Approximate Solutions of Hypersingular Integral Equations of the First Kind Using Chebyshev Polynomials of the Second Kind in Matrix-Vector Versions

E. S. Shoukralla and B. M. Ahmed

Abstract—**In this study, we focus on obtaining approximate solutions to hypersingular Fredholm integral equations of the first kind that appear in water wave dynamics, elasticity, acoustics, fluid mechanics, and others. The proposed method is based on Chebyshev polynomials of the second kind in matrix-vector forms. The analytical method was used to remove the singularity of the unknown function, whereas the asymptotic recurrence formula was used to remove the singularity of the hypersingular integral. A matrix equation whose solution is equivalent to the solution of the integral equation was obtained. Without using the collocation approach, this equivalent matrix equation has been transformed into a linear system of algebraic equations. A square coefficient matrix necessary for the algebraic system's solution has been extracted from the matrix equation. The solutions to four examples with tables and figures were provided. As it turns out, the derived approximate vector-matrix polynomials solutions significantly converge to the exact ones. The absolute errors were uniform and symmetric, and they also demonstrated proximity to zero as the degree of approximation was increased. The presented method included only a few simple steps and was highly accurate and innovative.**

Index Terms— **Hypersingular integral equations; chebyshev polynomials; computational method; wave dynamics; elasticity; acoustics; fluid mechanics.**

I. Introduction

In many fields, such as nanotechnology, artificial intelligence, water wave dynamics, elasticity, acoustics, fluid mechanics, cracks problems, and others, papers have been published that present various methods and techniques for solving hypersingular integral equations [1-3]. Most of the boundary value problems of the Laplace and Helmholtz equations are transformed into equivalent boundary integral equations under the influence of Dirichlet or Niemann conditions. These integral equations are singular if the boundaries are open. This singularity may be of the Cauchy type, the weakly singular type, or the hypersingular type. Zhong Chen et al. [3] created a method for solving first-kind hypersingular integral equations in reproducing kernel spaces by improving the standard reproducing kernel method.

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E. S. Shoukralla is a Professor of Engineering Mathematics, Faculty of Electronic Engineering, Menofia University, Egypt. (e-mail: shoukralla@el-eng.menofia.edu.eg).

B. M. Ahmed is a Lecturer of Engineering Mathematics, Faculty of Engineering and Technology, Future University in Egypt, Cairo, Egypt. (Corresponding author, phone: +201152157861; e-mail: Basma.magdy@fue.edu.eg).

This method's solution techniques are laborious. Using Chebyshev polynomials of the second kind (CPS). Gerardo Iovane et al. [4] developed an efficient direct numerical collocation method. Mahiub et al. [5] created a method for approximating only the unknown function. The smooth kernel, the badly-behaved function, and the regular unknown are all approximated in the proposed technique by applying a newly established version of (CPS) in matrix-vector forms.

The hypersingular kernel is replaced by an asymptotic recurrence expression based on the Chebyshev polynomials of the first kind. Shoukralla et al. [6-10] used several approaches and techniques to solve weakly singular Fredholm equations of the first kind with singular logarithmic kernel and singular unknown function. In [6,7], the author applied the Chebyshev and shifted Chebyshev polynomials to approximate the singular unknown function. The kernel's singularity is removed by numerical integration. In [8], the authors applied the economized monic Chebyshev polynomials to solve Fredholm integral equations with weakly singular kernels. In [9,10], the authors solved Fredholm's integral equation of the first kind with logarithmic kernels. In the first article, they employed monic Chebyshev polynomials, and in the second article, they used a different technique through the Vandermonde matrix.

However, we are more interested in investigating the application of the normalized second-kind Chebyshev polynomials in matrix-vector forms as a new method for solving hypersingular integral equations of the first kind. We must consider the numerous publications [11–20] that have been published to solve the second-kind regular and weakly singular linear Volterra and Fredholm integral equations. Rules for the distribution of the interpolation nodes were created to guarantee the removal of the integral equation's singularity. We begin by obtaining the coefficients from each Chebyshev polynomial and creating a square coefficient matrix. Second, using three matrices, we express the regular unknown function, the poorly known behaved function, the given data function, and the smooth kernel.

The monomial basis row matrix is the first; the square known matrix is the second, and the unknown functional value matrix is the third. As a result, we are given a matrix equation equivalent to the solution of the hypersingular integral equation without applying the collocation method. Then, we convert the matrix equation into an algebraic linear system whose solution provides values to the unknown coefficients of the approximated unknown function. Moreover, we obtain the continuous vector-matrix approximate polynomial solution for the integral equation.

The resulting vector-matrix approximate polynomials with fewer degrees significantly converge to the exact ones, as seen in the provided tables and figures, demonstrating the novelty and viability of the proposed approach.

II. Matrix-Vector Chebyshev polynomials Method

Consider the hypersingular Fredholm integral equation of the first kind

$$\int_{-1}^{1} k_1(x,t)\vartheta(t)\,dt + \int_{-1}^{1} k_2(x,t)\vartheta(t)\,dt = \phi(x)\ ; -1 \le x \le 1 \quad (1)$$

Where $k_2(x,t)$ is a regular square-integrable function of t and x, and $\phi(x)$ is a smooth given function. The Hadamard finite part concept is used to explain the first integral, where $k_1(x,t) = \frac{1}{(t-x)^2}$ is the hypersingular kernel. The unknown function $\vartheta(x)$ becomes zero at the endpoints of the integration domain that is $\vartheta(\pm 1) = 0$. To remove the singularity of the unknown function, we put it in the form

$$\vartheta(t) = \mu(t)\varphi(t); \varphi(t) = \sqrt{1-t^2} \quad (2)$$

Where $\mu(x)$ is a smooth, regular, unknown function to be determined. Here, the given kernel $k_2(x,t)$ is a regular smooth kernel defined on the square $\{(x,t), -1 \le x,t \le 1\}$. The Chebyshev polynomials of the second kind $\{U_n(t)\}_0^m$; $m \ge 0$ on the interval $[-1,1]$ are defined by

$$U_n(t) = 2^n \prod_{k=1}^{n}\left[t - \cos\left(\frac{k\pi}{n+1}\right)\right];$$
$$\int_{-1}^{1} \sqrt{1-t^2}\ U_i(t)U_j(t)\,dt = \begin{cases} 0\ ;\ i \ne j \\ \frac{\pi}{2}\ ;\ i = j \end{cases} \quad (3)$$

