Impulsive Synchronization of a New Supply Chain System

Yang Peng, Zhen Yang, Xuhang Li, Changhan Li

***Abstract*—This study investigates impulsive synchronization control in supply chain systems by proposing a distributed cooperative strategy based on Lyapunov stability theory and the analysis of the maximum Lyapunov exponent. A dynamic supply chain model with time-varying characteristics is first developed, in which impulsive differential equations are employed to mathematically describe the discretized information interactions among nodal components. Parametric relationships between impulse intensity and control timing are systematically established. Utilizing the Lyapunov function method, sufficient conditions for ensuring exponential synchronization are rigorously derived through formal stability analysis. Furthermore, the theoretical framework incorporates the computation of the maximum Lyapunov exponent to quantify the system's sensitivity to initial perturbations, thereby offering quantitative guidance for parameter optimization. The study concludes with a numerical case study that empirically validates the effectiveness of the proposed synchronization mechanism.**

***Index Terms*—Supply chain; Impulsive synchronization; Maximum Lyapunov exponent; Chaos**

I. Introduction

Supply chain systems are complex entities characterized by diverse nonlinear behaviors arising from both exogenous and endogenous influences. As defined by Ninlawan et al. [1], traditional supply chain management (SCM) involves "the coordination and administration of intricate network activities required to deliver finished products to end users or customers," encompassing the complete lifecycle—from resource extraction, manufacturing, utilization, reuse, and recycling [2], to final disposal. Each operational phase imposes significant environmental impacts.

Structurally, supply chains represent value-adding networks composed of raw material suppliers, manufacturers, distributors, retailers, and end consumers, with processes spanning demand-initiated product delivery cycles [3]. Notably, in supply chain finance contexts, the system architecture often simplifies into a two-tier decision-making structure between suppliers and retailers. Contract parameters (e.g., wholesale pricing, buyback rates) directly influence the credit risk assessments of financial institutions by affecting collateral valuation. For example, retailers utilizing newsvendor-model-based demand forecasts may leverage procurement contracts to secure loans, while banks must dynamically assess the impact of supply chain decisions, such as supplier buyback policies, on residual collateral value in determining loan-to-value ratios [4].

The paramount challenge in SCM lies in effectively integrating activities across heterogeneous organizations within the supply chain, especially when these activities involve environmental externalities. Conventional linear management frameworks often fail to capture such systemic dynamics [5], [6]. For decades, mathematical modeling and analysis of supply chain systems have attracted significant scholarly attention. In particular, dynamic systems theory has proven effective in studying the evolving behaviors of supply chain systems [7], offering unique advantages in simulating nonlinear phenomena such as resource flows, waste accumulation, and environmental cost propagation.

In the domain of mathematical modeling, dynamic systems theory provides a powerful paradigm for revealing intrinsic behavioral patterns within supply chain networks. In recent years, with the intensification of global supply chain complexity, traditional static models have become inadequate for addressing the challenges of inventory coordination and optimization in dynamic environments. Scholars such as Thotappa and Ravindranath [8] have proposed integrating data mining and evolutionary algorithms into dynamic modeling frameworks. By reconstructing temporal inventory characteristics using the exponential moving average (EMA), identifying inter-node inventory associations via association rule mining, and optimizing inventory decision rules through genetic algorithms, they demonstrated a data-driven approach to dynamic inventory management.

Similarly, in supply chain finance models, the dynamic adjustment of loan-to-value ratios by banks is seen as a form of risk control. This is achieved by constructing closed-loop optimization models based on game-theoretic interactions between upstream and downstream entities—such as suppliers' wholesale pricing strategies and retailers' ordering decisions—thereby incorporating supply chain parameters into core financial risk assessment frameworks [4]. These methods demonstrate not only the efficacy of dynamic systems theory in modeling supply chain behavior but also establish a methodological foundation for managing inventory oscillations under demand uncertainty through dynamic rule iteration mechanisms [9].

