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Optimal Control and Qualitative Analysis of a Microbial Insecticide Model

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Abstract—**The use of entomopathogenic nematodes in agriculture is not only an environmentally friendly option, but also a sustainable pest management strategy that can help promote the development of ecologically friendly agriculture. The mathematical model of the biological control factors of insect pathogenic nematodes with environmental tolerance of pests is proposed and studied in this paper. Under certain assumptions, the existence and types of possible equilibria are discussed. In particular, it is found that when the system has a unique positive equilibrium, it is an interesting degenerate point, which is degenerate cusp with codimension 1 or the saddle-node depending on the value of parameters. The saddle-node bifurcation is further explored by use of the Sotomayor's theorem. The existence of a limit cycle is proved when the system has two equilibria. Finally, we propose the optimal control. While reducing the number of pests, the use of pesticides should be minimized and the total cost is the lowest.**

Index Terms—**optimal control, limit cycle, bifurcation analysis, microbial insecticide model.**

I. INTRODUCTION

CROP pest control is an important technical measure for high and stable production. For a long time, chemical pesticides have been used as an effective means of controlling pests and ensuring crop yields due to their high efficiency and rapid killing effects. However, with the widespread use of chemical pesticides, a series of potential dangers and disadvantages, such as increased pesticide residues, soil and water pollution, accelerated pest resistance development and resurgence of pests, have become increasingly apparent. In order to meet the growing demand of human beings for a good ecological environment and food safety, it is imperative to develop new pest control measures.

Developing and applying biological control technology can not only protect natural enemies but also reduce the selective pressure of pesticides, build effective ways for modern agricultural development and ecological governance, and promote sustainable and high-quality agricultural development. Deputy Director of the Sichuan Academy of Agricultural Sciences, Professor Ren Guangjun said that insect pathogenic nematodes are specialized parasitic natural enemies of insects, and are microbial insecticides with the characteristics of both natural enemies and pathogenic microorganisms. They are an important factor in insect pest biological control and can efficiently control harmful organisms while being safe for non-target organisms and the environment. Therefore, they have great potential for sustainable pest control.

The entomopathogenic nematode and their symbiotic bacteria are a golden pair that neither can do without the other. The two good brothers support and cooperate with each other to explore the world and defeat enemies. The symbiotic bacteria cannot survive alone in the soil but exist in the intestines of the nematode. The nematode can protect the symbiotic bacteria from the adverse effects of the external environment. As a carrier, the nematode carries the symbiotic bacteria into the target insect and then releases them into the blood cavity of the host insect. During this process, the nematode also needs to protect the symbiotic bacteria from being recognized by the host's immune response. So, under the protection of its good brother, the symbiotic bacteria began to show their capabilities. They multiplied in large numbers in the insect's blood cavity and produced antibacterial substances and toxins, causing the insect to die from sepsis. Symbiotic bacteria provide nutrient sources and a favorable environment for the growth, development and reproduction of nematodes by secreting various extracellular enzymes and antibiotics. Thus, after several generations of reproduction, the new nematode-symbiotic bacteria complex emerges in a free state from the carcass of the host insect, once again searching for a new insect host and embarking on a new journey.

As the use of insect pathogenic nematodes becomes more widespread, some studies suggest that nematodes play a role in regulating insect populations and suppressing pests in nature, and their commercial importance is increasing, second only to bacteria. The biggest difference between bacteria and nematodes is that bacteria cannot move, but nematodes can. Famous bacterial insect-killing bacteria such as Bacillus thuringiensis cannot move, so they can only kill insects in the sprayed area, and insects outside the sprayed area cannot be killed; nematodes can move, so they can actively attack their hosts and can kill insects in a wider range.

Based on the advantages of insect pathogenic nematodes in biological pest control, many scholars have conducted extensive research and discussion on their control mechanisms and reproduction in recent years [1]-[13]. We studied insect pathogenic nematodes through mathematical modeling based on their feeding characteristics, and obtained certain research results [14]-[17]. In this paper, we consider the case where insects have an environmental carrying capacity and establish the following mathematical model

$$\begin{cases} \dfrac{dx}{dt} = rx\left(1-\dfrac{x}{k}\right) - \alpha x I, \\[2ex] \dfrac{dI}{dt} = \alpha x I^2 - \beta I, \end{cases} \tag{1}$$

where $x(t)$ and $I(t)$ denote the densities of pests and the

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entomopathogenic nematode at time t, respectively; $r > 0$ and $k > 0$ describe the intrinsic growth rate and the carrying capacity of pests, respectively; $\alpha > 0$ is the attachment rate of the entomopathogenic nematode, $\beta > 0$ is the death rate of the entomopathogenic nematode.