Applying $\{U_n(t)\}_0^m$; $m \ge 0$ on the interval $[-1,1]$ to approximate both $\mu(t)$ and $\varphi(t)$, yields the approximate functions $\mu_n(t)$ and $\varphi_n(t)$ each of degree at most n in the forms

$$\mu_n(t) = \sum_{i=0}^{n} A_i U_i(t) = \mathrm{X}(t)\mathrm{C}^T\mathrm{A},\ \ \varphi_n(t) = \mathrm{X}(t)\mathrm{C}^T\mathrm{F} \quad (4)$$

We have extracted the coefficients of each Chebyshev polynomials $\{U_i(t)\}_{i=0}^n$ and created a square coefficient matrix C, in ascending powers of t; the matrix $\mathrm{A}=[A_i]_{i=0}^n$ is the unknown coefficients column matrix to be determined; the matrix $\mathrm{F}=[\varphi_i]_{i=0}^n$ is the known coefficients column matrix of the badly behaved function $\varphi(t)$, which can be calculated by using (3), and $\mathrm{X}(t) = \left[t^i\right]_{i=0}^n$ is a row matrix of the monomial basis functions. Hence, we find the matrix-vector approximate Chebyshev polynomials $\vartheta_n(x)$ in the form

$$\vartheta_n(t) = \mathrm{F}^T\mathrm{C}\tilde{\mathrm{X}}(t)\mathrm{C}^T\mathrm{A}\ ;\ \tilde{\mathrm{X}}(t) = \mathrm{X}^T(t)\mathrm{X}(t) \quad (5)$$

The given function $\phi(t)$ is approximated similar to $\mu_n(t)$ via the known coefficients column matrix Φ in the form

$$\phi_n(x) = \mathrm{X}(x)\mathrm{C}^T\Phi\ ;\ \Phi=[\phi_i]_{i=0}^n\ ;$$
$$\phi_i = \frac{2}{\pi}\int_{-1}^{1}\sqrt{1-x^2}\ U_i(x)\phi(x)\,dx \quad (6)$$

The kernel $k_2(x,t)$ is approximated with regard to the variable t so that it is transformed into a product of three matrices; the third one is a column matrix with each element representing one of the kernel's values, which is a function of the second variable x. Thus, we get the single matrix-vector approximate form of the kernel $k_2(x,t)$ denoted by $k_{2,n}(x,t)$ via $(n+1)\times 1$ column matrix $\mathrm{E}_2(x)$ in the form

$$k_{2,n}(x,t) = \mathrm{X}(t)\mathrm{C}^T\mathrm{E}_2(x)\ ;\ \mathrm{E}_2(x) = [e_i(x)]_{i=0}^n,$$
$$e_i(x) = \frac{2}{\pi}\int_{-1}^{1}\sqrt{1-t^2}\ k_2(x,t)U_i(t)\,dt \quad (7)$$

Moreover, we get

$$\int_{-1}^{1} k_{2,n}(x,t)\vartheta_n(t)\,dt = \Psi(x)\mathrm{C}^T\mathrm{A}\ ;$$
$$\Psi(x) = \int_{-1}^{1}\mathrm{X}(t)\mathrm{C}^T\mathrm{E}_2(x)\mathrm{F}^T\mathrm{C}\tilde{\mathrm{X}}(t)\,dt \quad (8)$$

For the first integral with the kernel $k_1(x,t)$, we replace $\vartheta(t)$ with $\vartheta_n(t) = \varphi(t)U_i(t)\mathrm{A}$. Hence, we get [3]

$$\int_{-1}^{1} k_{1,n}(x,t)\vartheta(t)\,dt = \int_{-1}^{1}\frac{\sqrt{1-t^2}}{(t-x)^2}\ U_i(t)\mathrm{A}\,dt$$
$$= -\pi(i+1)U_i(x)\mathrm{A} \quad (9)$$

By substituting (9), (8), and (6) into (1), we obtain the following matrix equation

$$\left(-\pi(i+1)U_i(x) + \Psi(x)\mathrm{C}^T\right)\mathrm{A} = \mathrm{X}(x)\mathrm{C}^T\Phi \quad (10)$$

Where $\left(-\pi(i+1)U_i(x) + \Psi(x)\mathrm{C}^T\right)$ is a $1\times(n+1)$ row matrix. By extracting the coefficients of each element of $\left(-\pi(i+1)U_i(x) + \Psi(x)\mathrm{C}^T\right)$ and create a square matrix H of order $(n+1)\times(n+1)$; by converting the matrix equation (12) into an algebraic linear system, the unknown coefficient matrix A can be obtained by

$$A = H^{-1}C^{T}\Phi \quad (11)$$

Substituting A from (11) into (5), we get the required solution $\vartheta_n(t)$ of integral equation (1) in the following matrix form

$$\vartheta_n(t) = F^{T}C\tilde{X}C^{T}H^{-1}C^{T}\Phi \quad (12)$$

III. Computational Results

We designed a MATLAB R2019a code for the solution of four examples to find the approximate matrix-vector polynomial solutions $\vartheta_n(t)$. The approximate solutions $\vartheta_n(t_i)$ are found at the set of points $t_i = -1{:}0.2{:}1$. The absolute errors are denoted by $R_n(t_i) = |\vartheta(t_i) - \vartheta_n(t_i)|$. The first kind's four hypersingular Fredholm integral equations are with algebraic and non-algebraic data functions and the smooth kernels.

The obtained matrix-vector approximate solutions strongly converge to the exact ones. We make a comparison of CPU time between different examples 1,2,3, and 4 in Table 9.

