Functional Analytic Approximations via Generalized Variational Iteration for Incompressible Time-Asymmetric Evolution Systems

Abdulhameed Qahtan Abbood Altai

Abstract—**This work presents an iterative method for solving time-asymmetric partial differential equation systems analytically. In this procedure, two symmetric systems in the spatial variable are constructed, and the associated correction functionals are then generated. Applicability of this iterative method is demonstrated in solving the governing equations for both incompressible fluid flow and incompressible micropolar fluid flow.**

Index Terms—Generalized variational iteration method, incompressible fluid flow, incompressible micropolar fluid flow, correction functional.

I. INTRODUCTION

THE variational iteration method (VIM) has been widely applied to solve various types of differential equations [19–28]. Its novelty lies in providing faster successive approximations of solutions—both exact and numerical-than Adomian's decomposition method [1, 2, 43]. Furthermore, unlike perturbation techniques, it does not rely on small parameters [20]. While VIM has been applied to resolve symmetric systems of partial differential equations (PDEs) involving time [3,44], its application to time-asymmetric systems remains unexplored. We define a system of PDEs as time-asymmetric if, for any unknown μ_k $(k=1,\cdots,m)$ and ρ, the equations include $\frac{\partial\mu_k}{\partial\tau}$ but exclude $\frac{\partial\rho}{\partial\tau}$.

In this paper, we address time-asymmetric systems containing both the Laplacian operator $\Delta\mu_k (k=1,\cdots,m)$ and the gradient operator $\nabla\rho$. We construct two spatially symmetric systems regarding the spatial variable $\xi=(\xi_1,\cdots,\xi_d), d\geq 2$. The first system is derived by rescaling the coefficients of $\Delta\mu_k$ $(k=1,\cdots,m)$, while the second system is formulated by applying the gradient operator ∇ and the outward unit normal $\boldsymbol{n}$ to $\partial\mho$ to the equations containing $\nabla\rho$. Finally, we employ the generalized variational iteration method (GVIM) to solve both systems simultaneously for μ_k $(k=1,\cdots,m)$, and ρ.

II. GENERALIZED VARIATIONAL ITERATION METHOD (GVIM)

Let $\boldsymbol{\mu}=(\mu_1,\cdots,\mu_m):\mho_{\mathrm{T}}\to\mathcal{R}^m$ and $\rho:\mho_{\mathrm{T}}\to\mathcal{R}$, where $\mho_{\mathrm{T}}=\mho\times[0,\mathrm{T}], \mho\subset\mathcal{R}^d (d\geq 2)$. Consider these partial differential equations:

$$\begin{cases}\mathfrak{L}_\tau\mu_1+\mathfrak{R}_1(\Delta\mu_1,\mu_2,\cdots,\mu_m,\nabla\rho)+\mathcal{N}_1(\mu_1,\cdots,\mu_m)=\varrho_1,\\ \vdots\\ \mathfrak{L}_\tau\mu_m+\mathfrak{R}_m(\mu_1,\cdots,\mu_{m-1},\Delta\mu_m,\nabla\rho)+\mathcal{N}_m(\mu_1,\cdots,\mu_m)=\varrho_m,\end{cases}\quad(1)$$

satisfying the divergence-free condition $\nabla\cdot\mu=0$ together with the initial condition $\mu(\xi,0)=\mu_0(\xi)$.

Here, $\mathfrak{L}_\tau$ is a partial differential operator of a first-order, the partial differential operators $\mathfrak{R}_k$ and $\mathcal{N}_k$ $(k=1,\cdots,m)$ are linear and nonlinear, respectively, and ϱ_k $(k=1,\cdots,m)$ are source terms.

Since the Laplacian terms $\Delta\mu_k$ $(k=1,\cdots,m)$ are linear, we rewrite system (1) as:

$$\begin{cases}\Delta\mu_1 +\mathfrak{R}_1(\mu_1,\cdots,\mu_m,\nabla\rho) +\mathcal{N}_1(\mu_1,\dots,\mu_m) =\mathfrak{h}_1,\\ \vdots\\ \Delta\mu_m +\mathfrak{R}_m(\mu_1,\cdots,\mu_m,\nabla\rho) +\mathcal{N}_m(\mu_1,\dots,\mu_m) =\mathfrak{h}_m.\end{cases}\quad(2)$$

applying the divergence operator $(\nabla\cdot)$ to both sides of (2), we obtain a Nuemman problem for pressure:

$$\begin{cases}\Delta\rho+\nabla\cdot\mathfrak{R}(\mu_1,\cdots,\mu_m)+\nabla\cdot\mathcal{N}(\mu_1,\dots,\mu_m)=\nabla\cdot\varrho,\\ \frac{\partial\rho}{\partial n}=(\mathfrak{L}_\tau\mu+\mathfrak{R}(\boldsymbol{\mu},\rho)+\mathcal{N}(\boldsymbol{\mu})-\varrho)\cdot\boldsymbol{n},\end{cases}\quad(3)$$

where $\boldsymbol{n}$ denotes the outward unit normal to $\partial\mho$.Then, the correction functionals for the systems (2) and (3.1) are:

$$\begin{aligned}&\mu_{1,n+1}(\xi,\tau)\\&=\mu_{1,n}(\xi,\tau)\end{aligned}+\int_\mho \lambda_1(\Xi)\begin{pmatrix}\Delta\mu_1\\+\mathfrak{R}_1(\tilde{\mu}_1,\dots,\tilde{\mu}_m,\nabla\tilde{\rho})\\+\mathcal{N}_1(\tilde{\mu}_1,\dots,\tilde{\mu}_m)\\-\mathfrak{h}_1\end{pmatrix}(\Xi)d\Xi,$$

$$\vdots$$

$$\begin{aligned}&\mu_{m,n+1}(\xi,\tau)\\&=\mu_{m,n}(\xi,\tau)\end{aligned}+\int_\mho \lambda_m(\Xi)\begin{pmatrix}\Delta\mu_m\\+\mathfrak{R}_m(\tilde{\mu}_1,\dots,\tilde{\mu}_m,\nabla\tilde{\rho})\\+\mathcal{N}_m(\tilde{\mu}_1,\dots,\tilde{\mu}_m)\\-\mathfrak{h}_m\end{pmatrix}(\Xi)d\Xi,\quad(4)$$

$$\begin{aligned}&\rho_{n+1}(\xi,\tau)\\&=\rho_n(\xi,\tau)\end{aligned}+\int_\mho \mu(\Xi)\begin{pmatrix}\Delta\rho\\+\nabla\mathfrak{R}(\tilde{\mu}_1,\dots,\tilde{\mu}_m)\\+\nabla\mathcal{N}(\tilde{\mu}_1,\dots,\tilde{\mu}_m)\\-\nabla\varrho\end{pmatrix}(\Xi)d\Xi.$$

III. GVIM FOR INCOMPRESSIBLE FLUID FLOW SYSTEMS

The system of incompressible fluid flow is governed by the Navier-Stokes equations [10, 11, 18, 29, 36, 39, 40, 42]: Given an external body force $\boldsymbol{\varrho}\in\mathfrak{L}^2(0,\mathrm{T},\boldsymbol{\mathfrak{H}}^{-1}(\mho))$, boundary data $\boldsymbol{\vartheta}\in\mathfrak{H}^1\left(0,\mathrm{T},\boldsymbol{\mathfrak{H}}^{\frac{1}{2}}(\partial\mho)\right)$, initial data $\boldsymbol{\mu}_0$, and time $\mathrm{T}>0$, find:

- Velocity field: $\boldsymbol{\mu}:\mho_{\mathrm{T}}\to\mathcal{R}^d$ $(d=2,3)$ belonging to $\mathfrak{L}^\infty(0,\mathrm{T},\mathfrak{H}^{\mathbf{1}}(\mho))\cap\mathfrak{L}^2(0,\mathrm{T},\boldsymbol{V}_{\boldsymbol{\vartheta}}(\mho))$, where
$$\boldsymbol{V}_{\boldsymbol{\vartheta}}(\mho)=\{\boldsymbol{\mu}\in\mathfrak{H}^{\mathbf{2}}(\mho):\boldsymbol{\mu}|_{\partial\mho}=\boldsymbol{\vartheta},\nabla\cdot\boldsymbol{\mu}=0\text{ in }\mho\},$$
- Pressure field: $\rho:\mho_{\mathrm{T}}\to\mathcal{R}$ belonging to $\mathfrak{L}^2(0,\mathrm{T},\mathfrak{H}^1(\mho))$, such that:

$$\begin{cases}\dfrac{\partial\boldsymbol{\mu}}{\partial\tau}+(\boldsymbol{\mu}.\nabla)\boldsymbol{\mu}-\mathfrak{v}\Delta\boldsymbol{\mu}=-\nabla\rho+\boldsymbol{\varrho} & \text{in }\mho_T,\\ \nabla\cdot\boldsymbol{\mu}=0 & \text{in }\mho_T,\\ \boldsymbol{\mu}=\boldsymbol{\vartheta} & \text{on }\partial\mho_T,\\ \boldsymbol{\mu}(\xi,0)=\boldsymbol{\mu}_0(\xi) & \text{in }\mho_T,\end{cases}\quad(5)$$

where $\mathfrak{v}(>0)$ is the kinematic viscosity coefficient.

Manuscript received October 30, 2024, revised August 16, 2025.

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Theorem 1. The correction functionals for the system (5) are defined as follows:

For $d = 2$:

$$\mu_{1,n+1}(\xi_1,\xi_2,\tau) = \mu_{1,n}(\xi_1,\xi_2,\tau) + \int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_{\mu_1}}(\Xi_1,\Xi_2)-\gamma_{2,\lambda_1}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\mu_1}}}}\Bigg(\Delta\mu_{1,n} - \frac{1}{\nu}\Big(\frac{\partial\mu_{1,n}}{\partial\tau} + (\boldsymbol{\mu}_n\cdot\nabla)\mu_{1,n} + \frac{\partial\rho_n}{\partial\Xi_1} - \varrho_1\Big)\Bigg)(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2, \quad (6)$$

$$\mu_{2,n+1}(\xi_1,\xi_2,\tau) = \mu_{2,n}(\xi_1,\xi_2,\tau) + \int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_{\mu_2}}(\Xi_1,\Xi_2)-\gamma_{2,\lambda_{\mu_2}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\mu_2}}}}\Bigg(\Delta\mu_{2,n} - \frac{1}{\nu}\Big(\frac{\partial\mu_{2,n}}{\partial\tau} + (\boldsymbol{\mu}_n\cdot\nabla)\mu_{2,n} + \frac{\partial\rho_n}{\partial\Xi_2} - \varrho_2\Big)\Bigg)(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2, \quad (7)$$

$$\rho_{n+1}(\xi_1,\xi_2,\tau) = \rho_n(\xi_1,\xi_2,\tau) + \int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_\rho}(\Xi_1,\Xi_2)-\gamma_{2,\lambda_\rho}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_\rho}}}\Bigg(\Delta\rho_n + \frac{\partial}{\partial x_1}\Big((\boldsymbol{\mu}_n\cdot\nabla)\mu_{1,n} - \varrho_1\Big) + \frac{\partial}{\partial x_2}\Big((\boldsymbol{\mu}_n\cdot\nabla)\mu_{2,n} - \varrho_2\Big)\Bigg)(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2. \quad (8)$$

For $d = 3$:

$$\mu_{1,n+1}(\xi_1,\xi_2,\xi_3,\tau) = \mu_{1,n}(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_1}}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_{\mu_1}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\mu_1}}}}\Bigg(\Delta\mu_{1,n} - \frac{1}{\mathfrak{v}}\Big(\frac{\partial\mu_{1,n}}{\partial\tau} + (\boldsymbol{\mu}_n.\nabla)\mu_{1,n} + \frac{\partial\rho_n}{\partial\Xi_1} - \varrho_1\Big)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3, \quad (9)$$

$$\mu_{2,n+1}(\xi_1,\xi_2,\xi_3,\tau) = \mu_{2,n}(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_2}}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_{\mu_2}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\mu_2}}}}\Bigg(\Delta\mu_{2,n} - \frac{1}{\mathfrak{v}}\Big(\frac{\partial\mu_{2,n}}{\partial\tau} + (\boldsymbol{\mu}_n.\nabla)\mu_{2,n} + \frac{\partial\rho_n}{\partial\Xi_2} - \varrho_2\Big)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3, \quad (10)$$

$$\mu_{3,n+1}(\xi_1,\xi_2,\xi_3,\tau) = \mu_{3,n}(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_3}}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_{\mu_3}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\mu_3}}}}\Bigg(\Delta\mu_{3,n} - \frac{1}{\mathfrak{v}}\Big(\frac{\partial\mu_{3,n}}{\partial\tau} + (\boldsymbol{\mu}_n.\nabla)\mu_{3,n} + \frac{\partial\rho_n}{\partial\Xi_3} - \varrho_3\Big)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3, \quad (11)$$

$$\rho_{n+1}(\xi_1,\xi_2,\xi_3,\tau) = \rho_n(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_\rho}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_\rho}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_\rho}}}\Bigg(\Delta\rho_n + \frac{\partial}{\partial x_1}\Big((\boldsymbol{\mu}_n\cdot\nabla)\mu_{1,n} - \varrho_1\Big) + \frac{\partial}{\partial x_2}\Big((\boldsymbol{\mu}_n.\nabla)\mu_{2,n} - \varrho_2\Big) + \frac{\partial}{\partial x_3}\Big((\boldsymbol{\mu}_n\cdot\nabla)\mu_{3,n} - \varrho_3\Big)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3. \quad (12)$$

Proof. We rewrite the system (5) as:

$$\begin{cases} \Delta\boldsymbol{\mu} = \dfrac{1}{\mathfrak{v}}\Big(\dfrac{\partial\boldsymbol{\mu}}{\partial\tau} + (\boldsymbol{\mu}.\nabla)\boldsymbol{\mu} + \nabla\rho - \boldsymbol{\varrho}\Big) & \text{in } \mho_{\mathrm{T}}, \\ \nabla.\boldsymbol{\mu} = 0 & \text{in } \mho_{\mathrm{T}}, \\ \boldsymbol{\mu} = \boldsymbol{\vartheta} & \text{on } \partial\mho_{\mathrm{T}}, \\ \boldsymbol{\mu}(\boldsymbol{x},0) = \boldsymbol{\mu}_0(\boldsymbol{x}) & \text{in } \mho. \end{cases} \quad (13)$$

Since $\boldsymbol{\mu}$ and ρ are not time-symmetric (with ρ the enforcing incompressibility via $\nabla\cdot\boldsymbol{\mu} = 0$ [18], we apply the divergence operator $(\nabla\cdot)$ to (5.1) to derive the Neumann problem for ρ:

$$\begin{cases} \Delta\rho = -\nabla\cdot\big((\boldsymbol{\mu}\cdot\nabla)\boldsymbol{\mu} - \boldsymbol{\varrho}\big) & \text{in } \mho, \\ \dfrac{\partial\rho}{\partial n} = \Big(\mathfrak{v}\Delta\boldsymbol{\mu} - \dfrac{\partial\boldsymbol{\mu}}{\partial\tau} - (\boldsymbol{\mu}\cdot\nabla)\boldsymbol{\mu} + \boldsymbol{\varrho}\Big)\cdot\boldsymbol{n} & \text{on } \partial\mho, \end{cases} \quad (14)$$

where $\boldsymbol{n}$ is the outward unit normal to $\partial\mho$.

For $d = 2$, expanding (13.1) and (14.1) yields:

$$\Delta\mu_1 = \frac{1}{\mathfrak{v}}\Big(\frac{\partial\mu_1}{\partial\tau} + (\boldsymbol{\mu}.\nabla)\mu_1 + \frac{\partial\rho}{\partial\xi_1} - \varrho_1\Big), \quad (15)$$

$$\Delta\mu_2 = \frac{1}{\mathfrak{v}}\Big(\frac{\partial\mu_2}{\partial\tau} + (\boldsymbol{\mu}.\nabla)\mu_2 + \frac{\partial\rho}{\partial\xi_2} - \varrho_2\Big), \quad (16)$$

$$\Delta\rho = -\frac{\partial}{\partial x_1}\big((\boldsymbol{\mu}.\nabla)\mu_1 - \varrho_1\big) - \frac{\partial}{\partial x_2}\big((\boldsymbol{\mu}.\nabla)\mu_2 - \varrho_2\big). \quad (17)$$

For $d = 3$, analogous steps give:

$$\Delta\mu_1 = \frac{1}{\mathfrak{v}}\Big(\frac{\partial\mu_1}{\partial\tau} + (\boldsymbol{\mu}.\nabla)\mu_1 + \frac{\partial\rho}{\partial\xi_1} - \varrho_1\Big), \quad (18)$$

$$\Delta\mu_2 = \frac{1}{\mathfrak{v}}\Big(\frac{\partial\mu_2}{\partial\tau} + (\boldsymbol{\mu}.\nabla)\mu_2 + \frac{\partial\rho}{\partial\xi_2} - \varrho_2\Big), \quad (19)$$

$$\Delta\mu_3 = \frac{1}{\mathfrak{v}}\Big(\frac{\partial\mu_3}{\partial\tau} + (\boldsymbol{\mu}.\nabla)\mu_3 + \frac{\partial\rho}{\partial\xi_3} - \varrho_3\Big), \quad (20)$$

$$\Delta\rho = -\frac{\partial}{\partial x_1}\big((\boldsymbol{\mu}.\nabla)\mu_1 - \varrho_1\big) - \frac{\partial}{\partial x_2}\big((\boldsymbol{\mu}.\nabla)\mu_2 - \varrho_2\big) - \frac{\partial}{\partial x_3}\big((\boldsymbol{\mu}.\nabla)\mu_3 - \varrho_3\big). \quad (21)$$

The correction functionals for (15)-(17) on $\mho = [0,\xi_1]\times[0,\xi_2]$ are:

$$\mu_{1,n+1}(\xi_1,\xi_2,\tau) = \mu_{1,n}(\xi_1,\xi_2,\tau) + \int_0^{\xi_2}\int_0^{\xi_1} {}_2\lambda_{\mu_1}(\Xi_1,\Xi_2)\Bigg(\Delta\mu_{1,n} - \frac{1}{\mathfrak{v}}\Big(\frac{\partial\tilde{\mu}_{1,n}}{\partial\tau} + (\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\mu}_{1,n} + \frac{\partial\tilde{\rho}_n}{\partial\Xi_1} - \varrho_1\Big)\Bigg)(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2, \quad (22)$$

$$\mu_{2,n+1}(\xi_1,\xi_2,\tau) = \mu_{2,n}(\xi_1,\xi_2,\tau) + \int_0^{\xi_2}\int_0^{\xi_1} {}_2\lambda_{\mu_2}(\Xi_1,\Xi_2)\Bigg(\Delta\mu_{2,n} - \frac{1}{\mathfrak{v}}\left(\frac{\partial\tilde{\mu}_{2,n}}{\partial\tau} + (\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\mu}_{2,n} + \frac{\partial\tilde{\rho}_n}{\partial\Xi_2} - \varrho_2\right)\Bigg)(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2, \tag{23}$$

$$\rho_{n+1}(\xi_1,\xi_2,\tau) = \rho_n(\xi_1,\xi_2,\tau) + \int_0^{\xi_2}\int_0^{\xi_1} {}_2\lambda_{\rho}(\Xi_1,\Xi_2)\Bigg(\Delta\rho_n + \frac{\partial}{\partial x_1}\Big((\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\mu}_{1,n} - \varrho_1\Big) + \frac{\partial}{\partial x_2}\Big((\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\mu}_{2,n} - \varrho_2\Big)\Bigg)(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2. \tag{24}$$

The correction functionals for (18)-(21) on $\mho = [0,\xi_1]\times[0,\xi_2]\times[0,\xi_3]$ are:

$$\mu_{1,n+1}(\xi_1,\xi_2,\xi_3,\tau) = \mu_{1,n}(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} {}_3\lambda_{\mu_1}(\Xi_1,\Xi_2,\Xi_3)\Bigg(\Delta\mu_{1,n} - \frac{1}{\mathfrak{v}}\left(\frac{\partial\tilde{\mu}_{1,n}}{\partial\tau} + (\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\mu}_{1,n} + \frac{\partial\tilde{\rho}_n}{\partial\Xi_1} - \varrho_1\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3, \tag{25}$$

$$\mu_{2,n+1}(\xi_1,\xi_2,\xi_3,\tau) = \mu_{2,n}(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} {}_3\lambda_{\mu_2}(\Xi_1,\Xi_2,\Xi_3)\Bigg(\Delta\mu_{2,n} - \frac{1}{\mathfrak{v}}\left(\frac{\partial\tilde{\mu}_{2,n}}{\partial\tau} + (\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\mu}_{2,n} + \frac{\partial\tilde{\rho}_n}{\partial\Xi_2} - \varrho_2\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3, \tag{26}$$

$$\mu_{3,n+1}(\xi_1,\xi_2,\xi_3,\tau) = \mu_{3,n}(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} {}_3\lambda_{\mu_3}(\Xi_1,\Xi_2,\Xi_3)\Bigg(\Delta\mu_{3,n} - \frac{1}{\mathfrak{v}}\left(\frac{\partial\tilde{\mu}_{3,n}}{\partial\tau} + (\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\mu}_{3,n} + \frac{\partial\tilde{\rho}_n}{\partial\Xi_3} - \varrho_3\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3, \tag{27}$$

$$\rho_{n+1}(\xi_1,\xi_2,\xi_3,\tau) = \rho_n(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} {}_3\lambda_{\rho}(\Xi_1,\Xi_2,\Xi_3)\Bigg(\Delta\rho_n + \frac{\partial}{\partial x_1}\Big((\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\mu}_{1,n} - \varrho_1\Big) + \frac{\partial}{\partial x_2}\Big((\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\mu}_{2,n} - \varrho_2\Big) + \frac{\partial}{\partial x_3}\Big((\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\mu}_{3,n} - \varrho_3\Big)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3. \tag{28}$$

Here $\tilde{\mu}_{i,n}(i = 1,2), \tilde{\mu}_{j,n}(j = 1,2,3)$, and $\tilde{\rho}_n$ are restricted variations [16], i.e., $\delta\tilde{\mu}_{i,n} = 0, \delta\tilde{\mu}_{j,n} = 0$, and $\delta\tilde{\rho}_n = 0$. By imposing stationarity on the correction functionals (22)–(24) and (25)–(28) through Green's first identity, we derive the stationary conditions:

For (22)-(24):

$$\begin{aligned}\delta\mu_{1,n+1}(\xi_1,\xi_2,\tau) &= \delta\mu_{1,n}(\xi_1,\xi_2,\tau) + {}_2\lambda_{\mu_1}(\Xi_1,\Xi_2)\big(\delta\nabla\mu_{1,n}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\tau)\big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)} \\ &- \big(\nabla_2\lambda_{\mu_1}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2)\delta\mu_{1,n}(\Xi_1,\Xi_2,\tau)\big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)} \\ &+ \int_0^{\xi_2}\int_0^{\xi_1}\Delta_2\lambda_{\mu_1}(\Xi_1,\Xi_2)\delta\mu_{1,n}(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2 = 0,\end{aligned} \tag{29}$$

$$\begin{aligned}\delta\mu_{2,n+1}(\xi_1,\xi_2,\tau) &= \delta\mu_{2,n}(\xi_1,\xi_2,\tau) + {}_2\lambda_{\mu_2}(\Xi_1,\Xi_2)\big(\delta\nabla\mu_{2,n}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\tau)\big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)} \\ &- \big(\nabla_2\lambda_{\mu_2}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2)\delta\mu_{2,n}(\Xi_1,\Xi_2,\tau)\big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)} \\ &+ \int_0^{\xi_2}\int_0^{\xi_1}\Delta_2\lambda_{\mu_2}(\Xi_1,\Xi_2)\delta\mu_{2,n}(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2 = 0,\end{aligned} \tag{30}$$

$$\begin{aligned}\delta\rho_{n+1}(\xi_1,\xi_2,\tau) &= \delta\rho_n(\xi_1,\xi_2,\tau) + {}_2\lambda_{\rho}(\Xi_1,\Xi_2,\tau)(\delta\nabla\rho_n\cdot\boldsymbol{n})(\Xi_1,\Xi_2,\tau)\big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)} \\ &- \big(\nabla_2\lambda_{\rho}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2)\delta\rho_n(\Xi_1,\Xi_2,\tau)\big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)} \\ &+ \int_0^{\xi_2}\int_0^{\xi_1}\Delta_2\lambda_{\rho}(\Xi_1,\Xi_2)\delta\rho_n(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2 = 0.\end{aligned} \tag{31}$$

For (25)-(28):

$$\begin{aligned}\delta\mu_{1,n+1}(\xi_1,\xi_2,\xi_3,\tau) &= \delta\mu_{1,n}(\xi_1,\xi_2,\xi_3,\tau) + {}_3\lambda_{\mu_1}(\Xi_1,\Xi_2,\Xi_3)\big(\delta\nabla\mu_{1,n}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)} \\ &- \big(\nabla_3\lambda_{\mu_1}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3)\delta\mu_{1,n}(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)} \\ &+ \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1}\begin{matrix}\Delta_3\lambda_{\mu_1}(\Xi_1,\Xi_2,\Xi_3)\\ \times\,\delta\mu_{1,n}(\Xi_1,\Xi_2,\Xi_3,\tau)\end{matrix}d\Xi_1\,d\Xi_2\,d\Xi_3 = 0,\end{aligned} \tag{32}$$

$$\begin{aligned}
&\delta\mu_{2,n+1}(\xi_1,\xi_2,\xi_3,\tau)\\
&= \delta\mu_{2,n}(\xi_1,\xi_2,\xi_3,\tau)\\
&+ {}_3\lambda_{\mu_2}(\Xi_1,\Xi_2,\Xi_3)\big(\delta\nabla\mu_{2,n}\\
&\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)}\\
&- \big(\nabla_3\lambda_{\mu_2}\\
&\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3)\delta\mu_{2,n}(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)}\\
&+ \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1}\begin{pmatrix}\Delta_3\lambda_{\mu_2}(\Xi_1,\Xi_2,\Xi_3)\\ \times\,\delta\mu_{2,n}(\Xi_1,\Xi_2,\Xi_3,\tau)\end{pmatrix}d\Xi_1\,d\Xi_2\,d\Xi_3 = 0,
\end{aligned}\tag{33}$$

$$\begin{aligned}
&\delta\mu_{3,n+1}(\xi_1,\xi_2,\xi_3,\tau)\\
&= \delta\mu_{3,n}(\xi_1,\xi_2,\xi_3,\tau)\\
&+ {}_3\lambda_{\mu_3}(\Xi_1,\Xi_2,\Xi_3)\big(\delta\nabla\mu_{3,n}\\
&\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)}\\
&- \big(\nabla_3\lambda_{\mu_3}\\
&\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3)\delta\mu_{3,n}(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)}\\
&+ \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1}\begin{pmatrix}\Delta_3\lambda_{\mu_3}(\Xi_1,\Xi_2,\Xi_3)\\ \times\,\delta\mu_{3,n}(\Xi_1,\Xi_2,\Xi_3,\tau)\end{pmatrix}d\Xi_1\,d\Xi_2\,d\Xi_3 = 0,
\end{aligned}\tag{34}$$

$$\begin{aligned}
&\delta\rho_{n+1}(\xi_1,\xi_2,\xi_3,\tau)\\
&= \delta\rho_n(\xi_1,\xi_2,\xi_3,\tau)\\
&+ {}_3\lambda_{\rho}(\Xi_1,\Xi_2,\Xi_3,\tau)(\delta\nabla\rho_n\\
&\cdot\boldsymbol{n})(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)}\\
&- \big(\nabla_3\lambda_{\rho}\\
&\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3)\delta\rho_n(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)}\\
&+ \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1}\begin{pmatrix}\Delta_3\lambda_{\rho}(\Xi_1,\Xi_2,\Xi_3)\\ \times\,\delta\rho_n(\Xi_1,\Xi_2,\Xi_3,\tau)\end{pmatrix}d\Xi_1\,d\Xi_2\,d\Xi_3 = 0.
\end{aligned}\tag{35}$$