To better characterize the dynamic behavior of multi-tier supply chain coordination, recent research has proposed refined mathematical models grounded in dynamic systems theory. These models define differential equations across supply chain echelons to quantitatively capture the

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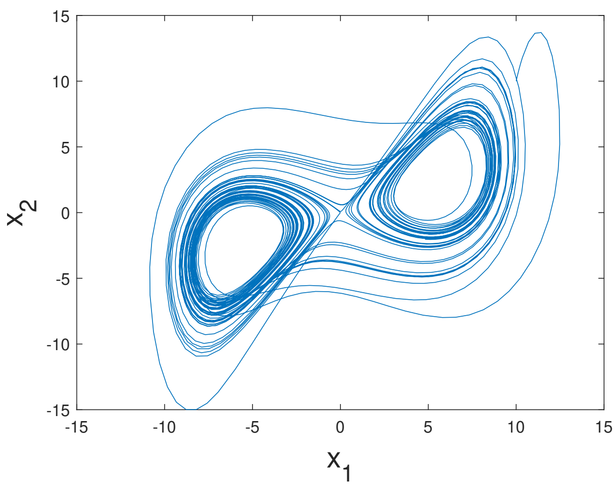
nonlinear coupling among inventory fluctuations, production delays, and demand feedback. Particularly in addressing multi-echelon inventory optimization, the models require joint consideration of vertical material transmission delays across hierarchical levels and horizontal decision interaction effects among network nodes. For instance, when supplier production is regulated by downstream distributor orders, the configuration of production threshold parameters (r) and safety stock coefficients (b) critically determines the system's resilience to demand perturbations. At the same time, the dynamic alignment between distributor delivery coefficients (m) and retailer demand fulfillment rates (n) influences the magnitude of the bullwhip effect across the supply chain. These nonlinear interdependencies not only validate the classification of supply chains as complex adaptive systems but also lay a structural foundation for subsequent stability analyses and the design of effective control strategies [10].

Let x_1, x_2, x_3 denote, respectively, the quantity demanded by the retailer, the quantity supplied by the distributor, and the quantity produced in the current period based on received orders. The parameters are defined as follows: m represents the distributor's delivery coefficient, n is the retailer's customer demand satisfaction rate, r denotes the production threshold, and b is the manufacturer's safety stock factor. Anne, Chedjou, and Kyamakya [11] proposed a three-echelon supply chain model, which can be formulated as:

$$\begin{cases} \dot{x}_1 = mx_2 - (n+1)x_1, \\ \dot{x}_2 = x_1(r - x_3) - x_2, \\ \dot{x}_3 = x_1x_2 + (b-1)x_3, \end{cases} \tag{1.1}$$

As noted by Peng et al. [17], the supply chain system described by (1.1) exhibits complex dynamic behavior and is highly sensitive to external disturbances.



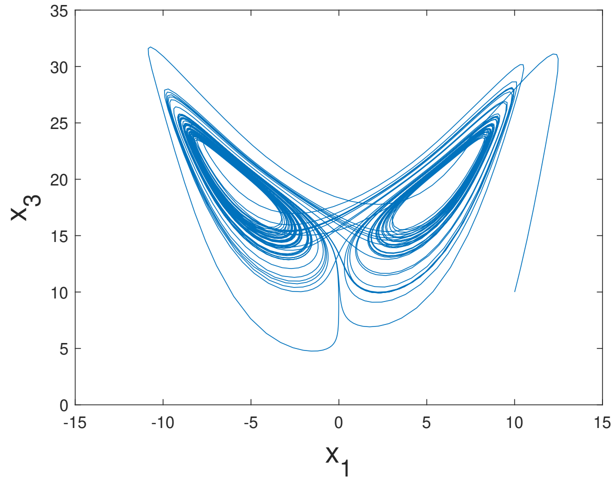


Fig. 1: Some chaotic attractors of supply chain system (1.3).