Example 1

Consider the Fredholm integral equation of the second kind

$$\int_{-1}^{1} \frac{\vartheta(t)}{(t-x)^2}\,dt + 16\int_{-1}^{1} t^3x^3\vartheta(t)\,dt = -\pi\left(31x^3 - 16x\right) \quad (13)$$

Whose exact solution [4] is $\vartheta(t) = \sqrt{1-t^2}\left(8t^3 - 4t\right)$.

Table 1 compares the exact $\vartheta(t_i)$ and obtained approximate solutions $\vartheta_n(t_i)$ for $n = 3,5,15,25$ and $n = 30$ at $t_i = -1{:}0.2{:}1$. Table 2 displays the absolute errors for $n = 3,5,15,25$ and $n = 30$. Figure 1 illustrates the graphs of the exact solution and the approximate solution for $n = 30$. Figure 2 illustrates the approximate solutions for $n = 3,5,15,25$ and $n = 30$. The CPU total time are 7.883 sec., 9.679 sec., 24.489 sec., 50.548 sec., and 68.498 sec. respectively.

Example 2

Consider the Fredholm integral equation of the second kind

$$\int_{-1}^{1} \frac{\vartheta(t)}{(t-x)^2}\,dt + \int_{-1}^{1} t^2e^{x}\vartheta(t)\,dt = -\pi\left(12x^2 - \frac{e^{x}}{8} - 3\right) \quad (14)$$

Whose exact solution [5] is $\vartheta(t) = \left(4t^2 - 1\right)\sqrt{1-t^2}$. Table 3 compares the exact $\vartheta(t_i)$ and obtained approximate solutions $\vartheta_n(t_i)$ for $n = 5,10,15$ and $n = 30$ at $t_i = -1{:}0.2{:}1$. Table 4 displays the absolute errors for $n = 5,10,15$ and $n = 30$. Figure 3 illustrates the graphs of the exact solution and the approximate solution for $n = 30$. Figure 4 illustrates the approximate solutions for $n = 5,10,15$ and $n = 30$. The CPU total time are 18.916 sec., 171.595 sec., 182.947 sec., and 1173.969 sec. respectively.

Example 3

Consider the Fredholm integral equation of the second kind

$$\int_{-1}^{1} \frac{\vartheta(t)}{(t-x)^2}\,dt + \int_{-1}^{1} tx\vartheta(t)\,dt = -8\pi x^3 + \frac{17\pi x}{8} - \pi \quad (15)$$

Whose exact solution [5] is $\vartheta(t) = \left(1 + 2t^3\right)\sqrt{1-t^2}$. Table 5 compares the exact $\vartheta(t_i)$ and obtained approximate solutions $\vartheta_n(t_i)$ for $n = 5,10,15$ and $n = 30$ at $t_i = -1{:}0.2{:}1$. Table 6 displays the absolute errors for $n = 5,10,15$ and $n = 30$. Figure 5 illustrates the graphs of the exact solution and the approximate solution for $n = 30$. Figure 6 illustrates the approximate solutions for $n = 5,10,15$ and $n = 30$. The CPU total time for $n = 5,10,15$ and $n = 30$ are 10.376 sec., 20.010 sec., 27.503 sec., and 87.723 sec. respectively.

Example 4

Consider the Fredholm integral equation of the second kind

$$\int_{-1}^{1} \frac{\vartheta(t)}{(t-x)^2}\,dt + \int_{-1}^{1} t^4\sin(x)\vartheta(t)\,dt = -\pi\left(5\left(16x^4 - 12x^2 + 1\right)\right) - \frac{\sin(x)}{32} \quad (16)$$

Whose exact solution [5] is $\vartheta(t) = \left(16x^4 - 12x^2 + 1\right)\sqrt{1-t^2}$. Table 7 compares the exact $\vartheta(t_i)$ and obtained approximate solutions $\vartheta_n(t_i)$ at $t_i = -1{:}0.2{:}1$ for $n = 15,25$ and $n = 35$. Table 8 displays the absolute errors for $n = 15,25$ and $n = 35$. Figure 7 illustrates the graphs of the exact solution and the approximate solution for $n = 35$. Figure 8 illustrates the approximate solutions for $n = 15,25$ and $n = 35$. The CPU total time for $n = 15,25$, and $n = 35$ are 102.516 sec., 407.342 sec., and 1063.603 sec. respectively.

IV Conclusion

The hypersingular integral equations of the first kind are solved using a novel approach based on using matrix-vector versions of second-kind Chebyshev polynomials. Two square coefficient matrices are created to obtain a continuous approximate polynomial solution. The elements of the first matrix stand for the Chebyshev polynomials' coefficients, which are ordered by increasing power. The second matrix is constructed without using the collocation method to obtain a linear algebraic system equivalent to the hypersingular equation's solution. The solutions to the four examples demonstrate the viability and efficacy of the proposed method.

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Table 1. A comparison between the exact solutions $\vartheta(t_i)$ and the obtained approximate solutions $\vartheta_n(t_i)$ of example 1 for $n=3,5,15,25,$ and $n=30$ at $t_i=-1:0.2:1$.

t_i	$\vartheta(t_i)$	$\vartheta_3(t_i)$	$\vartheta_5(t_i)$	$\vartheta_{15}(t_i)$	$\vartheta_{25}(t_i)$	$\vartheta_{30}(t_i)$
-1	0	-1.4252	-0.88459	-0.31974	-0.1962	-0.15932
-0.8	-0.5376	-0.59343	-0.54483	-0.53783	-0.53768	-0.5377
-0.6	0.5376	0.46758	0.52593	0.53681	0.53749	0.53751
-0.4	0.99717	0.95836	0.99492	0.99717	0.99711	0.99716
-0.2	0.72113	0.71558	0.71624	0.72074	0.72106	0.72114
0	0	0	0	0	0	0
0.2	-0.72113	-0.71558	-0.71624	-0.72074	-0.72106	-0.72114
0.4	-0.99717	-0.95836	-0.99492	-0.99717	-0.99711	-0.99716
0.6	-0.5376	-0.46758	-0.52593	-0.53681	-0.53749	-0.53751
0.8	0.5376	0.59343	0.54483	0.53783	0.53768	0.5377
1	0	1.4252	0.88459	0.31974	0.1962	0.15932

Table 2. The absolute errors of the obtained matrix-vector approximate solutions $\vartheta_n(t_i)$ of example 1 for $n=3,5,15,25,$ and $n=30$ at $t_i=-1:0.2:1$.

