Engineering and Technology, Faculty of Engineering and Technology, SRM Institute of Science and Technology, SRM Nagar, Kattankulathur 603203, Kanchipuram, Tamil Nadu, India. (*Corresponding author to provide phone: +91-9942287309; e-mail: aruljoseph.alex@gmail.com

Wardowski [17] introduced a concept of $\digamma$-contraction in metric spaces. Recently, Kari et al. [18] proposed the notion of θ-ϕ-contraction within $\mathtt{B}_b\mathtt{MS}$, presenting a fixed-point theorem for such metric spaces. In 2018, Wardowski [19] investigated a variation of the Banach fixed-point theorem, studying a novel class of contraction maps termed $(\phi, \digamma)$-contraction on metric spaces. This study was influenced by the research of both Wardowski and Kari et al. Furthermore, Rossafi et al. [20] recently explored the concept of $(\phi, \digamma)$-contraction type maps in rectangular $\mathfrak{b}$-metric spaces, building upon previous work in the field.

In 2017, Eshaghi Gordji et al. [21] introduced the notion of orthogonality and presented a full structure for further developments. Building upon this initial study, Eshaghi Gordji and Habibi [22] expanded the research, demonstrating fixed point theorems in generalized orthogonal metric spaces. For additional insights into orthogonal concepts and related results, refer to ([23]- [32]).

In this paper, we expand upon several fixed-point theorems concerning different types of orthogonal $(\phi, \digamma)$ contraction of a map type in orthogonal complete $\mathtt{B}_b\mathtt{MS}$. Our proof technique, employed in this study, effectively establishes the existence of fixed points, enhancing the robustness of our recent findings. Additionally, we bolster our results by providing both an illustrative example and an application. Moreover, we present an application that focuses on verifying the existence and uniqueness of an integral equation, along with demonstrating the validity of the analytical solutions.

II. Preliminaries

Definition 1. *[14] Let $\mho$ be a non-empty set, $\omega_\wr \geq 1$ be a given real number, and let $\mathsf{B}_{\dot{\mathfrak{b}}} : \mho \times \mho \to [0, +\infty)$ be a map such that for all $\tilde{ß}, \propto \in \mho$ and all distinct points $\mathrm{q}, \aleph \in \mho$, each distinct from $\tilde{ß}$ and $\propto$:*

1) $\mathsf{B}_{\dot{\mathfrak{b}}}(\tilde{ß}, \propto) = 0 \iff \tilde{ß} = \propto$;
2) $\mathsf{B}_{\dot{\mathfrak{b}}}(\tilde{ß}, \propto) = \mathsf{B}_{\dot{\mathfrak{b}}}(\propto, \tilde{ß})$;
3) $\mathsf{B}_{\dot{\mathfrak{b}}}(\tilde{ß}, \propto) \leq \omega_\wr[\mathsf{B}_{\dot{\mathfrak{b}}}(\tilde{ß}, \mathrm{q}) + \mathsf{B}_{\dot{\mathfrak{b}}}(\mathrm{q}, \aleph) + \mathsf{B}_{\dot{\mathfrak{b}}}(\aleph, \propto)]$.

Then, $(\mho, \mathsf{B}_{\dot{\mathfrak{b}}})$ is called a $\mathtt{B}_b\mathtt{MS}$.

Lemma 1. *[16] Let $(\mho, \mathsf{B}_{\dot{\mathfrak{b}}})$ be a $\mathtt{B}_b\mathtt{MS}$:*

1) *Suppose that sequences $\{\tilde{ß}_\imath\}$ and $\{\propto_\imath\}$ in $\mho$ are such that $\tilde{ß}_\imath \to \tilde{ß}$ and $\propto_\imath \to \propto$ as $\imath \to \infty$, with $\tilde{ß} \neq \propto$, $\tilde{ß}_\imath \neq \tilde{ß}$ and $\propto_\imath \neq \propto$ for all $\imath \in \mathbf{N}$. Then, we have*
$$\frac{1}{\omega_\wr}\mathsf{B}_{\dot{\mathfrak{b}}}(\tilde{ß}, \propto) \leq \lim_{\imath \to \infty} \inf \mathsf{B}_{\dot{\mathfrak{b}}}(\tilde{ß}_\imath, \propto_\imath) \leq \lim_{\imath \to \infty} \sup \mathsf{B}_{\dot{\mathfrak{b}}}(\tilde{ß}_\imath, \propto_\imath) \leq \omega_\wr \mathsf{B}_{\dot{\mathfrak{b}}}(\tilde{ß}, \propto).$$
2) *If $\propto \in \mho$ and $\{\tilde{ß}_\imath\}$ is a Cauchy sequence in $\mho$ with $\tilde{ß}_\imath \neq \tilde{ß}_j$ for every $j, \imath \in \mathbf{N}$, $j \neq \imath$, converging to $\tilde{ß} \neq \propto$, then*

$$\frac{1}{\omega_\wr}\mathsf{B}_{\dot{b}}(\tilde{\ss},\propto) \le \lim_{\mathsf{i}\to\infty}\inf \mathsf{B}_{\dot{b}}(\tilde{\ss}_{\mathsf{i}},\propto) \le \lim_{\mathsf{i}\to\infty}\sup \mathsf{B}_{\dot{b}}(\tilde{\ss}_{\mathsf{i}},\propto) \le \omega_\wr \mathsf{B}_{\dot{b}}(\tilde{\ss},\propto),\ \textit{for all}\ \ \tilde{\ss}\in\mho.$$

Lemma 2. *[18] Let* $(\mho,\mathsf{B}_{\dot{b}})$ *be a* $\mathtt{B}_b\mathtt{MS}$ *and let* $\{\tilde{\ss}_{\mathsf{i}}\}$ *be a sequence in* $\mho$ *such that*

$$\lim_{\mathsf{i}\to\infty}\mathsf{B}_{\dot{b}}(\tilde{\ss}_{\mathsf{i}},\tilde{\ss}_{\mathsf{i}+1}) = \lim_{\mathsf{i}\to\infty}\ \mathsf{B}_{\dot{b}}(\tilde{\ss}_{\mathsf{i}},\tilde{\ss}_{\mathsf{i}+2}) = 0.$$

If $\{\tilde{\ss}_{\mathsf{i}}\}$ *is not a Cauchy sequence, then there exist* $\xi > 0$ *and two sequences* $\{j(\sigma)\}$ *and* $\{\mathsf{i}(\sigma)\}$ *of positive integers such that*

$$\begin{aligned}
\xi &\le \lim_{\sigma\to\infty}\inf \mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)}},\tilde{\ss}_{\mathsf{i}_{(\sigma)}})\\
&\le \lim_{\sigma\to\infty}\sup \mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)}},\tilde{\ss}_{\mathsf{i}_{(\sigma)}}) \le \omega_\wr\xi.\\
\xi &\le \lim_{\sigma\to\infty}\inf \mathsf{B}_{\dot{b}}(\tilde{\ss}_{\mathsf{i}_{(\sigma)}},\tilde{\ss}_{j_{(\sigma)+1}})\\
&\le \lim_{\sigma\to\infty}\sup \mathsf{B}_{\dot{b}}(\tilde{\ss}_{\mathsf{i}_{(\sigma)}},\tilde{\ss}_{j_{(\sigma)+1}}) \le \omega_\wr\xi.\\
\xi &\le \lim_{\sigma\to\infty}\inf \mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)}},\tilde{\ss}_{\mathsf{i}_{(\sigma)+1}})\\
&\le \lim_{\sigma\to\infty}\sup \mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)}},\tilde{\ss}_{\mathsf{i}_{(\sigma)+1}}) \le \omega_\wr\xi.\\
\frac{\xi}{\omega_\wr} &\le \lim_{\sigma\to\infty}\inf \mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)+1}},\tilde{\ss}_{\mathsf{i}_{(\sigma)+1}})\\
&\le \lim_{\sigma\to\infty}\sup \mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)+1}},\tilde{\ss}_{\mathsf{i}_{(\sigma)+1}}) \le \omega_\wr^2\xi.
\end{aligned}$$

The subsequent definition, as introduced by Wardowski [19], will serve as the foundation for proving our result.

Definition 2. *[19] Let* χ *be the set of all functions* $\hat{\digamma}:\mathsf{R}^+\to\mathsf{R}$ *and* $\phi:(0,+\infty)\to(0,+\infty)$ *holds the axioms as follows:*

1) $\digamma$ *is strictly increasing;*
2) *for every sequence* $\{\tilde{\ss}_{\mathsf{i}}\}_{\mathsf{i}\in\mathbf{N}}$ *is positive integers*
$$\lim_{\mathsf{i}\to\infty}\tilde{\ss}_{\mathsf{i}} = 0 \iff \lim_{\mathsf{i}\to\infty}\digamma(\tilde{\ss}_{\mathsf{i}}) = -\infty;$$
3) $\liminf_{\omega_\wr\to\alpha^+}\phi(\omega_\wr) > 0$ *for each* $\omega_\wr > 0$;
4) *there exists* $\sigma\in(0,1)$ *such that*
$$\lim_{\tilde{\ss}\to 0^+}\tilde{\ss}^\sigma\digamma(\tilde{\ss}) = 0.$$

Rossafi et al. initiate a new class of $(\phi,\digamma)$-contraction.

Definition 3. *[20] Let* χ *be the set of all functions* $\digamma:\mathsf{R}^+\to\mathsf{R}$ *and* Π *be the set of all functions* $\phi:(0,+\infty)\to(0,+\infty)$ *holds the conditions as follows:*

1) $\digamma$ *is strictly increasing;*
2) *for every sequence* $\{\tilde{\ss}_{\mathsf{i}}\}_{\mathsf{i}\in\mathbf{N}}$ *of positive integers*
$$\lim_{\mathsf{i}\to\infty}\tilde{\ss}_{\mathsf{i}} = 0 \iff \lim_{\mathsf{i}\to\infty}\digamma(\tilde{\ss}_{\mathsf{i}}) = -\infty;$$
3) $\liminf_{\omega_\wr\to\alpha^+}\phi(\omega_\wr) > 0$ *for all* $\omega_\wr > 0$;
4) $\digamma$ *is continuous.*

Definition 4. *[20] Let* $(\mho,\mathsf{B}_{\dot{b}})$ *be a* $\mathtt{B}_b\mathtt{MS}$ *with coefficient* $\omega_\wr > 1$ *and* $\mathcal{W}:\mho\to\mho$ *be a map:*

1) $\mathcal{W}$ *is called a* $(\phi,\digamma)$*-contraction of a map type if there exists* $\digamma\in\chi$ *and* $\phi\in\Pi$ *such that* $\mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{\ss},\mathcal{W}\propto) > 0 \to$
$$\digamma[\omega_\wr^2\mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{\ss},\mathcal{W}\propto)] + \phi(\mathsf{B}_{\dot{b}}(\tilde{\ss},\propto)) \le \digamma[\mathcal{M}(\tilde{\ss},\propto)],$$
where
$$\mathcal{M}(\tilde{\ss},\propto) = \max\{\mathsf{B}_{\dot{b}}(\tilde{\ss},\propto),\mathsf{B}_{\dot{b}}(\tilde{\ss},\mathcal{W}\tilde{\ss}),\mathsf{B}_{\dot{b}}(\propto,\mathcal{W}\propto),\mathsf{B}_{\dot{b}}(\propto,\mathcal{W}\tilde{\ss})\}.$$
2) $\mathcal{W}$ *is called a* $(\phi,\digamma)$*-Kannan contraction of a map type if there exists* $\digamma\in\chi$ *and* $\phi\in\Pi$ *such that* $\mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{\ss},\mathcal{W}\propto) > 0$, *we have*
$$\digamma[\omega_\wr^2\mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{\ss},\mathcal{W}\propto))] + \phi(\mathsf{B}_{\dot{b}}(\tilde{\ss},\propto)) \le \digamma\Big(\frac{\mathsf{B}_{\dot{b}}(\tilde{\ss},\mathcal{W}\tilde{\ss}) + \mathsf{B}_{\dot{b}}(\propto,\mathcal{W}\propto)}{2}\Big).$$
3) $\mathcal{W}$ *is called a* $(\phi,\digamma)$*-Reich contraction of a map type if there exists* $\digamma\in\chi$ *and* $\phi\in\Pi$ *such that* $\mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{\ss},\mathcal{W}\propto) > 0$, *we have*
$$\digamma[\omega_\wr^2\mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{\ss},\mathcal{W}\propto))] + \phi(\mathsf{B}_{\dot{b}}(\tilde{\ss},\propto)) \le \digamma\Big(\frac{\mathsf{B}_{\dot{b}}(\tilde{\ss},\propto) + \mathsf{B}_{\dot{b}}(\tilde{\ss},\mathcal{W}\tilde{\ss}) + \mathsf{B}_{\dot{b}}(\propto,\mathcal{W}\propto)}{3}\Big).$$

The subsequent definition of orthogonality served as the cornerstone for the remainder of our research.