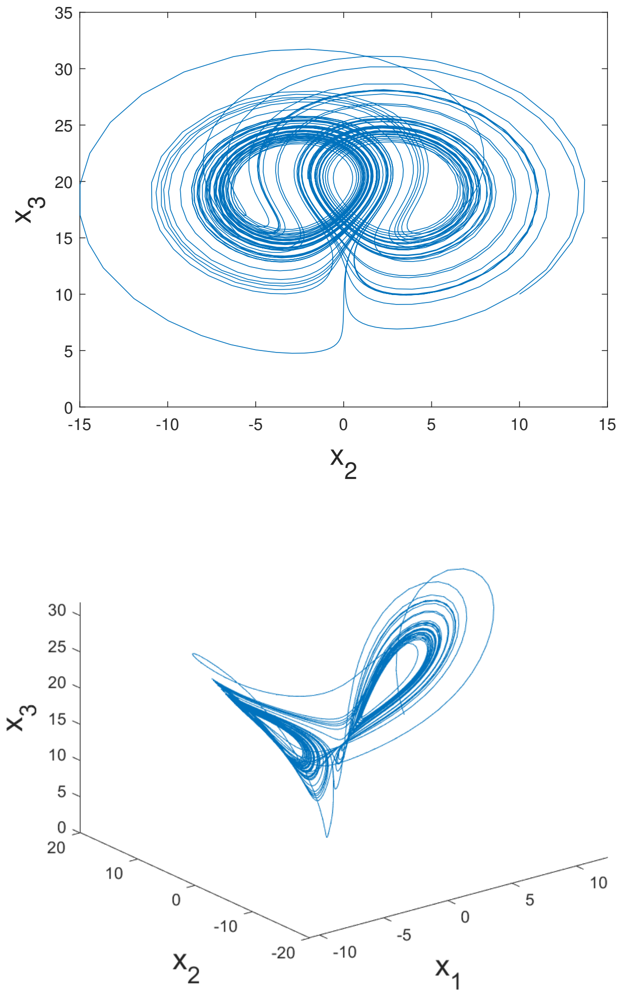


Fig. 2: Some chaotic attractors of supply chain system (1.3).

Taking into account the fact that product demand does not increase monotonically with rising inventory levels, Mondal [15] proposed an improved supply chain model, expressed as follows:

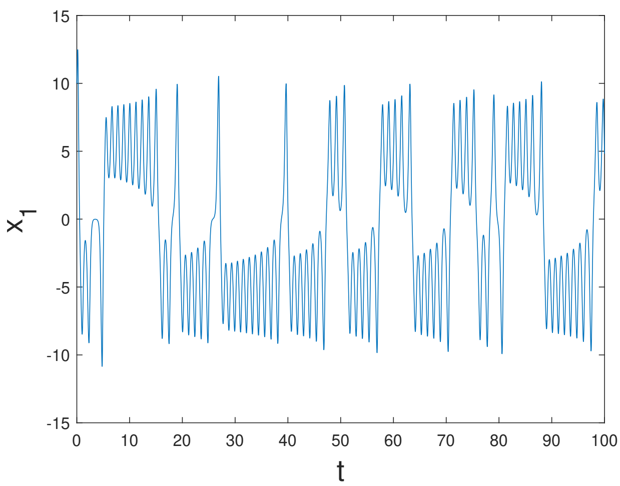
$$\begin{cases} \dot{x}_1 = \dfrac{mx_2}{1+ax_2} - (n+1)x_1, \\ \dot{x}_2 = x_1(r - x_3) - \dfrac{px_2}{1+ax_2}, \\ \dot{x}_3 = x_1x_2 + (b-1)x_3, \end{cases} \tag{1.2}$$

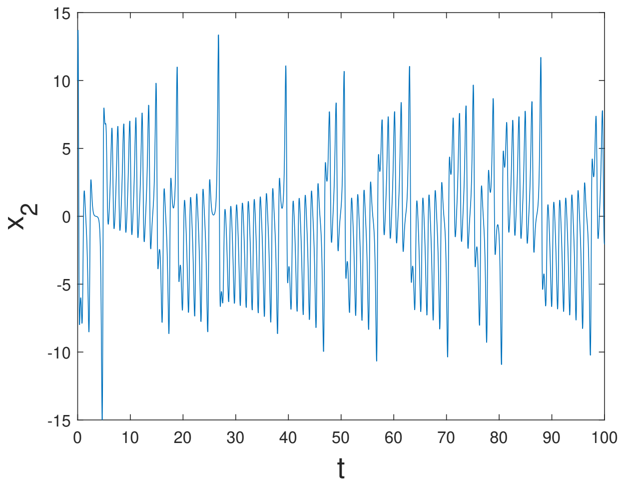
where the parameter a characterizes the saturation rate of demand as inventory increases, and p represents the sensitivity of inventory to demand fluctuations. The author also analyzed the synchronization behavior of two coupled identical supply chain models under both unidirectional and bidirectional coupling schemes.

However, system (1.2) exhibits a singularity when $x_2 = -\frac{1}{a}$, and the second equation is considered physically unreasonable, as pointed out by Anne, Chedjou, and Kyamakya [11]. To address this issue, Peng et al. [17] proposed a revised supply chain model that combines the formulations of systems (1.1) and (1.2), resulting in the following system:

$$\begin{cases} \dot{x}_1 = \dfrac{mx_2}{\sqrt{1+a^2x_2^2}} - (n+1)x_1, \\ \dot{x}_2 = x_1(r - x_3) - x_2, \\ \dot{x}_3 = x_1x_2 + (b-1)x_3, \end{cases} \tag{1.3}$$

It is noteworthy that when $a = 0$, system (1.3) reduces to system (1.1), preserving the structure of the original model proposed by Anne et al. Furthermore, when choosing the parameters $m = 10, n = 3, r = 18, b = \frac{3}{7}, a = 0.3$, system (1.3) exhibits chaotic behavior. The chaotic dynamics of the system under the initial condition $(x_1, x_2, x_3)^T = (10, 10, 10)^T$ are illustrated in Figure 3.