II. ANALYSIS OF EQUILIBRIA OF SYSTEM (1)

According to the practical significance of system (1), we only consider region $x \geq 0, I \geq 0$. It is easy to see that system (1) takes the straight line $x = 0$ (I-axis) and $I = \frac{r}{\alpha}\left(1 - \frac{x}{k}\right)$ as the vertical isoclines and the straight line $I = 0$ (x-axis) and a hyperbola $I = \frac{\beta}{\alpha} \cdot \frac{1}{x}$ as the horizontal isoclines. Among them, the hyperbola $y = \frac{\beta}{\alpha} \cdot \frac{1}{x}$ has the straight line I-axis and x-axis as the straight and horizontal asymptotes, respectively. And the right branch of the hyperbola intersects with the straight line $I = \frac{r}{\alpha}\left(1 - \frac{x}{k}\right)$ at one point, two points or not at all within the first quadrant, entirely depending on the range of values for r, β, k. This leads to the situation that in system (1), except for the coordinate origin $O(0, 0)$ and the point $A(k, 0)$ as the equilibria, one, two positive equilibria or no positive equilibrium may occur as the values of r, β, k change.

By calculating the positive equilibrium of the system (1), it is obtained the following theorem.

Theorem 2.1 Let $\Delta = r\left(r - \frac{4\beta}{k}\right)$, then

(i) When $\Delta > 0$, i.e. $r > \frac{4\beta}{k}$, system (1) has two positive equilibria $N_i\left(x_i, I_i\right)(i = 1, 2)$:

$$x_{1,2} = \frac{k}{2}\left(1 \pm \sqrt{1 - \frac{4\beta}{kr}}\right)$$

and

$$I_{1,2} = \frac{r}{2\alpha}\left(1 \mp \sqrt{1 - \frac{4\beta}{kr}}\right),$$

where x_i is the two positive real roots of the algebraic quadratic equation $rx^2 - krx + \beta k = 0$.

(ii) When $\Delta = 0$, i.e. $r = \frac{4\beta}{k}$, system (1) has only one positive equilibrium $M\left(\frac{k}{2}, \frac{2\beta}{k\alpha}\right)$.

(iii) There is no positive equilibrium for system (1) for all other values of r, β, k except (i) and (ii).

Regarding the structure of the type of equilibrium of system (1) in the first quadrant, it is easy to determine that the coordinate origin O is a saddle and the boundary equilibrium A is a stable node. The phase protrait is given in Fig.1.

Theorem 2.2 When $\Delta = 0$ (i.e., $r = \frac{4\beta}{k}$) and $k \neq 2$, the positive equilibrium $M\left(\frac{k}{2}, \frac{2\beta}{k\alpha}\right)$ of system (1) is a saddle-node. When $k = 2$, M is a degenerate cusp.

Proof Due to

$$q = \begin{vmatrix} -\dfrac{2\beta}{k} & -\dfrac{k\alpha}{2} \\ \dfrac{4\beta^2}{k^2\alpha} & \beta \end{vmatrix} = 0, \quad p = -\left(\beta - \frac{2\beta}{k}\right), \tag{2}$$

it is easy to find the characteristic roots as $\lambda_1 = 0, \lambda_2 = \beta - \frac{2\beta}{k}$.

When $k \neq 2$, it is obtained that $\lambda_1 = 0, \lambda_2 = \beta - \frac{2\beta}{k} \neq 0$. Therefore, system (1) belongs to the case studied by Theorem 4.10 in the literature [18].

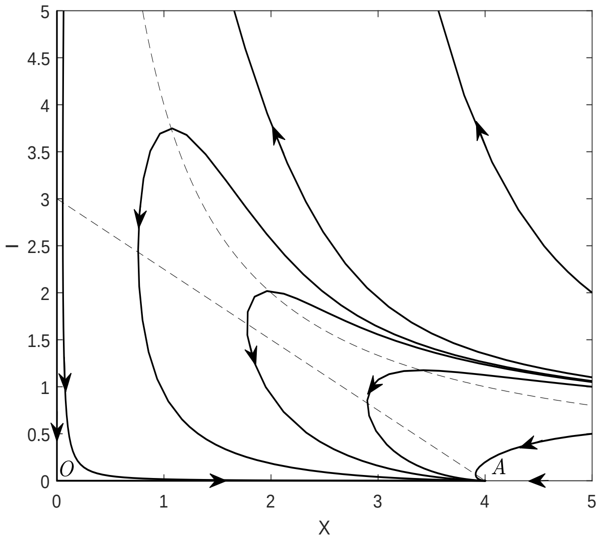

Fig. 1: Two boundary equilibria: O is a saddle and A is a stable node. The parameters are taken as $r = 3, k = 4, \alpha = 1$ and $\beta = 4$, respectively.