The systems of stationary conditions:

For $d = 2$: For $i = 1,2$,

$$\begin{cases}\begin{cases}\delta\mu_{i,n} & :\Delta_2\lambda_{\mu_i}(\Xi_1,\Xi_2) = 0,\\ \delta\nabla\mu_{i,n}\cdot\boldsymbol{n} & :{}_2\lambda_{\mu_i}(\Xi_1,\Xi_2)\big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)} = 0,\\ \delta\mu_{i,n} & :\big(\nabla_2\lambda_{\mu_i}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2)\big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)} = 1,\end{cases}\\ \begin{cases}\delta\rho_n & :\Delta_2\lambda_{\rho}(\Xi_1,\Xi_2) = 0,\\ \delta\nabla\rho_n\cdot\boldsymbol{n} & :{}_2\lambda_{\rho}(\Xi_1,\Xi_2)\big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)} = 0,\\ \delta\rho_n & :\big(\nabla_2\lambda_{\rho}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2)\big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)} = 1.\end{cases}\end{cases}\tag{36}$$

For $d = 3$: For $j = 1,2,3$,

$$\begin{cases}\begin{cases}\delta\mu_{j,n} & :\Delta_3\lambda_{\mu_j}(\Xi_1,\Xi_2,\Xi_3) = 0,\\ \delta\nabla\mu_{j,n}\cdot\boldsymbol{n} & :{}_3\lambda_{\mu_j}(\Xi_1,\Xi_2,\Xi_3)\big|_{\substack{(\Xi_1,\Xi_2,\Xi_3)\\=(\xi_1,\xi_2,\xi_3)}} = 0,\\ \delta\mu_{j,n} & :\Big(\nabla_3\lambda_{\mu_j}\cdot\boldsymbol{n}\Big)(\Xi_1,\Xi_2,\Xi_3)\big|_{\substack{(\Xi_1,\Xi_2,\Xi_3)\\=(\xi_1,\xi_2,\xi_3)}} = 1,\end{cases}\\ \begin{cases}\delta\rho_n & :\Delta_3\lambda_{\rho}(\Xi_1,\Xi_2,\Xi_3) = 0,\\ \delta\nabla\rho_n\cdot\boldsymbol{n} & :{}_3\lambda_{\rho}(\Xi_1,\Xi_2,\Xi_3)\big|_{\substack{(\Xi_1,\Xi_2,\Xi_3)\\=(\xi_1,\xi_2,\xi_3)}} = 0,\\ \delta\rho_n & :\big(\nabla_3\lambda_{\rho}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3)\big|_{\substack{(\Xi_1,\Xi_2,\Xi_3)\\=(\xi_1,\xi_2,\xi_3)}} = 1.\end{cases}\end{cases}\tag{37}$$

The solutions to (36) and (37) are non-unique:

For $d = 2$: For $i = 1,2$,

$$\begin{cases}{}_2\lambda_{\mu_j}(\Xi_1,\Xi_2) = \dfrac{\gamma_{2,\lambda_{\mu_j}}(\Xi_1,\Xi_2) - \gamma_{2,\lambda_{\mu_j}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\mu_j}}}},\\[2ex] {}_2\lambda_{\rho}(\Xi_1,\Xi_2) = \dfrac{\gamma_{2,\lambda_{\rho}}(\Xi_1,\Xi_2) - \gamma_{2,\lambda_{\rho}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\rho}}}},\end{cases}\tag{38}$$

For $d = 3$: For $j = 1,2,3$,

$$\begin{cases}{}_3\lambda_{\mu_j}(\Xi_1,\Xi_2,\Xi_3) = \dfrac{1}{a_{\gamma_{3,\lambda_{\mu_j}}}}\begin{pmatrix}\gamma_{3,\lambda_{\mu_j}}(\Xi_1,\Xi_2,\Xi_3)\\ -\gamma_{3,\lambda_{\mu_j}}(\xi_1,\xi_2,\xi_3)\end{pmatrix},\\[2ex] {}_3\lambda_{\rho}(\Xi_1,\Xi_2,\Xi_3) = \dfrac{1}{a_{\gamma_{3,\lambda_{\rho}}}}\begin{pmatrix}\gamma_{3,\lambda_{\rho}}(\Xi_1,\Xi_2,\Xi_3)\\ -\gamma_{3,\lambda_{\rho}}(\xi_1,\xi_2,\xi_3)\end{pmatrix}.\end{cases}\tag{39}$$

Here $\gamma_{2,\lambda_{\mu_j}}(j = 1,2), \gamma_{2,\lambda_\rho}, \gamma_{3,\lambda_{\mu_j}}(j = 1,2,3)$, and γ_{3,λ_ρ} are harmonic functions in $\mho$ satisfying the boundary conditions:

- For $d = 2$: For $i = 1,2$,
$$\nabla\gamma_{2,\lambda_{\mu_i}}\cdot n\big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)} = a_{\gamma_{2,\lambda_\rho}} \neq 0,$$
$$\nabla\gamma_{2,\lambda_{\rho}}\cdot n\big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)} = a_{\gamma_{2,\lambda_\rho}} \neq 0.$$
- For $d = 3$: For $j = 1,2,3$,
$$\nabla\gamma_{3,\lambda_{\mu_j}}\cdot n\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)} = a_{\gamma_{3,\lambda_{\mu_j}}} \neq 0,$$
$$\nabla\gamma_{3,\lambda_{\rho}}\cdot n\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)} = a_{\gamma_{3,\lambda_\rho}} \neq 0.$$

The proof is completed by substituting (38) and (39) into (22)-(24) and (25)-(28), respectively.

System (5) solutions' existence and uniqueness can be studied through the following theorems.

Theorem 2. Let the operators

$$\mathrm{M}: \mathfrak{L}^\infty\big(0,\mathrm{T},\mathfrak{H}^1(\mho)\big)\cap\mathfrak{L}^2\big(0,\mathrm{T},\boldsymbol{\mathcal{V}_\vartheta}(\mho)\big) \to \mathfrak{L}^\infty\big(0,\mathrm{T},\mathfrak{H}^1(\mho)\big)\cap\mathfrak{L}^2(0,\mathrm{T},\boldsymbol{\mathcal{V}_\vartheta}(\mho))$$

and

$$\mathrm{P}: \mathfrak{L}^2(0,\mathrm{T},\mathfrak{H}^1(\mho)) \to \mathfrak{L}^2(0,\mathrm{T},\mathfrak{H}^1(\mho))$$

be defined as follows:

For $d = 2$: let $\mathrm{M}[\boldsymbol{\mu}] = (\mathrm{M}[\mu_1],\mathrm{M}[\mu_2])$, where

$$\begin{aligned}
&\mathrm{M}_1[\mu_1](\xi_1,\xi_2,\tau)\\
&= \mu_1(\xi_1,\xi_2,\tau)\\
&+ \int_0^{\xi_2}\int_0^{\xi_1}\frac{\gamma_{2,\lambda_{\mu_1}}(\Xi_1,\Xi_2) - \gamma_{2,\lambda_{\mu_1}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\mu_1}}}}\Bigg(\Delta\mu_1\\
&-\frac{1}{\mathfrak{v}}\Big(\frac{\partial\mu_1}{\partial\tau} + (\boldsymbol{\mu}.\nabla)\mu_1 + \frac{\partial\rho}{\partial\Xi_1} - \varrho_1\Big)\Bigg)d\Xi_1\,d\Xi_2,
\end{aligned}\tag{40}$$

$$\begin{aligned}
&\mathrm{M}_2[\mu_2](\xi_1,\xi_2,\tau)\\
&= \mu_2(\xi_1,\xi_2,\tau)\\
&+ \int_0^{\xi_2}\int_0^{\xi_1}\frac{\gamma_{2,\lambda_{\mu_2}}(\Xi_1,\Xi_2) - \gamma_{2,\lambda_{\mu_2}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\mu_2}}}}\Bigg(\Delta\mu_2\\
&-\frac{1}{\mathfrak{v}}\Big(\frac{\partial\mu_2}{\partial\tau} + (\boldsymbol{\mu}.\nabla)\mu_2 + \frac{\partial\rho}{\partial\Xi_2} - \varrho_2\Big)\Bigg)d\Xi_1\,d\Xi_2,
\end{aligned}\tag{41}$$

$$\begin{aligned}
\mathrm{P}[\rho](\xi_1,\xi_2,\tau) = \rho(\xi_1,\xi_2,\tau)&\\
+ \int_0^{\xi_2}\int_0^{\xi_1}&\frac{\gamma_{2,\lambda_\rho}(\Xi_1,\Xi_2) - \gamma_{2,\lambda_\rho}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_\rho}}}\Bigg(\Delta\rho\\
&+\frac{\partial}{\partial x_1}((\boldsymbol{\mu}.\nabla)\mu_1 - \varrho_1)\\
&+\frac{\partial}{\partial x_2}((\boldsymbol{\mu}.\nabla)\mu_2 - \varrho_2)\Bigg)d\Xi_1\,d\Xi_2.
\end{aligned}\tag{42}$$

For $d = 3$: Let $\mathrm{M}[\boldsymbol{\mu}] = (\mathrm{M}_1[\mu_1], \mathrm{M}_2[\mu_2], \mathrm{M}_3[\mu_3])$, where

$$\mathrm{M}_1[\mu_1](\xi_1,\xi_2,\xi_3,\tau) = \mu_1(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_1}}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_{\mu_1}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\mu_1}}}} \Bigg(\Delta\mu_1 - \frac{1}{\mathfrak{v}}\left(\frac{\partial\mu_1}{\partial\tau} + (\boldsymbol{\mu}.\nabla)\mu_1 + \frac{\partial\rho}{\partial\Xi_1} - \varrho_1\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)\,d\Xi_1\,d\Xi_2\,d\Xi_3, \quad (43)$$

$$\mathrm{M}_2[\mu_2](\xi_1,\xi_2,\xi_3,\tau) = \mu_2(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_2}}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_{\mu_2}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\mu_2}}}} \Bigg(\Delta\mu_2 - \frac{1}{\mathfrak{v}}\left(\frac{\partial\mu_2}{\partial\tau} + (\boldsymbol{\mu}.\nabla)\mu_2 + \frac{\partial\rho}{\partial\Xi_2} - \varrho_2\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)\,d\Xi_1\,d\Xi_2\,d\Xi_3, \quad (44)$$

$$\mathrm{M}_3[\mu_3](\xi_1,\xi_2,\xi_3,\tau) = \mu_3(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_3}}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_{\mu_3}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\mu_3}}}} \Bigg(\Delta\mu_3 - \frac{1}{\mathfrak{v}}\left(\frac{\partial\mu_3}{\partial\tau} + (\mu.\nabla)\mu_3 + \frac{\partial\rho}{\partial\Xi_3} - \varrho_3\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)\,d\Xi_1\,d\Xi_2\,d\Xi_3, \quad (45)$$

$$\mathrm{P}[\rho](\xi_1,\xi_2,\xi_3,\tau) = \rho(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_\rho}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_\rho}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_\rho}}} \Bigg(\Delta\rho + \frac{\partial}{\partial x_1}\big((\boldsymbol{\mu}.\nabla)\mu_1 - \varrho_1\big) + \frac{\partial}{\partial x_2}\big((\boldsymbol{\mu}.\nabla)\mu_2 - \varrho_2\big) + \frac{\partial}{\partial x_3}\big((\boldsymbol{\mu}.\nabla)\mu_3 - \varrho_3\big)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)\,d\Xi_1\,d\Xi_2\,d\Xi_3. \quad (46)$$

If there exist constants $\mathfrak{C}_\mathrm{M} < 1$ and $\mathfrak{C}_\mathrm{P} < 1$ such that:

$$\|\mathrm{M}[\boldsymbol{\mu}] - \mathrm{M}[\tilde{\boldsymbol{\mu}}]\|_{\mathfrak{L}^2(0,\mathrm{T},\mathfrak{H}^2(\mho))} \leq \mathfrak{C}_\mathrm{M}\|\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}\|_{\mathfrak{L}^2(0,\mathrm{T},\mathfrak{H}^2(\mho))}, \quad (47)$$

$$\|\mathrm{P}[\rho] - \mathrm{P}[\tilde{\rho}]\|_{\mathfrak{L}^2(0,\mathrm{T},\mathfrak{H}^1(\mho))} \leq \mathfrak{C}_\mathrm{P}\|\rho - \tilde{\rho}\|_{\mathfrak{L}^2(0,\mathrm{T},\mathfrak{H}^1(\mho))}, \quad (48)$$

for all $\boldsymbol{\mu}, \tilde{\boldsymbol{\mu}} \in \mathfrak{L}^\infty(0,\mathrm{T},\mathfrak{H}^1(\mho)) \cap \mathfrak{L}^2(0,\mathrm{T},\boldsymbol{\mathcal{V}}_{\boldsymbol{\vartheta}}(\mho))$ and $\rho, \tilde{\rho} \in \mathfrak{L}^2(0,\mathrm{T},\mathfrak{H}^1(\mho))$, then system (5) has a unique solution.

Proof. The result follows directly from Banach's Fixed Point Theorem.

Theorem 3. Assume the operators $\mathrm{M}: \mathcal{K}_\mathrm{M} \to \mathcal{K}_\mathrm{M}$ and $\mathrm{P}: \mathcal{K}_\mathrm{P} \to \mathcal{K}_\mathrm{P}$ in Theorem 2 are defined on compact, convex subsets $\mathcal{K}_\mathrm{M}$ and $\mathcal{K}_\mathrm{P}$ of the spaces $\mathfrak{L}^\infty(0,\mathrm{T},\mathfrak{H}^1(\mho)) \cap \mathfrak{L}^2(0,\mathrm{T},\boldsymbol{\mathcal{V}}_{\boldsymbol{\vartheta}}(\mho))$ and $\mathfrak{L}^2(0,\mathrm{T},\mathfrak{H}^1(\mho))$, respectively. If M and P are continuous, then system (5) admits at least one solution.

Proof. The result follows directly from Schauder's fixed-point theorem.

Adopting a comparable methodology, we derive the following results for steady-state fluid flow.

Theorem 4. The correction functionals for the stationary fluid governed by

$$\begin{cases} (\boldsymbol{\mu}.\nabla)\boldsymbol{\mu} - \mathfrak{v}\Delta\boldsymbol{\mu} = -\nabla\rho + \boldsymbol{\varrho} & \text{in } \mho, \\ \nabla\cdot\boldsymbol{\mu} = 0 & \text{in } \mho, \\ \boldsymbol{\mu} = \boldsymbol{\vartheta} & \text{on } \partial\mho. \end{cases} \quad (49)$$

are defined as follows:

For $d = 2$:

$$\mu_{1,n+1}(\xi_1,\xi_2) = \mu_{1,n}(\xi_1,\xi_2) + \int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_{\mu_1}}(\Xi_1,\Xi_2) - \gamma_{2,\lambda_{\mu_1}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\mu_1}}}} \Bigg(\Delta\mu_{1,n} - \frac{1}{\mathfrak{v}}\left((\boldsymbol{\mu}_n\cdot\nabla)\mu_{1,n} + \frac{\partial\rho_n}{\partial\Xi_1} - \varrho_1\right)\Bigg)(\Xi_1,\Xi_2)\,d\Xi_1\,d\Xi_2, \quad (50)$$

$$\mu_{2,n+1}(\xi_1,\xi_2) = \mu_{2,n}(\xi_1,\xi_2) + \int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_{\mu_2}}(\Xi_1,\Xi_2) - \gamma_{2,\lambda_{\mu_2}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\mu_2}}}} \Bigg(\Delta\mu_{2,n} - \frac{1}{\mathfrak{v}}\left((\boldsymbol{\mu}_n\cdot\nabla)\mu_{2,n} + \frac{\partial\rho_n}{\partial\Xi_2} - \varrho_2\right)\Bigg)(\Xi_1,\Xi_2)\,d\Xi_1\,d\Xi_2, \quad (51)$$

$$\rho_{n+1}(\xi_1,\xi_2) = \rho_n(\xi_1,\xi_2) + \int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_\rho}(\Xi_1,\Xi_2) - \gamma_{2,\lambda_\rho}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_\rho}}} \Bigg(\Delta\rho_n + \frac{\partial}{\partial x_1}\big((\boldsymbol{\mu}_n\cdot\nabla)\mu_{1,n} - \varrho_1\big) + \frac{\partial}{\partial x_2}\big((\boldsymbol{\mu}_n\cdot\nabla)\mu_{2,n} - \varrho_2\big)\Bigg)(\Xi_1,\Xi_2)\,d\Xi_1\,d\Xi_2, \quad (52)$$

For $d = 3$:

$$\mu_{1,n+1}(\xi_1,\xi_2,\xi_3) = \mu_{1,n}(\xi_1,\xi_2,\xi_3) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_1}}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_{\mu_1}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\mu_1}}}} \Bigg(\Delta\mu_{1,n} - \frac{1}{\mathfrak{v}}\left((\boldsymbol{\mu}_n\cdot\nabla)\mu_{1,n} + \frac{\partial\rho_n}{\partial\Xi_1} - \varrho_1\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3)\,d\Xi_1\,d\Xi_2\,d\Xi_3, \quad (53)$$

$$\mu_{2,n+1}(\xi_1,\xi_2,\xi_3) = \mu_{2,n}(\xi_1,\xi_2,\xi_3) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_2}}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_{\mu_2}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\mu_2}}}}\Bigg(\Delta\mu_{2,n} - \frac{1}{\mathfrak{v}}\bigg((\boldsymbol{\mu}_n\cdot\nabla)\mu_{2,n} + \frac{\partial\rho_n}{\partial\Xi_2} - \varrho_2\bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3)d\Xi_1\, d\Xi_2\, d\Xi_3, \tag{54}$$

$$\mu_{3,n+1}(\xi_1,\xi_2,\xi_3) = \mu_{3,n}(\xi_1,\xi_2,\xi_3) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_3}}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_{\mu_3}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\mu_3}}}}\Bigg(\Delta\mu_{3,n} - \frac{1}{\mathfrak{v}}\bigg((\boldsymbol{\mu}_n.\nabla)\mu_{3,n} + \frac{\partial\rho_n}{\partial\Xi_3} - \varrho_3\bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3)d\Xi_1\, d\Xi_2\, d\Xi_3, \tag{55}$$

$$\rho_{n+1}(\xi_1,\xi_2,\xi_3) = \rho_n(\xi_1,\xi_2,\xi_3) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_\rho}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_\rho}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_\rho}}}\Bigg(\Delta\rho_n + \frac{\partial}{\partial x_1}\Big((\boldsymbol{\mu}_n\cdot\nabla)\mu_{1,n} - \varrho_1\Big) + \frac{\partial}{\partial x_2}\Big((\boldsymbol{\mu}_n\cdot\nabla)\mu_{2,n} - \varrho_2\Big) + \frac{\partial}{\partial x_3}\Big((\boldsymbol{\mu}_n\cdot\nabla)\mu_{3,n} - \varrho_3\Big)\Bigg)(\Xi_1,\Xi_2,\Xi_3)d\Xi_1\, d\Xi_2\, d\Xi_3. \tag{56}$$

Theorem 5. Let the operators
$\mathrm{M}: \mathfrak{H}^1(\mho) \cap \boldsymbol{\mathcal{V}}_{\boldsymbol{\vartheta}}(\mho) \to \mathfrak{H}^1(\mho) \cap \boldsymbol{\mathcal{V}}_{\boldsymbol{\vartheta}}(\mho)$, $\mathrm{P}: \mathfrak{H}^1(\mho) \to \mathfrak{H}^1(\mho)$
and

$$\mathrm{P}: \mathfrak{H}^1(\mho) \to \mathfrak{H}^1(\mho)$$

be defined as follows:
For $d = 2$:
$\mathrm{M}[\boldsymbol{\mu}] = (\mathrm{M}_1[\mu_1], \mathrm{M}_2[\mu_2])$,
where

$$\mathrm{M}_1[\mu_1](\xi_1,\xi_2) = \mu_1(\xi_1,\xi_2) + \int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_{\mu_1}}(\Xi_1,\Xi_2) - \gamma_{2,\lambda_{\mu_1}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\mu_1}}}}\Big(\Delta\mu_1 - \frac{1}{v}\big((\boldsymbol{\mu}.\nabla)\mu_1 + \frac{\partial\rho}{\partial\Xi_1} - \varrho_1\Big)(\Xi_1,\Xi_2)d\Xi_1\, d\Xi_2, \tag{57}$$

$$\mathrm{M}_2[\mu_2](\xi_1,\xi_2) = \mu_2(\xi_1,\xi_2) + \int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_{\mu_2}}(\Xi_1,\Xi_2) - \gamma_{2,\lambda_{\mu_2}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\mu_2}}}}\Big(\Delta\mu_2 - \frac{1}{\mathfrak{v}}\big((\boldsymbol{\mu}.\nabla)\mu_2 + \frac{\partial\rho}{\partial\Xi_2} - \varrho_2\Big)(\Xi_1,\Xi_2)d\Xi_1\, d\Xi_2, \tag{58}$$

$$\mathrm{P}[\rho](\xi_1,\xi_2) = \rho(\xi_1,\xi_2) + \int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_\rho}(\Xi_1,\Xi_2) - \gamma_{2,\lambda_\rho}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_\rho}}}\Bigg(\Delta\rho + \frac{\partial}{\partial x_1}\big((\boldsymbol{\mu}.\nabla)\mu_1 - \varrho_1\big) + \frac{\partial}{\partial x_2}\big((\boldsymbol{\mu}.\nabla)\mu_2 - \varrho_2\big)\Bigg)(\Xi_1,\Xi_2)d\Xi_1\, d\Xi_2. \tag{59}$$

For $d = 3$:

$$\mathrm{M}[\boldsymbol{\mu}] = (\mathrm{M}_1[\mu_1], \mathrm{M}_2[\mu_2], \mathrm{M}_3[\mu_3]),$$

where,

$$\mathrm{M}_1[\mu_1](\xi_1,\xi_2,\xi_3) = \mu_1(\xi_1,\xi_2,\xi_3) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_1}}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_{\mu_1}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3\lambda_{\mu_1}}}}\Bigg(\Delta\mu_1 - \frac{1}{\mathfrak{v}}\bigg(\frac{\partial\mu_1}{\partial\tau} + (\boldsymbol{\mu}\cdot\nabla)\mu_1 + \frac{\partial\rho}{\partial\Xi_1} - \varrho_1\bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3)d\Xi_1\, d\Xi_2\, d\Xi_3, \tag{60}$$

$$\mathrm{M}_2[\mu_2](\xi_1,\xi_2,\xi_3) = \mu_2(\xi_1,\xi_2,\xi_3) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_2}}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_{\mu_2}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\mu_2}}}}\Bigg(\Delta\mu_2 - \frac{1}{\mathfrak{v}}\bigg(\frac{\partial\mu_2}{\partial\tau} + (\boldsymbol{\mu}.\nabla)\mu_2 + \frac{\partial\rho}{\partial\Xi_2} - \varrho_2\bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3)d\Xi_1\, d\Xi_2\, d\Xi_3, \tag{61}$$

$$\mathrm{M}_3[\mu_3](\xi_1,\xi_2,\xi_3) = \mu_3(\xi_1,\xi_2,\xi_3) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_3}}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_{\mu_3}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\mu_3}}}}\Bigg(\Delta\mu_3 - \frac{1}{\mathfrak{v}}\bigg(\frac{\partial\mu_3}{\partial\tau} + (\boldsymbol{\mu}.\nabla)\mu_3 + \frac{\partial\rho}{\partial\Xi_3} - \varrho_3\bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3)d\Xi_1\, d\Xi_2\, d\Xi_3, \tag{62}$$

$$\mathrm{P}[\rho](\xi_1,\xi_2,\xi_3) = \rho(\xi_1,\xi_2,\xi_3) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_\rho}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_\rho}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_\rho}}}\Bigg(\Delta\rho + \frac{\partial}{\partial x_1}\big((\boldsymbol{\mu}.\nabla)\mu_1 - \varrho_1\big) + \frac{\partial}{\partial x_2}\big((\boldsymbol{\mu}.\nabla)\mu_2 - \varrho_2\big) + \frac{\partial}{\partial x_3}\big((\boldsymbol{\mu}.\nabla)\mu_3 - \varrho_3\big)\Bigg)(\Xi_1,\Xi_2,\Xi_3)d\Xi_1\, d\Xi_2\, d\Xi_3. \tag{63}$$

Suppose there exist $\mathfrak{C}_\mathrm{M} < 1$ and $\mathfrak{C}_\mathrm{P} < 1$ such that:

$$\|\mathrm{M}[\boldsymbol{\mu}] - \mathrm{M}[\tilde{\boldsymbol{\mu}}]\|_{\mathfrak{H}^2(\mho)} \le \mathfrak{C}_\mathrm{M}\|\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}\|_{\mathfrak{H}^2(\mho)}, \forall \boldsymbol{\mu}, \tilde{\boldsymbol{\mu}} \in \mathfrak{H}^1(\mho) \cap \boldsymbol{\mathcal{V}}_{\boldsymbol{\vartheta}}(\mho), \tag{64}$$

$$\|\mathrm{P}[\rho] - \mathrm{P}[\tilde{\rho}]\|_{\mathfrak{H}^1(\mho)} \le \mathfrak{C}_\mathrm{P}\|\rho - \tilde{\rho}\|_{\mathfrak{H}^1(\mho)}, \forall \rho, \tilde{\rho} \in \mathfrak{H}^1(\mho). \tag{65}$$

Then the system (49) has a unique solution.