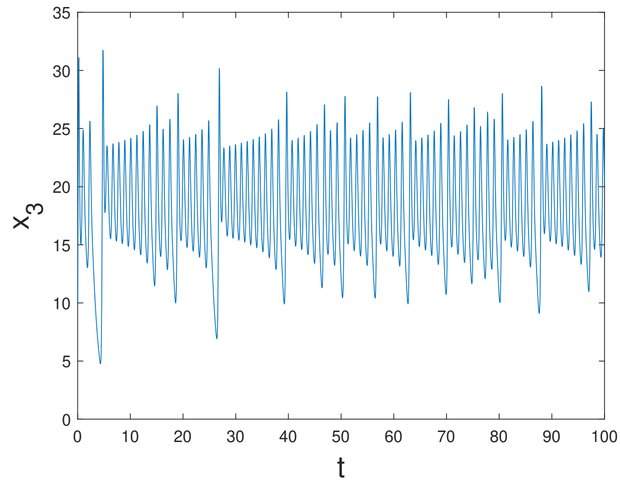


Fig. 3: Time series of supply chain system (1.3).

Synchronization is a critical strategy for mitigating the adverse effects of uncertainties and disruptions within supply chains. Peng [17] presented a sufficient condition for impulsive synchronization of the supply chain system (1.3). For a more comprehensive understanding of the impulsive synchronization method and its associated benefits, readers are encouraged to refer to [12], [13], [16], [18], [19], [20] and the references cited therein.

In this paper, we propose a novel impulsive synchronization strategy for the supply chain system (1.3), based on Lyapunov stability theory and the maximum Lyapunov exponent. The paper concludes with a numerical example demonstrating the effectiveness of the proposed approach.

II. Impulsive Synchronization of Supply Chain System (1.3)

By decomposing the linear and nonlinear components of the system in (1.3), it can be reformulated as follows:

$$\dot{X} = AX + \phi(X),$$

where

$$A = \begin{pmatrix} -n-1 & 0 & 0 \\ r & -1 & 0 \\ 0 & 0 & b-1 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and

$$\phi(X)=\begin{pmatrix}\frac{mx_2}{\sqrt{1+a^2x_2^2}}\\ -x_1x_3\\ x_1x_2\end{pmatrix}.$$

Suppose that system (1.3) is the driving system and the driven system is defined as

$$\begin{cases}\dfrac{dY}{dt}=AY+\phi(Y),t\neq\tau_k,\\ \Delta Y=Be,t=\tau_k,k=1,2,\cdots,\end{cases}\tag{2.1}$$

where $Y=(y_1,y_2,y_3)^T$ and B is impulsive control gain matrix. The symbol e denotes system errors and $e=(e_1,e_2,e_3)^T=(y_1-x_1,y_2-x_2,y_3-x_3)^T$. There will be a jump in system (2.2) when $t=\tau_k$ and $t_0=\tau_0<\tau_1<\tau_2<\cdots$, $\lim\limits_{k\to\infty}\tau_k=\infty$.

System (2.1) minus system (1.3), we get

$$\begin{cases}\dfrac{de}{dt}=Ae+\psi(X,Y),t\neq\tau_k,\\ \Delta e=Be,t=\tau_k,k=1,2,\cdots,\end{cases}\tag{2.2}$$

where $\psi(X,Y)=\phi(Y)-\phi(X)$.

Next, we present a sufficient condition for the stability of error system (2.2).

Theorem 2.1 Let λ be the largest eigenvalues $\left(I+B^T\right)(I+B)$. Supposed that L is the maximum Lyapunov exponent of system (2.1). If

$$\begin{aligned}&\tau_k-\tau_{k-1}<1/L,\\ &\tau_k-\tau_{k-1}<-\tfrac{\log\lambda}{2L},\end{aligned}$$

hold for all $k=1,2,\cdots$, then the origin of impulsive synchronization error system (2.2) is asymptotically stable.

Proof. It is shown in [14] that the maximum predictable time of chaotic motion is $1/L$. Therefore, the impulsive control interval $\tau_k-\tau_{k-1},k=1,2,\cdots$ must satisfy $\tau_k-\tau_{k-1}<1/L$.