$R_3(t_i)$	$R_5(t_i)$	$R_{15}(t_i)$	$R_{25}(t_i)$	$R_{30}(t_i)$
1.4252	0.88459	0.31974	0.1962	0.15932
0.05583	0.00723	0.00023	8E-05	1E-04
0.07002	0.01167	0.00079	0.00011	9E-05
0.03881	0.00225	0	6E-05	1E-05
0.00555	0.00489	0.00039	7E-05	1E-05
0	0	0	0	0
0.00555	0.00489	0.00039	7E-05	1E-05
0.03881	0.00225	0	6E-05	1E-05
0.07002	0.01167	0.00079	0.00011	9E-05
0.05583	0.00723	0.00023	8E-05	1E-04
1.4252	0.88459	0.31974	0.1962	0.15932

Table 3. A comparison between the exact solutions $\vartheta(t_i)$ and the obtained matrix-vector approximate solutions $\vartheta_n(t_i)$ of example 2 for $n=5,10,15,$ and $n=30$ at $t_i=-1:0.2:1$.

t_i	$\vartheta(t_i)$	$\vartheta_5(t_i)$	$\vartheta_{10}(t_i)$	$\vartheta_{15}(t_i)$	$\vartheta_{30}(t_i)$
-1	0	0.65555	0.32058	0.23968	0.11948
-0.8	0.936	0.91799	0.93747	0.93508	0.93601
-0.6	0.352	0.35716	0.35235	0.35204	0.35201
-0.4	-0.32995	-0.32798	-0.32958	-0.32987	-0.32993
-0.2	-0.82303	-0.81697	-0.82199	-0.82253	-0.82303
0	-1	-0.98907	-1.0001	-1.0001	-1
0.2	-0.82303	-0.81597	-0.82187	-0.82249	-0.82303
0.4	-0.32995	-0.32608	-0.32936	-0.32978	-0.32992
0.6	0.352	0.35969	0.35264	0.35216	0.35203
0.8	0.936	0.9205	0.93777	0.9352	0.93602
1	0	0.65676	0.32065	0.2397	0.11949

Table 4. The absolute errors of the obtained matrix-vector approximate solutions $\vartheta_n(t_i)$ of example 2 for $n=5,10,15,$ and $n=30$ at $t_i=-1:0.2:1$.

$R_5(t_i)$	$R_{10}(t_i)$	$R_{15}(t_i)$	$R_{30}(t_i)$
0.65555	0.32058	0.23968	0.11948
0.01801	0.00147	0.00092	1E-05
0.00516	0.00035	4E-05	1E-05
0.00197	0.00037	8E-05	2E-05
0.00606	0.00104	0.0005	0
0.01093	1E-04	1E-04	0
0.00706	0.00116	0.00054	0
0.00387	0.00059	0.00017	3E-05
0.00769	0.00064	0.00016	3E-05
0.0155	0.00177	0.0008	2E-05
0.65676	0.32065	0.2397	0.11949

Table 5. A comparison between the exact solutions $\vartheta(t_i)$ and the obtained approximate solutions $\vartheta_n(t_i)$ of example 3 for $n=5,10,15,$ and $n=30$ at $t_i=-1:0.2:1$.

t_i	$\vartheta(t_i)$	$\vartheta_5(t_i)$	$\vartheta_{10}(t_i)$	$\vartheta_{15}(t_i)$	$\vartheta_{30}(t_i)$
-1	0	-0.21859	-0.10686	-0.0799	-0.03983
-0.8	-0.0144	-0.01477	-0.0145	-0.01442	-0.0144
-0.6	0.4544	0.45617	0.45429	0.45422	0.45439
-0.4	0.7992	0.80373	0.79938	0.79947	0.79923
-0.2	0.96412	0.96226	0.96351	0.96379	0.96416
0	1	0.99434	1.0007	1.0003	1
0.2	0.99547	0.99413	0.99491	0.99516	0.99551
0.4	1.0338	1.0409	1.0342	1.0342	1.0339
0.6	1.1456	1.1525	1.1456	1.1453	1.1456
0.8	1.2144	1.1889	1.2161	1.2131	1.2144
1	0	0.65513	0.32055	0.23967	0.11948

Table 6. The absolute errors of the obtained matrix-vector approximate solutions $\vartheta_n(t_i)$ of example 3 for $n=5,10,15,$ and $n=30$ at $t_i=-1:0.2:1$.

$R_5(t_i)$	$R_{10}(t_i)$	$R_{15}(t_i)$	$R_{30}(t_i)$
0.21859	0.10686	0.079895	0.039828
0.000374	0.0001	1.8E-05	4E-06
0.00177	0.00011	0.00018	1E-05
0.00453	0.00018	0.00027	3E-05
0.00186	0.00061	0.00033	4E-05
0.00566	0.0007	0.0003	0
0.00134	0.00056	0.00031	4E-05
0.0071	0.0004	0.0004	1E-04
0.0069	0	0.0003	0
0.0255	0.0017	0.0013	0
0.65513	0.32055	0.23967	0.11948

Table 7. A comparison between the exact solutions $\vartheta(t_i)$ and the obtained matrix-vector approximate solutions $\vartheta_n(t_i)$ of example 4 for $n=15,25,$ and $n=35$ at $t_i=-1:0.2:1$.

t_i	$\vartheta(t_i)$	$\vartheta_{15}(t_i)$	$\vartheta_{25}(t_i)$	$\vartheta_{35}(t_i)$
-1	0	0.39869	0.24476	0.17664
-0.8	-0.07584	-0.08047	-0.08048	-0.08048
-0.6	-0.99712	-1.0016	-1.002	-1.0019
-0.4	-0.46779	-0.47172	-0.47151	-0.47148
-0.2	0.53458	0.53238	0.53255	0.53259
0	1	1.0003	0.99993	1
0.2	0.53458	0.53643	0.53656	0.53658
0.4	-0.46779	-0.46421	-0.46409	-0.46408
0.6	-0.99712	-0.99197	-0.99241	-0.99234
0.8	-0.07584	-0.07105	-0.07117	-0.07119
1	0	0.40021	0.24568	0.17731

Table 8. The absolute errors of the obtained matrix-vector approximate solutions $\vartheta_n(t_i)$ of example 4 for $n=15,25,$ and $n=35$ at $t_i=-1:0.2:1$.