Theorem 6. Assume the operators $\mathrm{M}: \mathcal{K}_{\mathrm{M}} \to \mathcal{K}_{\mathrm{M}}$ and $\mathrm{P}: \mathcal{K}_{\mathrm{P}} \to \mathcal{K}_{\mathrm{P}}$ in Theorem 5 are defined on compact and convex subsets $\mathcal{K}_{\mathrm{M}}$ and $\mathcal{K}_{\mathrm{P}}$ of $\mathfrak{H}^1(\mho) \cap V_g(\mho)$ and $\mathfrak{H}^1(\mho)$, respectively. If U and P are continuous, then system (49) has a solution.

IV. GVIM FOR INCOMPRESSIBLE MICROPOLAR FLUID FLOW SYSTEMS

The system of incompressible micropolar fluid flow is modeled using Navier-Stokes equations and microrotational velocity equations as follows [15, 17, 30, 32, 33, 36]: Given the external body force and moment (torques) $\boldsymbol{\varrho}_1, \boldsymbol{\varrho}_2 \in \mathfrak{L}^2\left(0, \mathrm{T}, \mathfrak{H}^{-1}(\mho)\right)$, boundary data $\boldsymbol{\vartheta}, \boldsymbol{\vartheta} \in \mathfrak{H}^1\left(0, \mathrm{T}, \mathfrak{H}^{\frac{1}{2}}(\partial\mho)\right)$, initial data $\boldsymbol{\mu}_0, \boldsymbol{\omega}_0$, and time $\mathrm{T} > 0$, find:

- Velocity field: $\boldsymbol{\mu}: \mho_{\mathrm{T}} \to \mathcal{R}^d$ in $\mathfrak{L}^\infty\left(0, \mathrm{T}, \mathfrak{H}^1(\mho)\right) \cap \mathfrak{L}^2\left(0, \mathrm{T}, \boldsymbol{\mathcal{V}}_{\boldsymbol{\vartheta}}(\mho)\right)$, where
$$\boldsymbol{\mathcal{V}}_{\boldsymbol{\vartheta}}(\mho) = \{\boldsymbol{\mu} \in \mathfrak{H}^2(\mho): \boldsymbol{\mu}|_{\partial\mho} = \boldsymbol{\vartheta}, \nabla \cdot \boldsymbol{\mu} = 0 \text{ in } \mho\},$$
- Microrotation field: $\boldsymbol{\omega}: \mho_{\mathrm{T}} \to \mathcal{R}^d$ in $\mathfrak{L}^\infty\left(0, \mathrm{T}, \mathfrak{H}^1(\mho)\right) \cap \mathfrak{L}^2\left(0, \mathrm{T}, \mathfrak{H}^2_{\boldsymbol{\vartheta}}(\mho)\right)$, where
$$\mathfrak{H}^2_{\boldsymbol{\vartheta}}(\mho) = \{\boldsymbol{\omega} \in \mathfrak{H}^2(\mho): \boldsymbol{\omega}|_{\partial\mho} = \boldsymbol{\vartheta}\},$$
- Pressure field: $\rho: \mho_{\mathrm{T}} \to \mathcal{R}$ in $\mathfrak{L}^2(0, \mathrm{T}, \mathfrak{H}^1(\mho))$,

such that, for $d = 2$:
$$\begin{cases} \dfrac{\partial \boldsymbol{\mu}}{\partial \tau} - (\mathfrak{v} + \mathfrak{v}_{\mathrm{r}})\Delta\boldsymbol{\mu} + (\boldsymbol{\mu} \cdot \nabla)\boldsymbol{\mu} + \nabla\rho = 2\mathfrak{v}_{\mathrm{r}}\nabla \times \boldsymbol{\omega} + \boldsymbol{\varrho}_1 & \text{in } \mho_{\mathrm{T}}, \\ \dfrac{\partial \boldsymbol{\omega}}{\partial \tau} - (\mathfrak{c}_{\mathrm{a}} + \mathfrak{c}_{\mathfrak{d}})\Delta\boldsymbol{\omega} + (\boldsymbol{\mu} \cdot \nabla)\boldsymbol{\omega} + 4\mathfrak{v}_{\mathrm{r}}\boldsymbol{\omega} = 2\mathfrak{v}_{\mathrm{r}}\nabla \times \boldsymbol{\mu} + \boldsymbol{\varrho}_2 & \text{in } \mho_{\mathrm{T}}, \\ \nabla \cdot \boldsymbol{\mu} = 0 & \text{in } \mho_{\mathrm{T}}, \end{cases} \tag{66}$$
and for $d = 3$:
$$\begin{cases} \dfrac{\partial \boldsymbol{\mu}}{\partial \tau} - (\mathfrak{v} + \mathfrak{v}_{\mathrm{r}})\Delta\boldsymbol{\mu} + (\boldsymbol{\mu} \cdot \nabla)\boldsymbol{\mu} + \nabla\rho = 2\mathfrak{v}_{\mathrm{r}}\nabla \times \boldsymbol{\omega} + \boldsymbol{\varrho}_1 & \text{in } \mho_{\mathrm{T}}, \\ \dfrac{\partial \boldsymbol{\omega}}{\partial \tau} - (\mathfrak{c}_{\mathrm{a}} + \mathfrak{c}_{\mathfrak{d}})\Delta\boldsymbol{\omega} - (\mathfrak{c}_0 + \mathfrak{c}_{\mathfrak{d}} - \mathfrak{c}_{\mathrm{a}})\nabla(\nabla \cdot \boldsymbol{\omega}) + (\boldsymbol{\mu} \cdot \nabla)\boldsymbol{\omega} + 4\mathfrak{v}_{\mathrm{r}}\boldsymbol{\omega} = 2\mathfrak{v}_{\mathrm{r}}\nabla \times \boldsymbol{\mu} + \boldsymbol{\varrho}_2 & \text{in } \mho_{\mathrm{T}}, \\ \nabla \cdot \boldsymbol{\mu} = 0 & \text{in } \mho_{\mathrm{T}}, \end{cases} \tag{67}$$
with the boundary conditions imposed:
$$\begin{cases} \boldsymbol{\mu} = \boldsymbol{\vartheta}_1, \boldsymbol{\omega} = \boldsymbol{\vartheta}_2 & \text{on } \partial\mho_{\mathrm{T}}, \\ \boldsymbol{\mu}(\boldsymbol{x}, 0) = \boldsymbol{\mu}_0(\boldsymbol{x}), \boldsymbol{\omega}(\boldsymbol{x}, 0) = \boldsymbol{\omega}_0(\boldsymbol{x}) & \text{in } \mho, \end{cases}$$
where:

- For $d = 2$:
$\boldsymbol{\mu}(\xi_1, \xi_2, \tau) = (\mu_1(\xi_1, \xi_2, \tau), \mu_2(\xi_1, \xi_2, \tau), 0), \boldsymbol{\omega}(\xi_1, \xi_2, \tau) = \left(0,0, \omega_3(\xi_1, \xi_2, \tau)\right), \rho(\xi_1, \xi_2, \tau)$.
- For $d = 3$:
$\boldsymbol{\mu}(\xi_1, \xi_2, \xi_3, \tau) = \left(\mu_1(\xi_1, \xi_2, \xi_3, \tau), \mu_2(\xi_1, \xi_2, \xi_3, \tau), \mu_3(\xi_1, \xi_2, \xi_3, \tau)\right)$,
$\boldsymbol{\omega}(\xi_1, \xi_2, \xi_3, \tau) = (\omega_1(\xi_1, \xi_2, \xi_3, \tau), \omega_2(\xi_1, \xi_2, \xi_3, \tau), \omega_3(\xi_1, \xi_2, \xi_3, \tau),)$,
$\rho(\xi_1, \xi_2, \xi_3, \tau)$.

Here, the positive constants $\mathfrak{v}, \mathfrak{v}_{\mathrm{r}}, \mathfrak{c}_0, \mathfrak{c}_{\mathrm{a}}, \mathfrak{c}_{\mathfrak{d}}$ represent viscosity coefficients, where $\mathfrak{v}$ is the Newtonian viscosity and $\mathfrak{v}_{\mathrm{r}}$ is the microrotation viscosity. The constants $\mathfrak{c}_0, \mathfrak{c}_{\mathfrak{d}}$ and $\mathfrak{c}_{\mathrm{a}}$ satisfy $\mathfrak{c}_0 + \mathfrak{c}_{\mathfrak{d}} > \mathfrak{c}_{\mathrm{a}}$.

Theorem 7. The correction functionals for the systems (66) and (67) are as follows:
For $d = 2$:
$$\begin{aligned} &\mu_{1,n+1}(\xi_1, \xi_2, \tau) \\ &= \mu_{1,n}(\xi_1, \xi_2, \tau) \\ &+ \int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_{\mu_1}}(\Xi_1, \Xi_2) - \gamma_{2,\lambda_{\mu_1}}(\xi_1, \xi_2)}{a_{\gamma_{2,\lambda_{\mu_1}}}}\Bigg(\Delta\mu_{1,n} \\ &- \frac{1}{(\mathfrak{v} + \mathfrak{v}_{\mathrm{r}})}\left(\frac{\partial \mu_{1,n}}{\partial \tau} + (\boldsymbol{\mu}_n \cdot \nabla)\mu_{1,n} + \frac{\partial \rho_n}{\partial \Xi_1} - 2\mathfrak{v}_{\mathrm{r}}\frac{\partial \omega_{3,n}}{\partial \Xi_2} \right. \\ &\left. - \varrho_{1,1}\right)\Bigg)(\Xi_1, \Xi_2, \tau)d\Xi_1\, d\Xi_2, \end{aligned} \tag{68}$$

$$\begin{aligned} &\mu_{2,n+1}(\xi_1, \xi_2, \tau) \\ &= \mu_{2,n}(\xi_1, \xi_2, \tau) \\ &+ \int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_{\mu_2}}(\Xi_1, \Xi_2) - \gamma_{2,\lambda_{\mu_2}}(\xi_1, \xi_2)}{a_{\gamma_{2,\lambda_{\mu_2}}}}\Bigg(\Delta\mu_{2,n} \\ &- \frac{1}{(\mathfrak{v} + \mathfrak{v}_{\mathrm{r}})}\left(\frac{\partial \mu_{2,n}}{\partial \tau} + (\boldsymbol{\mu}_n \cdot \nabla)\mu_{2,n} + \frac{\partial \rho_n}{\partial \Xi_2} + 2\mathfrak{v}_{\mathrm{r}}\frac{\partial \omega_{3,n}}{\partial \Xi_1} \right. \\ &\left. - \varrho_{1,2}\right)\Bigg)(\Xi_1, \Xi_2, \tau)d\Xi_1\, d\Xi_2, \end{aligned} \tag{69}$$

$$\begin{aligned} &\omega_{3,n+1}(\xi_1, \xi_2, \tau) \\ &= \omega_{3,n}(\xi_1, \xi_2, \tau) \\ &+ \int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_{\omega_3}}(\Xi_1, \Xi_2) - \gamma_{2,\lambda_{\omega_3}}(\xi_1, \xi_2)}{a_{\gamma_{2,\lambda_{\omega_3}}}}\Bigg(\Delta\omega_{3,n} \\ &- \frac{1}{(\mathfrak{c}_{\mathrm{a}} + \mathfrak{c}_{\mathfrak{d}})}\left(\frac{\partial \omega_{3,n}}{\partial \tau} + (\boldsymbol{\mu}_n \cdot \nabla)\omega_{3,n} + 4\mathfrak{v}_{\mathrm{r}}\omega_{3,n} \right. \\ &\left. - 2\mathfrak{v}_{\mathrm{r}}\left(\frac{\partial \mu_{2,n}}{\partial \Xi_1} - \frac{\partial \mu_{1,n}}{\partial \Xi_2}\right) - \varrho_{2,3}\right)\Bigg)(\Xi_1, \Xi_2, \tau)d\Xi_1\, d\Xi_2, \end{aligned} \tag{70}$$

$$\begin{aligned} \rho_{n+1}(\xi_1, \xi_2, \tau) = \rho_n(\xi_1, \xi_2, \tau) &+ \int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_\rho}(\Xi_1, \Xi_2) - \gamma_{2,\lambda_\rho}(\xi_1, \xi_2)}{a_{\gamma_{2,\lambda_\rho}}}\Bigg(\Delta\rho_n \\ &+ \frac{\partial}{\partial x_1}\left((\boldsymbol{\mu}_n \cdot \nabla)\mu_{1,n} - \varrho_{1,1}\right) \\ &+ \frac{\partial}{\partial x_2}\left((\boldsymbol{\mu}_n \cdot \nabla)\mu_{2,n} \right. \\ &\left. - \varrho_{1,2}\right)\Bigg)(\Xi_1, \Xi_2, \tau)d\Xi_1\, d\Xi_2. \end{aligned} \tag{71}$$

For $d = 3$:
$$\begin{aligned} &\mu_{1,n+1}(\xi_1, \xi_2, \xi_3, \tau) \\ &= \mu_{1,n}(\xi_1, \xi_2, \xi_3, \tau) \\ &+ \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_1}}(\Xi_1, \Xi_2, \Xi_3) - \gamma_{3,\lambda_{\mu_1}}(\xi_1, \xi_2, \xi_3)}{a_{\gamma_{3,\lambda_{\mu_1}}}}\Bigg(\Delta\mu_{1,n} \\ &- \frac{1}{(\mathfrak{v} + \mathfrak{v}_{\mathrm{r}})}\left(\frac{\partial \mu_{1,n}}{\partial \tau} + (\boldsymbol{\mu}_n \cdot \nabla)\mu_{1,n} + \frac{\partial \rho_n}{\partial \Xi_1} \right. \\ &- 2\mathfrak{v}_{\mathrm{r}}\left(\frac{\partial \omega_{3,n}}{\partial \Xi_2} - \frac{\partial \omega_{2,n}}{\partial \Xi_3}\right) \\ &\left. - \varrho_{1,1}\right)\Bigg)(\Xi_1, \Xi_2, \Xi_3, \tau)d\Xi_1\, d\Xi_2\, d\Xi_3, \end{aligned} \tag{72}$$

$$\mu_{2,n+1}(\xi_1,\xi_2,\xi_3,\tau) = \mu_{2,n}(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_2}}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_{\mu_2}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\mu_2}}}} \Bigg(\Delta\mu_{2,n} - \frac{1}{(\mathfrak{v}+\mathfrak{v}_r)}\left(\frac{\partial\mu_{2,n}}{\partial\tau} + (\boldsymbol{\mu}_n\cdot\nabla)\mu_{2,n} + \frac{\partial\rho_n}{\partial\Xi_2} - 2\,\mathfrak{v}r_r\left(\frac{\partial\omega_{1,n}}{\partial\Xi_3} - \frac{\partial\omega_{3,n}}{\partial\Xi_1}\right) - \varrho_{1,2}\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3, \quad (73)$$

$$\mu_{3,n+1}(\xi_1,\xi_2,\xi_3,\tau) = \mu_{3,n}(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_3}}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_{\mu_3}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\mu_3}}}} \Bigg(\Delta\mu_{3,n} - \frac{1}{(\mathfrak{v}+\mathfrak{v}_r)}\left(\frac{\partial\mu_{3,n}}{\partial\tau} + (\boldsymbol{\mu}_n\cdot\nabla)\mu_{3,n} + \frac{\partial\rho_n}{\partial\Xi_3} - 2\,\mathfrak{v}_r\left(\frac{\partial\omega_{2,n}}{\partial\Xi_1} - \frac{\partial\omega_{1,n}}{\partial\Xi_2}\right) - \varrho_{1,3}\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3, \quad (74)$$

$$\omega_{1,n+1}(\xi_1,\xi_2,\xi_3,\tau) = \omega_{1,n}(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\omega_1}}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_{\omega_1}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\omega_1}}}} \Bigg(\Delta\omega_{1,n} - \frac{1}{(\mathfrak{c}_a+\mathfrak{c}_b)}\left(\frac{\partial\omega_{1,n}}{\partial\tau} - (\mathfrak{c}_0+\mathfrak{c}_b-\mathfrak{c}_a)\frac{\partial}{\partial\Xi_1}(\nabla\cdot\boldsymbol{\omega}_n) + (\boldsymbol{\mu}_n\cdot\nabla)\omega_{1,n} + 4\mathfrak{v}_r\omega_{1,n} - 2\,\mathfrak{v}_r\left(\frac{\partial\mu_{3,n}}{\partial\Xi_2} - \frac{\partial\mu_{2,n}}{\partial\Xi_3}\right) - \varrho_{2,1}\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3, \quad (75)$$

$$\omega_{2,n+1}(\xi_1,\xi_2,\xi_3,\tau) = \omega_{2,n}(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\omega_2}}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_{\omega_2}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\omega_2}}}} \Bigg(\Delta\omega_{2,n} - \frac{1}{(\mathfrak{c}_a+\mathfrak{c}_b)}\left(\frac{\partial\omega_{2,n}}{\partial\tau} - (\mathfrak{c}_0+\mathfrak{c}_b-\mathfrak{c}_a)\frac{\partial}{\partial\Xi_2}(\nabla\cdot\boldsymbol{\omega}_n) + (\boldsymbol{\mu}_n\cdot\nabla)\omega_{2,n} + 4\mathfrak{v}_r\omega_{2,n} - 2\,\mathfrak{v}r_r\left(\frac{\partial\mu_{1,n}}{\partial\Xi_3} - \frac{\partial\mu_{3,n}}{\partial\Xi_1}\right) - \varrho_{2,2}\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3, \quad (76)$$

$$\omega_{3,n+1}(\xi_1,\xi_2,\xi_3,\tau) = \omega_{3,n}(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\omega_3}}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_{\omega_3}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_3\lambda_{\omega_3}}} \Bigg(\Delta\omega_{3,n} - \frac{1}{(\mathfrak{c}_a+\mathfrak{c}_b)}\left(\frac{\partial\omega_{3,n}}{\partial\tau} - (\mathfrak{c}_0+\mathfrak{c}_b-\mathfrak{c}_a)\frac{\partial}{\partial\Xi_3}(\nabla\cdot\boldsymbol{\omega}_n) + (\boldsymbol{\mu}_n\cdot\nabla)\omega_{3,n} + 4\mathfrak{v}_r\omega_{3,n} - 2\mathfrak{v}_r\left(\frac{\partial\mu_{2,n}}{\partial\Xi_1} - \frac{\partial\mu_{1,n}}{\partial\Xi_2}\right) - \varrho_{2,3}\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3, \quad (77)$$

$$\rho_{n+1}(\xi_1,\xi_2,\xi_3,\tau) = \rho_n(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_\rho}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_\rho}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_\rho}}} \Bigg(\Delta\rho_n + \frac{\partial}{\partial x_1}\Big((\boldsymbol{\mu}_n\cdot\nabla)\mu_{1,n} - \varrho_{1,1}\Big) + \frac{\partial}{\partial x_2}\Big((\boldsymbol{\mu}_n\cdot\nabla)\mu_{2,n} - \varrho_{1,2}\Big) + \frac{\partial}{\partial x_3}\Big((\boldsymbol{\mu}_n\cdot\nabla)\mu_{3,n} - \varrho_{1,3}\Big)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3. \quad (78)$$

Proof. We rewrite both systems (66) and (67) as:

$$\begin{cases} \Delta\boldsymbol{\mu} = \dfrac{1}{(\mathfrak{v}+\mathfrak{v}_r)}\begin{pmatrix}\dfrac{\partial\boldsymbol{\mu}}{\partial\tau} + (\boldsymbol{\mu}\cdot\nabla)\boldsymbol{\mu} + \nabla\rho \\ -2\mathfrak{v}_r\nabla\times\boldsymbol{\omega} - \boldsymbol{\varrho}_1\end{pmatrix} & \text{in } \mho_T, \\ \triangle\,\boldsymbol{\omega} = \dfrac{1}{(\mathfrak{c}_a+\mathfrak{c}_b)}\begin{pmatrix}\dfrac{\partial\boldsymbol{\omega}}{\partial\tau} + (\boldsymbol{\mu}\cdot\nabla)\boldsymbol{\omega} + 4\mathfrak{v}_r\boldsymbol{\omega} \\ -2\mathfrak{v}_r\nabla\times\boldsymbol{\mu} - \boldsymbol{\varrho}_2\end{pmatrix} & \text{in } \mho_T, \\ \nabla\cdot\boldsymbol{\mu} = 0 & \text{in } \mho_T, \end{cases} \quad (79)$$

and

$$\begin{cases} \Delta\boldsymbol{\mu} = \dfrac{1}{(\mathfrak{v}+\mathfrak{v}_r)}\begin{pmatrix}\dfrac{\partial\boldsymbol{\mu}}{\partial\tau} + (\boldsymbol{\mu}\cdot\nabla)\boldsymbol{\mu} + \nabla\rho \\ -2\mathfrak{v}_r\nabla\times\boldsymbol{\omega} - \boldsymbol{\varrho}_1\end{pmatrix} & \text{in } \mho_T, \\ \triangle\,\boldsymbol{\omega} = \dfrac{1}{(\mathfrak{c}_a+\mathfrak{c}_b)}\begin{pmatrix}\dfrac{\partial\boldsymbol{\omega}}{\partial\tau} - (\mathfrak{c}_0+\mathfrak{c}_b-\mathfrak{c}_a)\nabla(\nabla\cdot\boldsymbol{\omega}) \\ +(\boldsymbol{\mu}\cdot\nabla)\boldsymbol{\omega} + 4\mathfrak{v}_r\boldsymbol{\omega} \\ -2\mathfrak{v}_r\nabla\times\boldsymbol{\mu} - \boldsymbol{\varrho}_2\end{pmatrix} & \text{in } \mho_T, \\ \nabla\cdot\boldsymbol{\mu} = 0 & \text{in } \mho_T, \end{cases} \quad (80)$$

respectively. Since $\boldsymbol{\mu}, \boldsymbol{\omega}$ and ρ do not appear in a symmetric manner with respect to time, we apply the divergence operator $(\nabla\,\cdot)$ to (66.1) (and similarly for (67.1)) to derive the Neumann problem for ρ:

$$\begin{cases} \triangle\,\rho = -\nabla\cdot\left((\boldsymbol{\mu}\cdot\nabla)\boldsymbol{\mu} - \boldsymbol{\varrho}_1\right) & \text{in } \mho_T, \\ \frac{\partial\rho}{\partial\boldsymbol{n}} = \begin{pmatrix}(\mathfrak{v}+\mathfrak{v}_r)\Delta\boldsymbol{\vartheta} - \frac{\partial\boldsymbol{\vartheta}}{\partial\tau} - (\boldsymbol{\vartheta}\cdot\nabla)\boldsymbol{\vartheta} \\ +2\mathfrak{v}_r\nabla\times\boldsymbol{\vartheta} + \boldsymbol{\varrho}_1\end{pmatrix}\cdot\boldsymbol{n} & \text{on } \partial\mho_T, \end{cases} \quad (81)$$

where $\boldsymbol{n}$ is the outward unit normalto $\partial\mho$.