If system (2.2) operates without control, then after a short period of time, we have

$$\sqrt{e^T(t)e(t)}=\|e(t)\|\le\|e(t_0)\|e^{L(t-t_0)}.$$

Let us consider the following Lyapunov function:

$$V(e(t))=e^T(t)e(t)=\|e(t)\|^2.$$

For $t\in[t_0,\tau_1)$, it follows that

$$\begin{aligned}V(e(t))&=\|e(t)\|^2\\ &\le\|e(t_0)\|^2e^{2L(t-t_0)}\\ &=V(e(t_0))e^{2L(t-t_0)}.\end{aligned}\tag{2.3}$$

When $t=\tau_1$, it follows from system (2.2) and inequality (2.3) that

$$\begin{aligned}V(e(\tau_1))&=\left((I+B)e(\tau_1^-)\right)^T(I+B)e(\tau_1^-)\\ &=e^T(\tau_1^-)(I+B^T)(I+B)e(\tau_1^-)\\ &\le\lambda\, e^T(\tau_1^-)e(\tau_1^-)\\ &\le\lambda V(e(t_0))e^{2L(\tau_1-t_0)}.\end{aligned}\tag{2.4}$$

Similarly, for $t\in[\tau_1,\tau_2)$, we have

$$V(e(t))\le V(e(\tau_1))e^{2L(t-\tau_1)}.\tag{2.5}$$

It then follows from (2.4) and (2.5) that

$$V(e(t))\le\lambda V(e(t_0))e^{2L(t-t_0)}.\tag{2.6}$$

When $t=\tau_2$, by the same method used to derive inequalities (2.4) and (2.6), we obtain

$$\begin{aligned}V(e(\tau_2))&\le\lambda V(e(\tau_2^-))\\ &\le\lambda^2V(e(t_0))e^{2L(\tau_2-t_0)},\end{aligned}$$

which implies

$$\begin{aligned}V(e(t))&\le V(e(\tau_2))e^{2L(t-\tau_2)}\\ &\le\lambda^2V(e(t_0))e^{2L(\tau_2-t_0)}e^{2L(t-\tau_2)}\\ &=\lambda^2V(e(t_0))e^{2L(t-t_0)},\quad t\in[\tau_2,\tau_3).\end{aligned}\tag{2.7}$$

By the repeatability of the proof process, we know that for $t\in[\tau_{k-1},\tau_k)$, we have

$$\begin{aligned}V(e(t))&\le\lambda^{k-1}V(e(t_0))e^{2L(t-t_0)}\\ &=V(e(t_0))e^{2L(t-t_0)}e^{(k-1)\log\lambda}\\ &=V(e(t_0))e^{\log\lambda+2L(\tau_1-t_0)}e^{\log\lambda+2L(\tau_2-\tau_1)}\\ &\quad\cdots e^{2L(t-\tau_{k-1})}.\end{aligned}$$

Thus, we obtain the following expression:

$$\begin{aligned}V(e(t))=\lambda^{-1}V(e(t_0))e^{\log\lambda+2L(\tau_1-t_0)}e^{\log\lambda+2L(\tau_2-\tau_1)}\cdots\\ \cdots e^{\log\lambda+2L(t-\tau_{k-1})}.\end{aligned}\tag{2.8}$$

Since the supply chain system (1.3) is chaotic, we have $L>0$, and therefore

$$V(e(t))\le\lambda^{-1}V(e(t_0))e^{\sum_{i=1}^{k}(\log\lambda+2L(\tau_i-\tau_{i-1}))}.$$

Hence, if $\log\lambda+2L(\tau_k-\tau_{k-1})<0$, for $k=1,2,\cdots$, then $V(e(t))\to0$. This completes the proof.

III. Simulation experiments

This paper concludes with a numerical example that illustrates the effectiveness of our method. As stated in Section 1, when $m=10$, $n=3$, $r=18$, $b=\frac{3}{7}$, and $a=0.3$, system (1.3) exhibits chaotic behavior.

The initial values for the driving and driven systems are chosen as $(x_1,x_2,x_3)^T=(10,10,10)^T$ and $(y_1,y_2,y_3)^T=(11,12,13)^T$, respectively. A small calculation shows that the maximum Lyapunov exponent of system (1.3) is $L=0.4296$.