Making the substitution $\bar{x} = x - \frac{k}{2}, \bar{I} = I - \frac{2\beta}{k\alpha}$ reduces the system (1) to:

$$\begin{cases} \dfrac{d\bar{x}}{dt} = -\dfrac{2\beta}{k}\bar{x} - \dfrac{k\alpha}{2}\bar{I} - \dfrac{r}{k}\bar{x}^2 - \alpha\bar{x}\bar{I}, \\ \dfrac{d\bar{I}}{dt} = \dfrac{4\beta^2}{k^2\alpha}\bar{x} + \beta\bar{I} + \dfrac{k\alpha}{2}\bar{I}^2 + \dfrac{4\beta}{k}\bar{x}\bar{I} + \alpha\bar{x}\bar{I}^2. \end{cases} \tag{3}$$

It is not difficult to find non-singular linear transformations

$$\begin{cases} \bar{x} = -\dfrac{k^2\alpha}{4\beta}\xi - \dfrac{k\alpha}{2\beta}\eta, \\ \bar{I} = \xi + \eta, \end{cases} \tag{4}$$

and the time transformation $d\tau = \frac{\beta(k-2)}{k}dt$, reducing (3) to the form required by Theorem 4.10.

$$\begin{cases} \dfrac{d\xi}{d\tau} = & \frac{k^2\alpha}{\beta(k-2)^2}\xi^2 + \frac{k(6-k)\alpha}{\beta(k-2)^2}\xi\eta - \frac{\left(k^2-2k-4\right)\alpha}{\beta(k-2)^2}\eta^2 \\ & + \frac{k^3\alpha^2}{2\beta^2(k-2)^2}\xi^3 + \frac{(k+1)k^2\alpha^2}{\beta^2(k-2)^2}\xi^2\eta + \frac{(k+4)k^2\alpha^2}{2\beta^2(k-2)^2}\xi\eta^2 \\ & + \frac{k^2\alpha^2}{\beta^2(k-2)}\eta^3 \triangleq \Phi\left(\xi, \eta\right), \\ \dfrac{d\eta}{d\tau} = & \eta - \frac{k^3\alpha}{2\beta(k-2)^2}\xi^2 - \frac{k\alpha(k+2)}{\beta(k-2)^2}\xi\eta \\ & + \frac{\left(k^3-4k^2+4k-8\right)\alpha}{2\beta(k-2)^2}\eta^2 - \frac{k^4\alpha^2}{4\beta^2(k-2)^2}\xi^3 \\ & - \frac{k^3\alpha^2(k+1)}{2\beta^2(k-2)^2}\xi^2\eta - \frac{k^3\alpha^2(k+4)}{4\beta^2(k-2)^2}\xi\eta^2 \\ & - \frac{k^3\alpha^2}{2\beta^2(k-2)^2} \triangleq \eta + \Psi\left(\xi, \eta\right). \end{cases} \tag{5}$$

Solve for $\eta = \eta\left(\xi\right)$ from the equation $\eta + \Psi\left(\xi, \eta\right) = 0$ and substitute into $\Phi(\xi,\eta)$ to get

$$\Phi(\xi,\eta(\xi)) = \frac{k^2\alpha}{\beta(k-2)^2}\xi^2 + o\left(\xi^3\right).$$

By comparison with Theorem 4.10 in literature [18], it can be known that $m = 2$ and $g = \frac{k^2\alpha}{\beta(k-2)^2} > 0 (k \neq 2)$. Therefore, the point $(0,0)$ of system (3) is a saddle-node, and thus $M\left(\frac{k}{2}, \frac{2\beta}{k\alpha}\right)$ is known to be a saddle-node (here $k \neq 2$). The phase portrait is given in Fig.2.