For $d = 2$, expanding (79.1), (79.2), and (81.1) yields:

$$\Delta\mu_1 = \frac{1}{(\mathfrak{v}+\mathfrak{v}_r)}\left(\frac{\partial\mu_1}{\partial\tau} + (\boldsymbol{\mu}\cdot\nabla)\mu_1 + \frac{\partial\rho}{\partial\xi_1} + 2\mathfrak{v}_r\frac{\partial\omega_3}{\partial\xi_2} - \varrho_{1,1}\right), \quad (82)$$

$$\Delta\mu_2 = \frac{1}{(\mathfrak{v}+\mathfrak{v}_r)}\left(\frac{\partial\mu_2}{\partial\tau} + (\boldsymbol{\mu}\cdot\nabla)\mu_2 + \frac{\partial\rho}{\partial\xi_2} + 2\mathfrak{v}_r\frac{\partial\omega_3}{\partial\xi_1} - \varrho_{1,2}\right), \quad (83)$$

$$\Delta\omega_3 = \frac{1}{(\mathfrak{c}_a+\mathfrak{c}_b)}\left(\frac{\partial\omega_3}{\partial\tau} + (\boldsymbol{\mu}\cdot\nabla))\omega_3 + 4\mathfrak{v}_r\omega_3 - 2\mathfrak{v}_r\left(\frac{\partial\mu_2}{\partial\xi_1} - \frac{\partial\mu_1}{\partial\xi_2}\right) - \varrho_{2,3}\right), \quad (84)$$

$$\Delta\rho = -\frac{\partial}{\partial x_1}\Big((\boldsymbol{\mu}\cdot\nabla)\mu_1 - \varrho_{1,1}\Big) - \frac{\partial}{\partial x_2}\Big((\boldsymbol{\mu}\cdot\nabla)\mu_2 - \varrho_{1,2}\Big). \quad (85)$$

For $d = 3$, expanding (80.1), (80.2), and (81.1) yields:

$$\Delta\mu_1 = \frac{1}{(\mathfrak{v}+\mathfrak{v}_r)}\left(\frac{\partial\mu_1}{\partial\tau} + (\boldsymbol{\mu}\cdot\nabla)\mu_1 + \frac{\partial\rho}{\partial\xi_1} - 2\mathfrak{v}_r\left(\frac{\partial\omega_3}{\partial\xi_2} - \frac{\partial\omega_2}{\partial\xi_3}\right) - \varrho_{1,1}\right), \quad (86)$$

$$\Delta\mu_2 = \frac{1}{(\mathfrak{v}+\mathfrak{v}_{\mathrm{r}})}\Big(\frac{\partial\mu_2}{\partial\tau} + (\boldsymbol{\mu}\cdot\nabla)\mu_2 + \frac{\partial\rho}{\partial\xi_2} - 2\mathfrak{v}_{\mathrm{r}}\Big(\frac{\partial\omega_1}{\partial\xi_3} - \frac{\partial\omega_3}{\partial\xi_1}\Big) - \varrho_{1,2}\Big), \quad (87)$$

$$\Delta\mu_3 = \frac{1}{(\mathfrak{v}+\mathfrak{v}_{\mathrm{r}})}\Big(\frac{\partial\mu_3}{\partial\tau} + (\boldsymbol{\mu}\cdot\nabla)\mu_3 + \frac{\partial\rho}{\partial\xi_3} - 2\mathfrak{v}_{\mathrm{r}}\Big(\frac{\partial\omega_2}{\partial\xi_1} - \frac{\partial\omega_1}{\partial\xi_2}\Big) - \varrho_{1,3}\Big), \quad (88)$$

$$\Delta\omega_1 = \frac{1}{(\mathfrak{c}_{\mathrm{a}}+\mathfrak{c}_{\mathfrak{d}})}\Big(\frac{\partial\omega_1}{\partial\tau} - (\mathfrak{c}_0+\mathfrak{c}_{\mathfrak{d}}-\mathfrak{c}_{\mathrm{a}})\frac{\partial}{\partial\xi_1}(\nabla\cdot\boldsymbol{\omega}) + (\boldsymbol{\mu}\cdot\nabla)\omega_1 + 4\mathfrak{v}_{\mathrm{r}}\omega_1 - 2\mathfrak{v}_{\mathrm{r}}\Big(\frac{\partial\mu_3}{\partial\xi_2} - \frac{\partial\mu_2}{\partial\xi_3}\Big) - \varrho_{2,1}\Big), \quad (89)$$

$$\Delta\omega_2 = \frac{1}{(\mathfrak{c}_{\mathrm{a}}+\mathfrak{c}_{\mathfrak{d}})}\Big(\frac{\partial\omega_2}{\partial\tau} - (\mathfrak{c}_0+\mathfrak{c}_{\mathfrak{d}}-\mathfrak{c}_{\mathrm{a}})\frac{\partial}{\partial\xi_2}(\nabla\cdot\boldsymbol{\omega}) + (\boldsymbol{\mu}\cdot\nabla)\omega_2 + 4\mathfrak{v}_{\mathrm{r}}\omega_2 - 2\mathfrak{v}_{\mathrm{r}}\Big(\frac{\partial\mu_1}{\partial\xi_3} - \frac{\partial\mu_3}{\partial\xi_1}\Big) - \varrho_{2,2}\Big), \quad (90)$$

$$\Delta\omega_3 = \frac{1}{(\mathfrak{c}_{\mathrm{a}}+\mathfrak{c}_{\mathfrak{d}})}\Big(\frac{\partial\omega_3}{\partial\tau} - (\mathfrak{c}_0+\mathfrak{c}_{\mathfrak{d}}-\mathfrak{c}_{\mathrm{a}})\frac{\partial}{\partial\xi_3}(\nabla\cdot\boldsymbol{\omega}) + (\boldsymbol{\mu}\cdot\nabla)\omega_3 + 4\mathfrak{v}_{\mathrm{r}}\omega_3 - 2\mathfrak{v}_{\mathrm{r}}\Big(\frac{\partial\mu_2}{\partial\xi_1} - \frac{\partial\mu_1}{\partial\xi_2}\Big) - \varrho_{2,3}\Big), \quad (91)$$

$$\Delta\rho = -\frac{\partial}{\partial x_1}\big((\boldsymbol{\mu}\cdot\nabla)\mu_1 - \varrho_{1,1}\big) - \frac{\partial}{\partial x_2}\big((\boldsymbol{\mu}\cdot\nabla)\mu_2 - \varrho_{1,2}\big) - \frac{\partial}{\partial x_3}\big((\boldsymbol{\mu}\cdot\nabla)\mu_3 - \varrho_{1,3}\big). \quad (92)$$

The correction functionals for (82)-(85) on $\mho = [0,\xi_1]\times[0,\xi_2]$ are:

$$\mu_{1,n+1}(\xi_1,\xi_2,\tau) = \mu_{1,n}(\xi_1,\xi_2,\tau) + \int_0^{\xi_2}\int_0^{\xi_1} {}_2\lambda_{\mu_1}(\Xi_1,\Xi_2)\Big(\Delta\mu_{1,n} - \frac{1}{(\mathfrak{v}+\mathfrak{v}_{\mathrm{r}})}\Big(\frac{\partial\tilde{\mu}_{1,n}}{\partial\tau} + (\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\mu}_{1,n} + \frac{\partial\tilde{\rho}_n}{\partial\Xi_1} - 2\mathfrak{v}_{\mathrm{r}}\frac{\partial\tilde{\omega}_{3,n}}{\partial\Xi_2} - \varrho_{1,1}\Big)\Big)(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2, \quad (93)$$

$$\mu_{2,n+1}(\xi_1,\xi_2,\tau) = \mu_{2,n}(\xi_1,\xi_2,\tau) + \int_0^{\xi_2}\int_0^{\xi_1} {}_2\lambda_{\mu_2}(\Xi_1,\Xi_2)\Big(\Delta\mu_{2,n} - \frac{1}{(\mathfrak{v}+\mathfrak{v}_{\mathrm{r}})}\Big(\frac{\partial\tilde{\mu}_{2,n}}{\partial\tau} + (\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\mu}_{2,n} + \frac{\partial\tilde{\rho}_n}{\partial\Xi_2} + 2\mathfrak{v}_{\mathrm{r}}\frac{\partial\tilde{\omega}_{3,n}}{\partial\Xi_1} - \varrho_{1,2}\Big)\Big)(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2, \quad (94)$$

$$\omega_{3,n+1}(\xi_1,\xi_2,\tau) = \omega_{3,n}(\xi_1,\xi_2,\tau) + \int_0^{\xi_2}\int_0^{\xi_1} {}_2\lambda_{\omega_3}(\Xi_1,\Xi_2)\Big(\Delta\omega_{3,n} - \frac{1}{(c_{\mathrm{a}}+c_{\mathfrak{d}})}\Big(\frac{\partial\tilde{\omega}_{3,n}}{\partial\tau} + (\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\omega}_{3,n} + 4\mathfrak{v}_{\mathrm{r}}\tilde{\omega}_{3,n} - 2\mathfrak{v}_{\mathrm{r}}\Big(\frac{\partial\tilde{\mu}_{2,n}}{\partial\Xi_1} - \frac{\partial\tilde{\mu}_{1,n}}{\partial\Xi_2}\Big) - \varrho_{2,3}\Big)\Big)(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2, \quad (95)$$

$$\rho_{n+1}(\xi_1,\xi_2,\tau) = \rho_n(\xi_1,\xi_2,\tau) + \int_0^{\xi_2}\int_0^{\xi_1} {}_2\lambda_{\rho}(\Xi_1,\Xi_2)\Big(\Delta\rho_n + \frac{\partial}{\partial x_1}\big((\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\mu}_{1,n} - \varrho_{1,1}\big) + \frac{\partial}{\partial x_2}\big((\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\mu}_{2,n} - \varrho_{1,2}\big)\Big)(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2. \quad (96)$$

The correction functionals for (86)-(92) on $\mho = [0,\xi_1]\times[0,\xi_2]\times[0,\xi_3]$ are:

$$\mu_{1,n+1}(\xi_1,\xi_2,\xi_3,\tau) = \mu_{1,n}(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} {}_3\lambda_{\mu_1}(\Xi_1,\Xi_2,\Xi_3)\Big(\Delta\mu_{1,n} - \frac{1}{(\mathfrak{v}+\mathfrak{v}_{\mathrm{r}})}\Big(\frac{\partial\tilde{\mu}_{1,n}}{\partial\tau} + (\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\mu}_{1,n} + \frac{\partial\tilde{\rho}_n}{\partial\Xi_1} - 2\mathfrak{v}_{\mathrm{r}}\Big(\frac{\partial\tilde{\omega}_{3,n}}{\partial\Xi_2} - \frac{\partial\tilde{\omega}_{2,n}}{\partial\Xi_3}\Big) - \varrho_{1,1}\Big)\Big)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3, \quad (97)$$

$$\mu_{2,n+1}(\xi_1,\xi_2,\xi_3,\tau) = \mu_{2,n}(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} {}_3\lambda_{\mu_2}(\Xi_1,\Xi_2,\Xi_3)\Big(\Delta\mu_{2,n} - \frac{1}{(\mathfrak{v}+\mathfrak{v}_{\mathrm{r}})}\Big(\frac{\partial\tilde{\mu}_{2,n}}{\partial\tau} + (\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\mu}_{2,n} + \frac{\partial\tilde{\rho}_n}{\partial\Xi_2} - 2\mathfrak{v}_{\mathrm{r}}\Big(\frac{\partial\tilde{\omega}_{1,n}}{\partial\Xi_3} - \frac{\partial\tilde{\omega}_{3,n}}{\partial\Xi_1}\Big) - \varrho_{1,2}\Big)\Big)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3, \quad (98)$$

$$\mu_{3,n+1}(\xi_1,\xi_2,\xi_3,\tau) = \mu_{3,n}(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} {}_3\lambda_{\mu_3}(\Xi_1,\Xi_2,\Xi_3)\Big(\Delta\mu_{3,n} - \frac{1}{(\mathfrak{v}+\mathfrak{v}_{\mathrm{r}})}\Big(\frac{\partial\tilde{\mu}_{3,n}}{\partial\tau} + (\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\mu}_{3,n} + \frac{\partial\tilde{\rho}_n}{\partial\Xi_3} - 2\mathfrak{v}_{\mathrm{r}}\Big(\frac{\partial\tilde{\omega}_{2,n}}{\partial\Xi_1} - \frac{\partial\tilde{\omega}_{1,n}}{\partial\Xi_2}\Big) - \varrho_{1,3}\Big)\Big)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3, \quad (99)$$

$$\omega_{1,n+1}(\xi_1,\xi_2,\xi_3,\tau) = \omega_{1,n}(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} {}_3\lambda_{\omega_1}(\Xi_1,\Xi_2,\Xi_3)\Bigg(\Delta\omega_{1,n} - \frac{1}{(c_{\mathrm{a}}+\mathfrak{c}_{\mathrm{b}})}\bigg(\frac{\partial\tilde{\omega}_{1,n}}{\partial\tau} - (\mathfrak{c}_0+\mathfrak{c}_{\mathrm{b}}-\mathfrak{c}_{\mathrm{a}})\frac{\partial}{\partial\Xi_1}(\nabla\cdot\tilde{\boldsymbol{\omega}}_n) + (\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\omega}_{1,n} + 4\mathfrak{v}_{\mathrm{r}}\tilde{\omega}_{1,n} - 2\mathfrak{v}_{\mathrm{r}}\left(\frac{\partial\tilde{\mu}_{3,n}}{\partial\Xi_2} - \frac{\partial\tilde{\mu}_{2,n}}{\partial\Xi_3}\right) - \varrho_{2,1}\bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)\,d\Xi_1\,d\Xi_2\,d\Xi_3, \tag{100}$$

$$\omega_{2,n+1}(\xi_1,\xi_2,\xi_3,\tau) = \omega_{2,n}(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} {}_3\lambda_{\omega_2}(\Xi_1,\Xi_2,\Xi_3)\Bigg(\Delta\omega_{2,n} - \frac{1}{(c_{\mathrm{a}}+\mathfrak{c}_{\mathrm{b}})}\bigg(\frac{\partial\tilde{\omega}_{2,n}}{\partial\tau} - (\mathfrak{c}_0+\mathfrak{c}_{\mathrm{b}}-\mathfrak{c}_{\mathrm{a}})\frac{\partial}{\partial\Xi_2}(\nabla\cdot\tilde{\boldsymbol{\omega}}_n) + (\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\omega}_{2,n} + 4\mathfrak{v}_{\mathrm{r}}\tilde{\omega}_{2,n} - 2\mathfrak{v}_{\mathrm{r}}\left(\frac{\partial\tilde{\mu}_{1,n}}{\partial\Xi_3} - \frac{\partial\tilde{\mu}_{3,n}}{\partial\Xi_1}\right) - \varrho_{2,2}\bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)\,d\Xi_1\,d\Xi_2\,d\Xi_3, \tag{101}$$

$$\omega_{3,n+1}(\xi_1,\xi_2,\xi_3,\tau) = \omega_{3,n}(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} {}_3\lambda_{\omega_3}(\Xi_1,\Xi_2,\Xi_3)\Bigg(\Delta\omega_{3,n} - \frac{1}{(c_{\mathrm{a}}+\mathfrak{c}_{\mathrm{b}})}\bigg(\frac{\partial\tilde{\omega}_{3,n}}{\partial\tau} - (\mathfrak{c}_0+\mathfrak{c}_{\mathrm{b}}-\mathfrak{c}_{\mathrm{a}})\frac{\partial}{\partial\Xi_3}(\nabla\cdot\tilde{\boldsymbol{\omega}}_n) + (\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\omega}_{3,n} + 4\mathfrak{v}_{\mathrm{r}}\tilde{\omega}_{3,n} - 2\mathfrak{v}_{\mathrm{r}}\left(\frac{\partial\tilde{\mu}_{2,n}}{\partial\Xi_1} - \frac{\partial\tilde{\mu}_{1,n}}{\partial\Xi_2}\right) - \varrho_{2,3}\bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)\,d\Xi_1\,d\Xi_2\,d\Xi_3, \tag{102}$$

$$\rho_{n+1}(\xi_1,\xi_2,\xi_3,\tau) = \rho_n(\xi_1,\xi_2,\xi_3,\tau) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} {}_3\lambda_{\rho}(\Xi_1,\Xi_2,\Xi_3)\Bigg(\Delta\rho_n + \frac{\partial}{\partial x_1}\Big((\tilde{\boldsymbol{\mu}}_n.\nabla)\tilde{\mu}_{1,n} - \varrho_{1,1}\Big) + \frac{\partial}{\partial x_2}\Big((\tilde{\boldsymbol{\mu}}_n.\nabla)\tilde{\mu}_{2,n} - \varrho_{1,2}\Big) + \frac{\partial}{\partial x_3}\Big((\tilde{\boldsymbol{\mu}}_n\cdot\nabla)\tilde{\mu}_{3,n} - \varrho_{1,3}\Big)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)\,d\Xi_1\,d\Xi_2\,d\Xi_3. \tag{103}$$

Here, $\tilde{\mu}_{i,n}(i=1,2), \tilde{\mu}_{j,n}, \tilde{\omega}_{j,n}(j=1,2,3)$, and $\tilde{\rho}_n$ are restricted variations (see [16]), i.e., $\delta\tilde{\mu}_{i,n}=0, \delta\tilde{\mu}_{j,n}=0, \delta\tilde{\omega}_{j,n}=0$, and $\delta\tilde{\rho}_n=0$. By rendering the correction functionals (93)-(96) and (97)-(103) stationary and applying Green's first identity, we derive the stationary conditions:

For $d=2$:

$$\delta\mu_{1,n+1}(\xi_1,\xi_2,\tau) = \delta\mu_{1,n}(\xi_1,\xi_2,\tau) + {}_2\lambda_{\mu_1}(\Xi_1,\Xi_2)\big(\delta\nabla\mu_{1,n}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\tau)\big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)} - \big(\nabla_2\lambda_{\mu_1}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2)\delta\mu_{1,n}(\Xi_1,\Xi_2,\tau)\big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)} + \int_0^{\xi_2}\int_0^{\xi_1} \Delta_2\lambda_{\mu_1}(\Xi_1,\Xi_2) \times \delta\mu_{1,n}(\Xi_1,\Xi_2,\tau)\,d\Xi_1\,d\Xi_2 = 0, \tag{104}$$

$$\delta\mu_{2,n+1}(\xi_1,\xi_2,\tau) = \delta\mu_{2,n}(\xi_1,\xi_2,\tau) + {}_2\lambda_{\mu_2}(\Xi_1,\Xi_2)\big(\delta\nabla\mu_{2,n}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\tau)\big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)} - \big(\nabla_2\lambda_{\mu_2}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2)\delta\mu_{2,n}(\Xi_1,\Xi_2,\tau)\big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)} + \int_0^{\xi_2}\int_0^{\xi_1} \Delta_2\lambda_{\mu_2}(\Xi_1,\Xi_2) \times \delta\mu_{2,n}(\Xi_1,\Xi_2,\tau)\,d\Xi_1\,d\Xi_2 = 0, \tag{105}$$

$$\delta\omega_{3,n+1}(\xi_1,\xi_2,\tau) = \delta\omega_{3,n}(\xi_1,\xi_2,\tau) + {}_2\lambda_{\omega_3}(\Xi_1,\Xi_2)\big(\delta\nabla\omega_{3,n}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\tau)\big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)} - \big(\nabla_2\lambda_{\omega_3}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2)\delta\omega_{3,n}(\Xi_1,\Xi_2,\tau)\big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)} + \int_0^{\xi_2}\int_0^{\xi_1} \Delta_2\lambda_{\omega_3}(\Xi_1,\Xi_2) \times \delta\omega_{3,n}(\Xi_1,\Xi_2,\tau)\,d\Xi_1\,d\Xi_2 = 0, \tag{106}$$

$$\delta\rho_{n+1}(\xi_1,\xi_2,\tau) = \delta\rho_n(\xi_1,\xi_2,\tau) + {}_2\lambda_{\rho}(\Xi_1,\Xi_2)(\delta\nabla\rho_n\cdot\boldsymbol{n})(\Xi_1,\Xi_2,\tau)\big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)} - (\nabla\beta_3\cdot\boldsymbol{n})(\Xi_1,\Xi_2)\delta\rho_n(\Xi_1,\Xi_2,\tau)|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)} + \int_0^{\xi_2}\int_0^{\xi_1} \Delta_2\lambda_{\rho}(\Xi_1,\Xi_2) \times \delta\rho_n(\Xi_1,\Xi_2,\tau)\,d\Xi_1\,d\Xi_2 = 0. \tag{107}$$

For $d=3$:

$$\delta\mu_{1,n+1}(\xi_1,\xi_2,\xi_3,\tau) = \delta\mu_{1,n}(\xi_1,\xi_2,\xi_3,\tau) + {}_3\lambda_{\mu_1}(\Xi_1,\Xi_2,\Xi_3)\big(\delta\nabla\mu_{1,n}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)} - \big(\nabla_3\lambda_{\mu_1}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3)\delta\mu_{1,n}(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)} + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \Delta_3\lambda_{\mu_1}(\Xi_1,\Xi_2,\Xi_3) \times \delta\mu_{1,n}(\Xi_1,\Xi_2,\Xi_3,\tau)\,d\Xi_1\,d\Xi_2\,d\Xi_3 = 0, \tag{108}$$

$$\delta\mu_{2,n+1}(\xi_1,\xi_2,\xi_3,\tau) = \delta\mu_{2,n}(\xi_1,\xi_2,\xi_3,\tau) + {}_3\lambda_{\mu_2}(\Xi_1,\Xi_2,\Xi_3)\big(\delta\nabla\mu_{2,n}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)} - \big(\nabla_3\lambda_{\mu_2}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3)\delta\mu_{2,n}(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)} + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \Delta_3\lambda_{\mu_2}(\Xi_1,\Xi_2,\Xi_3) \times \delta\mu_{2,n}(\Xi_1,\Xi_2,\Xi_3,\tau)\,d\Xi_1\,d\Xi_2\,d\Xi_3 = 0, \tag{109}$$

$$
\begin{aligned}
&\delta\mu_{3,n+1}(\xi_1,\xi_2,\xi_3,\tau)\\
&=\delta\mu_{3,n}(\xi_1,\xi_2,\xi_3,\tau)\\
&+\ {}_3\lambda_{\mu_3}(\Xi_1,\Xi_2,\Xi_3)\big(\delta\nabla\mu_{3,n}\\
&\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)}\\
&-\big(\nabla_3\lambda_{\mu_3}\\
&\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3)\delta\mu_{3,n}(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)}\\
&+\int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1}\begin{pmatrix}\Delta_3\lambda_{\mu_3}(\Xi_1,\Xi_2,\Xi_3)\\ \times\,\delta\mu_{3,n}(\Xi_1,\Xi_2,\Xi_3,\tau)\end{pmatrix}d\Xi_1\,d\Xi_2\,d\Xi_3=0,
\end{aligned}\tag{110}
$$

$$
\begin{aligned}
&\delta\omega_{1,n+1}(\xi_1,\xi_2,\xi_3,\tau)\\
&=\delta\omega_{1,n}(\xi_1,\xi_2,\xi_3,\tau)\\
&+\ {}_3\lambda_{\omega_1}(\Xi_1,\Xi_2,\Xi_3)\big(\delta\nabla\omega_{1,n}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)}\\
&-\big(\nabla_3\lambda_{\omega_1}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3)\delta\omega_{1,n}(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)}\\
&+\int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1}\begin{pmatrix}\Delta_3\lambda_{\omega_1}(\Xi_1,\Xi_2,\Xi_3)\\ \times\,\delta\omega_{1,n}(\Xi_1,\Xi_2,\Xi_3,\tau)\end{pmatrix}d\Xi_1\,d\Xi_2\,d\Xi_3=0,
\end{aligned}\tag{111}
$$

$$
\begin{aligned}
&\delta\omega_{2,n+1}(\xi_1,\xi_2,\xi_3,\tau)\\
&=\delta\omega_{2,n}(\xi_1,\xi_2,\xi_3,\tau)\\
&+\ {}_3\lambda_{\omega_2}(\Xi_1,\Xi_2,\Xi_3)\big(\delta\nabla\omega_{2,n}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)}\\
&-\big(\nabla_3\lambda_{\omega_2}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3)\delta\omega_{2,n}(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)}\\
&+\int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1}\begin{pmatrix}\Delta_3\lambda_{\omega_2}(\Xi_1,\Xi_2,\Xi_3)\\ \times\,\delta\omega_{2,n}(\Xi_1,\Xi_2,\Xi_3,\tau)\end{pmatrix}d\Xi_1\,d\Xi_2\,d\Xi_3=0,
\end{aligned}\tag{112}
$$

$$
\begin{aligned}
&\delta\omega_{3,n+1}(\xi_1,\xi_2,\xi_3,\tau)\\
&=\delta\omega_{3,n}(\xi_1,\xi_2,\xi_3,\tau)\\
&+\ {}_3\lambda_{\omega_3}(\Xi_1,\Xi_2,\Xi_3)\big(\delta\nabla\omega_{3,n}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)}\\
&-\big(\nabla_3\lambda_{\omega_3}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3)\delta\omega_{3,n}(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)}\\
&+\int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1}\begin{pmatrix}\Delta_3\lambda_{\omega_3}(\Xi_1,\Xi_2,\Xi_3)\\ \times\,\delta\omega_{3,n}(\Xi_1,\Xi_2,\Xi_3,\tau)\end{pmatrix}d\Xi_1\,d\Xi_2\,d\Xi_3=0,
\end{aligned}\tag{113}
$$

$$
\begin{aligned}
&\delta\rho_{n+1}(\xi_1,\xi_2,\xi_3,\tau)\\
&=\delta\rho_{n}(\xi_1,\xi_2,\xi_3,\tau)\\
&+\ {}_3\lambda_{\rho}(\Xi_1,\Xi_2,\Xi_3)(\delta\nabla\rho_{n}.\boldsymbol{n})(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)}\\
&-\big(\nabla_3\lambda_{\rho}.\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3)\delta\rho_{n}(\Xi_1,\Xi_2,\Xi_3,\tau)\big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)}\\
&+\int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1}\begin{pmatrix}\Delta_3\lambda_{\rho}(\Xi_1,\Xi_2,\Xi_3)\\ \times\,\delta\rho_{n}(\Xi_1,\Xi_2,\Xi_3,\tau)\end{pmatrix}d\Xi_1\,d\Xi_2\,d\Xi_3=0.
\end{aligned}\tag{114}
$$

The systems of stationary conditions:

For $d=2$: For $i=1,2$,

$$
\begin{cases}
\begin{cases}
\delta\mu_{i,n} & :\Delta_2\lambda_{\mu_i}(\Xi_1,\Xi_2)=0,\\
\delta\nabla\mu_{i,n}\cdot\boldsymbol{n}_{\mu_{i,n}} & :\ {}_2\lambda_{\mu_i}(\Xi_1,\Xi_2)\big|_{\substack{(\Xi_1,\Xi_2)\\=(\xi_1,\xi_2)}}=0,\\
\delta\mu_{i,n} & :\big(\nabla_2\lambda_{\mu_i}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2)\big|_{\substack{(\Xi_1,\Xi_2)\\=(\xi_1,\xi_2)}}=1,
\end{cases}\\
\begin{cases}
\delta\omega_{3,n} & :\Delta_2\lambda_{\omega_3}(\Xi_1,\Xi_2)=0,\\
\delta\nabla\omega_{3,n}\cdot\boldsymbol{n}_{\omega_{3,n}} & :\ {}_2\lambda_{\omega_3}(\Xi_1,\Xi_2)\big|_{\substack{(\Xi_1,\Xi_2)\\=(\xi_1,\xi_2)}}=0,\\
\delta\omega_{3,n} & :\big(\nabla_2\lambda_{\omega_{3,n}}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2)\big|_{\substack{(\Xi_1,\Xi_2)\\=(\xi_1,\xi_2)}}=1,
\end{cases}\\
\begin{cases}
\delta\rho_{n} & :\Delta_2\lambda_{\rho}(\Xi_1,\Xi_2)=0,\\
\delta\nabla\rho_{n}\cdot\boldsymbol{n} & :\ {}_2\lambda_{\rho}(\Xi_1,\Xi_2)\big|_{\substack{(\Xi_1,\Xi_2)\\=(\xi_1,\xi_2)}}=0,\\
\delta\rho_{n} & :\big(\nabla_2\lambda_{\rho}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2)\big|_{\substack{(\Xi_1,\Xi_2)\\=(\xi_1,\xi_2)}}=1,
\end{cases}
\end{cases}\tag{115}
$$