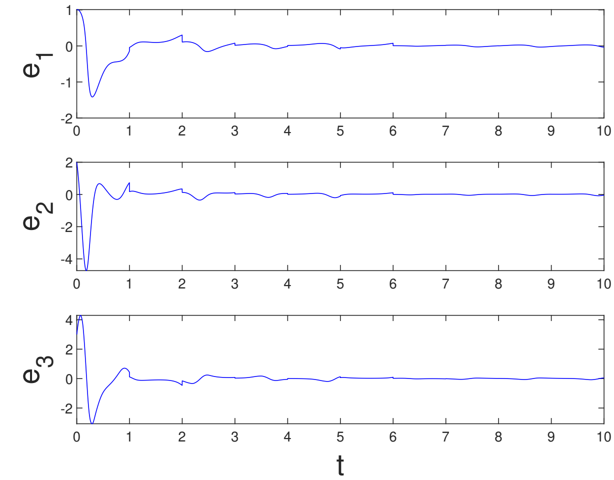
In accordance with Theorem 2.1, through straightforward calculations, we can determine that if the intensity and time interval of the impulsive control satisfy the following inequalities:

$$\begin{aligned}\tau_k-\tau_{k-1}&<\frac{1}{0.4296}=2.3277,\\ \tau_k-\tau_{k-1}&<-\frac{\log\lambda}{2\times0.4296},\end{aligned}\tag{3.1}$$

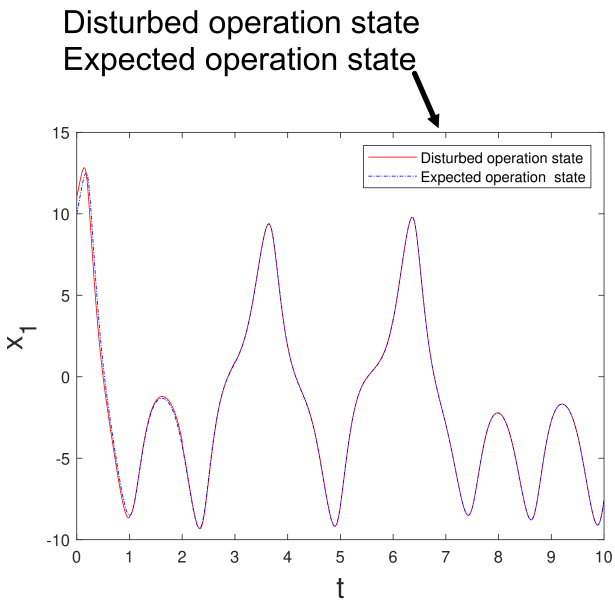
then the perturbed operational state will return to the desired trajectory after the application of appropriate impulsive control. To satisfy the inequality above, we select the impulsive control gain matrix $B=-0.7I$. Thus, we have:

$$\begin{aligned}\tau_k-\tau_{k-1}&<\frac{1}{0.4296}=2.3277,\\ \tau_k-\tau_{k-1}&<-\frac{\log(0.09)}{2\times0.4296}=2.8025,\end{aligned}\tag{3.2}$$

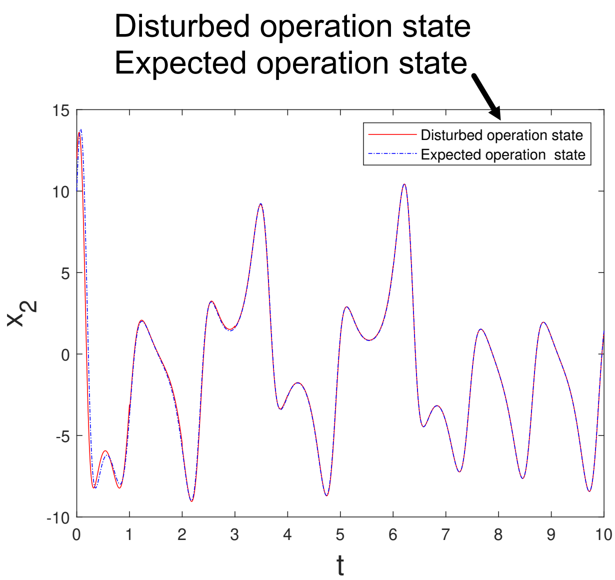
which implies that the time interval of regulation must satisfy $\tau_k-\tau_{k-1}<2.3277$. Suppose that $\tau_k-\tau_{k-1}=1.000$. The numerical simulation results are shown in Figure 4.