$R_{15}(t_i)$	$R_{25}(t_i)$	$R_{35}(t_i)$
0.39869	0.24476	0.17664
0.004628	0.004642	0.004644
0.00448	0.00488	0.00478
0.00393	0.00372	0.00369
0.0022	0.00203	0.00199
0.0003	7E-05	0
0.00185	0.00198	0.002
0.00358	0.0037	0.00371
0.00515	0.00471	0.00478
0.004795	0.004674	0.004653
0.40021	0.24568	0.17731

Table 9. Comparison CPU time between different examples 1,2,3, and 4.

Examples	$\phi(x)$	$k_1(x,t)$	$k_2(x,t)$	CPU time (sec)
Example 1 for $n=3,5,15,25$ and $n=30$	$-\pi\left(31x^3-16x\right)$	$\frac{1}{(t-x)^2}$	t^3x^3	7.883, 9.679, 24.489,50.548 and 68.498
Example 2 for $n=5,10,15$ and $n=30$	$-\pi\left(12x^2-\frac{e^x}{8}-3\right)$	$\frac{1}{(t-x)^2}$	t^2e^x	18.916, 171.595, 182.947, and 1173.969
Example 3 for $n=5,10,15$ and $n=30$	$-8\pi x^3+\frac{17\pi x}{8}-\pi$	$\frac{1}{(t-x)^2}$	tx	10.376, 20.010, 27.503, and 87.723
Example 4 for $n=15,25$ and $n=35$	$-\pi\left(5\left(16x^4-12x^2+1\right)\right)-\frac{\sin(x)}{32}$	$\frac{1}{(t-x)^2}$	$t^4\sin(x)$	102.516, 407.342, and 1063.603

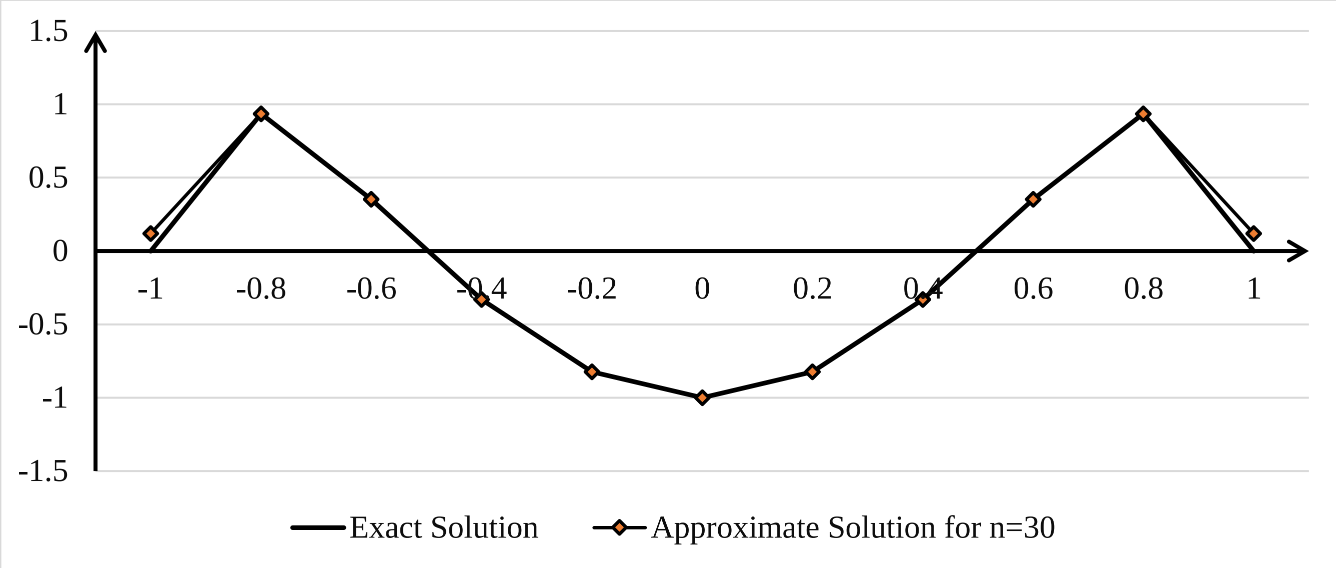

Fig. 1. The exact solution and the approximate solution for $n=30$.