Definition 5. *[21] Let* $\mho$ *be a non-empty and* $\perp\subseteq\mho\times\mho$ *be an binary relation. If* $\perp$ *satisfies the below condition:*

$$\exists\ \delta_0 : (\propto\perp\delta_0)\ \ \textit{or}\ \ (\delta_0\perp\propto),\ \forall\ \propto\in\mho,$$

then, $(\mho,\perp)$ *is called an orthogonal set (O-set).*

Definition 6. *[21] Let* $(\mho,\perp)$ *be an O-set. A sequence* $\{\tilde{\ss}_{\mathsf{i}}\}$ *is called an orthogonal sequence (briefly, O-sequence) if*

$$(\tilde{\ss}_{\mathsf{i}}\perp\tilde{\ss}_{\mathsf{i}+1})\quad\textit{or}\quad(\tilde{\ss}_{\mathsf{i}+1}\perp\tilde{\ss}_{\mathsf{i}}),\ \forall\ \mathsf{i}\in\mathbf{N}.$$

Definition 7. *Let* $(\mho,\perp,\mathsf{B}_{\dot{b}})$ *be an orthogonal* $\mathtt{B}_b\mathtt{MS}$ *if* $(\mho,\perp)$ *is an O-set and* $(\mho,\mathsf{B}_{\dot{b}})$ *is a* $\mathtt{B}_b\mathtt{MS}$.

Definition 8. *Let* $(\mho,\perp,\mathsf{B}_{\dot{b}})$ *be an orthogonal* $\mathtt{B}_b\mathtt{MS}$.

(1) *A map* $\mathcal{W}:\mho\to\mho$ *is said to be an O-continuous in* $\tilde{\ss}\in\mho$ *if for each O-sequence* $\{\tilde{\ss}_{\mathsf{i}}\}_{\mathsf{i}\in\mathbf{N}}$ *in* $\mho$ *with* $\tilde{\ss}_{\mathsf{i}}\to\tilde{\ss}$, *we have* $\mathcal{W}(\tilde{\ss}_{\mathsf{i}})\to\mathcal{W}(\tilde{\ss})$. *Also,* $\mathcal{W}$ *is said to be an O-continuous on* $\mho$ *if* $\mathcal{W}$ *is an O-continuous in each* $\tilde{\ss}\in\mho$.
(2) *A set* $\mho$ *is called an orthogonal complete if every orthogonal Cauchy sequence is convergent.*

III. Main Results

In this section, we construct fixed point results for orthogonal $\mathtt{B}_b\mathtt{MS}$ and discuss the notion of $(\phi,\digamma)$-contraction types in these areas.

Definition 9. *Let* $(\mho,\perp,\mathsf{B}_{\dot{b}})$ *be an orthogonal* $\mathtt{B}_b\mathtt{MS}$ *with coefficient* $\omega_\wr > 1$ *and a map* $\mathcal{W}:\mho\to\mho$;

1) $\mathcal{W}$ *is called an orthogonal* $(\phi,\digamma)$*-contraction of a map type if there exists* $\digamma\in\chi$ *and* $\phi\in\Pi$ *such that* $\mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{\ss},\mathcal{W}\propto) > 0$ *with* $\tilde{\ss}\perp\propto$, *we have*
$$\begin{aligned}&\digamma[\omega_\wr^2\mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{\ss},\mathcal{W}\propto)] + \phi(\mathsf{B}_{\dot{b}}(\tilde{\ss},\propto))\\ &\qquad\le \digamma[\mathcal{M}(\tilde{\ss},\propto)],\ \forall\ \tilde{\ss},\propto\in\mho,\end{aligned}\tag{1}$$
where
$$\mathcal{M}(\tilde{\ss},\propto) = \max\{\mathsf{B}_{\dot{b}}(\tilde{\ss},\propto),\mathsf{B}_{\dot{b}}(\tilde{\ss},\mathcal{W}\tilde{\ss}),\mathsf{B}_{\dot{b}}(\propto,\mathcal{W}\propto),\mathsf{B}_{\dot{b}}(\propto,\mathcal{W}\tilde{\ss})\}.$$
2) $\mathcal{W}$ *is called an orthogonal* $(\phi,\digamma)$*-Kannan contraction of a map type if there exists* $\digamma\in\chi$ *and* $\phi\in\Pi$ *such that* $\mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{\ss},\mathcal{W}\propto) > 0$ *with* $\tilde{\ss}\perp\propto$, *we have*
$$\digamma[\omega_\wr^2\mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{\ss},\mathcal{W}\propto))] + \phi(\mathsf{B}_{\dot{b}}(\tilde{\ss},\propto)) \le \digamma\Big(\frac{\mathsf{B}_{\dot{b}}\tilde{\ss},\mathcal{W}\tilde{\ss}) + \mathsf{B}_{\dot{b}}(\propto,\mathcal{W}\propto)}{2}\Big).$$
3) $\mathcal{W}$ *is called an orthogonal* $(\phi,\digamma)$*-Reich contraction of a map type if there exists* $\digamma\in\chi$

and $\phi \in \Pi$ such that $\mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{ß}, \mathcal{W} \propto) > 0$ with $\tilde{ß} \perp \propto$, we have

$$\mathcal{F}[\omega_{\wr}^2 \mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{ß}, \mathcal{W} \propto))] + \phi(\mathsf{B}_{\dot{b}}(\tilde{ß}, \propto)) \leq \mathcal{F}(\frac{\mathsf{B}_{\dot{b}}(\tilde{ß}, \propto) + \mathsf{B}_{\dot{b}}(\tilde{ß}, \mathcal{W}\tilde{ß}) + \mathsf{B}_{\dot{b}}(\propto, \mathcal{W} \propto)}{3}).$$

Lemma 3. *Let $(\mho, \perp, \mathsf{B}_{\dot{b}})$ be an orthogonal $\mathtt{B}_b\mathtt{MS}$.*

1) *Suppose that the O-sequences $\{\tilde{ß}_{i}\}$ and $\{\propto_{i}\}$ in $\mho$ are such that $\tilde{ß}_{i} \to \tilde{ß}$ and $\propto_{i} \to \propto$ as $i \to \infty$, with $\tilde{ß} \neq \propto$, $\tilde{ß}_{i} \neq \tilde{ß}$ and $\propto_{i} \neq \propto$ for all $i \in \mathbf{N}$. Then, we have*
$$\frac{1}{\omega_{\wr}} \mathsf{B}_{\dot{b}}(\tilde{ß}, \propto) \leq \lim_{i \to \infty} \inf \mathsf{B}_{\dot{b}}(\tilde{ß}_{i}, \propto_{i}) \leq \lim_{i \to \infty} \sup \mathsf{B}_{\dot{b}}(\tilde{ß}_{i}, \propto_{i}) \leq \omega_{\wr} \mathsf{B}_{\dot{b}}(\tilde{ß}, \propto).$$
2) *If $\propto \in \mho$ and $\{\tilde{ß}_{i}\}$ is an orthogonal Cauchy sequence in $\mho$ with $\tilde{ß}_{i} \neq \tilde{ß}_j$ for any $j, i \in \mathbf{N}$, $j \neq i$, converging to $\tilde{ß} \neq \propto$, then*
$$\frac{1}{\omega_{\wr}} \mathsf{B}_{\dot{b}}(\tilde{ß}, \propto) \leq \lim_{i \to \infty} \inf \mathsf{B}_{\dot{b}}(\tilde{ß}_{i}, \propto) \leq \lim_{i \to \infty} \sup \mathsf{B}_{\dot{b}}(\tilde{ß}_{i}, \propto) \leq \omega_{\wr} \mathsf{B}_{\dot{b}}(\tilde{ß}, \propto), \text{ for all } \tilde{ß} \in \mho.$$

Lemma 4. *Let $(\mho, \perp, \mathsf{B}_{\dot{b}})$ be an orthogonal $\mathtt{B}_b\mathtt{MS}$, and let $\{\tilde{ß}_{i}\}$ be an O-sequence in $\mho$ such that*
$$\lim_{i \to \infty} \mathsf{B}_{\dot{b}}(\tilde{ß}_{i}, \tilde{ß}_{i+1}) = \lim_{i \to \infty} \mathsf{B}_{\dot{b}}(\tilde{ß}_{i}, \tilde{ß}_{i+2}) = 0. \tag{2}$$
If $\{\tilde{ß}_{i}\}$ is not an orthogonal Cauchy sequence, then there exist $\xi > 0$ and two O-sequences $\{j(\sigma)\}$ and $\{i(\sigma)\}$ of positive integers such that
$$\xi \leq \lim_{\sigma \to \infty} \inf \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}}, \tilde{ß}_{i_{(\sigma)}}) \leq \lim_{\sigma \to \infty} \sup \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}}, \tilde{ß}_{i_{(\sigma)}}) \leq \omega_{\wr}\xi.$$
$$\xi \leq \lim_{\sigma \to \infty} \inf \mathsf{B}_{\dot{b}}(\tilde{ß}_{i_{(\sigma)}}, \tilde{ß}_{j_{(\sigma)+1}}) \leq \lim_{\sigma \to \infty} \sup \mathsf{B}_{\dot{b}}(\tilde{ß}_{i_{(\sigma)}}, \tilde{ß}_{j_{(\sigma)+1}}) \leq \omega_{\wr}\xi.$$
$$\xi \leq \lim_{\sigma \to \infty} \inf \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}}, \tilde{ß}_{i_{(\sigma)+1}}) \leq \lim_{\sigma \to \infty} \sup \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}}, \tilde{ß}_{i_{(\sigma)+1}}) \leq \omega_{\wr}\xi.$$
$$\frac{\xi}{\omega_{\wr}} \leq \lim_{\sigma \to \infty} \inf \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)+1}}, \tilde{ß}_{i_{(\sigma)+1}}) \leq \lim_{\sigma \to \infty} \sup \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)+1}}, \tilde{ß}_{i_{(\sigma)+1}}) \leq \omega_{\wr}^2\xi.$$