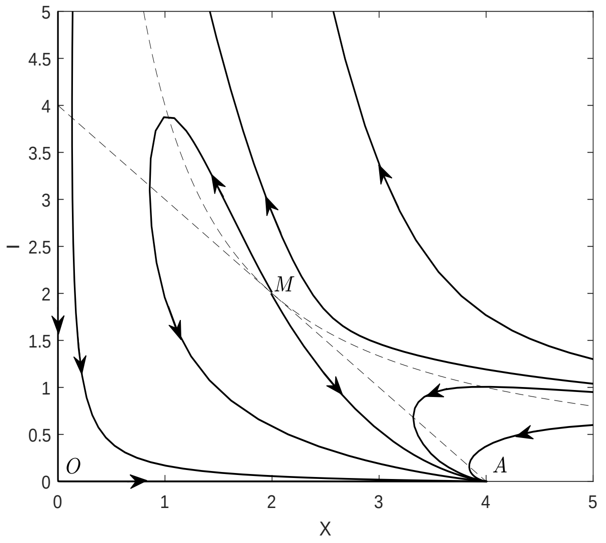

Fig. 2: Two boundary equilibria and a positive equilibrium: O is a saddle, A is a stable node and M is a saddle-node. The parameters are taken as $r = 4, k = 4, \alpha = 1$ and $\beta = 4$, respectively.

When $k = 2$, we have $\lambda_1 = \lambda_2 = 0$. System (1) belongs to the case studied by Theorem 4.1 in the literature [19].

Let $\hat{x} = x - 1, \hat{I} = I - \frac{\beta}{\alpha}$, System (1) is transformed into (still noting x, I in terms of $\hat{x}, \hat{I}$):

$$\begin{cases} \dfrac{dx}{dt} = -\beta x - \alpha I - \dfrac{r}{2}x^2 - \alpha x I, \\ \dfrac{dI}{dt} = \dfrac{\beta^2}{\alpha}x + \beta I + \alpha I^2 + 2\beta x I + \alpha x I^2. \end{cases} \tag{6}$$

Then let $x_1 = x, I_1 = -\beta x - \alpha I$, the system (6) reduces to:

$$\begin{cases} \dfrac{dx_1}{dt} = I_1 + \left(\beta - \dfrac{r}{2}\right)x_1^2 + x_1 I_1, \\ \dfrac{dI_1}{dt} = \dfrac{\beta r}{2}x_1^2 - I_1^2 - \beta x_1 I_1 - \beta^2 x_1^3 - 2\beta x_1^2 I_1 - x_1 I_1^2. \end{cases} \tag{7}$$

When $k = 2$, $\Delta = r^2 - 2\beta r = 0$, that is $\beta = \dfrac{r}{2}$. Substituting into the above equation, we have

$$\begin{cases} \dfrac{dx_1}{dt} = I_1 + x_1 I_1, \\ \dfrac{dI_1}{dt} = \beta^2 x_1^2 - I_1^2 - \beta x_1 I_1 - \beta^2 x_1^3 - 2\beta x_1^2 I_1 - x_1 I_1^2. \end{cases} \tag{8}$$

Making the substitution $x_2 = x_1, I_2 = I_1 + x_1 I_2$ changes (8) to:

$$\begin{cases} \dfrac{dx_2}{dt} = I_2, \\ \dfrac{dI_2}{dt} = \beta^2 x_2^2\left(1 - x_2^2\right) - \beta x_2 I_2\left(1 + 2x_2\right) - \dfrac{x_2}{1 + x_2}I_2^2 \\ \qquad \triangleq Q_2\left(x_2, I_2\right), \end{cases} \tag{9}$$

where $h(x_2) = -x_2^2, g(x_2) = 2x_2, f(x_2, I_2) = -\dfrac{x_2}{1 + x_2}$ are analytic in the first quadrant, $h(0) = g(0) = 0$, $r = 2$, $a_r = \beta^2 \neq 0$ and $n = 1$ are natural number. From Theorem 4.1 of Chapter 7 of the literature [19], it follows that $(0,0)$ of system (9) is a dual singularity. Also since $b_n = -\beta \neq 0$, $n = m = 1$, then $(0,0)$ of (9) is a degenerate singularity. Therefore, the positive equilibrium $M\left(1, \frac{\beta}{\alpha}\right)$ of system (1) is a degenerate cusp when $\Delta = 0$ (i.e. $r = \frac{4\beta}{k}$) and $k = 2$. The phase portrait is given in Fig.3.