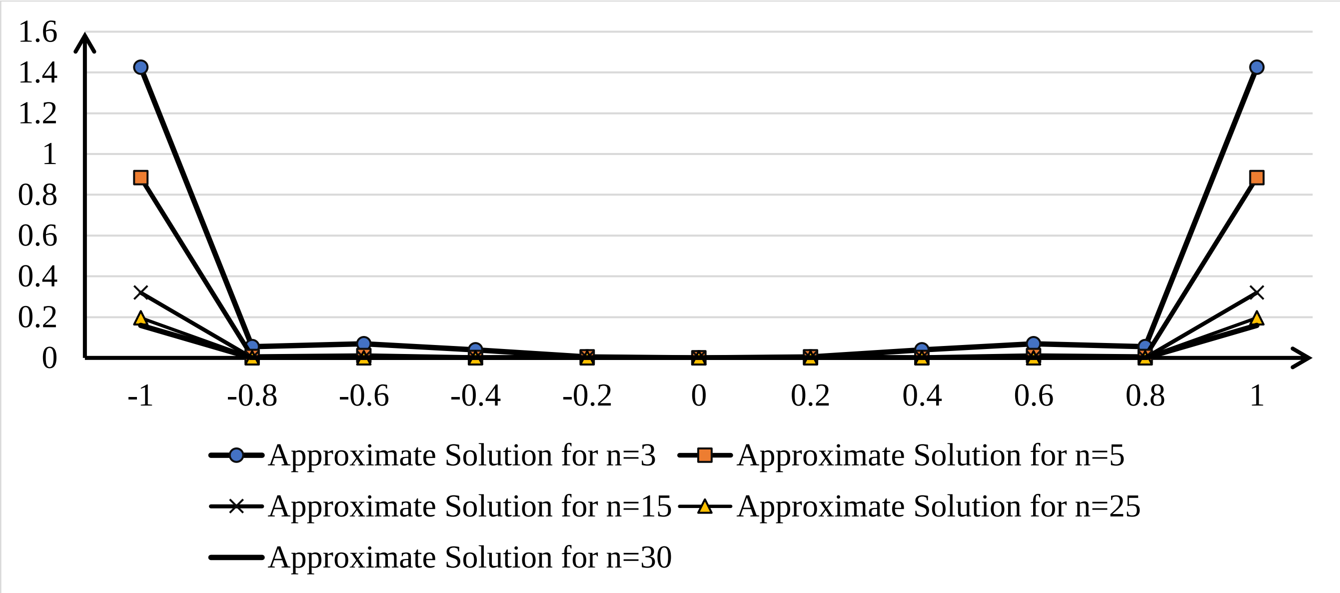

Fig. 2. The approximate solutions for $n=3,5,15,25,$ and $n=30$.

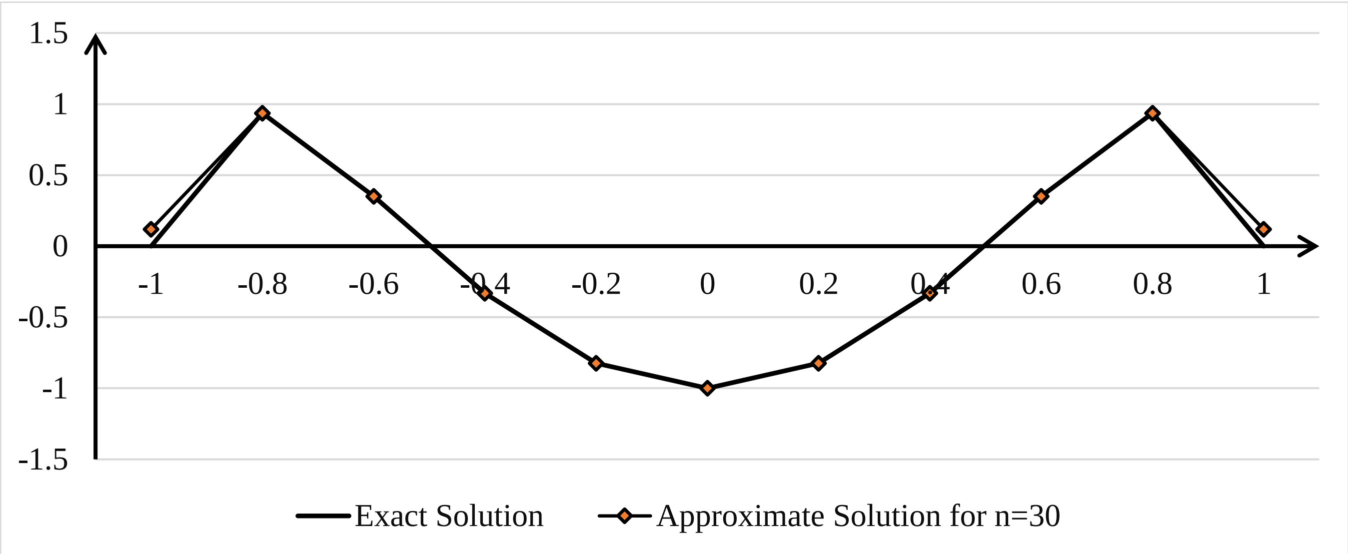

Fig. 3. The exact solution and the approximate solution for $n=30$.