Proof: If $\{\tilde{ß}_{i}\}$ is not an orthogonal Cauchy sequence, then there exist $\xi > 0$ and two sequences $\{j_{(\sigma)}\}$ and $\{i_{(\sigma)}\}$ of positive integers such that $j_{(\sigma)} > i_{(\sigma)} > \sigma$,
$$\xi \leq \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}}, \tilde{ß}_{i_{(\sigma)}}) \text{ and } \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}-1}, \tilde{ß}_{i_{(\sigma)}}) < \xi, \tag{3}$$
for all positive integers σ. By the $\mathfrak{b}$-rectangular inequality, we have
$$\xi \leq \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}}, \tilde{ß}_{i_{(\sigma)}}) \leq \omega_{\wr}[\mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}}, \tilde{ß}_{j_{(\sigma)}+1}) + \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}+1}, \tilde{ß}_{j_{(\sigma)}-1}) + \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}-1}, \tilde{ß}_{i_{(\sigma)}})]. \tag{4}$$
Taking the upper and lower limits as $\sigma \to \infty$ in (4) and using (2) and (3), we obtain
$$\xi \leq \lim_{\sigma \to \infty} \inf \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}}, \tilde{ß}_{i_{(\sigma)}}) \leq \lim_{\sigma \to \infty} \sup \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}}, \tilde{ß}_{i_{(\sigma)}}) \leq \omega_{\wr}\xi. \tag{5}$$
Using the $\mathfrak{b}$-rectangular inequality again, we have
$$\xi \leq \mathsf{B}_{\dot{b}}(\tilde{ß}_{i_{(\sigma)}}, \tilde{ß}_{j_{(\sigma)}+1}) \leq \omega_{\wr}[\mathsf{B}_{\dot{b}}(\tilde{ß}_{i_{(\sigma)}}, \tilde{ß}_{j_{(\sigma)}-1}) + \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}-1}, \tilde{ß}_{j_{(\sigma)}}) + \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}}, \tilde{ß}_{j_{(\sigma)}+1})]. \tag{6}$$
Taking the upper and lower limits as $\sigma \to \infty$ in (6) and using (2) and (3), we obtain
$$\xi \leq \lim_{\sigma \to \infty} \inf \mathsf{B}_{\dot{b}}(\tilde{ß}_{i_{(\sigma)}}, \tilde{ß}_{j_{(\sigma)}+1}) \leq \lim_{\sigma \to \infty} \sup \mathsf{B}_{\dot{b}}(\tilde{ß}_{i_{(\sigma)}}, \tilde{ß}_{j_{(\sigma)}+1}) \leq \omega_{\wr}\xi. \tag{7}$$
Using the $\mathfrak{b}$-rectangular inequality again, we have
$$\xi \leq \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}}, \tilde{ß}_{i_{(\sigma)}+1}) \leq \omega_{\wr}[\mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}}, \tilde{ß}_{j_{(\sigma)}-1}) + \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}-1}, \tilde{ß}_{i_{(\sigma)}}) + \mathsf{B}_{\dot{b}}(\tilde{ß}_{i_{(\sigma)}}, \tilde{ß}_{i_{(\sigma)}+1})]. \tag{8}$$
Taking the upper and lower limits as $\sigma \to \infty$ in (8) and using (2) and (3), we obtain
$$\xi \leq \lim_{\sigma \to \infty} \inf \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}}, \tilde{ß}_{i_{(\sigma)}+1}) \leq \lim_{\sigma \to \infty} \sup \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}}, \tilde{ß}_{i_{(\sigma)}+1}) \leq \omega_{\wr}\xi. \tag{9}$$
Using the $\mathfrak{b}$-rectangular inequality again, we have
$$\mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}+1}, \tilde{ß}_{i_{(\sigma)}+1}) \leq \omega_{\wr}[\mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}+1}, \tilde{ß}_{j_{(\sigma)}}) + \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}}, \tilde{ß}_{i_{(\sigma)}}) + \mathsf{B}_{\dot{b}}(\tilde{ß}_{i_{(\sigma)}}, \tilde{ß}_{i_{(\sigma)}+1})], \tag{10}$$
$$\xi \leq \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}}, \tilde{ß}_{i_{(\sigma)}}) \leq \omega_{\wr}[\mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}}, \tilde{ß}_{j_{(\sigma)}+1}) + \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}+1}, \tilde{ß}_{i_{(\sigma)}+1}) + \mathsf{B}_{\dot{b}}(\tilde{ß}_{i_{(\sigma)}+1}, \tilde{ß}_{i_{(\sigma)}})]. \tag{11}$$
Taking the upper and lower limits as $\sigma \to \infty$ in (10) and (11) and using (2) and (5), we obtain
$$\frac{\xi}{\omega_{\wr}} \leq \lim_{\sigma \to \infty} \inf \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}+1}, \tilde{ß}_{i_{(\sigma)}+1}) \leq \lim_{\sigma \to \infty} \sup \mathsf{B}_{\dot{b}}(\tilde{ß}_{j_{(\sigma)}+1}, (\tilde{ß}_{i_{(\sigma)}+1}) \leq \omega_{\wr}^2\xi.$$
■

Theorem 1. *Let $(\mho, \perp, \mathsf{B}_{\dot{b}})$ be an orthogonal complete $\mathtt{B}_b\mathtt{MS}$ with an orthogonal element $\tilde{ß}_0$ and constant $\omega_{\wr} > 1$, and let $\mathcal{W}$ be a self-map on $\mho$ which satisfies:*

(i) *$\mathcal{W}$ is $\perp$-preserving,*
(ii) *$\mathcal{W}$ is orthogonal $(\phi, \mathcal{F})$-contraction of a map type,*
(iii) *$\mathcal{W}$ is O-continuous.*

Then, $\mathcal{W}$ has a unique fixed point.

Proof: According to the concept of orthogonality, if $(\mho, \perp)$ is an orthogonal set, then there exists
$$\tilde{ß}_0 \in \mho : \forall\ \tilde{ß} \in \mho,\ \tilde{ß} \perp \tilde{ß}_0 \quad (or) \quad \forall\ \tilde{ß} \in \mho,\ \tilde{ß}_0 \perp \tilde{ß}.$$
It follows that $\tilde{ß}_0 \perp \mathcal{W}\tilde{ß}_0$ or $\mathcal{W}\tilde{ß}_0 \perp \tilde{ß}_0$. Let
$$\tilde{ß}_1 = \mathcal{W}\tilde{ß}_0,\ \tilde{ß}_2 = \mathcal{W}\tilde{ß}_1 = \mathcal{W}^2\tilde{ß}_0 \cdots \tilde{ß}_{i} = \mathcal{W}\tilde{ß}_{i-1} = \mathcal{W}^{i}\tilde{ß}_0,$$
$$\tilde{ß}_{i+1} = \mathcal{W}\tilde{ß}_{i} = \mathcal{W}^{i+1}\tilde{ß}_0, \quad \forall\ i \in \mathbf{N}.$$
For any $\tilde{ß}_0 \in \mho$, set $\tilde{ß}_{i} = \mathcal{W}\tilde{ß}_{i-1}$.

Next, we will examine the two possibilities offered:

(i) If $\tilde{ß}_{i} = \tilde{ß}_{i+1}$ for any $i \in \mathbf{N} \cup \{0\}$, then we have $\mathcal{W}\tilde{ß}_{i} = \tilde{ß}_{i}$.
It is easy to see that $\tilde{ß}_{i}$ is a fixed point of $\mathcal{W}$.
Hence, the proof is complete.
(ii) If $\tilde{ß}_{i} \neq \tilde{ß}_{i+1}$ for any $i \in \mathbf{N} \cup \{0\}$, then we get $\mathsf{B}_{\dot{b}}(\tilde{ß}_{i+1}, \tilde{ß}_{i}) > 0$, for each $i \in \mathbf{N}$.

Since $\mathcal{W}$ is $\perp$-preserving, we obtain

$$\tilde{ß}_i \perp \tilde{ß}_{i+1} \quad \text{(or)} \quad \tilde{ß}_{i+1} \perp \tilde{ß}_i.$$

This implies that $\{\tilde{ß}_i\}$ is an O-sequence.

Since $\mathcal{W}$ is an orthogonal $(\phi, \digamma)$-contraction map of a map type, by substituting $\tilde{ß} = \tilde{ß}_{i-1}$ and $\propto = \tilde{ß}_i$ into (1), for all $i \in \mathbf{N}$, we have

$$\begin{aligned}\digamma[\mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+1})] &\leq \digamma[\omega_\wr^2 \mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+1})] + \phi(\mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_i)) \\ &\leq \digamma(\mathcal{M}(\tilde{ß}_{i-1}, \tilde{ß}_i)), \forall\, i \in \mathbf{N}, \end{aligned} \tag{12}$$

where

$$\begin{aligned}\mathcal{M}(\tilde{ß}_{i-1}, \tilde{ß}_i) &= \max(\mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_i), \mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_i), \mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+1}), \\ &\quad \mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_i)\} \\ &= \max\{\mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_i), \mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+1})\}.\end{aligned}$$

If $\mathcal{M}(\tilde{ß}_{i-1}, \tilde{ß}_i) = \mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+1})$, by (12), we have

$$\begin{aligned}\digamma(\mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+1})) &\leq \digamma(\mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+1})) - \phi(\mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_i)) \\ &< \digamma(\mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+1})).\end{aligned}$$

Since $\digamma$ is increasing, we get

$$\mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+1}) < \mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_i). \tag{13}$$

It is a contradiction.

Hence, $\mathcal{M}(\tilde{ß}_{i-1}, \tilde{ß}_i) = \mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_i)$.

Thus,

$$\digamma(\mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+1})) \leq \digamma(\mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_i)) - \phi(\mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_i)).$$

Repeating this step, we conclude that

$$\begin{aligned}\digamma(\mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+1})) &\leq \digamma(\mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_i)) - \phi(\mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_i)) \\ &\leq \digamma(\mathsf{B}_{\dot{b}}(\tilde{ß}_{i-2}, \tilde{ß}_{i-1})) - \phi(\mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_i)) \\ &\quad - \phi(\mathsf{B}_{\dot{b}}(\tilde{ß}_{i-2}, \tilde{ß}_{i-1})) \\ &\leq \cdots \leq \digamma(\mathsf{B}_{\dot{b}}(\tilde{ß}_0, \tilde{ß}_1)) - \sum_{i=0}^{i} \phi(\mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+1})).\end{aligned}$$

Since $\liminf_{\alpha \to \omega_\wr^+} \phi(\alpha) > 0$.

If we obtain $\liminf_{i \to \infty} \phi(\mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_i)) > 0$, then by the definition of the limit, there exists $i_0 \in \mathbf{N}$ and $\mathcal{H} > 0$ such that for all $i \geq i_0$, $\phi(\mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_i)) > \mathcal{H}$.

Hence,

$$\begin{aligned}\digamma(\mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+1})) &\leq \digamma(\mathsf{B}_{\dot{b}}(\tilde{ß}_0, \tilde{ß}_1)) - \sum_{i=0}^{i_0-1} \phi(\mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+1})) \\ &\quad - \sum_{i=i_0-1}^{i} \phi(\mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+1})) \\ &\leq \digamma(\mathsf{B}_{\dot{b}}(\tilde{ß}_0, \tilde{ß}_1)) - \sum_{i=i_0-1}^{i} \mathcal{H} \\ &= \digamma(\mathsf{B}_{\dot{b}}(\tilde{ß}_0, \tilde{ß}_1)) - (i - i_0)\mathcal{H}, \quad \forall\, i \geq i_0.\end{aligned}$$

Letting the limit as $i \to \infty$ in the above inequality, we obtain

$$\lim_{i \to \infty} \digamma(\mathsf{B}_{\dot{b}} \tilde{ß}_i, \tilde{ß}_{i+1})) \leq \lim_{i \to \infty} [\digamma(\mathsf{B}_{\dot{b}}(\tilde{ß}_0, \tilde{ß}_1)) - (i - i_0)\mathcal{H}].$$

That is, $\lim_{i \to \infty} \digamma(\mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+1})) = -\infty$, then based on condition (ii) outlined in Definition 3, we can deduce that

$$\lim_{i \to \infty} \mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+1}) = 0. \tag{14}$$

Next, we shall prove that

$$\lim_{i \to \infty} \mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+2}) = 0.$$

We assume that $\tilde{ß}_i \neq \tilde{ß}_j$ for every $i, j \in \mathbf{N}$, $i \neq j$.