For $d=3$: For $j=1,2,3$,

$$
\begin{cases}
\begin{cases}
\delta\mu_{j,n} & :\Delta_3\lambda_{\mu_j}(\Xi_1,\Xi_2,\Xi_3)=0,\\
\delta\nabla\mu_{j,n}\cdot\boldsymbol{n}_{\mu_{j,n}} & :\ {}_3\lambda_{\mu_i}(\Xi_1,\Xi_2,\Xi_3)\big|_{\substack{(\Xi_1,\Xi_2,\Xi_3)\\=(\xi_1,\xi_2,\xi_3)}}=0,\\
\delta\mu_{j,n} & :\Big(\nabla_3\lambda_{\mu_j}\cdot\boldsymbol{n}\Big)(\Xi_1,\Xi_2,\Xi_3)\big|_{\substack{(\Xi_1,\Xi_2,\Xi_3)\\=(\xi_1,\xi_2,\xi_3)}}=1,
\end{cases}\\
\begin{cases}
\delta\omega_{j,n} & :\Delta_3\lambda_{\omega_j}(\Xi_1,\Xi_2,\Xi_3)=0,\\
\delta\nabla\omega_{j,n}\cdot\boldsymbol{n}_{\omega_{j,n}} & :\ {}_3\lambda_{\omega_j}(\Xi_1,\Xi_2,\Xi_3)\big|_{\substack{(\Xi_1,\Xi_2,\Xi_3)\\=(\xi_1,\xi_2,\xi_3)}}=0,\\
\delta\omega_{j,n} & :\Big(\nabla_3\lambda_{\omega_{j,n}}\cdot\boldsymbol{n}\Big)(\Xi_1,\Xi_2,\Xi_3)\big|_{\substack{(\Xi_1,\Xi_2,\Xi_3)\\=(\xi_1,\xi_2,\xi_3)}}=1,
\end{cases}\\
\begin{cases}
\delta\rho_{n} & :\Delta_3\lambda_{\rho}(\Xi_1,\Xi_2,\Xi_3)=0,\\
\delta\nabla\rho_{n}\cdot\boldsymbol{n} & :\ {}_3\lambda_{\rho}(\Xi_1,\Xi_2,\Xi_3)\big|_{\substack{(\Xi_1,\Xi_2,\Xi_3)\\=(\xi_1,\xi_2,\xi_3)}}=0,\\
\delta\rho_{n} & :\big(\nabla_3\lambda_{\rho}\cdot\boldsymbol{n}\big)(\Xi_1,\Xi_2,\Xi_3)\big|_{\substack{(\Xi_1,\Xi_2,\Xi_3)\\=(\xi_1,\xi_2,\xi_3)}}=1.
\end{cases}
\end{cases}\tag{116}
$$

The solutions to (115) and (116) are non-unique:

For $d=2$: For $i=1,2$,

$$
\begin{cases}
{}_2\lambda_{\mu_j}(\Xi_1,\Xi_2)=\dfrac{\gamma_{2,\lambda_{\mu_i}}(\Xi_1,\Xi_2)-\gamma_{2,\lambda_{\mu_i}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\mu_i}}}},\\
{}_2\lambda_{\omega_3}(\Xi_1,\Xi_2)=\dfrac{\gamma_{2,\lambda_{\omega_3}}(\Xi_1,\Xi_2)-\gamma_{2,\lambda_{\omega_3}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\omega_3}}}},\\
{}_2\lambda_{\rho}(\Xi_1,\Xi_2)=\dfrac{\gamma_{2,\lambda_{\rho}}(\Xi_1,\Xi_2)-\gamma_{2,\lambda_{\rho}}(\xi_1,\xi_2)}{a_{\gamma_2\lambda_{\rho}}},
\end{cases}\tag{117}
$$

For $d=3$: For $j=1,2,3$,

$$
\begin{cases}
{}_3\lambda_{\mu_j}(\Xi_1,\Xi_2)=\dfrac{\gamma_{3,\lambda_{\mu_j}}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_{\mu_j}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\mu_i}}}},\\
{}_3\lambda_{\omega_j}(\Xi_1,\Xi_2)=\dfrac{\gamma_{2,\lambda_{\omega_3}}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{2,\lambda_{\omega_3}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\omega_3}}}},\\
{}_3\lambda_{\rho}(\Xi_1,\Xi_2)=\dfrac{\gamma_{2,\lambda_{\rho}}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{2,\lambda_{\rho}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\rho}}}},
\end{cases}\tag{118}
$$

Here $\gamma_{2,\lambda_{\mu_i}}(i=1,2),\gamma_{2,\lambda_{\omega_3}},\gamma_{2,\lambda_\rho}$ and $\gamma_{3,\lambda_{\mu_j}},\gamma_{3,\lambda_{\omega_j}}(j=1,2,3),\gamma_{3,\lambda_\rho}$ are any harmonic functions in $\mho$ satisfying the boundary conditions:

- For $d=2$: For $i=1,2$,

$$
\begin{aligned}
\nabla\gamma_{2,\lambda_{\mu_i}}\cdot\boldsymbol{n}\Big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)}&=a_{\gamma_{2,\lambda_{\mu_i}}}\neq 0,\\
\nabla\gamma_{2,\lambda_{\omega_3}}\cdot\boldsymbol{n}\Big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)}&=a_{\gamma_{2,\lambda_{\omega_3}}}\neq 0,\\
\nabla\gamma_{2,\lambda_{\rho}}\cdot\boldsymbol{n}\Big|_{(\Xi_1,\Xi_2)=(\xi_1,\xi_2)}&=a_{\gamma_{2,\lambda_{\rho}}}\neq 0.
\end{aligned}
$$

- For $d=3$: For $j=1,2,3$,

$$
\begin{aligned}
\nabla\gamma_{3,\lambda_{\mu_j}}\cdot\boldsymbol{n}\Big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)}&=a_{\gamma_{3,\lambda_{\mu_j}}}\neq 0,\\
\nabla\gamma_{3,\lambda_{\omega_j}}\cdot\boldsymbol{n}\Big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)}&=a_{\gamma_{3,\lambda_{\omega_j}}}\neq 0,\\
\nabla\gamma_{3,\lambda_{\rho}}\cdot\boldsymbol{n}\Big|_{(\Xi_1,\Xi_2,\Xi_3)=(\xi_1,\xi_2,\xi_3)}&=a_{\gamma_{3,\lambda_{\rho}}}\neq 0.
\end{aligned}
$$

Substituting (117) and (118) into (93)-(96) and (97)-(103) completes the proof.

System (66) and system (67) solutions' existence and uniqueness can be studied through the following theorems.

Theorem 8. Let the operators

$$\mathrm{M}: \mathfrak{L}^{\infty}\big(0,\mathrm{T},\mathfrak{H}^1(\mho)\big) \cap \mathfrak{L}^2\big(0,\mathrm{T},\boldsymbol{\mathcal{V}}_{\vartheta}(\mho)\big) \to \mathfrak{L}^{\infty}\big(0,\mathrm{T},\mathfrak{H}^1(\mho)\big) \cap \mathfrak{L}^2\big(0,\mathrm{T},\boldsymbol{\mathcal{V}}_{\vartheta}(\mho)\big),$$

$$\Omega: \mathfrak{L}^{\infty}\big(0,\mathrm{T},\mathfrak{H}^1(\mho)\big) \cap \mathfrak{L}^2\left(0,\mathrm{T},\mathfrak{H}^2_{\vartheta}(\mho)\right) \to \mathfrak{L}^{\infty}\big(0,\mathrm{T},\mathfrak{H}^1(\mho)\big) \cap \mathfrak{L}^2\left(0,\mathrm{T},\mathfrak{H}^2_{\vartheta}(\mho)\right),$$

$$\mathrm{P}: \mathfrak{L}^2\big(0,\mathrm{T},\mathfrak{H}^1(\mho)\big) \to \mathfrak{L}^2\big(0,\mathrm{T},\mathfrak{H}^1(\mho)\big),$$

be defined as follows:

For $d = 2$:

$$\mathrm{M}[\boldsymbol{\mu}] = (\mathrm{M}_1[\mu_1], \mathrm{M}_2[\mu_2], 0), \Omega[\boldsymbol{\omega}] = (0,0,\Omega_3[\omega_3]),$$

where

$$\begin{aligned}&\mathrm{M}_1[\mu_1](\xi_1,\xi_2,\tau)\\&= \mu_1(\xi_1,\xi_2,\tau)\\&+\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_{\mu_1}}(\Xi_1,\Xi_2) - \gamma_{2,\lambda_{\mu_1}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\mu_1}}}}\Bigg(\Delta\mu_1\\&-\frac{1}{(\mathfrak{v}+\mathfrak{v}_{\mathrm{r}})}\left(\frac{\partial\mu_1}{\partial\tau} + (\boldsymbol{\mu}\cdot\nabla)\mu_1 + \frac{\partial\rho}{\partial\Xi_1} - 2\mathfrak{v}_{\mathrm{r}}\frac{\partial\omega_3}{\partial\Xi_2}\\&-\varrho_{1,1}\right)\Bigg)(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2,\end{aligned} \tag{119}$$

$$\begin{aligned}&\mathrm{M}_2[\mu_2](\xi_1,\xi_2,\tau)\\&= \mu_2(\xi_1,\xi_2,\tau)\\&+\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_{\mu_2}}(\Xi_1,\Xi_2) - \gamma_{2,\lambda_{\mu_2}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\mu_2}}}}\Bigg(\Delta\mu_2\\&-\frac{1}{(\mathfrak{v}+\mathfrak{v}_{\mathrm{r}})}\left(\frac{\partial\mu_2}{\partial\tau} + (\boldsymbol{\mu}\cdot\nabla)\mu_2 + \frac{\partial\rho}{\partial\Xi_2} + 2\mathfrak{v}_{\mathrm{r}}\frac{\partial\omega_3}{\partial\Xi_1}\\&-\varrho_{1,2}\right)\Bigg)(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2,\end{aligned} \tag{120}$$

$$\begin{aligned}&\Omega_3[\omega_3](\xi_1,\xi_2,\tau)\\&= \omega_3(\xi_1,\xi_2,\tau)\\&+\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_{\omega_3}}(\Xi_1,\Xi_2) - \gamma_{2,\lambda_{\omega_3}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\omega_3}}}}\Bigg(\Delta\omega_3\\&-\frac{1}{(\mathfrak{c}_{\mathrm{a}}+\mathfrak{c}_{\mathfrak{d}})}\left(\frac{\partial\omega_3}{\partial\tau} + (\boldsymbol{\mu}\cdot\nabla)\omega_3 + 4\mathfrak{v}_{\mathrm{r}}\omega_3 - 2\mathfrak{v}_{\mathrm{r}}\left(\frac{\partial\mu_2}{\partial\Xi_1} - \frac{\partial\mu_1}{\partial\Xi_2}\right)\\&-\varrho_{2,3}\right)\Bigg)(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2,\end{aligned} \tag{121}$$

$$\begin{aligned}\mathrm{P}[\rho](\xi_1,\xi_2,\tau) = \rho(\xi_1,\xi_2,\tau)&\\+\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_{\rho}}(\Xi_1,\Xi_2) - \gamma_{2,\lambda_{\rho}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\rho}}}}&\Bigg(\Delta\rho\\&+\frac{\partial}{\partial x_1}\big((\boldsymbol{\mu}.\nabla)\mu_1 - \varrho_{1,1}\big)\\&+\frac{\partial}{\partial x_2}\big((\boldsymbol{\mu}.\nabla)\mu_2\\&-\varrho_{1,2}\big)\Bigg)(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2.\end{aligned} \tag{122}$$

For $d = 3$:

$$\mathrm{M}[\boldsymbol{\mu}] = (\mathrm{M}_1[\mu_1], \mathrm{M}_2[\mu_2], \mathrm{M}_3[\mu_3]),$$
$$\Omega[\boldsymbol{\omega}] = (\Omega_1[\omega_1], \Omega_2[\omega_2], \Omega_3[\omega_3]),$$

where

$$\begin{aligned}&\mathrm{M}_1[\mu_1](\xi_1,\xi_2,\xi_3,\tau)\\&= \mu_1(\xi_1,\xi_2,\xi_3,\tau)\\&+\int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_1}}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_{\mu_1}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_3\lambda_{\mu_1}}}\Bigg(\Delta\mu_1\\&-\frac{1}{(\mathfrak{v}+\mathfrak{v}_{\mathrm{r}})}\left(\frac{\partial\mu_1}{\partial\tau} + (\boldsymbol{\mu}\cdot\nabla)\mu_1 + \frac{\partial\rho}{\partial\Xi_1} - 2\mathfrak{v}_{\mathrm{r}}\left(\frac{\partial\omega_3}{\partial\Xi_2} - \frac{\partial\omega_{2,n}}{\partial\Xi_3}\right)\\&-\varrho_{1,1}\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3,\end{aligned} \tag{123}$$

$$\begin{aligned}&\mathrm{M}_2[\mu_2](\xi_1,\xi_2,\xi_3,\tau)\\&= \mu_2(\xi_1,\xi_2,\xi_3,\tau)\\&+\int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_2}}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_{\mu_2}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_3,\lambda_{\mu_2}}}\Bigg(\Delta\mu_2\\&-\frac{1}{(\mathfrak{v}+\mathfrak{v}_{\mathrm{r}})}\left(\frac{\partial\mu_2}{\partial\tau} + (\boldsymbol{\mu}\cdot\nabla)\mu_2 + \frac{\partial\rho}{\partial\Xi_2} - 2\mathfrak{v}_{\mathrm{r}}\left(\frac{\partial\omega_1}{\partial\Xi_3} - \frac{\partial\omega_3}{\partial\Xi_1}\right)\\&-\varrho_{1,2}\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3,\end{aligned} \tag{124}$$

$$\begin{aligned}&\mathrm{M}_3[\mu_3](\xi_1,\xi_2,\xi_3,\tau)\\&= \mu_3(\xi_1,\xi_2,\xi_3,\tau)\\&+\int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_3}}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_{\mu_3}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_3,\lambda_{\mu_3}}}\Bigg(\Delta\mu_3\\&-\frac{1}{(\mathfrak{v}+\mathfrak{v}_{\mathrm{r}})}\left(\frac{\partial\mu_3}{\partial\tau} + (\boldsymbol{\mu}\cdot\nabla)\mu_3 + \frac{\partial\rho_n}{\partial\Xi_3} - 2\mathfrak{v}_{\mathrm{r}}\left(\frac{\partial\omega_2}{\partial\Xi_1} - \frac{\partial\omega_1}{\partial\Xi_2}\right)\\&-\varrho_{1,3}\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3,\end{aligned} \tag{125}$$

$$\begin{aligned}&\Omega_1[\omega_1](\xi_1,\xi_2,\xi_3,\tau)\\&= \omega_1(\xi_1,\xi_2,\xi_3,\tau)\\&+\int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\omega_1}}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_{\omega_1}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_3,\lambda_{\omega_1}}}\Bigg(\Delta\omega_1\\&-\frac{1}{(\mathfrak{c}_{\mathrm{a}}+\mathfrak{c}_{\mathfrak{d}})}\left(\frac{\partial\omega_1}{\partial\tau} - (\mathfrak{c}_0 + \mathfrak{c}_{\mathfrak{d}} - \mathfrak{c}_{\mathrm{a}})\frac{\partial}{\partial\Xi_1}(\nabla\cdot\boldsymbol{\omega}) + (\boldsymbol{\mu}\cdot\nabla)\omega_1\\&+4\mathfrak{v}_{\mathrm{r}}\omega_{1,n} - 2\mathfrak{v}_{\mathrm{r}}\left(\frac{\partial\mu_3}{\partial\Xi_2} - \frac{\partial\mu_2}{\partial\Xi_3}\right)\\&-\varrho_{2,1}\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3,\end{aligned} \tag{126}$$

$$\begin{aligned}&\Omega_2[\omega_2](\xi_1,\xi_2,\xi_3,\tau)\\&= \omega_2(\xi_1,\xi_2,\xi_3,\tau)\\&+\int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\omega_2}}(\Xi_1,\Xi_2,\Xi_3) - \gamma_{3,\lambda_{\omega_2}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_3\lambda_{\omega_2}}}\Bigg(\Delta\omega_2\\&-\frac{1}{(c_a+c_d)}\left(\frac{\partial\omega_2}{\partial\tau} - (c_0 + c_d - c_a)\frac{\partial}{\partial\Xi_2}(\nabla\cdot\boldsymbol{\omega})\\&+(\boldsymbol{\mu}\cdot\nabla)\omega_2 + 4\mathfrak{v}_{\mathrm{r}}\omega_2 - 2\mathfrak{v}_{\mathrm{r}}\left(\frac{\partial\mu_1}{\partial\Xi_3} - \frac{\partial\mu_3}{\partial\Xi_1}\right)\\&-\varrho_{2,2}\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3,\end{aligned} \tag{127}$$

$$\begin{aligned}&\Omega_3[\omega_3](\xi_1,\xi_2,\xi_3,\tau)\\&=\omega_3(\xi_1,\xi_2,\xi_3,\tau)\\&+\int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1}\frac{\gamma_{3,\lambda_{\omega_3}}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_{\omega_3}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_3\lambda_{\omega_3}}}\Bigg(\Delta\omega_3\\&-\frac{1}{(\mathfrak{c}_a+\mathfrak{c}_\mathfrak{d})}\bigg(\frac{\partial\omega_3}{\partial\tau}-(\mathfrak{c}_0+\mathfrak{c}_\mathfrak{d}-\mathfrak{c}_a)\frac{\partial}{\partial\Xi_3}(\nabla\cdot\boldsymbol{\omega})\\&+(\boldsymbol{\mu}\cdot\nabla)\omega_3+4\mathfrak{v}_r\omega_3-2\mathfrak{v}_r\left(\frac{\partial\mu_2}{\partial\Xi_1}-\frac{\partial\mu_1}{\partial\Xi_2}\right)\\&-\varrho_{2,3}\bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3,\end{aligned}\tag{128}$$

$$\begin{aligned}&\mathrm{P}[\rho](\xi_1,\xi_2,\xi_3,\tau)\\&=\rho(\xi_1,\xi_2,\xi_3,\tau)\\&+\int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1}\frac{\gamma_{3,\lambda_\rho}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_\rho}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_\rho}}}\Bigg(\Delta\rho\\&+\frac{\partial}{\partial x_1}\big((\boldsymbol{\mu}.\nabla)\mu_1-\varrho_{1,1}\big)+\frac{\partial}{\partial x_2}\big((\boldsymbol{\mu}.\nabla)\mu_2-\varrho_{1,2}\big)\\&+\frac{\partial}{\partial x_3}\big((\boldsymbol{\mu}.\nabla)\mu_3\\&-\varrho_{1,3}\big)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\,d\Xi_2\,d\Xi_3.\end{aligned}\tag{129}$$

If there exist constants $\mathfrak{C}_\mathrm{M}<1$, $\mathfrak{C}_\Omega<1$, and $\mathfrak{C}_\mathrm{P}<1$such that for all

$$\boldsymbol{\mu},\tilde{\boldsymbol{\mu}}\in\mathfrak{L}^\infty\big(0,\mathrm{T},\mathfrak{H}^1(\mho)\big)\cap\mathfrak{L}^2\big(0,\mathrm{T},\boldsymbol{\mathcal{V}}_{\boldsymbol{\vartheta}}(\mho)\big),$$
$$\boldsymbol{\omega},\tilde{\boldsymbol{\omega}}\in\mathfrak{L}^\infty\big(0,\mathrm{T},\mathfrak{H}^1(\mho)\big)\cap\mathfrak{L}^2\Big(0,\mathrm{T},\mathfrak{H}^2_{\boldsymbol{\vartheta}}(\mho)\Big),$$
$$\rho,\tilde{\rho}\in\mathfrak{L}^2(0,\mathrm{T},\mathfrak{H}^1(\mho)),$$

$$\begin{aligned}&\|\mathrm{M}[\boldsymbol{\mu}]-\mathrm{M}[\tilde{\boldsymbol{\mu}}]\|_{\mathfrak{L}^2\left(0,\mathrm{T},\mathfrak{H}^2(\mho)\right)}\\&\qquad\le\mathfrak{C}_\mathrm{M}\|\boldsymbol{\mu}-\tilde{\boldsymbol{\mu}}\|_{\mathfrak{L}^2\left(0,\mathrm{T},\mathfrak{H}^2(\mho)\right)},\end{aligned}\tag{130}$$

$$\begin{aligned}&\|\Omega[\boldsymbol{\omega}]-\Omega[\tilde{\boldsymbol{\mu}}]\|_{\mathfrak{L}^2\left(0,\mathrm{T},\mathfrak{H}^2(\mho)\right)}\\&\qquad\le\mathfrak{C}_\Omega\|\boldsymbol{\omega}-\tilde{\boldsymbol{\omega}}\|_{\mathfrak{L}^2\left(0,\mathrm{T},\mathfrak{H}^2(\mho)\right)},\end{aligned}\tag{131}$$

$$\begin{aligned}&\|\mathrm{P}[\rho]-\mathrm{P}[\tilde{\rho}]\|_{\mathfrak{L}^2\left(0,\mathrm{T},\mathfrak{H}^1(\mho)\right)}\\&\qquad\le\mathfrak{C}_\mathrm{P}\|\rho-\tilde{\rho}\|_{\mathfrak{L}^2\left(0,\mathrm{T},\mathfrak{H}^1(\mho)\right)},\end{aligned}\tag{132}$$

then systems (66) and (67) admit unique solutions.

Proof. The proof follows directly from Banach's Fixed Point Theorem.

Theorem 9. Assume the operator s
$\mathrm{M}:\mathcal{K}_\mathrm{M}\to\mathcal{K}_\mathrm{M}$, $\Omega:\mathcal{K}_\Omega\to\mathcal{K}_\Omega$ and $\mathrm{P}:\mathcal{K}_\mathrm{P}\to\mathcal{K}_\mathrm{P}$
from Theorem 8 are defined on compact, convex subsets $\mathcal{K}_\mathrm{M},\mathcal{K}_\Omega$ and $\mathcal{K}_\mathrm{P}$ of the spaces

$$\mathcal{K}_\mathrm{M}\subset\mathfrak{L}^\infty\Big(0,\mathrm{T},\mathfrak{H}^1(\mho)\Big)\cap\mathfrak{L}^2\Big(0,\mathrm{T},\boldsymbol{\mathcal{V}}_g(\mho)\Big),$$
$$\mathcal{K}_\Omega\subset\mathfrak{L}^\infty\Big(0,\mathrm{T},\mathfrak{H}^1(\mho)\Big)\cap\mathfrak{L}^2\Big(0,\mathrm{T},\mathfrak{H}^2_{\boldsymbol{\vartheta}}(\mho)\Big),$$
$$\mathcal{K}_\mathrm{P}\subset\mathfrak{L}^2\Big(0,\mathrm{T},\mathfrak{H}^1(\mho)\Big),$$

respectively. If $\mathrm{M},\Omega,$ and P are continuous, then the systems (66) and (67) admit solutions.

Proof. The proof follows directly from Schauder's Fixed Point Theorem.

In similar discussion, we introduce the following results for the stationary micropolar fluid flow.

Theorem 10. The correction functionals of the stationary micropolar fluid flow are defined as follows:
For $d=2$:
The governing equations are:

$$\begin{cases}\begin{matrix}-(\mathfrak{v}+\mathfrak{v}_r)\Delta\boldsymbol{\mu}\\+(\boldsymbol{\mu}\cdot\nabla)\boldsymbol{\mu}+\nabla\rho\end{matrix}=2\mathfrak{v}_r\nabla\times\boldsymbol{\omega}+\boldsymbol{\varrho}_1, & \text{in }\mho,\\ \begin{matrix}-(\mathfrak{c}_a+\mathfrak{c}_\mathfrak{d})\Delta\boldsymbol{\omega}\\+(\boldsymbol{\mu}\cdot\nabla)\boldsymbol{\omega}+4\mathfrak{v}_r\boldsymbol{\omega}\end{matrix}=2\mathfrak{v}_r\nabla\times\boldsymbol{\mu}+\boldsymbol{\varrho}_2, & \text{in }\mho,\\ \nabla\cdot\boldsymbol{\mu}=0, & \text{in }\mho,\\ \boldsymbol{\mu}=\boldsymbol{\vartheta},\boldsymbol{\omega}=\boldsymbol{\vartheta}, & \text{on }\partial\mho,\end{cases}\tag{133}$$

and the correction functionals are:

$$\begin{aligned}&\mu_{1,n+1}(\xi_1,\xi_2)\\&=\mu_{1,n}(\xi_1,\xi_2)\\&+\int_0^{\xi_2}\int_0^{\xi_1}\frac{\gamma_{2,\lambda_{\mu_1}}(\Xi_1,\Xi_2)-\gamma_{2,\lambda_{\mu_1}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\mu_1}}}}\Bigg(\Delta\mu_{1,n}\\&-\frac{1}{(\mathfrak{v}+\mathfrak{v}_\mathbf{r})}\bigg((\boldsymbol{\mu}_n\cdot\nabla)\mu_{1,n}+\frac{\partial\rho_n}{\partial\Xi_1}-2\mathfrak{v}_r\frac{\partial\omega_{3,n}}{\partial\Xi_2}\\&-\varrho_{1,1}\bigg)\Bigg)(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2,\end{aligned}\tag{134}$$

$$\begin{aligned}&\mu_{2,n+1}(\xi_1,\xi_2)\\&=\mu_{2,n}(\xi_1,\xi_2)\\&+\int_0^{\xi_2}\int_0^{\xi_1}\frac{\gamma_{2,\lambda_{\mu_2}}(\Xi_1,\Xi_2)-\gamma_{2,\lambda_{\mu_2}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\mu_2}}}}\Bigg(\Delta\mu_{2,n}\\&-\frac{1}{(\mathfrak{v}+\mathfrak{v}_r)}\bigg((\boldsymbol{\mu}_n\cdot\nabla)\mu_{2,n}+\frac{\partial\rho_n}{\partial\Xi_2}+2\mathfrak{v}_r\frac{\partial\omega_{3,n}}{\partial\Xi_1}\\&-\varrho_{1,2}\bigg)\Bigg)(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2,\end{aligned}\tag{135}$$

$$\begin{aligned}&\omega_{3,n+1}(\xi_1,\xi_2)\\&=\omega_{3,n}(\xi_1,\xi_2)\\&+\int_0^{\xi_2}\int_0^{\xi_1}\frac{\gamma_{2,\lambda_{\omega_3}}(\Xi_1,\Xi_2)-\gamma_{2,\lambda_{\omega_3}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\omega_3}}}}\Bigg(\Delta\omega_{3,n}\\&-\frac{1}{(\mathfrak{c}_a+\mathfrak{c}_\mathfrak{d})}\bigg((\boldsymbol{\mu}_n\cdot\nabla)\omega_{3,n}+4\mathfrak{v}_r\omega_{3,n}\\&-2\mathfrak{v}_r\left(\frac{\partial\mu_{2,n}}{\partial\Xi_1}-\frac{\partial\mu_{1,n}}{\partial\Xi_2}\right)\\&-\varrho_{2,3}\bigg)\Bigg)(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2,\end{aligned}\tag{136}$$

$$\begin{aligned}&\rho_{n+1}(\xi_1,\xi_2)\\&=\rho_n(\xi_1,\xi_2)\\&+\int_0^{\xi_2}\int_0^{\xi_1}\frac{\gamma_{2,\lambda_\rho}(\Xi_1,\Xi_2)-\gamma_{2,\lambda_\rho}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_\rho}}}\Bigg(\Delta\rho_n\\&+\frac{\partial}{\partial x_1}\big((\boldsymbol{\mu_n}.\nabla)\mu_{1,n}-\varrho_{1,1}\big)\\&+\frac{\partial}{\partial x_2}\big((\boldsymbol{\mu_n}.\nabla)\mu_{2,n}-\varrho_{1,2}\big)\Bigg)(\Xi_1,\Xi_2,\tau)d\Xi_1\,d\Xi_2.\end{aligned}\tag{137}$$