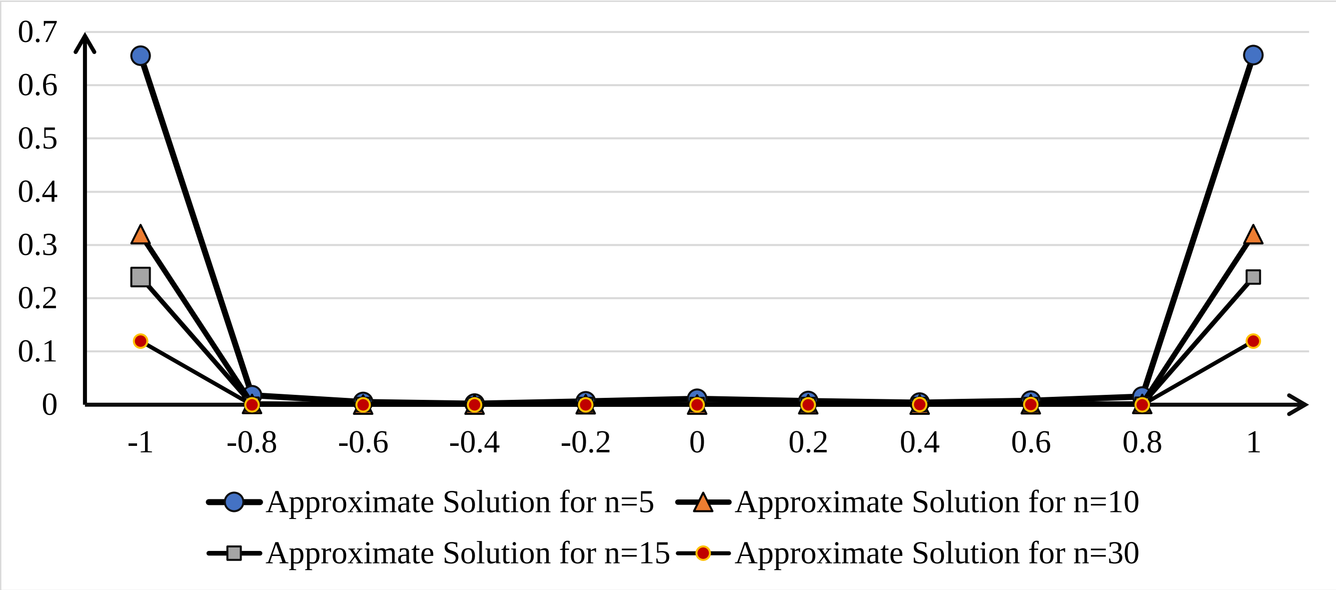

Fig. 4. The approximate solutions for $n=5,10,15,$ and $n=30$.

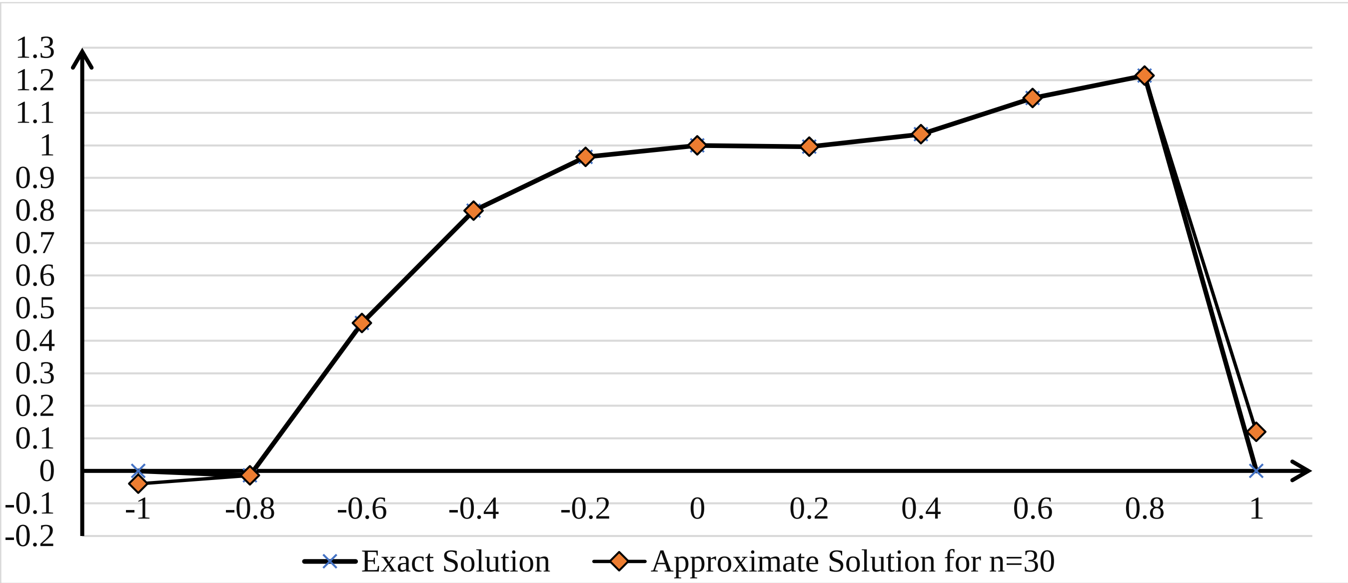

Fig. 5. The exact solution and the approximate solution for $n=30$.

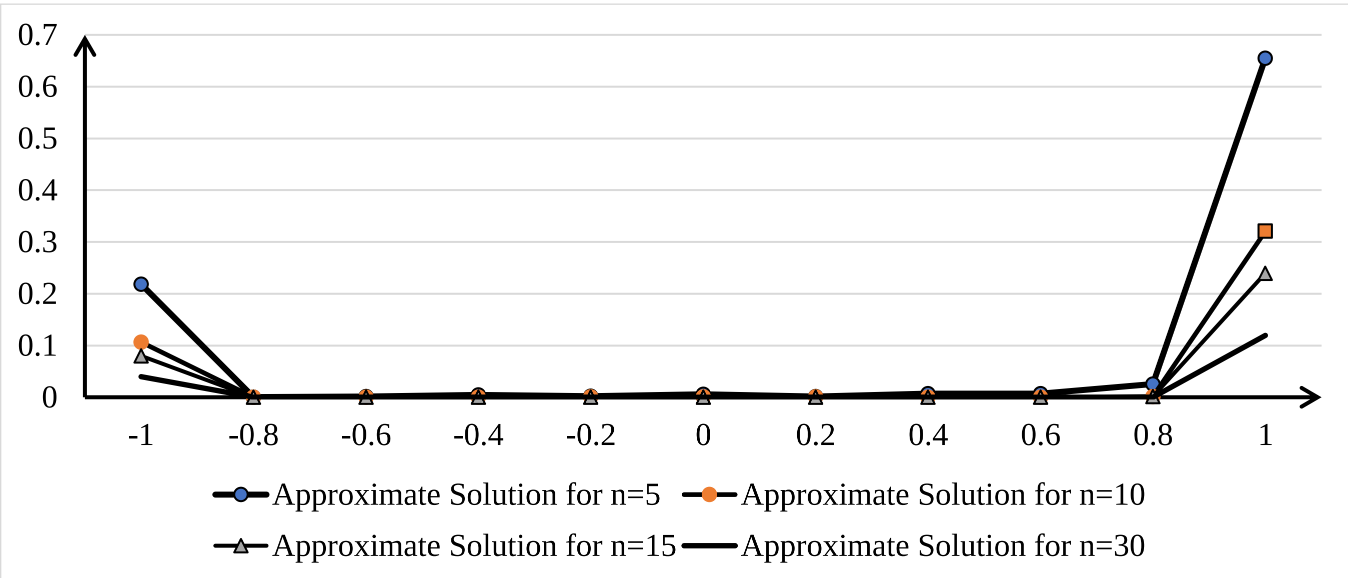

Fig. 6. The approximate solutions for $n=5,10,15,$ and $n=30$.