Indeed, suppose that $\tilde{ß}_i = \tilde{ß}_j$ for some $i = j + \sigma$ with $\sigma > 0$, and using (13), we have

$$\mathsf{B}_{\dot{b}}(\tilde{ß}_j, \tilde{ß}_{j+1}) = \mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+1}) < \mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_i).$$

Continuing this process, we have

$$\mathsf{B}_{\dot{b}}(\tilde{ß}_j, \tilde{ß}_{i+1}) = \mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+1}) < \mathsf{B}_{\dot{b}}(\tilde{ß}_j, \tilde{ß}_{j+1}).$$

It is a contradiction.

Therefore, $\mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_j) > 0$ for every $i, j \in \mathbf{N}$, $i \neq j$.

Now, applying (1) with $\tilde{ß} = \tilde{ß}_{i-1}$ and $\propto = \tilde{ß}_{i+1}$, we have

$$\begin{aligned}\digamma[\mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+2})] &= \digamma[\mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{ß}_{i-1}, \mathcal{W}\tilde{ß}_{i+1})] \\ &\leq \digamma[\omega_\wr^2 \mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{ß}_{i-1}, \mathcal{W}\tilde{ß}_{i+1})] \\ &\leq \digamma(\mathcal{M}(\tilde{ß}_{i-1}, \tilde{ß}_{i+1})) - \phi(\mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_{i+1})),\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}(\tilde{ß}_{i-1}, \tilde{ß}_{i+1}) &= \max\{\mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_{i+1}), \mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_i), \\ &\quad \mathsf{B}_{\dot{b}}(\tilde{ß}_{i+1}, \tilde{ß}_{i+2}), \mathsf{B}_{\dot{b}}(\tilde{ß}_{i+1}, \tilde{ß}_i)\} \\ &= \max\{\mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_{i+1}), \mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_i)\}.\end{aligned}$$

So, we get

$$\begin{aligned}&\digamma(\mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+2})) \\ &\quad \leq \digamma(\max\{\mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_i), \mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_{i+1})\}) \\ &\quad - \phi(\mathsf{B}_{\dot{b}}(\tilde{ß}_{i-1}, \tilde{ß}_{i+1})).\end{aligned} \tag{15}$$

Take $\gamma_i = \mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+2})$ and $\vartheta_i = \mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+1})$.

Thus, by (15), one can write

$$\digamma(\gamma_i) \leq \digamma(\max(\gamma_{i-1}, \vartheta_{i-1})) - \phi(\mathsf{B}_{\dot{b}}(\gamma_{i-1})). \tag{16}$$

Since $\digamma$ is increasing, we get

$$\gamma_i < \max\{\gamma_{i-1}, \vartheta_{i-1}\}.$$

By (13), we have

$$\vartheta_i \leq \vartheta_{i-1} \leq \max\{\gamma_{i-1}, \vartheta_{i-1}\},$$

which implies that

$$\max\{\gamma_i, \vartheta_i\} \leq \max\{\gamma_{i-1}, \vartheta_{i-1}\}, \ \forall\, i \in \mathbf{N}.$$

Hence, the O-sequence $\max\{\gamma_{i-1}, \vartheta_{i-1}\}_{i \in \mathbf{N}}$ is a non-negative decreasing O-sequence of real numbers.

Thus, there exists $\lambda \geq 0$ such that

$$\lim_{i \to \infty} \max\{\gamma_i, \vartheta_i\} = \lambda.$$

By (14), assume that $\lambda > 0$, we get

$$\begin{aligned}\lambda = \lim_{i \to \infty} \sup \gamma_i &= \lim_{i \to \infty} \sup \max\{\gamma_i, \vartheta_i\} \\ &= \lim_{i \to \infty} \max\{\gamma_i, \vartheta_i\}.\end{aligned}$$

Letting the $\limsup_{i \to \infty}$ in (16), and applying the contraction of $\digamma$ and the property of ϕ, we get

$$\begin{aligned}&\digamma(\lim_{i \to \infty} \sup \gamma_i) \\ &\quad \leq \digamma(\lim_{i \to \infty} \sup \max\{\gamma_{i-1}, \vartheta_{i-1}\}) - \lim_{i \to \infty} \sup \phi(\gamma_{i-1}) \\ &\quad \leq \digamma(\lim_{i \to \infty} \sup \max\{\gamma_{i-1}, \vartheta_{i-1}\}) - \lim_{i \to \infty} \inf \phi(\gamma_{i-1}) \\ &\quad < \digamma(\lim_{i \to \infty} \max\{\gamma_{i-1}, \vartheta_{i-1}\}).\end{aligned}$$

Therefore, $\digamma(\lambda) < \digamma(\lambda)$. It is a contradiction.

Hence,

$$\lim_{i \to \infty} \mathsf{B}_{\dot{b}}(\tilde{ß}_i, \tilde{ß}_{i+2}) = 0.$$

Next, we show that $\{\tilde{\ss}_{\dot\imath}\}_{\dot\imath\in\mathbf{N}}$ is an orthogonal Cauchy sequence, that is

$$\lim_{\dot\imath,j\to\infty} \mathsf{B}_{\dot{b}}(\tilde{\ss}_{\dot\imath},\tilde{\ss}_j)=0,\ \forall\ \dot\imath,j\in\mathbf{N}.$$

Suppose the opposite, according to Lemma 4, there are $\xi>0$ such that for any positive integer σ, there exist two O-sequences $\{\dot\imath_{(\sigma)}\}$ and $\{j_{(\sigma)}\}$, where

$$\begin{aligned}
\xi &\le \lim_{\sigma\to\infty}\inf \mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)}},\tilde{\ss}_{\dot\imath_{(\sigma)}})\\
&\le \lim_{\sigma\to\infty}\sup \mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)}},\tilde{\ss}_{\dot\imath_{(\sigma)}})\le \omega_{\iota}\xi.\\
\xi &\le \lim_{\sigma\to\infty}\inf \mathsf{B}_{\dot{b}}(\tilde{\ss}_{\dot\imath_{(\sigma)}},\tilde{\ss}_{j_{(\sigma)+1}})\\
&\le \lim_{\sigma\to\infty}\sup \mathsf{B}_{\dot{b}}(\tilde{\ss}_{\dot\imath_{(\sigma)}},\tilde{\ss}_{j_{(\sigma)+1}})\le \omega_{\iota}\xi.\\
\xi &\le \lim_{\sigma\to\infty}\inf \mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)}},\tilde{\ss}_{\dot\imath_{(\sigma)+1}})\\
&\le \lim_{\sigma\to\infty}\sup \mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)}},\tilde{\ss}_{\dot\imath_{(\sigma)+1}})\le \omega_{\iota}\xi.\\
\frac{\xi}{\omega_{\iota}} &\le \lim_{\sigma\to\infty}\inf \mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)+1}},\tilde{\ss}_{\dot\imath_{(\sigma)+1}})\\
&\le \lim_{\sigma\to\infty}\sup \mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)+1}},\tilde{\ss}_{\dot\imath_{(\sigma)+1}})\le \omega_{\iota}^2\xi. \qquad (17)
\end{aligned}$$

From (1), and by setting $\tilde{\ss}=\tilde{\ss}_{j_{(\sigma)}}$ and $\propto=\tilde{\ss}_{\dot\imath_{(\sigma)}}$ we have

$$\begin{aligned}
&\lim_{\sigma\to\infty}\mathcal{M}(\tilde{\ss}_{j_{(\sigma)}},\tilde{\ss}_{\dot\imath_{(\sigma)}})\\
&=\lim_{\sigma\to\infty}\max\{\mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)}},\tilde{\ss}_{\dot\imath_{(\sigma)}}),\mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)}},\tilde{\ss}_{j_{(\sigma)+1}}),\\
&\qquad \mathsf{B}_{\dot{b}}(\tilde{\ss}_{\dot\imath_{(\sigma)}},\tilde{\ss}_{\dot\imath_{(\sigma)+1}}),\mathsf{B}_{\dot{b}}(\tilde{\ss}_{\dot\imath_{(\sigma)}},\tilde{\ss}_{j_{(\sigma)+1}})\}\\
&\le \omega_{\iota}\xi. \qquad (18)
\end{aligned}$$

Now, applying (1) with $\tilde{\ss}=\tilde{\ss}_{j_{(\sigma)}}$ and $\propto=\tilde{\ss}_{\dot\imath_{(\sigma)}}$, we obtain

$$\begin{aligned}
&\mathcal{F}[\omega_{\iota}^2\mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)+1}},\tilde{\ss}_{j_{(\sigma)+1}})]\\
&\qquad\le \mathcal{F}(\mathcal{M}(\tilde{\ss}_{j_{(\sigma)}},\tilde{\ss}_{\dot\imath_{(\sigma)}}))-\phi(\mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)}},\tilde{\ss}_{\dot\imath_{(\sigma)}})).
\end{aligned}$$

Letting $\sigma\to\infty$ the above inequality and applying (18) and (17), we get

$$\begin{aligned}
\mathcal{F}(\frac{\xi}{\omega_{\iota}}\omega_{\iota}^2) &= \mathcal{F}(\xi\omega_{\iota})\le \mathcal{F}(\omega_{\iota}^2\lim_{\sigma\to\infty}\sup \mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)+1}},\tilde{\ss}_{\dot\imath_{(\sigma)+1}}))\\
&=\lim_{\sigma\to\infty}\sup \mathcal{F}(\omega_{\iota}^2\mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)+1}},\tilde{\ss}_{\dot\imath_{(\sigma+1}}))\\
&\le \lim_{\sigma\to\infty}\sup \mathcal{F}(\mathcal{M}(\tilde{\ss}_{j_{(\sigma)}},\tilde{\ss}_{\dot\imath_{(\sigma)}}))\\
&\qquad -\lim_{\sigma\to\infty}\sup \phi(\mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)}},\tilde{\ss}_{\dot\imath_{(\sigma)}}))\\
&=\mathcal{F}(\lim_{\sigma\to\infty}\sup \mathcal{M}(\tilde{\ss}_{j_{(\sigma)}},\tilde{\ss}_{\dot\imath_{(\sigma)}}))\\
&\qquad -\lim_{\sigma\to\infty}\sup \phi(\mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)}},\tilde{\ss}_{\dot\imath_{(\sigma)}}))\\
&\le \mathcal{F}(\lim_{\sigma\to\infty}\sup \mathcal{M}(\tilde{\ss}_{j_{(\sigma)}},\tilde{\ss}_{\dot\imath_{(\sigma)}}))\\
&\qquad -\lim_{\sigma\to\infty}\inf \phi(\mathsf{B}_{\dot{b}}(\tilde{\ss}_{j_{(\sigma)}},\tilde{\ss}_{\dot\imath_{(\sigma)}}))\\
&< \mathcal{F}(\lim_{\sigma\to\infty}\sup \mathcal{M}(\tilde{\ss}_{j_{(\sigma)}},\tilde{\ss}_{\dot\imath_{(\sigma)}}))\le \mathcal{F}(\omega_{\iota}\xi).
\end{aligned}$$

Therefore, the sequence $\{\tilde{\ss}_{\dot\imath}\}$ forms an orthogonal Cauchy sequence in $\mho$.