For $d = 3$:

The governing equations are:

$$\begin{cases} \begin{aligned} &-(\mathfrak{v}+\mathfrak{v}_{\mathrm{r}})\Delta\boldsymbol{\mu} \\ &+(\boldsymbol{\mu}\cdot\nabla)\boldsymbol{\mu}+\nabla\rho \end{aligned} = 2\mathfrak{v}_{\mathrm{r}}\nabla\times\boldsymbol{\omega}+\boldsymbol{\varrho}_1, & \text{in } \mho, \\ \begin{aligned} &-(\mathfrak{c}_{\mathrm{a}}+\mathfrak{c}_{\mathfrak{d}})\Delta\boldsymbol{\omega} \\ &-(\mathfrak{c}_0+\mathfrak{c}_{\mathfrak{d}}-\mathfrak{c}_{\mathrm{a}})\nabla(\nabla\cdot\boldsymbol{\omega}) \\ &+(\boldsymbol{\mu}\cdot\nabla)\boldsymbol{\omega}+4\mathfrak{v}_{\mathrm{r}}\boldsymbol{\omega} \end{aligned} = \begin{aligned} &2\mathfrak{v}_{\mathrm{r}}\nabla\times\boldsymbol{\mu} \\ &+\boldsymbol{\varrho}_2, \end{aligned} & \text{in } \mho, \\ \nabla\cdot\boldsymbol{\mu}=0, & \text{in } \mho, \\ \boldsymbol{\mu}=\boldsymbol{\vartheta}, \boldsymbol{\omega}=\boldsymbol{\vartheta}, & \text{on } \partial\mho, \end{cases} \tag{138}$$

and the correction functionals are:

$$\begin{aligned} &\mu_{1,n+1}(\xi_1,\xi_2,\xi_3) \\ &= \mu_{1,n}(\xi_1,\xi_2,\xi_3) \\ &+ \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_1}}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_{\mu_1}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\mu_1}}}} \Bigg(\Delta\mu_{1,n} \\ &- \frac{1}{(\mathfrak{v}+\mathfrak{v}_{\mathrm{r}})} \Bigg((\boldsymbol{\mu}_n\cdot\nabla)\mu_{1,n} + \frac{\partial\rho_n}{\partial\Xi_1} - 2\mathfrak{v}_{\mathrm{r}}\left(\frac{\partial\omega_{3,n}}{\partial\Xi_2}-\frac{\partial\omega_{2,n}}{\partial\Xi_3}\right) \\ &- \varrho_{1,1}\Bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)\,d\Xi_1\,d\Xi_2\,d\Xi_3, \end{aligned} \tag{139}$$

$$\begin{aligned} &\mu_{2,n+1}(\xi_1,\xi_2,\xi_3) \\ &= \mu_{2,n}(\xi_1,\xi_2,\xi_3) \\ &+ \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_2}}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_{\mu_2}}(x_1,x_2,x_3)}{a_{\gamma_{3,\lambda_{\mu_2}}}} \Bigg(\Delta\mu_{2,n} \\ &- \frac{1}{(\mathfrak{v}+\mathfrak{v}_{\mathrm{r}})} \Bigg((\boldsymbol{\mu}_n\cdot\nabla)\mu_{2,n} + \frac{\partial\rho_n}{\partial\Xi_2} - 2\mathfrak{v}_{\mathrm{r}}\left(\frac{\partial\omega_{1,n}}{\partial\Xi_3}-\frac{\partial\omega_{3,n}}{\partial\Xi_1}\right) \\ &- \varrho_{1,2}\Bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)\,d\Xi_1\,d\Xi_2\,d\Xi_3, \end{aligned} \tag{140}$$

$$\begin{aligned} &\mu_{3,n+1}(\xi_1,\xi_2,\xi_3) \\ &= \mu_{3,n}(\xi_1,\xi_2,\xi_3) \\ &+ \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_3}}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_{\mu_3}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_3\lambda_{\mu_3}}} \Bigg(\Delta\mu_{3,n} \\ &- \frac{1}{(\mathfrak{v}+\mathfrak{v}_{\mathrm{r}})} \Bigg((\boldsymbol{\mu}_n\cdot\nabla)\mu_{3,n} + \frac{\partial\rho_n}{\partial\Xi_3} - 2\mathfrak{v}_{\mathrm{r}}\left(\frac{\partial\omega_{2,n}}{\partial\Xi_1}-\frac{\partial\omega_{1,n}}{\partial\Xi_2}\right) \\ &- \varrho_{1,3}\Bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)\,d\Xi_1\,d\Xi_2\,d\Xi_3, \end{aligned} \tag{141}$$

$$\begin{aligned} &\omega_{1,n+1}(\xi_1,\xi_2,\xi_3) \\ &= \omega_{1,n}(\xi_1,\xi_2,\xi_3) \\ &+ \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\omega_1}}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_{\omega_1}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\omega_1}}}} \Bigg(\Delta\omega_{1,n} \\ &- \frac{1}{(\mathfrak{c}_{\mathrm{a}}+\mathfrak{c}_{\mathfrak{d}})} \Bigg((\mathfrak{c}_{\mathrm{a}}-\mathfrak{c}_0-\mathfrak{c}_{\mathfrak{d}})\frac{\partial}{\partial\Xi_1}(\nabla\cdot\boldsymbol{\omega}_n) + (\boldsymbol{\mu}_n\cdot\nabla)\omega_{1,n} \\ &+ 4\mathfrak{v}_{\mathrm{r}}\omega_{1,n} - 2\mathfrak{v}_{\mathrm{r}}\left(\frac{\partial\mu_{3,n}}{\partial\Xi_2}-\frac{\partial\mu_{2,n}}{\partial\Xi_3}\right) \\ &- \varrho_{2,1}\Bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)\,d\Xi_1\,d\Xi_2\,d\Xi_3, \end{aligned} \tag{142}$$

$$\begin{aligned} &\omega_{2,n+1}(\xi_1,\xi_2,\xi_3) \\ &= \omega_{2,n}(\xi_1,\xi_2,\xi_3) \\ &+ \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\omega_2}}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_{\omega_2}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\omega_2}}}} \Bigg(\Delta\omega_{2,n} \\ &- \frac{1}{(\mathfrak{c}_{\mathrm{a}}+\mathfrak{c}_{\mathfrak{d}})} \Bigg((\mathfrak{c}_{\mathrm{a}}-\mathfrak{c}_0-\mathfrak{c}_{\mathfrak{d}})\frac{\partial}{\partial\Xi_2}(\nabla\cdot\boldsymbol{\omega}_n) + (\boldsymbol{\mu}_n\cdot\nabla)\omega_{2,n} \\ &+ 4\mathfrak{v}_{\mathrm{r}}\omega_{2,n} - 2\mathfrak{v}_{\mathrm{r}}\left(\frac{\partial\mu_{1,n}}{\partial\Xi_3}-\frac{\partial\mu_{3,n}}{\partial\Xi_1}\right) \\ &- \varrho_{2,2}\Bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)\,d\Xi_1\,d\Xi_2\,d\Xi_3, \end{aligned} \tag{143}$$

$$\begin{aligned} &\omega_{3,n+1}(\xi_1,\xi_2,\xi_3) \\ &= \omega_{3,n}(\xi_1,\xi_2,\xi_3) \\ &+ \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\omega_3}}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_{\omega_3}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\omega_3}}}} \Bigg(\Delta\omega_{3,n} \\ &- \frac{1}{(\mathfrak{c}_{\mathrm{a}}+\mathfrak{c}_{\mathfrak{d}})} \Bigg((\mathfrak{c}_{\mathrm{a}}-\mathfrak{c}_0-\mathfrak{c}_{\mathfrak{d}})\frac{\partial}{\partial\Xi_3}(\nabla\cdot\boldsymbol{\omega}_n) + (\boldsymbol{\mu}_n\cdot\nabla)\omega_{3,n} \\ &+ 4\mathfrak{v}_{\mathrm{r}}\omega_{3,n} - 2\mathfrak{v}_{\mathrm{r}}\left(\frac{\partial\mu_{2,n}}{\partial\Xi_1}-\frac{\partial\mu_{1,n}}{\partial\Xi_2}\right) \\ &- \varrho_{2,3}\Bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)\,d\Xi_1\,d\Xi_2\,d\Xi_3, \end{aligned} \tag{144}$$

$$\begin{aligned} &\rho_{n+1}(\xi_1,\xi_2,\xi_3) \\ &= \rho_n(\xi_1,\xi_2,\xi_3) \\ &+ \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_\rho}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_\rho}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_\rho}}} \Bigg(\Delta\rho_n \\ &+ \frac{\partial}{\partial x_1}\left((\boldsymbol{\mu}_{\boldsymbol{n}}.\nabla)\mu_{1,n}-\varrho_{1,1}\right) + \frac{\partial}{\partial x_2}\left((\boldsymbol{\mu}_{\boldsymbol{n}}.\nabla)\mu_{2,n}-\varrho_{1,2}\right) \\ &+ \frac{\partial}{\partial x_3}\Big((\boldsymbol{\mu}_{\boldsymbol{n}}.\nabla)\mu_{3,n} \\ &- \varrho_{1,3}\Big)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)\,d\Xi_1\,d\Xi_2\,d\Xi_3. \end{aligned} \tag{145}$$

Theorem 11. Let the operators

$$\mathrm{M}: \mathfrak{H}^1(\mho)\cap\boldsymbol{\mathcal{V}}_{\boldsymbol{\vartheta}}(\mho) \to \mathfrak{H}^1(\mho)\cap\boldsymbol{\mathcal{V}}_{\boldsymbol{\vartheta}}(\mho),$$
$$\Omega: \mathfrak{H}^1(\mho)\cap\mathfrak{H}^2_{\boldsymbol{\vartheta}}(\mho) \to \mathfrak{H}^1(\mho)\cap\mathfrak{H}^2_{\boldsymbol{\vartheta}}(\mho),$$
$$\mathrm{P}: \mathfrak{H}^1(\mho) \to \mathfrak{H}^1(\mho)$$

be defined as follows:

For $d = 2$:

$$\mathrm{M}[\boldsymbol{\mu}] = (\mathrm{M}_1[\mu_1], \mathrm{M}_2[\mu_2], 0), \qquad \Omega[\boldsymbol{\omega}] = (0,0,\Omega_3[\omega_3]),$$

with

$$\begin{aligned} &\mathrm{M}_1[\mu_1](\xi_1,\xi_2) \\ &= \mu_1(\xi_1,\xi_2) \\ &+ \int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_{\mu_1}}(\Xi_1,\Xi_2)-\gamma_{2,\lambda_{\mu_1}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\mu_1}}}} \Bigg(\Delta\mu_1 \\ &- \frac{1}{(\mathfrak{v}+\mathfrak{v}_{\mathrm{r}})} \Bigg((\boldsymbol{\mu}.\nabla)\mu_1 + \frac{\partial\rho}{\partial\Xi_1} - 2\mathfrak{v}_{\mathrm{r}}\frac{\partial\omega_3}{\partial\Xi_2} \\ &- \varrho_{1,1}\Bigg)\Bigg)(\Xi_1,\Xi_2)\,d\Xi_1\,d\Xi_2, \end{aligned} \tag{146}$$

$$\mathrm{M}_2[\mu_2](\xi_1,\xi_2) = \mu_2(\xi_1,\xi_2) + \int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_{\mu_2}}(\Xi_1,\Xi_2)-\gamma_{2,\lambda_{\mu_2}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\mu_2}}}}\Bigg(\Delta\mu_2 - \frac{1}{(\mathfrak{v}+\mathfrak{v}_\mathrm{r})}\bigg((\boldsymbol{\mu}.\nabla)\mu_2 + \frac{\partial\rho}{\partial\Xi_2} + 2\mathfrak{v}_\mathrm{r}\frac{\partial\omega_3}{\partial\Xi_1} - \varrho_{1,2}\bigg)\Bigg)(\Xi_1,\Xi_2)d\Xi_1\, d\Xi_2, \tag{147}$$

$$\Omega_3[\omega_3](\xi_1,\xi_2) = \omega_3(\xi_1,\xi_2) + \int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_{\omega_3}}(\Xi_1,\Xi_2)-\gamma_{2,\lambda_{\omega_3}}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_{\omega_3}}}}\Bigg(\Delta\omega_3 - \frac{1}{(\mathfrak{c}_\mathrm{a}+\mathfrak{c}_\mathfrak{d})}\bigg((\boldsymbol{\mu}.\nabla)\omega_3 + 4\mathfrak{v}_\mathrm{r}\omega_3 - 2\mathfrak{v}_\mathrm{r}\left(\frac{\partial\mu_2}{\partial\Xi_1} - \frac{\partial\tilde{\mu}_1}{\partial\Xi_2}\right) - \varrho_{2,3}\bigg)\Bigg)(\Xi_1,\Xi_2)d\Xi_1\, d\Xi_2, \tag{148}$$

$$\mathrm{P}[\rho](\xi_1,\xi_2) = \rho(\xi_1,\xi_2) + \int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{2,\lambda_\rho}(\Xi_1,\Xi_2)-\gamma_{2,\lambda_\rho}(\xi_1,\xi_2)}{a_{\gamma_{2,\lambda_\rho}}}\Bigg(\Delta\rho + \frac{\partial}{\partial x_1}\big((\boldsymbol{\mu}.\nabla)\mu_1 - \varrho_{1,1}\big) + \frac{\partial}{\partial x_2}\big((\boldsymbol{\mu}.\nabla)\mu_2 - \varrho_{1,2}\big)\Bigg)(\Xi_1,\Xi_2)d\Xi_1\, d\Xi_2. \tag{149}$$

For $d = 3$:

$$\mathrm{M}[\boldsymbol{\mu}] = (\mathrm{M}_1[\mu_1], \mathrm{M}_2[\mu_2], \mathrm{M}_3[\mu_3]),$$
$$\Omega[\boldsymbol{\omega}] = (\Omega_1[\omega_1], \Omega_2[\omega_2], \Omega_3[\omega_3]),$$

with

$$\mathrm{M}_1[\mu_1](\xi_1,\xi_2,\xi_3) = \mu_1(\xi_1,\xi_2,\xi_3) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_1}}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_{\mu_1}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3\lambda_{\mu_1}}}}\Bigg(\Delta\mu_1 - \frac{1}{(\mathfrak{v}+\mathfrak{v}_\mathrm{r})}\bigg((\boldsymbol{\mu}\cdot\nabla)\mu_1 + \frac{\partial\rho}{\partial\Xi_1} - 2\mathfrak{v}_\mathrm{r}\left(\frac{\partial\omega_3}{\partial\Xi_2} - \frac{\partial\omega_2}{\partial\Xi_3}\right) - \varrho_{1,1}\bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\, d\Xi_2\, d\Xi_3, \tag{150}$$

$$\mathrm{M}_2[\mu_2](\xi_1,\xi_2,\xi_3) = \mu_2(\xi_1,\xi_2,\xi_3) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_2}}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_{\mu_2}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\mu_2}}}}\Bigg(\Delta\mu_2 - \frac{1}{(v+v_r)}\bigg((\boldsymbol{\mu}\cdot\nabla)\mu_2 + \frac{\partial\rho}{\partial\Xi_2} - 2\mathfrak{v}_\mathrm{r}\left(\frac{\partial\omega_1}{\partial\Xi_3} - \frac{\partial\omega_3}{\partial\Xi_1}\right) - \varrho_{1,2}\bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\, d\Xi_2\, d\Xi_3, \tag{151}$$

$$\mathrm{M}_3[\mu_3](\xi_1,\xi_2,\xi_3) = \mu_3(\xi_1,\xi_2,\xi_3) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\mu_3}}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_{\mu_3}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\mu_3}}}}\Bigg(\Delta\mu_3 - \frac{1}{(v+v_r)}\bigg((\boldsymbol{\mu}\cdot\nabla)\mu_3 + \frac{\partial\rho}{\partial\Xi_3} - 2\mathfrak{v}_\mathrm{r}\left(\frac{\partial\omega_2}{\partial\Xi_1} - \frac{\partial\omega_1}{\partial\Xi_2}\right) - \varrho_{1,3}\bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\, d\Xi_2\, d\Xi_3, \tag{152}$$

$$\Omega_1[\omega_1](\xi_1,\xi_2,\xi_3) = \omega_1(\xi_1,\xi_2,\xi_3) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\omega_1}}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_{\omega_1}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\omega_1}}}}\Bigg(\Delta\omega_1 - \frac{1}{(\mathfrak{c}_\mathrm{a}+\mathfrak{c}_\mathfrak{d})}\bigg((\mathfrak{c}_\mathrm{a}-\mathfrak{c}_0-\mathfrak{c}_\mathfrak{d})\frac{\partial}{\partial\Xi_1}(\nabla\cdot\boldsymbol{\omega}) + (\boldsymbol{\mu}\cdot\nabla)\omega_1 + 4\mathfrak{v}_\mathrm{r}\omega_1 - 2\mathfrak{v}_\mathrm{r}\left(\frac{\partial\mu_3}{\partial\Xi_2} - \frac{\partial\mu_2}{\partial\Xi_3}\right) - \varrho_{2,1}\bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\, d\Xi_2\, d\Xi_3, \tag{153}$$

$$\Omega_2[\omega_2](\xi_1,\xi_2,\xi_3) = \omega_2(\xi_1,\xi_2,\xi_3) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\omega_2}}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_{\omega_2}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\omega_2}}}}\Bigg(\Delta\omega_2 - \frac{1}{(\mathfrak{c}_\mathrm{a}+\mathfrak{c}_\mathfrak{d})}\bigg((\mathfrak{c}_\mathrm{a}-\mathfrak{c}_0-\mathfrak{c}_\mathfrak{d})\frac{\partial}{\partial\Xi_2}(\nabla\cdot\boldsymbol{\omega}) + (\boldsymbol{\mu}\cdot\nabla)\omega_2 + 4\mathfrak{v}_\mathrm{r}\omega_2 - 2\mathfrak{v}_\mathrm{r}\left(\frac{\partial\mu_1}{\partial\Xi_3} - \frac{\partial\mu_3}{\partial\Xi_1}\right) - \varrho_{2,2}\bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\, d\Xi_2\, d\Xi_3, \tag{154}$$

$$\Omega_3[\omega_3](\xi_1,\xi_2,\xi_3) = \omega_{3,n}(\xi_1,\xi_2,\xi_3) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_{\omega_3}}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_{\omega_3}}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_{3,\lambda_{\omega_3}}}}\Bigg(\Delta\omega_3 - \frac{1}{(\mathfrak{c}_\mathrm{a}+\mathfrak{c}_\mathfrak{d})}\bigg((\mathfrak{c}_\mathrm{a}-\mathfrak{c}_0-\mathfrak{c}_\mathfrak{d})\frac{\partial}{\partial\Xi_3}(\nabla\cdot\boldsymbol{\omega}) + (\boldsymbol{\mu}\cdot\nabla)\omega_3 + 4\mathfrak{v}_\mathrm{r}\omega_3 - 2\mathfrak{v}_\mathrm{r}\left(\frac{\partial\mu_2}{\partial\Xi_1} - \frac{\partial\mu_1}{\partial\Xi_2}\right) - \varrho_{2,3}\bigg)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\, d\Xi_2\, d\Xi_3, \tag{155}$$

$$\mathrm{P}[\rho](\xi_1,\xi_2,\xi_3) = \rho(\xi_1,\xi_2,\xi_3) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{3,\lambda_\rho}(\Xi_1,\Xi_2,\Xi_3)-\gamma_{3,\lambda_\rho}(\xi_1,\xi_2,\xi_3)}{a_{\gamma_3\lambda_\rho}}\Bigg(\Delta\rho + \frac{\partial}{\partial x_1}\big((\boldsymbol{\mu}\cdot\nabla)\mu_1 - \varrho_{1,1}\big) + \frac{\partial}{\partial x_2}\big((\boldsymbol{\mu}.\nabla)\mu_2 - \varrho_{1,2}\big) + \frac{\partial}{\partial x_3}\big((\boldsymbol{\mu}.\nabla)\mu_3 - \varrho_{1,3}\big)\Bigg)(\Xi_1,\Xi_2,\Xi_3,\tau)d\Xi_1\, d\Xi_2\, d\Xi_3. \tag{156}$$

If there are constants $\mathfrak{C}_{\mathrm{M}} < 1, \mathfrak{C}_{\Omega} < 1$ and $\mathfrak{C}_{\mathrm{P}} < 1$ such that:

$$\|\mathrm{M}[\boldsymbol{\mu}] - \mathrm{M}[\tilde{\boldsymbol{\mu}}]\|_{\mathfrak{H}^2(\mho)} \leq \mathfrak{C}_{\mathrm{M}}\|\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}\|_{\mathfrak{H}^2(\mho)}, \quad (157)$$

$$\|\Omega[\boldsymbol{\omega}] - \Omega[\tilde{\boldsymbol{\mu}}]\|_{\mathfrak{H}^2(\mho)} \leq \mathfrak{C}_{\Omega}\|\boldsymbol{\omega} - \tilde{\boldsymbol{\omega}}\|_{\mathfrak{H}^2(\mho)}, \quad (158)$$

$$\|\mathrm{P}[\rho] - \mathrm{P}[\tilde{\rho}]\|_{\mathfrak{H}^1(\mho)} \leq \mathfrak{C}_{\mathrm{P}}\|\rho - \tilde{\rho}\|_{\mathfrak{H}^1(\mho)}, \quad (159)$$

for all $\mu, \tilde{\mu} \in \mathfrak{H}^1(\mho) \cap V_g(\mho), \omega, \tilde{\omega} \in \mathfrak{H}^1(\mho) \cap \mathfrak{H}_q^2(\mho)$, and $\rho, \tilde{\rho} \in \mathfrak{H}^1(\mho)$, then systems (133) and (134) admit unique solutions.

Theorem 12. Assume the operators

$$\mathrm{M}: \mathcal{K}_{\mathrm{M}} \to \mathcal{K}_{\mathrm{M}}, \qquad \Omega: \mathcal{K}_{\Omega} \to \mathcal{K}_{\Omega}, \qquad \mathrm{P}: \mathcal{K}_{\mathrm{P}} \to \mathcal{K}_{\mathrm{P}}$$

Defined in Theorem 11 act on compact and convex subsets $\mathcal{K}_{\mathrm{M}}, \mathcal{K}_{\Omega}$ and $\mathcal{K}_{\mathrm{P}}$ of the spaces

$$\mathfrak{L}^{\infty}\left(0, \mathrm{T}, \mathfrak{H}^1(\mho)\right) \cap \mathfrak{L}^2\left(0, \mathrm{T}, \boldsymbol{\mathcal{V}}_g(\mho)\right),$$

$$\mathfrak{L}^{\infty}\left(0, \mathrm{T}, \mathfrak{H}^1(\mho)\right) \cap \mathfrak{L}^2\left(0, \mathrm{T}, \mathfrak{H}_{\boldsymbol{\vartheta}}^2(\mho)\right),$$

$$\mathfrak{L}^2(0, \mathrm{T}, \mathfrak{H}^1(\mho)),$$

respectively. If M, Ω and P are continuous, then systems (133) and (134) admit solutions.

V. APPLICATIONS

We now utilize our findings to solve the incomposible fluid and microrotational fluid flow systems.

Example 13 ([7, 12, 14]). Consider plane Couette flow with a velocity field $\boldsymbol{\mu} = (\mu_1(\xi_2), 0)$ and a pressure field $\rho(\xi_1)$ defined in the domain:

$$\mho = \{(\xi_1, \xi_2) \in \mathcal{R}^2 : \xi_2 \in [0, \mathfrak{h}]\}.$$

The governing system for the flow is:

$$\begin{cases} \mathfrak{v}\dfrac{d^2\mu_1}{d\xi_2^2} = \dfrac{d\rho}{d\xi_1} & \text{in } \mho, \\ \mu_1(0) = 0, \mu_1(\mathfrak{h}) = \mathrm{M}. \end{cases} \quad (160)$$

Rewriting (160.1) gives the simplified momentum equation:

$$\frac{d^2\mu_1}{d\xi_2^2} = \frac{1}{\mathfrak{v}}\frac{d\rho}{d\xi_1}. \quad (161)$$

Applying the operator $\frac{d}{d\xi_1}$ to equation (160.1) yields:

$$\begin{cases} \dfrac{d^2\rho}{d\xi_1^2} = 0, \\ \dfrac{\partial\rho}{\partial n} = 0. \end{cases} \quad (162)$$

The correction functionals for equations (161) and (162.1) are:

$$\mu_{1,n+1}(\xi_2) = \mu_{1,n}(\xi_2) + \int_0^{\xi_2}(\Xi_2 - \xi_2)\left(\frac{d^2\mu_{1,n}}{d\Xi_2^2} - \frac{1}{\mathfrak{v}}\frac{d\rho_n}{d\Xi_1}\right)(\Xi_2)d\Xi_2, \quad (163)$$

$$\rho_{n+1}(\xi_1) = \rho_n(\xi_1) + \int_0^{\xi_1}(\Xi_1 - \xi_1)\frac{d^2\rho_n}{d\Xi_1^2}(\Xi_1)d\Xi_1. \quad (164)$$

Starting with unconstrained initial guesses:

$$\mu_{1,0} = \mathfrak{A}_1\xi_2^2 + \mathfrak{B}_1\xi_2 + \mathfrak{C}_1, \quad (165)$$

$$\rho_0 = \mathfrak{A}_2\xi_1 + \mathfrak{B}_2. \quad (166)$$

where $\mathfrak{A}_i, \mathfrak{B}_i (i = 1,2), \mathfrak{C}_1$ are constants to be determined, the first iteration yields:

$$\mu_{1,1} = \frac{(\mathfrak{A}_2 - 2\mathfrak{A}_1\mathfrak{v})}{2\mathfrak{v}}\xi_2^2 + \mathfrak{A}_1\xi_2^2 + \mathfrak{B}_1\xi_2 + \mathfrak{C}_1, \quad (167)$$

$$\rho_1 = \mathfrak{A}_2\xi_1 + \mathfrak{B}_2. \quad (168)$$

Applying (160.2) and (162.2) requires that $\mathfrak{A}_2 = 2\mathfrak{A}_1\mathfrak{v}$, leading to:

$$\mathfrak{A}_1 = 0, \mathfrak{B}_1 = \frac{\mathrm{M}}{\mathfrak{h}}, \mathfrak{C}_1 = 0, \mathfrak{A}_2 = 0.$$

As a result, the exact solution is:

$$\mu_{1,1} = \left(\frac{\mathrm{M}}{\mathfrak{h}}\right)\xi_2, \quad (169)$$

$$\rho_1 = \mathfrak{B}_2, \quad (170)$$

where $\mathfrak{B}_2$ is an arbitrary constant.