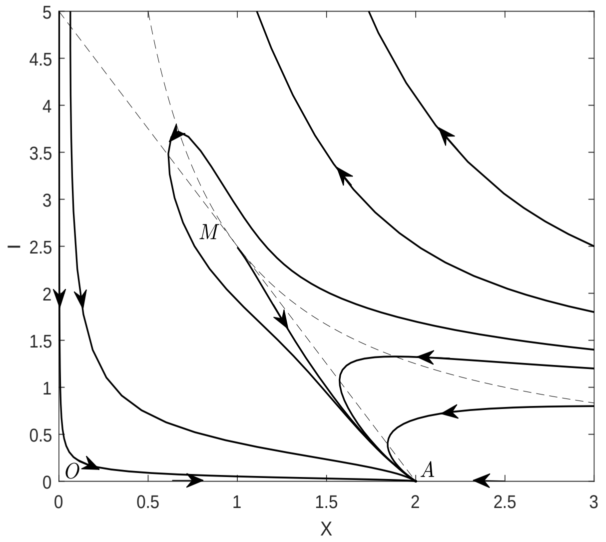

Fig. 3: Two boundary equilibria and a positive equilibrium: O is a saddle, A is a stable node and M is a degenerate cusp. The parameters are taken as $r = 5, k = 2, \alpha = 1$ and $\beta = 2.5$, respectively.

Theorem 2.3 When $\Delta > 0$ (i.e., $r > \frac{4\beta}{k}$) and $k \geq 2$, the two positive equilibria $N_1(x_1, I_1)$ of system (1) is a saddle and $N_2(x_2, I_2)$ is an unstable focus or node.

Proof Because at the equilibrium $N_i(x_i, I_i)(i = 1, 2)$, the jacobian matrix of system (1) is

$$J\left(N_i\right) = \begin{pmatrix} r\left(1 - \dfrac{x_i}{k}\right) - \dfrac{r}{k}x_i - \alpha I_i & -\alpha x_i \\ \alpha I_i^2 & 2\alpha x_i I_i - \beta \end{pmatrix}. \tag{10}$$

Since $detJ\left(N_1\right) = -\beta\sqrt{r\left(r - \frac{4\beta}{k}\right)} < 0$, N_1 is a saddle. $DetJ\left(N_2\right) = \beta\sqrt{r\left(r - \frac{4\beta}{k}\right)} > 0$ and $Tr\left(N_2\right) = x_2\frac{(k-2)r + kr\sqrt{1 - \frac{4\beta}{k}}}{2k} > 0 (k \geq 2)$, so N_2 is an unstable focus or node. The phase portrait is given in Fig.4.

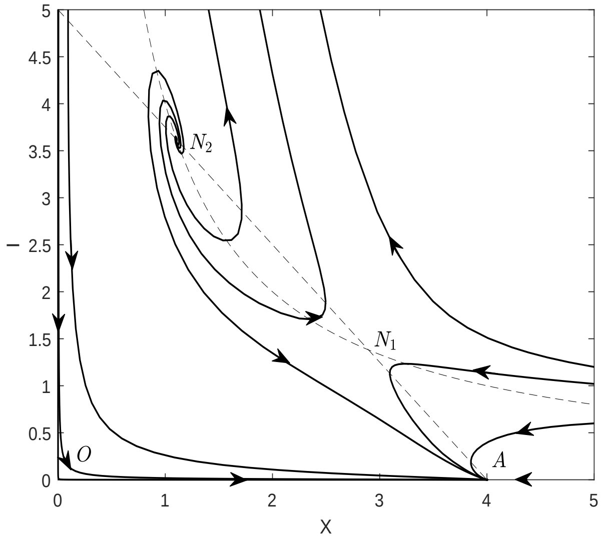

Fig. 4: Two boundary equilibria and two positive equilibria: O is a saddle, A is a stable node, N_1 is a saddle and N_2 is an unstable focus or node. The parameters are taken as $r = 5, k = 4, \alpha = 1$ and $\beta = 4$, respectively.

III. THE EXISTENCE OF LIMIT CYCLES

Theorem 3.1 When $r > \frac{4\beta}{k}$ and $k \neq 2$, system (1) has one limit cycle around the positive equilibrium $N_2\left(x_2, I_2\right)$.

Proof Under the condition of the theorem, the positive equilibrium $N_2\left(x_2, I_2\right)$ is an unstable focus or node, and $N_1\left(x_1, I_1\right)$ is a saddle. Construct a bounded region containing $N_2\left(x_2, I_2\right)$, whose boundary is bounded by N_1DCBN_1, as shown in Figure 5. Assume that L_1 and L_2 are the two dividing lines of saddle $N_1\left(x_1, I_1\right)$. Depending on the direction of the vector field of system (1), L_1 must be above the isocline $r(1-\dfrac{x}{k})-\alpha I=0$ and L_2 must be below the isocline $\alpha xI-\beta=0$.