Due to the completeness of $(\mho,\perp,\mathsf{B}_{\dot{b}})$, there exist $æ\in\mho$ such that

$$\lim_{\dot\imath\to\infty}\mathsf{B}_{\dot{b}}(\tilde{\ss}_{\dot\imath},æ)=0.$$

Currently, we demonstrate that $\mathsf{B}_{\dot{b}}(\mathcal{W}æ,æ)=0$ by applying a proof by contradiction.

We consider that

$$\mathsf{B}_{\dot{b}}(\mathcal{W}æ,æ)>0.$$

Since $\tilde{\ss}_{\dot\imath}\to æ$, as $\dot\imath\to\infty$. $\forall\ \dot\imath\in\mathbf{N}$, we can deduce from Lemma 4, we get

$$\begin{aligned}
\frac{1}{\omega_{\iota}}\mathsf{B}_{\dot{b}}(æ,\mathcal{W}æ) &\le \lim_{\dot\imath\to\infty}\sup \mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{\ss}_{\dot\imath},\mathcal{W}æ)\\
&\le \omega_{\iota}\mathsf{B}_{\dot{b}}(æ,\mathcal{W}æ). \qquad (19)
\end{aligned}$$

Now, applying (1) with $\tilde{\ss}=\tilde{\ss}_{\dot\imath}$ and $\propto=æ$, we have

$$\mathcal{F}(\omega_{\iota}^2\mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{\ss}_{\dot\imath},\mathcal{W}æ))\le \mathcal{F}(\mathcal{M}(\tilde{\ss}_{\dot\imath},æ))-\phi(\mathsf{B}_{\dot{b}}(\tilde{\ss}_{\dot\imath},æ)),$$

$\forall\ \dot\imath\in\mathbf{N}$, where

$$\begin{aligned}
\mathcal{M}(\tilde{\ss}_{\dot\imath},æ)=\max\{&\mathsf{B}_{\dot{b}}(\tilde{\ss}_{\dot\imath},æ),\mathsf{B}_{\dot{b}}(\tilde{\ss}_{\dot\imath},\mathcal{W}\tilde{\ss}_{\dot\imath}),\mathsf{B}_{\dot{b}}(æ,\mathcal{W}æ),\\
&\mathsf{B}_{\dot{b}}(æ,\mathcal{W}\tilde{\ss}_{\dot\imath})\},
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\dot\imath\to\infty}\sup\max\{\mathsf{B}_{\dot{b}}(\tilde{\ss}_{\dot\imath},æ),\mathsf{B}_{\dot{b}}(\tilde{\ss}_{\dot\imath},\mathcal{W}\tilde{\ss}_{\dot\imath}),\mathsf{B}_{\dot{b}}(æ,\mathcal{W}æ),\\
&\qquad \mathsf{B}_{\dot{b}}(æ,\mathcal{W}\tilde{\ss}_{\dot\imath})\}=\mathsf{B}_{\dot{b}}(æ,\mathcal{W}æ). \qquad (20)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{F}(\omega_{\iota}^2\mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{\ss}_{\dot\imath},\mathcal{W}æ))&\le \mathcal{F}(\max\{\mathsf{B}_{\dot{b}}(\tilde{\ss}_{\dot\imath},æ),\mathsf{B}_{\dot{b}}(\tilde{\ss}_{\dot\imath},\mathcal{W}\tilde{\ss}_{\dot\imath}),\\
&\mathsf{B}_{\dot{b}}(æ,\mathcal{W}æ),\mathsf{B}_{\dot{b}}(æ,\mathcal{W}\tilde{\ss}_{\dot\imath})\})-\phi(\mathsf{B}_{\dot{b}}(\tilde{\ss}_{\dot\imath},æ)). \qquad (21)
\end{aligned}$$

By letting $\dot\imath\to\infty$ in inequality (21), and utilizing equations (19) and (20), along with the contraction of $\mathcal{F}$, we obtain

$$\begin{aligned}
\mathcal{F}[\omega_{\iota}^2\frac{1}{\omega_{\iota}}\mathsf{B}_{\dot{b}}(æ,\mathcal{W}æ)] &= \mathcal{F}[\omega_{\iota}\mathsf{B}_{\dot{b}}(æ,\mathcal{W}æ)]\\
&\le \mathcal{F}[\omega_{\iota}^2\lim_{\dot\imath\to\infty}\sup \mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{\ss}_{\dot\imath},\mathcal{W}æ)]\\
&=\lim_{\dot\imath\to\infty}\sup \mathcal{F}[\omega_{\iota}^2\mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{\ss}_{\dot\imath},\mathcal{W}æ)]\\
&\le \lim_{\dot\imath\to\infty}\sup \mathcal{F}(\mathcal{M}(\tilde{\ss}_{\dot\imath},æ))-\lim_{\dot\imath\to\infty}\sup \phi(\mathsf{B}_{\dot{b}}(\tilde{\ss}_{\dot\imath},æ))\\
&=\mathcal{F}(\mathsf{B}_{\dot{b}}(\mathcal{W}æ,æ))-\lim_{\dot\imath\to\infty}\inf \phi(\mathsf{B}_{\dot{b}}(\tilde{\ss}_{\dot\imath},æ))\\
&< \mathcal{F}(\mathsf{B}_{\dot{b}}(æ,\mathcal{W}æ)).
\end{aligned}$$

Since $\mathcal{F}$ is increasing, we get

$$\omega_{\iota}\mathsf{B}_{\dot{b}}(æ,\mathcal{W}æ)<\mathsf{B}_{\dot{b}}(æ,\mathcal{W}æ).$$

This implies that

$$\mathsf{B}_{\dot{b}}(æ,\mathcal{W}æ)(\omega_{\iota}-1)<0\to\omega_{\iota}<1.$$

It is a contradiction.

Therefore, $\mathcal{W}æ=æ$.

Uniqueness: Assume that æ and q in $\mho$ are distinct fixed points of $\mathcal{W}$ such that $q\neq æ$.

Hence, we get

$$\mathsf{B}_{\dot{b}}(æ,q)=\mathsf{B}_{\dot{b}}(\mathcal{W}æ,\mathcal{W}q)>0.$$

By choice of $\tilde{\ss}^*$, we obtain

$$(\tilde{\ss}^*\perp q,\quad \tilde{\ss}^*\perp æ)\ \text{or}\ (q\perp\tilde{\ss}^*,\quad æ\perp\tilde{\ss}^*).$$

Since $\mathcal{W}$ is $\perp$-preserving, we have

$$\begin{aligned}
&(\mathcal{W}^{\dot\imath}\tilde{\ss}^*\perp\mathcal{W}^{\dot\imath}q,\ \mathcal{W}^{\dot\imath}\tilde{\ss}^*\perp\mathcal{W}^{\dot\imath}æ)\ \text{or}\\
&(\mathcal{W}^{\dot\imath}q\perp\mathcal{W}^{\dot\imath}\tilde{\ss}^*,\ \mathcal{W}^{\dot\imath}æ\perp\mathcal{W}^{\dot\imath}\tilde{\ss}^*),\ \forall\ \dot\imath\in\mathbf{N}.
\end{aligned}$$

Since $\mathcal{W}$ is an orthogonal $(\phi,\mathcal{F})$-contraction of a map type, and applying (1) with $\tilde{\ss}=æ$ and $\propto=q$, we obtain

$$\begin{aligned}
\mathcal{F}(\mathsf{B}_{\dot{b}}(æ,q))=\mathcal{F}(\mathsf{B}_{\dot{b}}(\mathcal{W}q,\mathcal{W}æ))&\le \mathcal{F}(\omega_{\iota}^2\mathsf{B}_{\dot{b}}(\mathcal{W}q,\mathcal{W}æ))\\
&\le \mathcal{F}(\mathcal{M}(æ,q))-\phi(\mathsf{B}_{\dot{b}}(æ,q)),
\end{aligned}$$

where

$$\mathcal{M}(æ, q) = \max\{B_{\dot{b}}(æ, q), B_{\dot{b}}(æ, \mathcal{W}æ), B_{\dot{b}}(q, \mathcal{W}q), B_{\dot{b}}(q, \mathcal{W}æ)\} = B_{\dot{b}}(æ, q).$$

Therefore, we get

$$F(B_{\dot{b}}(æ, q)) \leq F(B_{\dot{b}}(æ, q)) - \phi(B_{\dot{b}}(æ, q)) < F(B_{\dot{b}}(æ, q)).$$

This implies that

$$B_{\dot{b}}(æ, q) < B_{\dot{b}}(æ, q),$$

which is a contradiction.

Hence, $æ = q$. ■

Corollary 1. *Let $(\mho, \perp, B_{\dot{b}})$ be an orthogonal complete B_bMS and $\mathcal{W} : \mho \to \mho$ be a given map. Assume that there exists $F \in \chi$ and $\tau \in (0, \infty)$ such that for any $\tilde{ß}, \propto \in \mho$ with $\tilde{ß} \perp \propto$, we obtain*

$$B_{\dot{b}}\mathcal{W}\tilde{ß}, \mathcal{W}\propto) > 0 \to F[\omega_l^2 B_{\dot{b}}(\mathcal{W}\tilde{ß}, \mathcal{W}\propto)] + \tau \leq [F(\mathcal{M}(\tilde{ß}, \propto))],$$

where

$$\mathcal{M}(\tilde{ß}, \propto) = \max\{B_{\dot{b}}(\tilde{ß}, \propto), B_{\dot{b}}(\tilde{ß}, \mathcal{W}\tilde{ß}), B_{\dot{b}}(\propto, \mathcal{W}\propto), B_{\dot{b}}(\mathcal{W}\tilde{ß}, \propto)\}.$$

Then, $\mathcal{W}$ has a unique fixed point.

Expanding on Theorem 1, we establish fixed-point theorems for orthogonal (ϕ, F)-Kannan type and Reich type contraction maps.

Theorem 2. *Let $(\mho, \perp, B_{\dot{b}})$ be an orthogonal complete B_bMS, and $\mathcal{W} : \mho \to \mho$ be an orthogonal (ϕ, F)-Kannan contraction map of a map type. Then, $\mathcal{W}$ has a unique fixed point.*

Proof: According to the concept of orthogonality, if $(\mho, \perp)$ is an orthogonal set, then there exist

$$\tilde{ß}_0 \in \mho : \forall\ \tilde{ß} \in \mho,\ \tilde{ß} \perp \tilde{ß}_0\ (or)\ \forall\ \tilde{ß} \in \mho,\ \tilde{ß}_0 \perp \tilde{ß}.$$

It follows that $\tilde{ß}_0 \perp \mathcal{W}\tilde{ß}_0$ or $\mathcal{W}\tilde{ß}_0 \perp \tilde{ß}_0$. Let

$$\tilde{ß}_1 = \mathcal{W}\tilde{ß}_0,\ \tilde{ß}_2 = \mathcal{W}\tilde{ß}_1 = \mathcal{W}^2\tilde{ß}_0 \cdots \tilde{ß}_{\dot{\imath}} = \mathcal{W}\tilde{ß}_{\dot{\imath}-1} = \mathcal{W}^{\dot{\imath}}\tilde{ß}_0,$$
$$\tilde{ß}_{\dot{\imath}+1} = \mathcal{W}\tilde{ß}_{\dot{\imath}} = \mathcal{W}^{\dot{\imath}+1}\tilde{ß}_0,\quad \forall\ \dot{\imath} \in \mathbf{N}.$$

For any $\tilde{ß}_0 \in \mho$, set $\tilde{ß}_{\dot{\imath}} = \mathcal{W}\tilde{ß}_{\dot{\imath}-1}$.