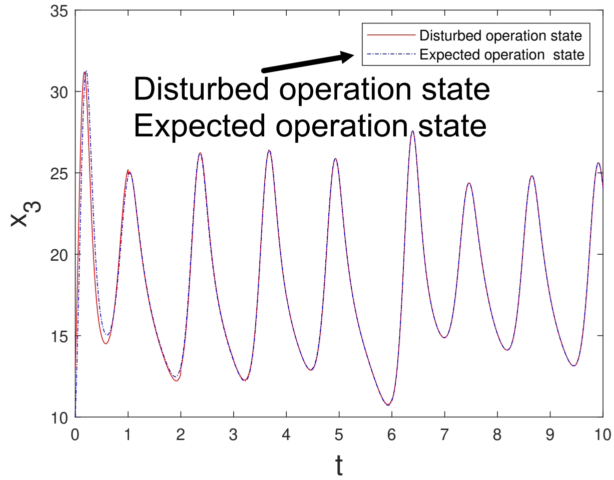
(a) Control error



(b) Satisfy the regulation theorem x1



(c) Satisfy the regulation theorem x2



(d) Satisfy the regulation theorem x3

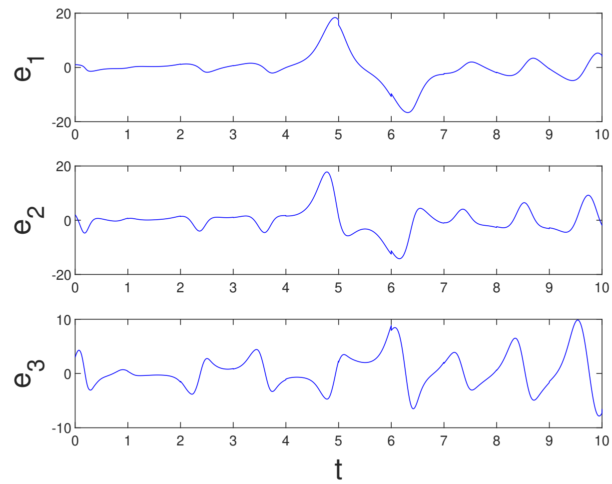
Fig. 4: Impulsive synchronization of supply chain system (1.3).

The experimental results demonstrate that when the pulse intensity and time interval satisfy Theorem 2.1, specifically Equation (3.1), the production quantity, distribution volume, and sales volume within the supply chain system can be restored to their expected states at $t = 3$, thereby achieving synchronization of the supply chain system (Figure 4(bcd)). The error plots further reveal that at $t = 6$, the errors of all components within the supply chain system converge to 0 (Figure 4(a)).

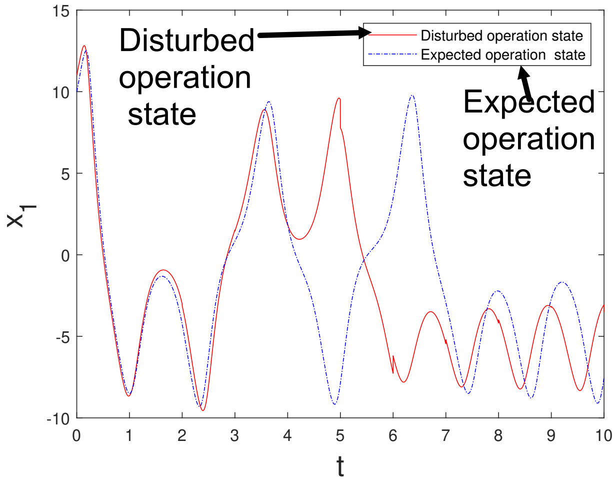
To satisfy inequality (3.3), we select the impulsive control gain matrix as $B = -0.1I$. Thus, we obtain the following conditions:

$$\begin{aligned} \tau_k - \tau_{k-1} &< \frac{1}{0.4296} = 2.3277, \\ \tau_k - \tau_{k-1} &< -\frac{\log(0.81)}{2 \times 0.4296} = 0.2453, \end{aligned} \tag{3.3}$$

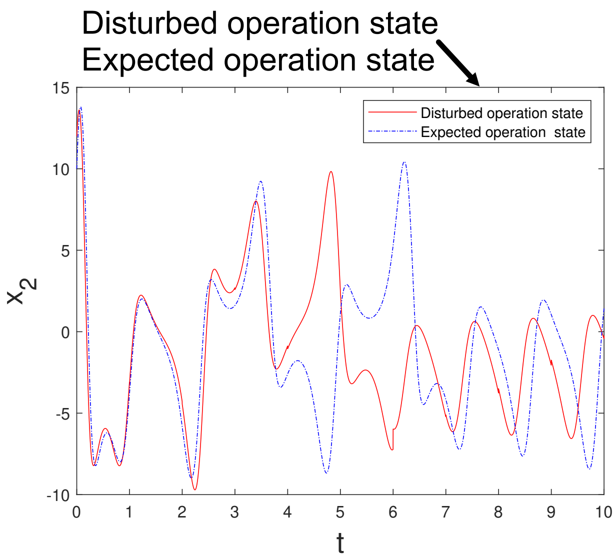
which implies that the time interval between successive impulses must satisfy $\tau_k - \tau_{k-1} < 0.2453$. Suppose that $\tau_k - \tau_{k-1} = 1.000$. The numerical simulation results under this condition are presented in Figure 5.