Examine the line $L_3: I+bx-c=0$, where $b>0, c>0$, there is

$$\frac{dL_3}{dt}|_{L_3=0} = \alpha b^2x^3 + (-2bc\alpha - \frac{br}{k} + b^2\alpha)x^2 + (\alpha c^2 + \beta b + br - bc\alpha)x - \beta c.$$

Let

$$G(x) = \alpha b^2x^3 + (-2bc\alpha - \frac{br}{k} + b^2\alpha)x^2 + (\alpha c^2 + \beta b + br - bc\alpha)x - \beta c.$$

When $x = 0, G(x) = -\beta c < 0$, $x \to +\infty, G(x) \to +\infty$, $x \to -\infty, G(x) \to -\infty$, the cubic curve $G(x) = 0$ may have one, two or three positive roots. If $G(x) = 0$ has only one positive root, then let it be $\bar{x}$. If there are two or three positive roots, then let the smallest of them be $\bar{x}$. Let $0 < x' < min\{\bar{x}, x_2\}$, then $f(x') < 0$, that is $\frac{dL_3}{dt}|_{L_3=0} < 0$. Point D is the intersection of the line L_3 and L_1, and point C is the intersection of L_3 and the curve $\alpha xI - \beta = 0$.

For the line $L_4: I - I' = 0$, as t increases, we have

$$\frac{dL_4}{dt}|_{L_4=0} = I'(\alpha xI' - \beta) < 0(x < x').$$

Point B is the intersection point of the negative direction of L_2 and L_4.

It can be seen from this that on BC and CD, when t increases, the direction in which the trajectory of system (1) passes is as shown in Fig.5, and BN_1 and N_1D are also trajectories. That is to say, on the outer boundary line of the circular domain, the trajectory of system (1) all point from the outside to the inside towards the positive equilibrium $N_2\left(x_2, I_2\right)$ when t increases or part of the outer boundary line is trajectory. So, by Poincare-Bendixson theorem [20], it is known that there is at least one limit cycle around $N_2\left(x_2, I_2\right)$.

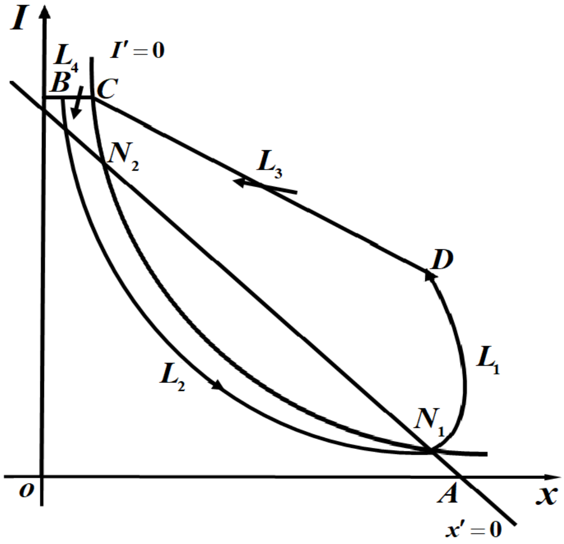

Fig. 5: Bounded domain.

IV. BIFURCATION

Theorem 4.1 Suppose that $\Delta = 0$ and $k \neq 2$, then the system (1) undergoes the saddle-node bifurcation at the threhold $r = r^*$, here $r^* = \frac{4\beta}{k}$.

Proof We will verify the transversality condition for the occurrence of saddle-node bifurcation at $r = r^*$ by use of Sotomayor's theorem [21]. Since $det(J_M) = \lambda_1\lambda_2 = 0$, J_M has one zero eigenvalue, that is λ_1. Let Q_1 and Q_2 be eigenvectors corresponding to the eigenvalue λ_1 for the matrices J_M and J_M^T, respectively. We can obtain

$$Q_1 = \begin{pmatrix} Q_{11} \\ Q_{12} \end{pmatrix} = \begin{pmatrix} 1 \\ -\dfrac{4\beta}{k\alpha} \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} Q_{21} \\ Q_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ \dfrac{k\alpha}{2\beta} \end{pmatrix}.$$

In addition, we get

$$F_r(M, r^*) = \begin{pmatrix} x - \dfrac{x^2}{k} \\ 0 \end{pmatrix}_{(M,r^*)} = \begin{pmatrix} \dfrac{k}{4} \\ 0 \end{pmatrix},$$

$$D^2F(M, r^*)(Q_1, Q_1) =$$

$$\begin{pmatrix} \dfrac{\partial^2 F_1}{\partial x^2}Q_{11}^2 + 2\dfrac{\partial^2 F_1}{\partial x\partial I}Q_{11}Q_{12} + \dfrac{\partial^2 F_1}{\partial I^2}Q_{12}^2 \\ \dfrac{\partial^2 F_2}{\partial x^2}Q_{11}^2 + 2\dfrac{\partial^2 F_2}{\partial x\partial I}Q_{11}Q_{12} + \dfrac{\partial^2 F_2}{\partial I^2}Q_{12}^2 \end{pmatrix}_{(M,r^*)} = \begin{pmatrix} 0 \\ \dfrac{16\beta^2(k-2)}{k^2\alpha} \end{pmatrix},$$

where $F_1(x,I) = rx\left(1-\frac{x}{k}\right) - \alpha xI, F_2(x,I) = \alpha xI^2 - \beta I$.