Now, let us consider $\tilde{ß}_{\dot{\imath}} = \tilde{ß}_{\dot{\imath}+1}$, then we have $\mathcal{W}\tilde{ß}_{\dot{\imath}} = \tilde{ß}_{\dot{\imath}}$.

It is easy to see that $\tilde{ß}_{\dot{\imath}}$ is a fixed point of $\mathcal{W}$.

Therefore, completing the proof.

If $\tilde{ß}_{\dot{\imath}} \neq \tilde{ß}_{\dot{\imath}+1}$ for any $\dot{\imath} \in \mathbf{N} \cup \{0\}$, then we have $B_{\dot{b}}(\tilde{ß}_{\dot{\imath}+1}, \tilde{ß}_{\dot{\imath}}) > 0$ for each $\dot{\imath} \in \mathbf{N}$.

Since $\mathcal{W}$ is $\perp$-preserving, we have

$$\tilde{ß}_{\dot{\imath}} \perp \tilde{ß}_{\dot{\imath}+1}\quad \text{(or)}\quad \tilde{ß}_{\dot{\imath}+1} \perp \tilde{ß}_{\dot{\imath}}.$$

This implies that $\{\tilde{ß}_{\dot{\imath}}\}$ is an O-sequence.

Since $\mathcal{W}$ is an orthogonal (ϕ, F)-Kannan contraction map of type, there exists $F \in \chi$ and $\phi \in \Pi$ such that

$$\begin{aligned} F[\omega_l^2 B_{\dot{b}}(\mathcal{W}\tilde{ß}, \mathcal{W}\propto)] + \phi(B_{\dot{b}}(\tilde{ß}, \propto)) &\leq F\Big(\frac{B_{\dot{b}}(\mathcal{W}\tilde{ß}, \tilde{ß}) + B_{\dot{b}}(\mathcal{W}\propto, \propto)}{2}\Big) \\ &\leq F(\max\{B_{\dot{b}}(\tilde{ß}, \mathcal{W}\tilde{ß}), B_{\dot{b}}(\propto, \mathcal{W}\propto)\}) \\ &\leq F(\max\{B_{\dot{b}}(\tilde{ß}, \propto), B_{\dot{b}}(\tilde{ß}, \mathcal{W}\tilde{ß}), B_{\dot{b}}(\propto, \mathcal{W}\propto), B_{\dot{b}}(\propto, \mathcal{W}\tilde{ß})\}). \end{aligned}$$

Therefore, $\mathcal{W}$ is an orthogonal (ϕ, F)-contraction map type.

As shown in the proof of Theorem 1, we conclude that $\mathcal{W}$ has a unique fixed point. ■

Theorem 3. *Let $(\mho, \perp, B_{\dot{b}})$ be an orthogonal complete B_bMS, and let $\mathcal{W} : \mho \to \mho$ be an orthogonal (ϕ, F)-Reich contraction map of type. Then, $\mathcal{W}$ has a unique fixed point.*

Proof: According to the concept of orthogonality, if $(\mho, \perp)$ is an orthogonal set, then there exist

$$\tilde{ß}_0 \in \mho : \tilde{ß} \perp \tilde{ß}_0\quad (or)\quad \tilde{ß}_0 \perp \tilde{ß},\quad \forall\ \tilde{ß} \in \mho.$$

It follows that $\tilde{ß}_0 \perp \mathcal{W}\tilde{ß}_0$ or $\mathcal{W}\tilde{ß}_0 \perp \tilde{ß}_0$. Let

$$\tilde{ß}_1 = \mathcal{W}\tilde{ß}_0,\ \tilde{ß}_2 = \mathcal{W}\tilde{ß}_1 = \mathcal{W}^2\tilde{ß}_0 \cdots \tilde{ß}_{\dot{\imath}} = \mathcal{W}\tilde{ß}_{\dot{\imath}-1} = \mathcal{W}^{\dot{\imath}}\tilde{ß}_0,$$
$$\tilde{ß}_{\dot{\imath}+1} = \mathcal{W}\tilde{ß}_{\dot{\imath}} = \mathcal{W}^{\dot{\imath}+1}\tilde{ß}_0,\quad \forall\ \dot{\imath} \in \mathbf{N}.$$

For any $\tilde{ß}_0 \in \mho$, set $\tilde{ß}_{\dot{\imath}} = \mathcal{W}\tilde{ß}_{\dot{\imath}-1}$.

Now, let us consider $\tilde{ß}_{\dot{\imath}} = \tilde{ß}_{\dot{\imath}+1}$.

In this case, we have $\mathcal{W}\tilde{ß}_{\dot{\imath}} = \tilde{ß}_{\dot{\imath}}$, indicating that $\tilde{ß}_{\dot{\imath}}$ is a fixed point of $\mathcal{W}$.

Thus, we complete the proof.

Alternatively, if $\tilde{ß}_{\dot{\imath}} \neq \tilde{ß}_{\dot{\imath}+1}$ for every $\dot{\imath} \in \mathbf{N} \cup \{0\}$, then we conclude that $B_{\dot{b}}(\tilde{ß}_{\dot{\imath}+1}, \tilde{ß}_{\dot{\imath}}) > 0$, for any $\dot{\imath} \in \mathbf{N}$.

Since $\mathcal{W}$ is $\perp$-preserving, we obtain

$$\tilde{ß}_{\dot{\imath}} \perp \tilde{ß}_{\dot{\imath}+1}\quad \text{(or)}\quad \tilde{ß}_{\dot{\imath}+1} \perp \tilde{ß}_{\dot{\imath}},$$

which implies that $\{\tilde{ß}_{\dot{\imath}}\}$ is an O-sequence.

Since $\mathcal{W}$ is an orthogonal (ϕ, F)-Reich contraction map of type, there exists $F \in \chi$ and $\phi \in \Pi$ such that

$$\begin{aligned} F[\omega_l^2 B_{\dot{b}}(\mathcal{W}\tilde{ß}, \mathcal{W}\propto)] + \phi(B_{\dot{b}}\tilde{ß}, \propto)) &\leq F\Big(\frac{B_{\dot{b}}(\tilde{ß}, \propto) + B_{\dot{b}}(\mathcal{W}\tilde{ß}, \tilde{ß}) + B_{\dot{b}}(\mathcal{W}\propto, \propto)}{3}\Big) \\ &\leq F(\max\{B_{\dot{b}}(\tilde{ß}, \propto), B_{\dot{b}}(\tilde{ß}, \mathcal{W}\tilde{ß}), B_{\dot{b}}(\propto, \mathcal{W}\propto), B_{\dot{b}}(\propto, \mathcal{W}\tilde{ß})\}). \end{aligned}$$

Therefore, $\mathcal{W}$ is (ϕ, F)-contraction map of type (χ).

Following a demonstration related to the proof of Theorem 1, we deduce that there exists only one fixed point for $\mathcal{W}$. ■

Example 4. *Consider the space $\mho = \mathcal{H} \cup \mathcal{Y}$, where $\mathcal{H} = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$ and $\mathcal{Y} = [1, 4]$, equipped with the Branciari $\mathfrak{b}$-metric $B_{\dot{b}}$ on $\mathbb{R}^+$.*

Let the orthogonal relation $\perp$ on $\mho$ be defined by $\tilde{ß} \perp \propto$ if $\tilde{ß}, \propto \geq 0$ for all $\tilde{ß}, \propto \in \mho$.

Define $B_{\dot{b}} : \mho \times \mho \to (0, \infty)$ as

$$B_{\dot{b}}(\tilde{ß}, \propto) = |\tilde{ß} - \propto|^2.$$

Then, $(\mho, \perp, B_{\dot{b}})$ is an orthogonal B_bMS with coefficient $\omega_l = 3$.

Define the map $\mathcal{W} : \mho \to \mho$ by

$$\mathcal{W}(\tilde{ß}) = \begin{cases} 1 & if\ \tilde{ß} \in [1, 4] \\ 2 & if\ \tilde{ß} \in \mathcal{H}. \end{cases}$$

Clearly, $\mathcal{W}$ is an $\perp$-preserving.

Let us consider $F(\beta) = \ln(\sqrt{\beta})$, $\phi(\beta) = \frac{1}{2+\beta}$.

It is obvious that $F \in \chi$ and $\phi \in \Pi$.

Consider $\tilde{ß} = 4, \propto = 0$ we get

$$\mathcal{W}(\tilde{ß}) = 1,\ \mathcal{W}(\propto) = 2,\ B_{\dot{b}}(\mathcal{W}\tilde{ß}, \mathcal{W}\propto) = (1-2)^2 = 1,$$
$$B_{\dot{b}}(\tilde{ß}, \propto) = 16,\ B_{\dot{b}}(\tilde{ß}, \mathcal{W}\tilde{ß}) = 9,\ B_{\dot{b}}(\propto, \mathcal{W}\propto) = 4,$$
$$B_{\dot{b}}(\propto, \mathcal{W}\tilde{ß}) = 1.$$

On the other hand

$$\digamma[\omega_\wr^2 \mathsf{B}_{\dot{\mathsf{b}}}(\mathcal{W}\tilde{\text{ß}}, \mathcal{W}\propto)] = \ln(3),$$
$$\phi[\mathsf{B}_{\dot{\mathsf{b}}}(\tilde{\text{ß}}, \propto)] = \frac{1}{2 + \mathsf{B}_{\dot{\mathsf{b}}}(\tilde{\text{ß}}, \propto)},$$

and

$$\begin{aligned}\mathcal{M}(\tilde{\text{ß}}, \propto) &= \max\{\mathsf{B}_{\dot{\mathsf{b}}}(\tilde{\text{ß}}, \propto), \mathsf{B}_{\dot{\mathsf{b}}}(\tilde{\text{ß}}, \mathcal{W}\tilde{\text{ß}}), \mathsf{B}_{\dot{\mathsf{b}}}(\propto, \mathcal{W}\propto), \\ &\quad \mathsf{B}_{\dot{\mathsf{b}}}(\propto, \mathcal{W}\tilde{\text{ß}})\} \\ &= \mathsf{B}_{\dot{\mathsf{b}}}(\tilde{\text{ß}}, \propto) = 16.\end{aligned}$$

Now, from (1)*, we obtain*

$$\begin{aligned}\digamma[\omega_\wr^2 \mathsf{B}_{\dot{\mathsf{b}}}(\mathcal{W}\tilde{\text{ß}}, \mathcal{W}\propto)] + \phi[\mathsf{B}_{\dot{\mathsf{b}}}(\tilde{\text{ß}}, \propto)] &\leq \digamma[\mathcal{M}(\tilde{\text{ß}}, \propto))] \\ \ln(3) + \frac{1}{2 + \mathsf{B}_{\dot{\mathsf{b}}}(\tilde{\text{ß}}, \propto)} &\leq \ln(4) \\ 1.0986 + 0.0555 &\leq 1.3862 \\ 1.1536 &\leq 1.3862.\end{aligned}$$

Thus, for all $\tilde{\text{ß}} \in [1, 4]$ *and* $\propto \in \mathcal{H}$*, we have*

$$\digamma[\omega_\wr^2 \mathsf{B}_{\dot{\mathsf{b}}}(\mathcal{W}\tilde{\text{ß}}, \mathcal{W}\propto)] + \phi[\mathsf{B}_{\dot{\mathsf{b}}}(\tilde{\text{ß}}, \propto)] \leq \digamma[\mathsf{B}_{\dot{\mathsf{b}}}(\tilde{\text{ß}}, \propto)].$$

Clearly, $\mathcal{W}$ *is O-continuous with* æ $= 1$*.*

Therefore, condition (1) *is fulfilled, and* $\mathcal{W}$ *has a fixed point.*

IV. Applications

By utilizing Theorem 1, we demonstrate the existence and uniqueness of the solution for the following integral equation:

$$\tilde{\text{ß}}(\beta) = \lambda \int_\gamma^\vartheta \mathcal{K}(\beta, \delta, \tilde{\text{ß}}(\delta))\mathsf{B}_{\dot{\mathsf{b}}}\delta, \tag{22}$$

where $\gamma, \vartheta \in \mathsf{R}$, $\tilde{\text{ß}} \in \mathcal{C}([\gamma, \vartheta], \mathsf{R})$ and $\mathcal{K} : [\gamma, \vartheta]^2 \times \mathsf{R} \to \mathsf{R}$ is a given continuous function.