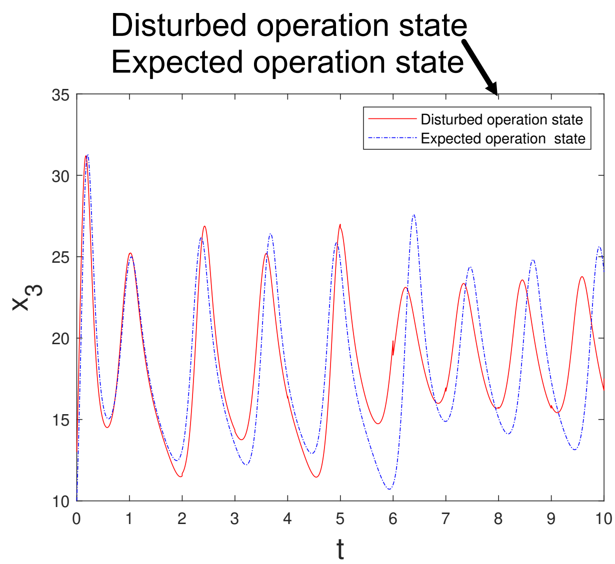
(a) Control error



(b) The regulation theorem x1 is not satisfied



(c) The regulation theorem x2 is not satisfied



(d) The regulation theorem x3 is not satisfied

Fig. 5: Impulsive synchronization of supply chain system (1.3).

The experimental results indicate that when the pulse intensity and time interval do not satisfy Theorem 2.1, namely when Equation (3.2) is not fulfilled, the production quantity, distribution volume, and sales volume within the supply chain system fail to return to their expected states (Figure 5(bcd)). As clearly illustrated in the error plots, the deviations of each component in the supply chain system begin to increase from $t = 4$ onward and continue to fluctuate within a certain range (Figure 5(a)).

As evidenced by the experimental results, failure to comply with the theorem's constraints on both the impulsive control intensity and the timing interval leads to suboptimal control performance. Under such conditions, all state variables exhibit stochastic divergence from their predefined equilibrium points, indicating a breakdown in deterministic convergence.

IV. Conclusion

On the other hand, according to the result presented in [17], we know that if

$$\tau_k - \tau_{k-1} < -\frac{\log \lambda}{98.3069},$$

then the perturbed operational state will return to the desired trajectory after the application of appropriate impulsive control. Figure 5 illustrates the stability region corresponding to different values of λ.

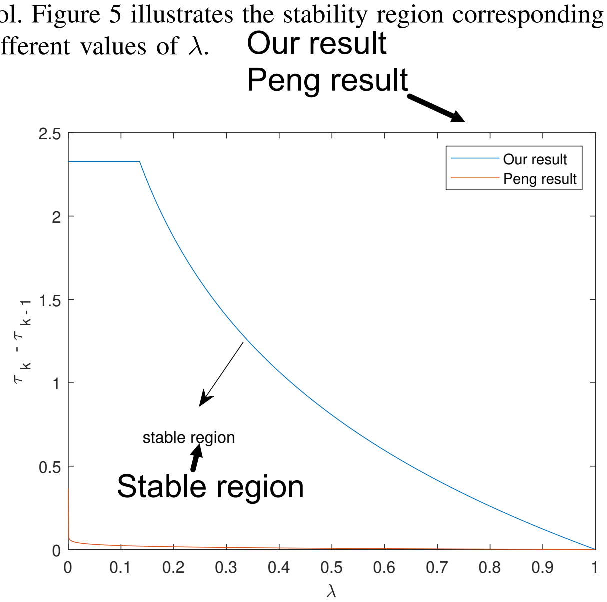


Fig. 6: The estimation of boundaries of stable region with different λ.

From Figure 6 we know that we get a larger stable region for supply chain system (1.3).

Data Availability

The Matlab code data used to support the findings of this study are available from the corresponding author upon request.

Author's contributions

All authors contributed equally to the writing of this paper.

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