We can calculate that Q_1 and Q_2 gratify the transversality conditions

$$Q_2^T F_r(M,r^*) = \frac{k}{4} \neq 0,$$

$$Q_2^T[D^2F(M,r^*)(Q_1,Q_1)] = \frac{8\beta(k-2)}{k} \neq 0,$$

in the case of saddle-node bifurcation at M when $r = r^*$. Accordingly, when the parameter r change, the equilibrium of system (1) changes from one to two.

V. OPTIMAL CONTROL

In this section, we present the optimal control problem, which consists of exploring how to actually spray pesticides over a certain period of time to reduce the number of pests at the lowest intervention cost. On the basis of system (1), we consider adding time-dependent interventions, that is, spraying pesticides $u_1(t)$. Spraying pesticides will kill pests, thereby reducing the number of pests. Then system (1) become

$$\begin{cases} \dfrac{dx}{dt} = rx\left(1-\dfrac{x}{k}\right) - \alpha xI - u_1(t)x, \\ \dfrac{dI}{dt} = \alpha xI^2 - \beta I, \end{cases} \tag{11}$$

where $u_1 \in \Omega$ is the control strategy variable and is a bounded Lebesgue production function, where the set Ω satisfies

$$\Omega = \{u_1 | 0 \leq u_1 \leq 1, t \in [0,T]\},$$

Our goal is to reduce the number of pests while minimizing the use of pesticides, which is necessary to reduce the opportunities for further pest development and minimize the overall control costs. The objective function under consideration is:

$$\Phi = \int_0^T \left[\varepsilon_1 x + \varepsilon_2 I + \frac{1}{2}\tau_1 u_1^2\right] dt \tag{12}$$

where $\varepsilon_i (i = 1; 2)$ are the weight constants of the number of pests and parasites. τ_1 is the relative cost of u_1 interventions. We represent the integrand $\Lambda = \varepsilon_1 x + \varepsilon_2 I + \frac{1}{2}\tau_1 u_1^2$ to find the optimal control function u^* such that

$$\Phi(u^*) = \min \Phi(u), u \in \Omega.$$

The model (11) is rewritten as:

$$\xi_t = \begin{bmatrix} \dot{x} \\ \dot{I} \end{bmatrix} = \mathrm{N}(t,\xi,u) = \begin{bmatrix} rx\left(1-\frac{x}{k}\right) - \alpha xI - u_1 x \\ \alpha xI^2 - \beta I \end{bmatrix} \tag{13}$$

Theorem 5.1 For a given initial value and objective function Φ of the control system (11), there exists an optimal control $u^* \in \Omega$ such that $\Phi(u^*) = \min \Phi(u)$.

Proof According to the definition of Ω, the control set Ω and the corresponding state variables are non-empty, and Ω is convex closed.

Let $l \in (0,1)$ and $d, v \in \Omega$, we have

$$\begin{aligned} &\Lambda(t,\xi(t),(1-l)d+lv) - (1-l)\Lambda(t,\xi(t),d) \\ &\qquad -l\Lambda(t,\xi(t),v) \\ &= \frac{\tau_i}{2}[(1-l)^2d^2 + l^2v^2 + 2l(1-l)dv] \\ &\qquad -\frac{\tau_i}{2}(1-l)d^2 - \frac{\tau_i}{2}lv^2 \\ &= \frac{\tau_i}{2}(l^2-l)(d-v)^2 < 0, \end{aligned} \tag{14}$$

which implies Λ is convex.

Then, we have

$$\begin{aligned} &|\mathrm{N}(\xi_1) - \mathrm{N}(\xi_2)| \\ &= |rx_1(1-\frac{x_1}{k}) - \alpha x_1 I_1 - u_1 x_1 - rx_2(1-\frac{x_2}{k}) \\ &\quad + \alpha x_2 I_2 + u_1 x_2 + \alpha x_1 I_1^2 - \beta I_1 - \alpha x_2 I_2^2 + \beta I_2| \\ &\leq (r+u_1)|x_2 - x_1| + \beta|I_2 - I_1|. \end{aligned}$$

Let $v = min(r+u_1, \beta)$, we obtain

$$\begin{aligned} |\mathrm{N}(\xi_1) - \mathrm{N}(\xi_2)| &\leq (r+u_1)|x_2-x_1| + \beta|I_2-I_1| \\ &\leq v(|x_2-x_1| + |I_2-I_1|). \end{aligned}$$

Thus, the right-hand side of the proposed model is bounded by a linear function in control variables and the state.