Theorem 5. *Assume that the kernel function* $\mathcal{K}$ *satisfies the condition*

$$\begin{aligned}&|\mathcal{K}(\beta, \delta, \tilde{\text{ß}}(\delta)) - \mathcal{K}(\beta, \delta, \propto(\delta))| \\ &\quad \leq \frac{1}{\sqrt[\omega_\wr]{\omega_\wr^2}} e^{-\frac{1}{|\tilde{\text{ß}}(\beta) - \propto(\beta)| + 1}} (|\tilde{\text{ß}}(\beta) - \propto(\beta)|),\end{aligned}$$

for all $\beta, \delta \in [\gamma, \vartheta]$ *and* $\tilde{\text{ß}}, \propto \in \mathbb{R}$*, then the equation* (22) *has a unique solution* $\tilde{\text{ß}} \in \mathcal{C}([\gamma, \vartheta])$ *for some parameter* λ*, determined by the constants* γ, ϑ*, and* $\omega_\wr$*.*

Proof: Define the binary relation $\perp$ on $\mho$ by

$$\tilde{\text{ß}} \perp \propto \iff \tilde{\text{ß}}(\beta) \propto (\beta) \geq \tilde{\text{ß}}(\beta) \text{ or } \tilde{\text{ß}}(\beta) \propto (\beta) \geq \propto (\beta),$$

for all $\beta \in [0, 1]$.

Define a function $\mathsf{B}_{\dot{\mathsf{b}}} : \mho \times \mho \to [0, \infty)$ by

$$\mathsf{B}_{\dot{\mathsf{b}}}(\tilde{\text{ß}}, \propto) = (\max_{\beta \in [\gamma, \vartheta]} |\tilde{\text{ß}}(\beta) - \propto(\beta)|)^{\omega_\wr}, \ \forall\ \tilde{\text{ß}}, \propto \in \mho,$$

It is evident that $(\mho, \perp, \mathsf{B}_{\dot{\mathsf{b}}})$ forms an orthogonal complete $\mathtt{B}_b\mathtt{MS}$.

Now, let us define a map $\mathcal{W} : \mho \to \mho$ as follows:

$$\mathcal{W}(\tilde{\text{ß}}(\beta)) = \lambda \int_\gamma^\vartheta \mathcal{K}(\beta, \delta, \tilde{\text{ß}}(\delta))\mathsf{B}_{\dot{\mathsf{b}}}\delta.$$

Now, we prove that $\mathcal{W}$ is $\perp$-preserving.

For every $\tilde{\text{ß}}, \propto \in \mho$ with $\tilde{\text{ß}} \perp \propto$ and $\beta \in \mho$, we obtain

$$\mathcal{W}(\tilde{\text{ß}}(\beta)) = \lambda \int_\gamma^\vartheta \mathcal{K}(\beta, \delta, \tilde{\text{ß}}(\delta)) \geq 1.$$

It follows that $[(\mathcal{W}\tilde{\text{ß}})(\beta)][(\mathcal{W}\propto)(\beta)] \geq (\mathcal{W}\propto)(\beta)$ and so $(\mathcal{W}\tilde{\text{ß}})(\beta) \perp (\mathcal{W}\propto)(\beta)$.

Thus, $\mathcal{W}$ is $\perp$-preserving.

Then, we get

$$\begin{aligned}&|\mathcal{W}\tilde{\text{ß}}(\beta) - \mathcal{W}\propto(\beta)|^{\omega_\wr} \\ &= |\lambda|^{\omega_\wr}\Big(\Big|\int_\gamma^\vartheta \mathcal{K}(\beta, \delta, \tilde{\text{ß}}(\delta))\mathsf{B}_{\dot{\mathsf{b}}}\delta \\ &\qquad - \int_\gamma^\vartheta \mathcal{K}(\beta, \delta, \propto(\delta))\mathsf{B}_{\dot{\mathsf{b}}}\delta\Big|\Big)^{\omega_\wr} \\ &= |\lambda|^{\omega_\wr}\Big|\int_\gamma^\vartheta \mathcal{K}(\beta, \delta, \tilde{\text{ß}}(\delta)) - \mathcal{K}(\beta, \delta, \propto(\delta))\mathsf{B}_{\dot{\mathsf{b}}}\delta\Big|^{\omega_\wr} \\ &\leq |\lambda|^{\omega_\wr}\int_\gamma^\vartheta |\mathcal{K}(\beta, \delta, \tilde{\text{ß}}(\delta)) - \mathcal{K}(\beta, \delta, \propto(\delta))\mathsf{B}_{\dot{\mathsf{b}}}\delta|^{\omega_\wr} \\ &\leq |\lambda|^{\omega_\wr}\int_\gamma^\vartheta \Big(\frac{1}{\sqrt[\omega_\wr]{\omega_\wr^2}} e^{-\frac{1}{|\tilde{\text{ß}}(\delta) - \propto(\delta)| + 1}} \\ &\qquad (|\tilde{\text{ß}}(\delta)| - |\propto(\delta)|)\mathsf{B}_{\dot{\mathsf{b}}}\delta\Big)^{\omega_\wr} \\ &= \frac{1}{\omega_\wr^2}|\lambda|^{\omega_\wr}\int_\gamma^\vartheta (e^{-\frac{1}{|\tilde{\text{ß}}(\delta) - \propto(\delta)| + 1}} \\ &\qquad (|\tilde{\text{ß}}(\delta)| - |\propto(\delta)|)\mathsf{B}_{\dot{\mathsf{b}}}\delta)^{\omega_\wr},\end{aligned}$$

which implies that

$$\begin{aligned}&\max_{\beta \in [\gamma, \vartheta]}(|\mathcal{W}\tilde{\text{ß}}(\beta) - \mathcal{W}\propto(\beta)|) \\ &= \max_{\beta \in [\gamma, \vartheta]} \frac{1}{\omega_\wr^2}|\lambda|^{\omega_\wr}\int_\gamma^\vartheta |\mathcal{K}(\beta, \delta, \tilde{\text{ß}}(\delta)) \\ &\qquad - \mathcal{K}(\beta, \delta, \propto(\delta))\mathsf{B}_{\dot{\mathsf{b}}}\delta|^{\omega_\wr} \\ &\leq \max_{\beta \in [\gamma, \vartheta]} \frac{1}{\omega_\wr^2}|\lambda|^{\omega_\wr}\int_\gamma^\vartheta (e^{-\frac{1}{|\tilde{\text{ß}}(\delta) - \propto(\delta)| + 1}} \\ &\qquad (|\tilde{\text{ß}}(\delta) - \propto(\delta)|)\mathsf{B}_{\dot{\mathsf{b}}}\delta)^{\omega_\wr} \\ &\leq |\lambda|^{\omega_\wr}\frac{1}{\omega_\wr^2}\int_\gamma^\vartheta (e^{-\max_{\delta \in [\gamma, \vartheta]} \frac{1}{|\tilde{\text{ß}}(\delta) - \propto(\delta)| + 1}} \\ &\qquad (\max_{\delta \in [\gamma, \vartheta]} |\tilde{\text{ß}}(\delta) - \propto(\delta)|)\mathsf{B}_{\dot{\mathsf{b}}}\delta)^{\omega_\wr}.\end{aligned}$$

Due to the definition of the orthogonal $\mathtt{B}_b\mathtt{MS}$, we have

$$\mathsf{B}_{\dot{\mathsf{b}}}(\mathcal{W}\tilde{\text{ß}}, \mathcal{W}\propto) > 0, \quad \text{and for any } \tilde{\text{ß}} \neq \propto.$$

Subsequently, we can apply the natural logarithm to both sides and obtain

$$\begin{aligned}&\ln[\omega_\wr^2 \mathsf{B}_{\dot{\mathsf{b}}}(\mathcal{W}\tilde{\text{ß}}, \mathcal{W}\propto)] \\ &\leq \ln[|\lambda|^{\omega_\wr}\int_\gamma^\vartheta (e^{-\max_{\delta \in [\gamma, \vartheta]} \frac{1}{|\tilde{\text{ß}}(\delta) - \propto(\delta)| + 1}} \\ &\qquad (\max_{\delta \in [\gamma, \vartheta]} |\tilde{\text{ß}}(\delta) - \propto(\delta)|)\mathsf{B}_{\dot{\mathsf{b}}}\delta)^{\omega_\wr}] \\ &= \ln[((\vartheta - \gamma)|\lambda|)^{\omega_\wr}] + \ln[\int_\gamma^\vartheta (e^{-\max_{\delta \in [\gamma, \vartheta]} \frac{1}{|\tilde{\text{ß}}(\delta) - \propto(\delta)| + 1}} \\ &\qquad (\max_{\delta \in [\gamma, \vartheta]} |\tilde{\text{ß}}(\delta) - \propto(\delta)|)\mathsf{B}_{\dot{\mathsf{b}}}\delta)]^{\omega_\wr} \\ &= \ln[((\vartheta - \gamma)|\lambda|)^{\omega_\wr}] + \ln((e^{-\max_{\beta \in [\gamma, \vartheta]} \frac{1}{|\tilde{\text{ß}}(\beta) - \propto(\beta)| + 1}})^{\omega_\wr}) \\ &\quad + \ln[\int_\gamma^\vartheta ((\max_{\delta \in [\gamma, \vartheta]} |\tilde{\text{ß}}(\delta) - \propto(\delta)|)\mathsf{B}_{\dot{\mathsf{b}}}\delta)^{\omega_\wr}] \\ &= \omega_\wr . \ln[(\vartheta - \gamma)|\lambda|] - \frac{1}{\max_{\beta \in [\gamma, \vartheta]} |\tilde{\text{ß}}(\beta) - \propto(\beta)| + 1} \\ &\quad + \ln[\int_\gamma^\vartheta ((\max_{\delta \in [\gamma, \vartheta]} |\tilde{\text{ß}}(\delta) - \propto(\delta)|)\mathsf{B}_{\dot{\mathsf{b}}}\delta)^{\omega_\wr}].\end{aligned}$$

Let $|\lambda|(\vartheta - \gamma) \leq \frac{1}{e^{\omega_\wr}}$, we get

$$\ln[\omega_\wr^2 \mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{\ss}, \mathcal{W} \propto)] \le -\frac{1}{\mathsf{B}_{\dot{b}}(\tilde{\ss},\propto)+1} + \ln(\mathsf{B}_{\dot{b}}(\tilde{\ss},\propto))$$
$$\le -\frac{1}{\mathsf{B}_{\dot{b}}(\tilde{\ss},\propto)+1} + \ln(\mathsf{B}_{\dot{b}}(\tilde{\ss},\propto)).$$

Hence,

$$\digamma(\omega_\wr^2 \mathsf{B}_{\dot{b}}(\mathcal{W}\tilde{\ss}, \mathcal{W} \propto)) + \phi(\mathsf{B}_{\dot{b}}(\tilde{\ss},\propto)) \le \digamma(\mathsf{B}_{\dot{b}}(\tilde{\ss},\propto)),$$

for all $\tilde{\ss}, \propto \in \mho$ with $\digamma(\beta) = ln(\beta)$ and $\phi(\beta) = \frac{1}{\beta+1}$.