Example 14 ([7, 12, 14]). Think about a flow that is rectilinear between parallel plates with a velocity field $\boldsymbol{\mu} = (\mu_1(\xi_3), \mu_2(\xi_3), 0)$ and a pressure field $\rho(\xi_1)$ defined in the domain:

$$\mho = \{(\xi_1, \xi_2, \xi_3) \in \mathcal{R}^3 : \xi_3 \in [-\mathfrak{h}, \mathfrak{h}]\}.$$

The following system regulates the flow:

$$\begin{cases} \mathfrak{v}\dfrac{d^2\mu_1}{d\xi_3^2} = \dfrac{d\rho}{d\xi_1} & \text{in } \mho, \\ \dfrac{d^2\mu_2}{d\xi_3^2} = 0 & \text{in } \mho \\ \mu_i(\mp\mathfrak{h}) = \mp\mathrm{M}, & i = 1,2 \end{cases} \quad (171)$$

Rearranging equation (171.1) yields the momentum equation:

$$\frac{d^2\mu_1}{d\xi_3^2} = \frac{1}{\mathfrak{v}}\frac{d\rho}{d\xi_1}. \quad (172)$$

Equation (171.2) remains:

$$\frac{d^2\mu_2}{d\xi_3^2} = 0. \quad (173)$$

Applying the operator $\frac{d}{d\xi_1}$ to equation (171.1) gives:

$$\begin{cases} \dfrac{d^2\rho}{d\xi_1^2} = 0, \\ \dfrac{\partial\rho}{\partial n} = 0. \end{cases} \quad (174)$$

The correction functionals for equations (172), (173), and (174.1) are:

$$\mu_{1,n+1}(\xi_3) = \mu_{1,n}(\xi_3) + \int_{-\mathfrak{h}}^{\xi_3}(\Xi_3 - \xi_3)\left(\frac{d^2\mu_{1,n}}{d\Xi_3^2} - \frac{1}{\mathfrak{v}}\frac{d\rho_n}{d\Xi_1}\right)(\Xi_3)d\Xi_3, \quad (175)$$

$$\mu_{2,n+1}(\xi_3) = \mu_{2,n}(\xi_3) + \int_{-\mathfrak{h}}^{\xi_3}(\Xi_3 - \xi_3)\frac{d^2\mu_{2,n}}{d\Xi_3^2}(\Xi_3)d\Xi_3, \quad (176)$$

$$\rho_{n+1}(\xi_1) = \rho_n(\xi_1) + \int_0^{\xi_1}(\Xi_1 - \xi_1)\frac{d^2\rho_n}{d\xi_1^2}(\Xi_1)d\Xi_1. \quad (177)$$

Starting with unconstrained initial guesses:

$$\mu_{1,0} = \mathfrak{A}_1\xi_3^2 + \mathfrak{B}_1\xi_3 + \mathfrak{C}_1, \quad (178)$$

$$\mu_{2,0} = \mathfrak{A}_2\xi_3 + \mathfrak{B}_2, \quad (179)$$

$$\rho_0 = \mathfrak{A}_3\xi_1 + \mathfrak{B}_3. \quad (180)$$

where the boundary conditions determine the constants $\mathfrak{A}_i, \mathfrak{B}_i (i = 1,2,3), \mathfrak{C}_1$, the first iteration produces:

$$\mu_{11} = \frac{(\mathfrak{A}_3 - 2\mathfrak{A}_1\mathfrak{v})}{2\mathfrak{v}}(\xi_3 + \mathfrak{h})^2 + \mathfrak{A}_1\xi_3^2 + \mathfrak{B}_1\xi_3 + \mathfrak{C}_1, (181)$$

$$\mu_{21} = \mathfrak{A}_2\xi_3 + \mathfrak{B}_2, \quad (182)$$

$$\rho_1 = \mathfrak{A}_3\xi_1 + \mathfrak{B}_3. \quad (183)$$

applying (171.3)-(171.4) requires $\mathfrak{A}_3 = 2\mathfrak{A}_1\mathfrak{v}$, leading to:

$$\mathfrak{A}_1 = 0, \mathfrak{B}_1 = \frac{\mathrm{M}}{\mathfrak{h}}, \mathfrak{C}_1 = 0, \mathfrak{A}_2 = \frac{\mathrm{M}}{\mathfrak{h}}, \mathfrak{B}_2 = 0.$$

Therefore, the exact solution is:

$$\mu_{1,1} = \left(\frac{\mathrm{M}}{\mathfrak{h}}\right)\xi_3, \quad (184)$$

$$\mu_{2,1} = \left(\frac{\mathrm{M}}{\mathfrak{h}}\right)\xi_3, \quad (185)$$

$$\rho_1 = \mathfrak{B}_3, \quad (186)$$

where $\mathfrak{B}_3$ is an arbitrary constant.

Example 15 ([7, 12, 14]). Assume the channel flow driven by the pulsating pressure gradient with a velocity field $\boldsymbol{\mu} = (\mu_1(\xi_3, \tau), 0,0)$ and a pressure field $\rho(\xi_1)$ in the domain:

$$\mho = \{(\xi_1, \xi_2, \xi_3, \tau): \xi_3 \in [0, \mathfrak{h}], \tau \in [0, \infty)\}.$$

The governing system is:

$$\begin{cases} \dfrac{\partial\mu_1}{\partial\tau} = -\dfrac{d\rho}{d\xi_1} + \mathfrak{v}\dfrac{\partial^2\mu_1}{\partial\xi_3^2}, & \text{in } \mho, \\ \dfrac{\partial\mu_1}{\partial\xi_3}(0,\tau) = \mathrm{M}_1, \mu_1(\mathfrak{h},\tau) = \mathrm{M}_2, & \forall\tau \geq 0. \end{cases} \quad (187)$$

We consider only the steady-state condition, i.e., $\frac{\partial\mu_1}{\partial\tau} = 0$. Under this assumption, equation (187.1) reduces to the simplified momentum equation:

$$\frac{\partial^2\mu_1}{\partial\xi_3^2} = \frac{1}{\mathfrak{v}}\frac{d\rho}{d\xi_1}. \quad (188)$$

Utilizing the operator $\frac{d}{d\xi_1}$ to (187.1) yields:

$$\begin{cases} \dfrac{d^2\rho}{d\xi_1^2} = 0, \\ \dfrac{\partial\rho}{\partial n} = 0. \end{cases} \quad (189)$$

The correction functionals for equations (188) and (189.1) are:

$$\mu_{1,n+1}(\xi_3, \tau) = \mu_{1,n}(\xi_3, \tau) + \int_0^{\xi_3}(\Xi_3 - \xi_3)\left(\frac{\partial^2\mu_{1,n}}{\partial\xi_3^2} - \frac{1}{\mathfrak{v}}\frac{d\rho_n}{d\xi_1}\right)(\Xi_3)d\Xi_3, \quad (190)$$

$$\rho_{n+1}(\xi_1, \tau) = \rho_n(\xi_1, \tau) + \int_0^{\xi_1}(\Xi_1 - \xi_1)\frac{d^2\rho_n}{d\xi_1^2}(\Xi_1)d\Xi_1. (191)$$

Starting with unconstrained initial guesses:

$$\mu_{1,0} = \mathfrak{A}_1\xi_3^2 + \mathfrak{B}_1\xi_3 + \mathfrak{C}_1, \quad (192)$$

$$\rho_0 = \mathfrak{A}_2\xi_1 + \mathfrak{B}_2, \quad (193)$$

where the boundary conditions determine the constants $\mathfrak{A}_i, \mathfrak{B}_i (i = 1,2), \mathfrak{C}_1$, the first iteration gives:

$$\mu_{1,1} = \frac{\mathfrak{A}_2}{2\mathfrak{v}}\xi_3^2 + \mathfrak{B}_1\xi_3 + \mathfrak{C}_1, \quad (194)$$

$$\rho_1 = \mathfrak{A}_2\xi_1 + \mathfrak{B}_2. \quad (195)$$

Implementing (187.2) and (187.2), we find:

$$\mathfrak{B}_1 = \mathrm{M}_1, \mathfrak{C} = \mathrm{M}_2 - \mathrm{M}_1\mathfrak{h}, \mathfrak{A}_2 = 0.$$

Hence, the exact solution becomes:

$$\mu_{11} = \mathrm{M}_1(\xi_3 - \mathfrak{h}) + \mathrm{M}_2, \quad (196)$$

$$\rho_1 = \mathfrak{B}_2, \quad (197)$$

where $\mathfrak{B}_2$ is an arbitrary constant.

Example 16 ([10, 31, 37, 41]). Think about the unsteady, decaying Taylor-Green vortex flow with velocity field $\boldsymbol{\mu} = \big(\mu_1(\xi_1, \xi_2, \tau), \mu_2(\xi_1, \xi_2, \tau)\big)$ and pressure field $\rho(\xi_1, \xi_2)$ within the space-time domain:

$$\mho_{\mathrm{T}} = \{(\xi_1, \xi_2, \tau): (\xi_1, \xi_2) \in \mho = [0,2\pi] \times [0,2\pi]; \tau \in [0, \mathrm{T}]\}.$$

The flow is governed by the incompressible Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial\mu_1}{\partial\tau} + \mu_1\dfrac{\partial\mu_1}{\partial\xi_1} + \mu_2\dfrac{\partial\mu_1}{\partial\xi_2} - \mathfrak{v}\Delta\mu_1 = -\dfrac{\partial\rho}{\partial\xi_1} & \text{in } \mho_{\mathrm{T}}, \\ \dfrac{\partial\mu_2}{\partial\tau} + \mu_1\dfrac{\partial\mu_2}{\partial\xi_1} + \mu_2\dfrac{\partial\mu_2}{\partial\xi_2} - \mathfrak{v}\Delta\mu_2 = -\dfrac{\partial\rho}{\partial\xi_2} & \text{in } \mho_{\mathrm{T}}, \\ \dfrac{\partial\mu_1}{\partial\xi_1} + \dfrac{\partial\mu_2}{\partial\xi_2} = 0 & \text{in } \mho_{\mathrm{T}}, \end{cases} (198)$$

under the periodic boundary conditions:

$$\begin{cases} \mu_i(0, \xi_2, \tau) = \mu_i(2\pi, \xi_2, \tau), \\ \mu_i(\xi_1, 0, \tau) = \mu_i(\xi_1, 2\pi, \tau), \end{cases} \quad \text{for } i = 1,2,3,$$

along with:

$$\mu_i(\xi_1, \xi_2, 0) = \mu_i^0(\xi_1, \xi_2,), \quad \text{in } \mho,$$

Rearranging the momentum equations (198.1) and (198.2), we obtain expressions for the Laplacians of the velocity components:

$$\Delta\mu_1 = \frac{1}{\mathfrak{v}}\left(\frac{\partial\mu_1}{\partial\tau} + \mu_1\frac{\partial\mu_1}{\partial\xi_1} + \mu_2\frac{\partial\mu_1}{\partial\xi_2} + \frac{\partial\rho}{\partial\xi_1}\right), \quad (199)$$

$$\Delta\mu_2 = \frac{1}{\mathfrak{v}}\left(\frac{\partial\mu_2}{\partial\tau} + \mu_1\frac{\partial\mu_2}{\partial\xi_1} + \mu_2\frac{\partial\mu_2}{\partial\xi_2} + \frac{\partial\rho}{\partial\xi_2}\right). \quad (200)$$

Making use of the divergence operator to the momentum equations yields a Neumann pressure problem:

$$\begin{cases} \Delta\rho = -\nabla\cdot((\boldsymbol{\mu}\cdot\nabla)\boldsymbol{\mu}) & \text{in } \mho_{\mathrm{T}}, \\ \dfrac{\partial\rho}{\partial n} = \left(\mathfrak{v}\Delta\boldsymbol{\mu} - \dfrac{\partial\boldsymbol{\mu}}{\partial\tau} - (\boldsymbol{\mu}\cdot\nabla)\boldsymbol{\mu}\right)\cdot\boldsymbol{n} & \text{on } \partial\mho_{\mathrm{T}}, \end{cases} \quad (201)$$

where $\boldsymbol{n}$ is the outward unit normal to $\partial\mho$.

The correction functionals for (199), (200), and (200.1) using

$$\gamma_{2,\lambda_j}(\Xi_1, \Xi_2) = a_{\gamma_{2,\lambda_{\mu_j}}}\Xi_j (j = 1,2),$$

$$\gamma_{2,\lambda_\rho}(\Xi_1, \Xi_2) = a_{\gamma_{2,\lambda_\rho}}\Xi_1,$$

are:

$$\mu_{1,n+1}(\xi_1, \xi_2, \tau) = \mu_{1,n}(\xi_1, \xi_2, \tau) + \int_0^{\xi_2}\int_0^{\xi_1}(\Xi_1 - \xi_1)\Bigg(\Delta\mu_1 - \frac{1}{\mathfrak{v}}\left(\frac{\partial\mu_1}{\partial\tau} + \mu_1\frac{\partial\mu_1}{\partial\xi_1} + \mu_2\frac{\partial\mu_1}{\partial\xi_2} + \frac{\partial\rho}{\partial\xi_1}\right)\Bigg)(\Xi_1, \Xi_2)d\Xi_1\, d\Xi_2, \quad (202)$$

$$\mu_{2,n+1}(\xi_1,\xi_2,\tau) = \mu_{2,n}(\xi_1,\xi_2,\tau) + \int_0^{\xi_2}\int_0^{\xi_1} (\Xi_2-\xi_2)\Bigg(\Delta\mu_2 - \frac{1}{\mathfrak{v}}\left(\frac{\partial\mu_2}{\partial\tau} + \mu_1\frac{\partial\mu_2}{\partial\xi_1} + \mu_2\frac{\partial\mu_2}{\partial\xi_2} + \frac{\partial\rho}{\partial\xi_2}\right)\Bigg)(\Xi_1,\Xi_2)d\Xi_1\,d\Xi_2, \tag{203}$$

$$\rho_{n+1}(\xi_1,\xi_2,\tau) = \rho_n(\xi_1,\xi_2,\tau) + \int_0^{\xi_2}\int_0^{\xi_1} (\Xi_1-\xi_1)\Bigg(\Delta\rho_n + \frac{\partial}{\partial\xi_1}\left(\mu_1\frac{\partial\mu_1}{\partial\xi_1} + \mu_2\frac{\partial\mu_1}{\partial\xi_2}\right) + \frac{\partial}{\partial\xi_2}\left(\mu_1\frac{\partial\mu_2}{\partial\xi_1} + \mu_2\frac{\partial\mu_2}{\partial\xi_2}\right)\Bigg)(\Xi_1,\Xi_2)d\Xi_1\,d\Xi_2. \tag{204}$$

The following initial approximations consistent with the initial conditions at $\tau = 0$ are used:

$$\mu_{1,0}(\xi_1,\xi_2,\tau) = -(1-2\mathfrak{v}\tau)\cos(\xi_1)\sin(\xi_2), \tag{205}$$

$$\mu_{2,0}(\xi_1,\xi_2,\tau) = (1-2\mathfrak{v}\tau)\sin(\xi_1)\cos(\xi_2), \tag{206}$$

$$\rho_0(\xi_1,\xi_2,\tau) = -\frac{(1-4\mathfrak{v}\tau+8\mathfrak{v}^2\tau^2)}{4}(\cos(2\xi_1) + \cos(2\xi_2)). \tag{207}$$

The approximate solutions after one iteration step are derived. The accuracy of these approximations improves with further iterations:

$$\mu_{1,\mathrm{ap}}\ (\xi_1,\xi_2,\tau) = -(1-2\mathfrak{v}\tau)\cos(\xi_1)\sin(\xi_2) + 4\tau\mathfrak{v}(\cos(\xi_1)-1)(\cos(\xi_2)-1) + \tau^2\mathfrak{v}\xi_2\left(\xi_1 - \frac{1}{2}\sin(\xi_1)\right), \tag{208}$$

$$\mu_{2,\mathrm{app}}\,(\xi_1,\xi_2,\tau) = (1-2\mathfrak{v}\tau)\sin(\xi_1)\cos(\xi_2) - 16\tau\mathfrak{v}\sin^2\left(\frac{\xi_1}{2}\right)\sin^2\left(\frac{\xi_2}{2}\right) + \tau^2\mathfrak{v}\xi_1\left(\xi_2 + \sin(\xi_2)\left(2\sin^2\left(\frac{\xi_2}{2}\right)-1\right)\right), \tag{209}$$

$$\rho_{\mathrm{app}}\,(\xi_1,\xi_2,\tau) = -\frac{(1-4\mathfrak{v}\tau+8\mathfrak{v}^2\tau^2)}{4}(\cos(2\xi_1) + \cos(2\xi_2)) - 2\tau^2\mathfrak{v}^2(\xi_2\sin^2(\xi_1) + x_1^2\cos(\xi_2)\sin(\xi_2)). \tag{210}$$

Contrasted with the exact solution:

$$\begin{cases}\mu_1(\xi_1,\xi_2,\tau) = -e^{-2\mathfrak{v}\tau}\cos(\xi_1)\sin(\xi_2)\,,\\ \mu_2(\xi_1,\xi_2,\tau) = e^{-2\mathfrak{v}\tau}\sin(\xi_1)\cos(\xi_2)\,,\\ \rho(\xi_1,\xi_2,\tau) = \dfrac{-e^{-4\mathfrak{v}\tau}}{4}\big(\cos(2\xi_1)+\cos(2\xi_2)\big),\end{cases} \tag{211}$$

yields the absolute errors presented in:

Table I: Absolute Errors in $\mathfrak{L}^2\big(0,\mathrm{T},\boldsymbol{\mathcal{V}}_g(\mho)\big)$ and $\mathfrak{L}^2(0,\mathrm{T},\mathfrak{H}^1(\mho))$ Norms.

Time within ten second	$\lVert\boldsymbol{\mu}-\boldsymbol{\mu}_{ap\rho}\rVert_{\mathfrak{L}^2(0,\mathrm{T},\mathfrak{H}^2(\mho))}$		$\lVert\rho-\rho_{ap\rho}\rVert_{\mathfrak{L}^2(0,\mathrm{T},\mathfrak{H}^1(\mho))}$	
	$\mathfrak{v}=\frac{1}{5000}$	$\mathfrak{v}=\frac{1}{10000}$	$\mathfrak{v}=\frac{1}{5000}$	$\mathfrak{v}=\frac{1}{10000}$
T = 0.00	0.0000×10^{-3}	0.0000×10^{-3}	0.1907×10^{-5}	0.2697×10^{-5}
T = 0.01	0.0093×10^{-3}	0.0047×10^{-3}	0.2369×10^{-5}	0.3410×10^{-5}
T = 0.02	0.0271×10^{-3}	0.0132×10^{-3}	0.1126×10^{-5}	0.3351×10^{-5}
T = 0.03	0.0498×10^{-3}	0.0252×10^{-3}	0.1557×10^{-5}	0.2510×10^{-5}
T = 0.04	0.0770×10^{-3}	0.0383×10^{-3}	0.3161×10^{-5}	0.1592×10^{-5}
T = 0.05	0.1078×10^{-3}	0.0540×10^{-3}	0.2208×10^{-5}	0.3327×10^{-5}
T = 0.06	0.1419×10^{-3}	0.0706×10^{-3}	0.2662×10^{-5}	0.2201×10^{-5}
T = 0.07	0.1790×10^{-3}	0.0896×10^{-3}	0.1728×10^{-5}	0.2026×10^{-5}
T = 0.08	0.2189×10^{-3}	0.1093×10^{-3}	0.2494×10^{-5}	0.4470×10^{-5}
T = 0.09	0.2616×10^{-3}	0.1310×10^{-3}	0.1497×10^{-5}	0.3563×10^{-5}
T = 0.10	0.3069×10^{-3}	0.1534×10^{-3}	0.2724×10^{-5}	0.3123×10^{-5}

Table II: Absolute Errors in $\mathfrak{L}^2\big(0,\mathrm{T},\boldsymbol{\mathcal{V}}_g(\mho)\big)$ and $\mathfrak{L}^2(0,\mathrm{T},\mathfrak{H}^1(\mho))$ Norms.

Time within one minute	$\lVert\boldsymbol{\mu}-\boldsymbol{\mu}_{ap\rho}\rVert_{\mathfrak{L}^2(0,\mathrm{T},\mathfrak{H}^2(\mho))}$		$\lVert\rho-\rho_{ap\rho}\rVert_{\mathfrak{L}^2(0,\mathrm{T},\mathfrak{H}^1(\mho))}$	
	$\mathfrak{v}=\frac{1}{5000}$	$\mathfrak{v}=\frac{1}{10000}$	$\mathfrak{v}=\frac{1}{5000}$	$\mathfrak{v}=\frac{1}{10000}$
T = 0.0	0.0000	0.0000	0.1907×10^{-5}	0.2697×10^{-5}
T = 0.1	0.0003	0.0002	0.2724×10^{-5}	0.3123×10^{-5}
T = 0.2	0.0009	0.0004	0.2455×10^{-5}	0.3852×10^{-5}
T = 0.3	0.0017	0.0008	0.2090×10^{-5}	0.4663×10^{-5}
T = 0.4	0.0027	0.0014	0.1996×10^{-5}	0.3471×10^{-5}
T = 0.5	0.0040	0.0020	0.1990×10^{-5}	0.3410×10^{-5}
T = 0.6	0.0057	0.0028	0.2701×10^{-5}	0.2956×10^{-5}
T = 0.7	0.0076	0.0038	0.1908×10^{-5}	0.1773×10^{-5}
T = 0.8	0.0099	0.0049	0.3846×10^{-5}	0.2823×10^{-5}
T = 0.9	0.0126	0.0063	0.3453×10^{-5}	0.2450×10^{-5}
T = 1.0	0.0157	0.0078	0.4595×10^{-5}	0.2814×10^{-5}

Example 17 ([4, 8, 9, 14]). Look at the Arnold-Beltrami-childress flow with velocity field $\boldsymbol{\mu}=\left(\mu_1(\xi_1,\xi_2,\xi_3,\tau),\mu_2(\xi_1,\xi_2,\xi_3,\tau),\mu_3(\xi_1,\xi_2,\xi_3,\tau)\right)$ and the pressure field $\rho(\xi_1,\xi_2,\xi_3)$ within the domain:

$$\mho_{\mathrm{T}}=\{(\xi_1,\xi_2,\xi_3,\tau):\xi_i\in[0,2\pi],i=1,2,3;\tau\in[0,\mathrm{T}]\}.$$

The governing Navier-Stokes equations in this domain are:

$$\begin{cases}\dfrac{\partial\mu_1}{\partial\tau}+\mu_1\dfrac{\partial\mu_1}{\partial\xi_1}+\mu_2\dfrac{\partial\mu_1}{\partial\xi_2}+\mu_3\dfrac{\partial\mu_1}{\partial\xi_3} -\mathfrak{v}\Delta\mu_1=-\dfrac{\partial\rho}{\partial\xi_1}, & \text{in } \mho_{\mathrm{T}}\\ \dfrac{\partial\mu_2}{\partial\tau}+\mu_1\dfrac{\partial\mu_2}{\partial\xi_1}+\mu_2\dfrac{\partial\mu_2}{\partial\xi_2}+\mu_3\dfrac{\partial\mu_2}{\partial\xi_3} -\mathfrak{v}\Delta\mu_2=-\dfrac{\partial\rho}{\partial\xi_2}, & \text{in } \mho_{\mathrm{T}}\\ \dfrac{\partial\mu_3}{\partial\tau}+\mu_1\dfrac{\partial\mu_3}{\partial\xi_1}+\mu_2\dfrac{\partial\mu_3}{\partial\xi_2}+\mu_3\dfrac{\partial\mu_3}{\partial\xi_3} -\mathfrak{v}\Delta\mu_3=-\dfrac{\partial\rho}{\partial\xi_3}, & \text{in } \mho_{\mathrm{T}}\\ \dfrac{\partial\mu_1}{\partial\xi_1}+\dfrac{\partial\mu_2}{\partial\xi_2}+\dfrac{\partial\mu_3}{\partial\xi_3}=0, & \text{in } \mho_{\mathrm{T}}\end{cases} \tag{212}$$

satisfying the periodic boundary conditions:

$$\begin{cases}\mu_i(0,\xi_2,\xi_3,\tau)=\mu_i(2\pi,\xi_2,\xi_3,\tau),\\ \mu_i(\xi_1,0,\xi_3,\tau)=\mu_i(\xi_1,2\pi,\xi_3,\tau), & \text{for } i=1,2,3,\\ \mu_i(\xi_1,\xi_2,0,\tau)=\mu_i(\xi_1,\xi_2,2\pi,\tau),\end{cases}$$

together with:

$$\mu_i(\xi_1,\xi_2,,\xi_3,0)=\mu_i^0(\xi_1,\xi_2,,\xi_3),\quad \text{in } \mho,$$

Rewriting the momentum equations (212.1), (212.2), and (212.3), we derive:

$$\Delta\mu_1=\frac{1}{\mathfrak{v}}\left(\frac{\partial\mu_1}{\partial\tau}+\mu_1\frac{\partial\mu_1}{\partial\xi_1}+\mu_2\frac{\partial\mu_1}{\partial\xi_2}+\mu_3\frac{\partial\mu_1}{\partial\xi_3}+\frac{\partial\rho}{\partial\xi_1}\right), \tag{213}$$

$$\Delta\mu_2=\frac{1}{\mathfrak{v}}\left(\frac{\partial\mu_2}{\partial\tau}+\mu_1\frac{\partial\mu_2}{\partial\xi_1}+\mu_2\frac{\partial\mu_2}{\partial\xi_2}+\mu_3\frac{\partial\mu_2}{\partial\xi_3}+\frac{\partial\rho}{\partial\xi_2}\right), \tag{214}$$

$$\Delta\mu_3=\frac{1}{\mathfrak{v}}\left(\frac{\partial\mu_3}{\partial\tau}+\mu_1\frac{\partial\mu_3}{\partial\xi_1}+\mu_2\frac{\partial\mu_3}{\partial\xi_2}+\mu_3\frac{\partial\mu_3}{\partial\xi_3}+\frac{\partial\rho}{\partial\xi_3}\right). \tag{215}$$

Applying the gradient operator ∇ to the momentum equations yields a Neumann pressure problem:

$$\begin{cases}\Delta\rho=-\nabla\cdot((\boldsymbol{\mu}\cdot\nabla)\boldsymbol{\mu}) & \text{in } \mho_{\mathrm{T}},\\ \dfrac{\partial\rho}{\partial n}=\left(\mathfrak{v}\Delta\boldsymbol{\mu}-\dfrac{\partial\boldsymbol{\mu}}{\partial\tau}-(\boldsymbol{\mu}\cdot\nabla)\boldsymbol{\mu}\right)\cdot\boldsymbol{n} & \text{on } \partial\mho_{\mathrm{T}},\end{cases} \tag{216}$$

where $\boldsymbol{n}$ is the outward unit normal to $\partial\mho$.