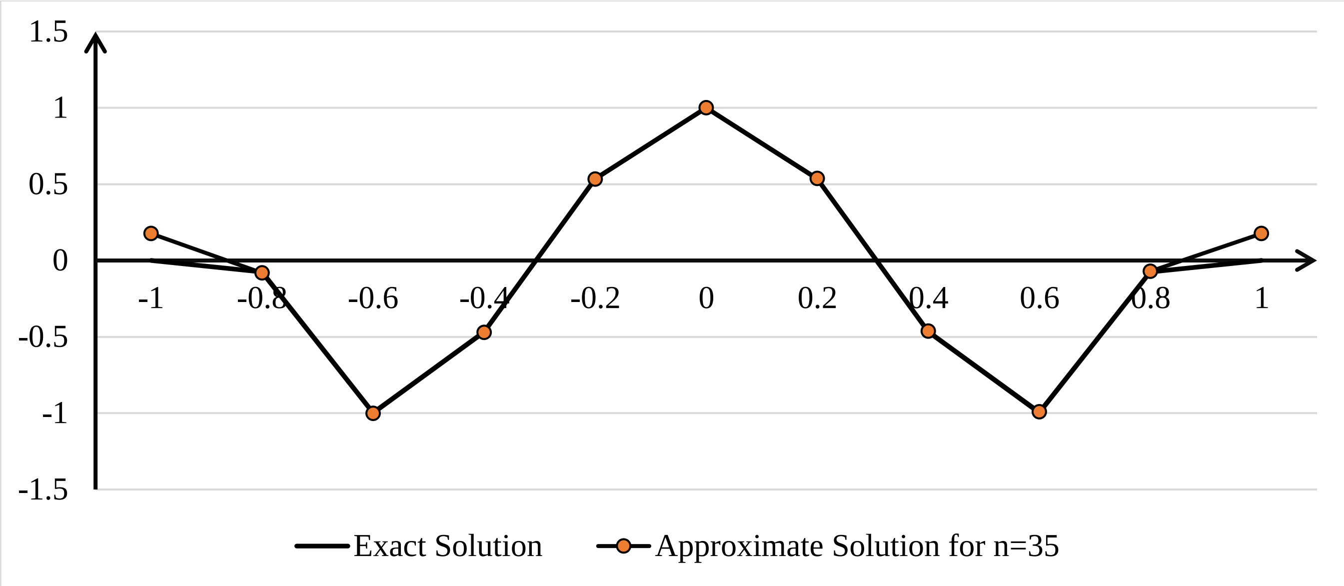

Fig. 7. The exact solution and the approximate solution for $n=35$.

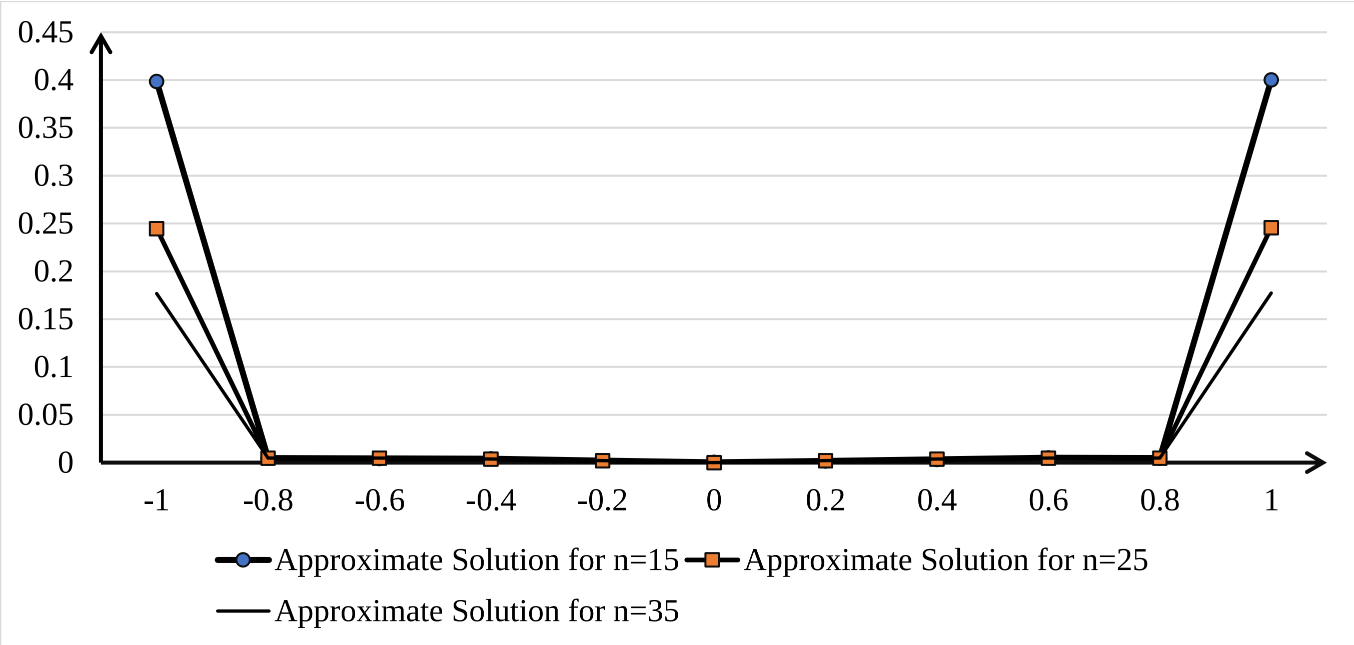

Fig. 8. The approximate solutions for $n=15,25,$ and $n=35$.