Finally, we have

$$\varepsilon_1 x + \varepsilon_2 I + \frac{1}{2}\tau_1 u_1^2 \geq \frac{1}{2}\tau_1 u_1^2 \geq h_1|u|^\rho - h_2,$$

where $h_1 = \frac{\tau_1}{2}, \rho = 2, h_2 > 0$.

Theorem 5.1 can be proved from the above discussion.

In the following, we will use Pontryagin's Maximum Principle to formulate the Hamiltonian function:

$$H(t,\xi,\lambda,u) = \varepsilon_1 x + \varepsilon_2 I + \frac{1}{2}\tau_1 u_1^2 + \sum_{j=1}^{2}\lambda_j \mathrm{N}_\mathrm{j}.$$

There exists a non-trivial vector function $\lambda = (\lambda_1, \lambda_2)$ satisfying the following equalities:

$$\frac{d\xi}{dt} = \frac{\partial H(t,\xi,\lambda,u)}{\partial \lambda}, 0 = \frac{\partial H(t,\xi,\lambda,u)}{\partial u},$$

$$\frac{d\lambda}{dt} = -\frac{\partial H(t,\xi,\lambda,u)}{\partial \xi}.$$

Theorem 5.2 Suppose two optimal control u_1 and (x^*, I^*) are optimal state solutions. There exists adjoint variables λ_1 and λ_2 satisfying:

$$\begin{cases} \dfrac{d\lambda_1}{dt} = -\varepsilon_1 - \lambda_1\left(r - \dfrac{2r}{k}x - \alpha I - u_1\right) - \lambda_2 \alpha I^2, \\ \dfrac{d\lambda_2}{dt} = -\varepsilon_2 - \lambda_2 \alpha x - \lambda_2(2\alpha xI - \beta), \end{cases} \tag{15}$$

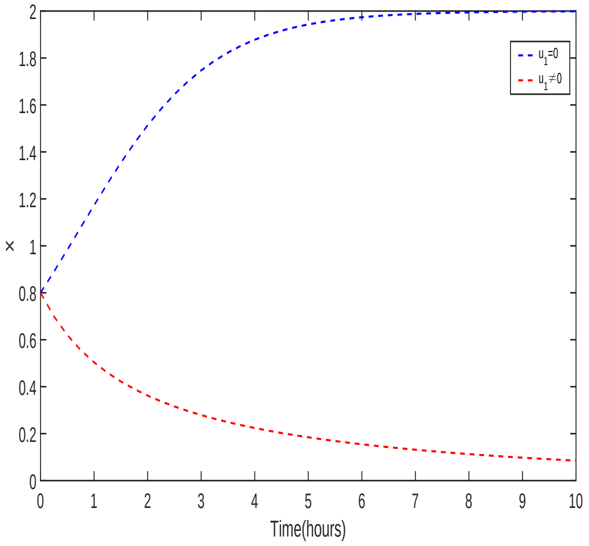

Fig. 6: The mechanisms of bacterial resistance.

and the terminal conditions $\lambda_1(T) = 0, \lambda_2(T) = 0$. The optimal control can be obtained by calculating:

$$u_1^*(t) = \max\left\{0, \min\left\{\frac{\lambda_1 x}{\tau_1}, 1\right\}\right\}.$$

VI. CONCLUSION

In summary, the ecological significance of the system can be explained. When $r > \frac{4\beta}{k}$ and $k \neq 2$, the system (1) has one limit cycle around the positive equilibrium $N_2(x_2, I_2)$. If the limit cycle is stable, all trajectory in the region tend to the stable cycle and achieve dynamic balance. If the limit cycle is unstable or the economic threshold is lower than the equilibrium region, we implement optimal control. While reducing the number of pests, the amount of pesticides used is minimized to minimize the total cost. In order to analyze the influence of the control strategy u_1 on pests and explore the effectiveness of the control strategy, we use the forward backward iterative algorithm to simulate the optimal control problem. The results of simulations without any control measures show a gradual increase in the number of pests. However, after the addition of insecticide, the number of pests dropped dramatically, as shown in Fig.6.

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