Therefore, all the conditions of Theorem 1 are satisfied.

Hence, there is a unique solution to equation (22). ■

Example 6. *Consider the nonlinear integral equation as follows:*

$$\tilde{\ss}(\beta) = \lambda \int_0^\eta \mathcal{K}(\beta, \delta, \tilde{\ss}(\delta))d\delta, \quad \beta \in [0, \eta], \ 0 \le \eta \le 1.$$

Here,

$$\tilde{\ss}(\beta) = 2 \int_0^\eta e^{2\beta - 2\delta}\tilde{\ss}(\delta)d\delta, \ \forall \ 0 \le \eta \le 1. \qquad (23)$$

Let us take $\mathcal{K}(\beta, \delta, \tilde{\ss}(\delta)) = e^{2\beta}$ as the exact solution (E. S) to equation (23) is determined.

Consequently, the absolute solution to the provided equation is $2\beta e^{2\beta}$ for $\beta > 0$.

A given table shows the numerical results as below:

TABLE I: NUMERIC SOLUTIONS

Iteration	A. S	E. S	Absolute Error
0.1	1.2214	0.2443	0.9771
0.2	1.4918	0.5967	0.8951
0.3	1.8221	1.0932	0.7289
0.4	2.2255	1.7804	0.5072
0.5	2.7183	2.7183	0.0000
0.6	3.3201	3.9841	0.6640
0.7	4.0552	5.6773	1.6221
0.8	4.9530	7.9249	2.9719
0.9	6.0496	10.8894	4.8398
1.0	7.3891	14.7781	7.3890

Table I shows that the fixed point of β is 0.5, and it is unique.

Comparison between approximate solution (A. S) and exact solution (E. S) shown in following Figure 1.

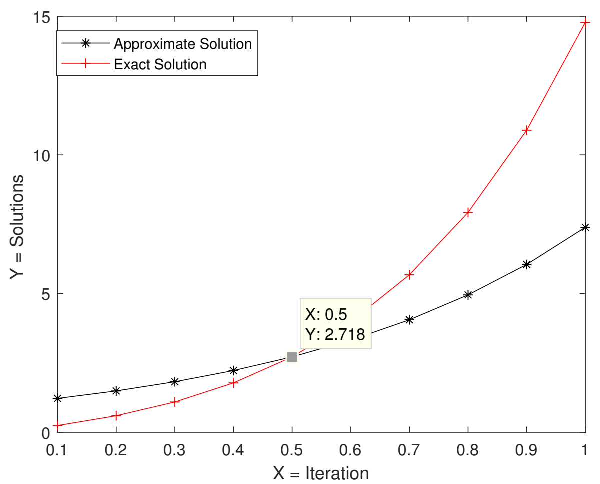

Fig. 1: shows that the fixed point of β is 0.5, which is unique.

V. Conclusions

In this article, we demonstrate several fixed-point theorems using distinct types of orthogonal $(\phi, \hat{\digamma})$-contraction maps in an orthogonal complete $\mathtt{B}_b\mathtt{MS}$. Our work expands upon and improves upon various recent results. Additionally, we provide an illustrative example to support our main findings and demonstrate an application of integral equations in resolving issues of existence and uniqueness, along with a comparison between analytical and numerical solutions.

References

[1] S. Banach, "Sur les operations Dans les ensembles abstraits et leur applications aux equations integrales," *Fundamenta Mathematicae*, 3, 133–181, 1922.

[2] F.E. Browder, "On the convergence of successive approximations for nonlinear functional equations," *Nederl. Akad. Wetensch. Proc. Ser. A Indag. Math.*, 30, 27–35, 1968.

[3] R. Kannan, "Some results on fixed points-II," *American Mathematical Monthly*, 76, 405–408, 1969.

[4] S. Reich, "Some remarks concerning contraction mappings," *Canadian Mathematical Bulletin*, 14, 121–124, 1971.

[5] S. Czerwik, "Contraction mappings in b-metric spaces," *Acta Mathematica Universitatis Ostraviensis*, 1, 5–11, 1993.

[6] H. Aydi, M. F. Bota, E. Karapinar, S. Moradi, "A common fixed point for weak ϕ-contractions on b-metric spaces," *Fixed Point Theory,* 13(2), 337–346, 2012.

[7] H. H. Alsulami, E. Karapinar, H. Piri, "Fixed points of generalised F-Suzuki type contraction in complete b-metric spaces," *Discrete Dynamics in Nature and Society,* p 8, 2015.

[8] M. Cosentino, M. Jleli, B. Samet, C. Vetro, "Solvability of integron differential problems via fixed point theory in b-metric spaces," *Fixed Point Theory and Applications*, 70, 2015.

[9] G. Nallaselli, A. J. Gnanaprakasam, G. Mani, Z. D. Mitrović, A. Aloqaily, N. Mlaiki, "Integral equation via fixed point theorems on a new type of convex contraction in b-metric and 2-metric spaces", *Mathematics*, 2023, 11(2):344.

[10] G. Mani, A. J. Gnanaprakasam, O. Ege, A. Aloqaily, N. Mlaiki, "Fixed point results in C^*-algebra-valued partial b-metric spaces with related application," *Mathematics*, 2023, 11(5):1158.

[11] G. Mani, A. J. Gnanaprakasam, L. Guran, R. George, Z. D. Mitrović, "Some results in fuzzy b-metric space with b-triangular property and applications to fredholm integral equations and dynamic programming", *Mathematics,* 2023, 11(19):4101.

[12] G. Mani, S. Haque S, A. J. Gnanaprakasam, O. Ege, N. Mlaiki, 'The study of bicomplex-valued controlled metric spaces with applications to fractional differential equations", *Mathematics*, 2023, 11(12):2742.

[13] A. Branciari, "A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces," *Publicationes Mathematicae Debrecen*, 57, 31–37, 2000.

[14] R. George, S. Radenovic, K.P. Reshma, S. Shukla, "Rectangular b-metric spaces and contraction principle," *Journal of Nonlinear Science and Applications,* 8, 1005–1013 (2015).

[15] H.S. Ding, V. Ozturk, S. Radenovic, "On some fixed point results in b-rectangular metric spaces," *Journal of Nonlinear Science and Applications,* 8, 378–386, 2015.

[16] J.R. Roshan, V. Parvaneh, Z. Kadelburg, N. Hussain, " New fixed point results in b-rectangular metric spaces," *Nonlinear Analysis: Modelling and Control,* 21(5), 614–634, 2016.

[17] D. Wardowski, "Fixed points of a new type of contractive mappings in complete metric spaces," *Fixed Point Theory and Applications,* 2012, 94, 2012.

[18] A. Kari, M. Rossafi, E. Marhrani, M. Aamri, "New fixed point theorems for $\theta - \phi -$contraction on complete rectangular $b-$metric spaces," *Abstract and Applied Analysis,* 2020, 12 (2020).

[19] D. Wardowski, "Solving existence problems via F-contractions," *Proceedings of the American Mathematical Society,* 146, 1585–1598, 2018.

[20] M. Rossafi, A. Kari, New fixed point theorems for (φ, F)-contraction on rectangular b-metric spaces, *Afrika Matematika,* 2023, 34:34.

[21] M. E. Gordji, M. Ramezani, M. De La Sen, and Y. J. Cho, "On orthogonal sets and Banach fixed point theorem," *Fixed Point Theory*, 18(2), 2017, 569 − 578.

[22] M. Eshaghi Gordji and H. Habibi, "Fixed point theory in generalized orthogonal metric space," *Journal of Linear and Topological Algebra,* 6(3), 2017, 251 − 260.

[23] A. J. Gnanaprakasam, G. Mani, J. R. Lee, C. Park, "Solving a nonlinear integral equation via orthogonal metric space," *AIMS Mathematics,* 2022, 7(1): 1198-1210.

[24] A. J. Gnanaprakasam, G. Mani, V. Parvaneh, H. Aydi, "Solving a Nonlinear Fredholm Integral Equation via an Orthogonal Metric," *Advances in Mathematical Physics*, vol. 2021, Article ID 1202527, 8 pages, 2021.

[25] S. K. Prakasam, A. J. Gnanaprakasam, N. Kausar, G. Mani, M. Munir and Salahuddin, "Solution of integral equation via orthogonally modified F-contraction maps on O-complete metric-like space", *International Journal of Fuzzy Logic and Intelligent Systems*, 2022, 22, 287–295.

[26] G. Nallaselli, A.S. Baazeem, A.J. Gnanaprakasam, G. Mani, K. Javed, E. Ameer, N. Mlaiki, "Fixed Point Theorems via Orthogonal Convex Contraction in Orthogonal b-Metric Spaces and Applications," *Axioms*, 2023, 12, 143.

[27] G. Mani, A.J. Gnanaprakasam, N. Kausar, M. Munir and Salahuddin, "Orthogonal F-Contraction map on orthogonal complete metric space with Applications," *International Journal of Fuzzy Logic and Intelligent Systems*, 2021, 21(3), 243–250.

[28] M. Dhanraj, A.J. Gnanaprakasam,, G. Mani, R. Ramaswamy, K. Hyatt Khan, Ola Ashour A. Abdelnaby and S. Radenovic, "Fixed point theorem on an orthogonal extended interpolative ψF-contraction," *AIMS Mathematics*, 2023, 8(7), 16151–16164.

[29] G. Mani, A.J. Gnanaprakasam, C. Park and S. Yun, "Orthogonal F-contractions on orthogonal complete b-metric space," *AIMS Mathematics*, 2021, 6(8), 8315–8330.

[30] A. J Gnanaprakasam, G. Mani, O.Ege, A. Aloqaily, N. Mlaiki, "New fixed point results in orthogonal B-metric spaces with related applications", *Mathematics*, 2023, 11(3):677.

[31] S. K. Prakasam, and A. J. Gnanaprakasam, "Solution of the volterra integral equation in orthogonal partial ordered metric spaces," *IAENG International Journal of Applied Mathematics*, vol. 53, no.2, pp 613-621, 2023.

[32] G. Nallaselli, and A. J. Gnanaprakasam, "Fixed point theorem for orthogonal (varphi, psi)-(Lambda, delta, Upsilon)-admissible multi-valued contractive mapping in orthogonal metric spaces," *IAENG International Journal of Applied Mathematics*, vol. 53, no.4, pp 1244-1252, 2023.

[33] L. Zhang, "Existence of positive solution for fractional differential equation with switched nonlinearity," *IAENG International Journal of Applied Mathematics*, vol. 54, no. 5, pp 945-949, 2024.

Menaha Dhanraj was born in Tamil Nadu, India. She graduated in Master of Science (2016) from Bharathiyar Arts and Science College for Women at Deviyakurichi, Attur, under Periyar University in Salem. Additionally, she completed a Master of Philosophy (2019) at Sri Chandrasekharendra Saraswathi Viswa Mahavidyalaya Deemed University in Kanchipuram. She is a full-time research scholar at SRM Institute of Science and Technology under the guidance of Dr. Arul Joseph Gnanaprakasam. Her research interests lie in fixed-point theory and its applications to differential and integral equations.

Arul Joseph Gnanaprakasam received the Ph.D. degree in Mathematics (2018) from the Madurai Kamaraj University, Madurai, Tamilnadu, India. He is currently an assistant professor of Mathematics in SRM Institute of Science and Technology, Kattankulathur, Kanchipuram, Tamilnadu, India. His research interests lie in fixed point theory and delay differential equations (DDEs).