The correction funcionals for (213)-(216) using:

$$\gamma_{3,\lambda_j}(\Xi_1,\Xi_2,\Xi_3)=a_{\gamma_{3,\lambda_{\mu_j}}}\Xi_j\ (j=1,2,3),$$

$$\gamma_{3,\lambda_\rho}(\Xi_1,\Xi_2,\Xi_3)=a_{\gamma_{3,\lambda_\rho}}\Xi_1,$$

are:

$$\begin{aligned}\mu_{1,n+1}&(\xi_1,\xi_2,\xi_3,\tau)\\ &=\mu_{1,n}(\xi_1,\xi_2,\xi_3,\tau)\\ &\quad+\int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1}(\Xi_1-\xi_1)\Bigg(\Delta\mu_1\\ &\quad-\frac{1}{\mathfrak{v}}\left(\frac{\partial\mu_1}{\partial\tau}+\mu_1\frac{\partial\mu_1}{\partial\Xi_1}+\mu_2\frac{\partial\mu_1}{\partial\Xi_2}+\mu_3\frac{\partial\mu_1}{\partial\Xi_3}\right.\\ &\quad\left.+\frac{\partial\rho}{\partial\xi_1}\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3)d\Xi_1\,d\Xi_2\,d\Xi_3,\end{aligned} \tag{217}$$

$$\begin{aligned}\mu_{2,n+1}&(\xi_1,\xi_2,\xi_3,\tau)\\ &=\mu_{2,n}(\xi_1,\xi_2,\xi_3,\tau)\\ &\quad+\int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1}(\Xi_2-\xi_2)\Bigg(\Delta\mu_2\\ &\quad-\frac{1}{\mathfrak{v}}\left(\frac{\partial\mu_2}{\partial\tau}+\mu_1\frac{\partial\mu_2}{\partial\Xi_1}+\mu_2\frac{\partial\mu_2}{\partial\Xi_2}+\mu_3\frac{\partial\mu_2}{\partial\Xi_3}\right.\\ &\quad\left.+\frac{\partial\rho}{\partial\Xi_2}\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3)d\Xi_1\,d\Xi_2\,d\Xi_3,\end{aligned} \tag{218}$$

$$\begin{aligned}\mu_{3,n+1}&(\xi_1,\xi_2,\xi_3,\tau)\\ &=\mu_{3,n}(\xi_1,\xi_2,\xi_3,\tau)\\ &\quad+\int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1}(\Xi_3-\xi_3)\Bigg(\Delta\mu_2\\ &\quad-\frac{1}{\mathfrak{v}}\left(\frac{\partial\mu_3}{\partial\tau}+\mu_1\frac{\partial\mu_3}{\partial\Xi_1}+\mu_2\frac{\partial\mu_3}{\partial\Xi_2}+\mu_3\frac{\partial\mu_3}{\partial\Xi_3}\right.\\ &\quad\left.+\frac{\partial\rho}{\partial\Xi_3}\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3)d\Xi_1\,d\Xi_2\,d\Xi_3,\end{aligned} \tag{219}$$

$$\begin{aligned}\rho_{n+1}(\xi_1,\xi_2,\xi_3,\tau)&=\rho_n(\xi_1,\xi_2,\xi_3,\tau)\\ &\quad+\int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1}(\Xi_1-\xi_1)\Bigg(\Delta\rho_n\\ &\quad+\frac{\partial}{\partial\xi_1}\left(\mu_1\frac{\partial\mu_1}{\partial\xi_1}+\mu_2\frac{\partial\mu_1}{\partial\xi_2}+\mu_3\frac{\partial\mu_1}{\partial\xi_3}\right)\\ &\quad+\frac{\partial}{\partial\xi_2}\left(\mu_1\frac{\partial\mu_2}{\partial\xi_1}+\mu_2\frac{\partial\mu_2}{\partial\xi_2}+\mu_3\frac{\partial\mu_2}{\partial\xi_3}\right)\\ &\quad+\frac{\partial}{\partial\xi_3}\left(\mu_1\frac{\partial\mu_3}{\partial\xi_1}+\mu_2\frac{\partial\mu_3}{\partial\xi_2}\right.\\ &\quad\left.+\mu_3\frac{\partial\mu_3}{\partial\xi_3}\right)\Bigg)(\Xi_1,\Xi_2,\Xi_3)d\Xi_1\,d\Xi_2\,d\Xi_3.\end{aligned} \tag{220}$$

The initial approximations are giving by:

$$\begin{cases}\mu_{1,0}(\xi_1,\xi_2,\xi_3,\tau)=e^{-\mathfrak{v}\tau}(\mathfrak{A}\sin(\xi_3)+\mathfrak{B}\cos(\xi_2)),\\ \mu_{2,0}(\xi_1,\xi_2,\xi_3,\tau)=e^{-\mathfrak{v}\tau}(\mathfrak{C}\sin(\xi_1)+\mathfrak{D}\cos(\xi_3)),\\ \mu_{3,0}(\xi_1,\xi_2,\xi_3,\tau)=e^{-\mathfrak{v}\tau}(\mathfrak{E}\sin(\xi_2)+\mathfrak{F}\cos(\xi_1)),\\ \rho_0(\xi_1,\xi_2,\xi_3,\tau)=e^{-2\mathfrak{v}\tau}\begin{pmatrix}\mathfrak{G}\cos(\xi_1)\sin(\xi_2)\\ +\mathfrak{H}\sin(\xi_1)\cos(\xi_3)\\ +\mathfrak{I}\sin(\xi_3)\cos(\xi_2)\end{pmatrix},\end{cases} \tag{221}$$

where the boundary conditions dictate $\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D},\mathfrak{E},\mathfrak{F},\mathfrak{G},\mathfrak{H},\mathfrak{I}$. From these, we derive the correction functionals:

$$\begin{aligned}\mu_{11}(\xi_1,\xi_2,\xi_3,\tau)&=\mathrm{e}^{-\mathfrak{v}\mathrm{t}}\left(\mathfrak{A}\sin(\xi_3)+\mathfrak{B}\cos(\xi_2)\right)\\ &\quad+\frac{e^{-2\mathfrak{v}\tau}}{\mathfrak{v}}\Bigg((\mathfrak{H}+\mathfrak{A}\mathfrak{F})(1\\ &\quad-\cos(\xi_1))\xi_2\sin(\xi_3)\\ &\quad+\frac{1}{2}(\mathfrak{A}\mathfrak{E}-\mathfrak{B}\mathfrak{D})\xi_1^2(1-\cos(\xi_2))\sin(\xi_3)\\ &\quad-(\mathfrak{G}+\mathfrak{B}\mathfrak{C})\xi_3(\xi_1-\sin(\xi_1))(1\\ &\quad-\cos(\xi_2))\Bigg),\end{aligned} \tag{222}$$

$$\begin{aligned}\mu_{21}(\xi_1,\xi_2,\xi_3,\tau)&=e^{-2\mathfrak{v}\tau}(\mathfrak{D}\cos(\xi_3)+\mathfrak{C}\sin(\xi_1))\\ &\quad+\frac{e^{-2\mathfrak{v}\tau}}{\mathfrak{v}}\Bigg((\mathfrak{G}+\mathfrak{B}\mathfrak{C})\xi_3(1\\ &\quad-\cos(\xi_2))\sin(\xi_1)\\ &\quad-(1\\ &\quad-\cos(\xi_2))\Big((\mathfrak{I}+\mathfrak{D}\mathfrak{E})\xi_1(\xi_2-\sin(\xi_2))\\ &\quad+\frac{1}{2}(\mathfrak{D}\mathfrak{F}-\mathfrak{A}\mathfrak{C})\xi_2^2\sin(\xi_1)\Big)\Bigg),\end{aligned} \tag{223}$$

$$\mu_{3,1}(\xi_1,\xi_2,\xi_3,\tau) = e^{-\mathfrak{v}\tau}(\mathfrak{E}\sin(\xi_2) + \mathfrak{F}\cos(\xi_1)) + \frac{e^{-2\mathfrak{v}\tau}}{\mathfrak{v}}\Big((\mathfrak{I} + \mathfrak{D}\mathfrak{E})\xi_1(1 - \cos(\xi_3))\sin(\xi_2) + (1-\cos(\xi_1))\Big(\frac{1}{2}(\mathfrak{C}\mathfrak{E} - \mathfrak{B}\mathfrak{F})\xi_3^2\sin(\xi_2) - (\mathfrak{H} + \mathfrak{A}\mathfrak{F})\xi_2(\xi_3 - \sin(\xi_3))\Big)\Big), \tag{224}$$

$$\rho_1(\xi_1,\xi_2,\xi_3,\tau) = e^{-2\upsilon\tau}\big(\mathfrak{G}\cos(\xi_1)\sin(\xi_2) + \mathfrak{H}\sin(\xi_1)\cos(\xi_3) + \sin(\xi_3)\cos(\xi_2) + 2(\mathfrak{G} + \mathfrak{B}\mathfrak{C})\xi_3(1-\cos(\xi_1))(1 - \cos(\xi_2)) + (\mathfrak{I} + \mathfrak{D}\mathfrak{E})\xi_1^2(1-\cos(\xi_3))\sin(\xi_2) + 2(\mathfrak{H} + \mathfrak{A}\mathfrak{F})\xi_2\sin(\xi_3)(\xi_1 - \sin(\xi_1))\big). \tag{225}$$

Applying periodicity conditions leads to the relationships:

$$\mathfrak{D} = \mathfrak{A}, \mathfrak{E} = \mathfrak{B}, \mathfrak{F} = \mathfrak{C}, \mathfrak{G} = -\mathfrak{B}\mathfrak{C}, \mathfrak{H} = -\mathfrak{A}\mathfrak{F}, \mathfrak{I} = -\mathfrak{D}\mathfrak{E}.$$

Thus, the exact solution is:

$$\begin{cases} \mu_{1,0}(\xi_1,\xi_2,\xi_3,\tau) = e^{-\mathfrak{v}\tau}(\mathfrak{A}\sin(\xi_3) + \mathfrak{B}\cos(\xi_2)), \\ \mu_{2,0}(\xi_1,\xi_2,\xi_3,\tau) = e^{-\mathfrak{v}\tau}(\mathfrak{C}\sin(\xi_1) + \mathfrak{A}\cos(\xi_3)), \\ \mu_{3,0}(\xi_1,\xi_2,\xi_3,\tau) = e^{-\mathfrak{v}\tau}(\mathfrak{B}\sin(\xi_2) + \mathfrak{C}\cos(\xi_1)), \\ \rho_0(\xi_1,\xi_2,\xi_3,\tau) = -e^{-2\mathfrak{v}\tau}\begin{pmatrix} \mathfrak{B}\mathfrak{C}\cos(\xi_1)\sin(\xi_2) \\ +\mathfrak{A}\mathfrak{C}\sin(\xi_1)\cos(\xi_3) \\ +\mathfrak{A}\mathfrak{B}\sin(\xi_3)\cos(\xi_2) \end{pmatrix}, \end{cases} \tag{226}$$

where $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ are arbitrary constants.

Example 18 ([1, 33, 37]). Take plane Poiseuille flow defined in the domain:

$$\mho = \{(\xi_1,\xi_2,\xi_3) \in \mathcal{R}^3 : \xi_2 \in (-\mathfrak{h},\mathfrak{h})\}.$$

with the velocity field $\boldsymbol{\mu} = (\mu_1(\xi_2), 0,0)$, the microrotation field $\boldsymbol{\omega} = (0,0,\omega_3(\xi_2))$, and the field of pressure $\rho(\xi_1)$.

The governing system is:

$$\begin{cases} (\mathfrak{v} + \mathfrak{v}_r)\dfrac{d^2\mu_1}{d\xi_2^2} + 2\mathfrak{v}_r\dfrac{d\omega_3}{d\xi_2} - \dfrac{d\rho}{d\xi_1} = 0 & \text{in } \mho, \\ (\mathfrak{c}_a + \mathfrak{c}_\mathfrak{d})\dfrac{d^2\omega_3}{d\xi_2^2} - 2\mathfrak{v}_r\left(2\omega_3 + \dfrac{d\mu_1}{d\xi_2}\right) = 0 & \text{in } \mho, \\ \mu_1(-\mathfrak{h}) = 0, \mu_1(\mathfrak{h}) = 0, \\ \omega_3(-\mathfrak{h}) = -\dfrac{1}{\mathfrak{h}}, \omega_3(\mathfrak{h}) = \dfrac{1}{\mathfrak{h}} \end{cases} \tag{227}$$

Rewriting (227.1) and (227.2) gives:

$$\frac{d^2\mu_1}{d\xi_2^2} = -\frac{1}{\mathfrak{v} + \mathfrak{v}_r}\left(2\mathfrak{v}_r\frac{d\omega_3}{d\xi_2} - \frac{d\rho}{d\xi_1}\right), \tag{228}$$

$$\frac{d^2\omega_3}{d\xi_2^2} = \frac{2\mathfrak{v}_r}{\mathfrak{c}_a + \mathfrak{c}_\mathfrak{d}}\left(2\omega_3 + \frac{d\mu\_1}{d\xi_2}\right). \tag{229}$$

Applying $\frac{d}{dx_1}$ to (227.1) shows that the pressure ρ satisfies the Neumann problem:

$$\begin{cases} \dfrac{d^2\rho}{d\xi_1^2} = 0 & \text{in } \mho, \\ \dfrac{\partial\rho}{\partial n} = 0 & \text{on } \partial\mho. \end{cases} \tag{230}$$

The correction functionals for (228), (229), and (230.1) are:

$$\mu_{1,n+1}(\xi_2) = \mu_{1,n}(\xi_2) + \int_{-\mathfrak{h}}^{\xi_2}(\Xi_2 - \xi_2)\left(\frac{d^2\mu_n}{d\Xi_2^2} + \frac{1}{\mathfrak{v} + \mathfrak{v}_r}\left(2\mathfrak{v}_r\frac{d\omega_n}{d\Xi_2} - \frac{d\rho}{d\Xi_1}\right)\right)(\Xi_2)d\Xi_2, \tag{231}$$

$$\omega_{3,n+1}(\xi_2) = \omega_{3,n}(\xi_2) + \int_{-\mathfrak{h}}^{\xi_2}\frac{\sqrt{\mathfrak{c}_a + \mathfrak{c}_\mathfrak{d}}}{2\sqrt{\mathfrak{v}_r}}\sinh\left(\frac{2\sqrt{\mathfrak{v}_r}}{\sqrt{\mathfrak{c}_a + \mathfrak{c}_\mathfrak{d}}}(\Xi_2 - \xi_2)\right)\left(\frac{d^2\omega_n}{d\Xi_2^2} - \frac{2\mathfrak{v}_r}{\mathfrak{c}_a + \mathfrak{c}_\mathfrak{d}}\left(2\omega_n + \frac{d\omega_n}{d\Xi_2}\right)\right)(\Xi_2)d\Xi_2, \tag{232}$$

$$\rho_{n+1}(\xi_1) = \rho_n(\xi_1) + \int_0^{\xi_1}(\Xi_1 - \xi_1)\frac{d^2\rho_n}{d\Xi_1^2}(\Xi_1)d\Xi_1. \tag{233}$$

Starting with the initial approximations:

$$\mu_{1,0} = \mathfrak{A} - \left(\frac{\xi_2}{\mathfrak{h}}\right)^2, \tag{234}$$

$$\omega_{3,0} = \mathfrak{B}\xi_2, \tag{235}$$

$$\rho_0 = \mathfrak{C}\xi_1, \tag{236}$$

where the boundary conditions subsequently determine the constants $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$:

$$\mu_{1,1} = \mathfrak{A} - \left(\frac{\xi_2}{\mathfrak{h}}\right)^2 + \frac{(\mathfrak{h} + \xi_2)^2\left(\mathfrak{v} + \mathfrak{v}_r + \frac{\mathfrak{C}\mathfrak{h}^2}{2} - \mathfrak{B}\mathfrak{h}^2\mathfrak{v}_r\right)}{\mathfrak{h}^2(\mathfrak{v} + \mathfrak{v}_r)}, \tag{237}$$

$$\omega_{3,1} = \mathfrak{B}\xi_2 - \frac{(\mathfrak{B}\mathfrak{h}^2 - 1)}{2\mathfrak{h}^2\sqrt{\mathfrak{v}_r}}\left(2\sqrt{\mathfrak{v}_r}\xi_2 - \sqrt{\mathfrak{c}_a + \mathfrak{c}_\mathfrak{d}}\sinh\left(2\sqrt{\frac{\mathfrak{v}_r}{\mathfrak{c}_a + \mathfrak{c}_\mathfrak{d}}}(\xi_2 + \mathfrak{h})\right) + 2\mathfrak{h}\sqrt{\mathfrak{v}_r}\cosh\left(2\sqrt{\frac{\mathfrak{v}_r}{\mathfrak{c}_a + \mathfrak{c}_\mathfrak{d}}}(\xi_2 + \mathfrak{h})\right)\right), \tag{238}$$

$$\rho_1 = \mathfrak{C}\xi_1. \tag{239}$$

Imposing $\mu(\pm\mathfrak{h}) = 0$ and $\omega(\pm\mathfrak{h}) = \pm 1/\mathfrak{h}$ on μ_1 and ω_1 determines the constants: $\mathfrak{A} = 1, \mathfrak{B} = 1/\mathfrak{h}^2$, and requires setting $\mathfrak{C} = -2\mathfrak{v}/\mathfrak{h}^2$. Substituting these values into (237) – (239) provides the exact solution for system (227):

$$\mu_1 = 1 - \left(\frac{\xi_2}{\mathfrak{h}}\right)^2, \tag{240}$$

$$\omega_3 = \frac{\xi_2}{\mathfrak{h}^2}, \tag{241}$$

$$\rho = -\frac{2\mathfrak{v}\xi_1}{\mathfrak{h}^2}. \tag{242}$$

Example 19. Observe the following system:

$$\begin{cases} \mathfrak{v}\frac{d^2\mu}{d\xi_2^2} + \mathfrak{v}_r\frac{d\omega}{d\xi_2} - \frac{d\rho}{d\xi_1} = 0 & \text{in } \mho, \\ (\mathfrak{c}_a + \mathfrak{c}_b)\frac{d^2\omega}{d\xi_2^2} - 2\mathfrak{v}_r\omega = 0 & \text{in } \mho, \\ \mu(-\mathfrak{h}) = 0, \mu(\mathfrak{h}) = 0, \\ \omega(-\mathfrak{h}) = 0, \omega(\mathfrak{h}) = 0, \end{cases} \tag{243}$$

with the same assumptions as Example 18. We first rewrite equations (243.1) and (243.2) as:

$$\frac{d^2\mu}{d\xi_2^2} = -\frac{1}{\mathfrak{v}}\left(\mathfrak{v}_r\frac{d\omega}{d\xi_2} - \frac{d\rho}{d\xi_1}\right), \tag{244}$$

$$\frac{d^2\omega}{d\xi_2^2} = \frac{2\mathfrak{v}_r}{\mathfrak{c}_a + \mathfrak{c}_b}\omega. \tag{245}$$

By means of the operator $\frac{d}{dx_1}$ to (243.1) reveals that the pressure ρ satisfies the Neumann problem:

$$\begin{cases} \frac{d^2\rho}{d\xi_1^2} = 0 & \text{in } \mho, \\ \frac{\partial\rho}{\partial n} = 0 & \text{on } \partial\mho. \end{cases} \tag{246}$$

The correction functionals for equation (244), (245), and (246.1) are:

$$\mu_{n+1}(\xi_2) = \mu_n(\xi_2) + \int_{-\mathfrak{h}}^{\xi_2}(\Xi_2 - \xi_2)\left(\frac{d^2\mu_n}{d\Xi_2^2} + \frac{1}{\mathfrak{v}}\left(\mathfrak{v}_r\frac{d\omega_n}{d\Xi_2} - \frac{d\rho_n}{d\Xi_1}\right)\right)(\Xi_2)d\Xi_2, \tag{247}$$

$$\omega_{n+1}(\xi_2) = \omega_n(\xi_2) + \int_{-\mathfrak{h}}^{\xi_2}\frac{\sqrt{\mathfrak{c}_a + \mathfrak{c}_b}}{2\sqrt{\mathfrak{v}_r}}\sinh\left(\frac{2\sqrt{\mathfrak{v}_r}}{\sqrt{\mathfrak{c}_a + \mathfrak{c}_b}}(\Xi_2 - \xi_2)\right)\left(\frac{d^2\omega_n}{d\Xi_2^2} - \frac{2\mathfrak{v}_r}{\mathfrak{c}_a + \mathfrak{c}_b}\omega_n\right)(\Xi_2)d\Xi_2, \tag{248}$$

$$\rho_{n+1}(\xi_1) = \rho_0(\xi_1) + \int_0^{\xi_1}(\Xi_1 - \xi_1)\frac{d^2\rho_n}{d\Xi_1^2}(\Xi_1)d\Xi_1. \tag{249}$$

Beginning with free initial approximations:

$$\mu_0 = \xi_2^2 - \mathfrak{A}, \tag{250}$$

$$\omega_0 = \mathfrak{B}\xi_2, \tag{251}$$

$$\rho_0 = \mathfrak{C}\xi_1, \tag{252}$$

where the boundary conditions determine the unknown constants $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, we apply the functionals (247)-(249) to obtain the first iterates:

$$\mu_1 = \xi_2^2 - \mathfrak{A} - \frac{1}{2\mathfrak{v}}(2\mathfrak{v} - \mathfrak{C} + \mathfrak{B}\mathfrak{v}_r)(\mathfrak{h} + \xi_2)^2, \tag{253}$$

$$\omega_1 = \frac{1}{2}\mathfrak{B}\xi_2 - \frac{1}{2}\mathfrak{B}\mathfrak{h}\cosh\left(2\sqrt{\frac{\mathfrak{v}_r}{\mathfrak{c}_a + \mathfrak{c}_b}}(\mathfrak{h} + \xi_2)\right) + \frac{1}{4}\sqrt{\frac{\mathfrak{c}_a + \mathfrak{c}_b}{\mathfrak{v}_r}}\mathfrak{B}\sinh\left(2\sqrt{\frac{\mathfrak{v}_r}{\mathfrak{c}_a + \mathfrak{c}_b}}(\mathfrak{h} + \xi_2)\right), \tag{254}$$

$$\rho_1 = \mathfrak{C}\xi_1. \tag{255}$$

Enforcing $\mu_1(\pm\mathfrak{h}) = 0$ and $\omega_1(\pm\mathfrak{h}) = 0$ requires setting $\mathfrak{C} = 2\mathfrak{v} + \mathfrak{B}\mathfrak{v}_r$ (from $\mu_1(\mathfrak{h}) = 0$) and leads to the values $\mathfrak{A} = \mathfrak{h}^2$ and $\mathfrak{B} = 0$. Consequently, $\mathfrak{c} = 2\mathfrak{v}$. Substituting these values into (253)-(255) yields the exact solution for the system:

$$\mu = \xi_2^2 - \mathfrak{h}^2, \tag{256}$$

$$\omega = 0, \tag{257}$$

$$\rho = 2\mathfrak{v}\xi_1. \tag{258}$$

Example 20 ([33, 35]). Regard the Couette flow of a micropolar fluid in the domain:

$$\mho_i = \{(\xi_1, \xi_2, \xi_3) \in \mathcal{R}^3 : \xi_2 = (i-1)\mathfrak{h}, \}, i = 1,2.$$

Assuming the velocity field $\boldsymbol{\mu} = (\mu(\xi_2), 0, 0)$, the microrotation field $\boldsymbol{\omega} = (0, 0, \omega(\xi_2))$, and the pressure $\rho = \rho(\xi_2)$, the governing system within $\mho$ is:

$$\begin{cases} (\mathfrak{v} + \mathfrak{v}_r)\frac{d^2\mu}{d\xi_2^2} + 2\mathfrak{v}_r\frac{d\omega}{d\xi_2} = 0 & \text{in } \mho, \\ (\mathfrak{c}_a + \mathfrak{c}_b)\frac{d^2\omega}{d\xi_2^2} - 2\mathfrak{v}_r\left(2\omega + \frac{d\mu}{d\xi_2}\right) = 0 & \text{in } \mho, \\ \mu(0) = 0, \mu(\mathfrak{h}) = \mathfrak{v}, \\ \omega(0) = -\frac{1}{2}\frac{d\mu}{d\xi_2}, \omega(\mathfrak{h}) = -\frac{1}{2}\frac{d\mu}{d\xi_2}, \end{cases} \tag{259}$$

Rewriting equations (259.1) and (259.2) explicitly for the highest derivatives gives:

$$\frac{d^2\mu}{d\xi_2^2} = -\frac{2\mathfrak{v}_r}{\mathfrak{v} + \mathfrak{v}_r}\frac{d\omega}{d\xi_2}, \tag{260}$$

$$\frac{d^2\omega}{d\xi_2^2} = \frac{2\mathfrak{v}_r}{\mathfrak{c}_a + \mathfrak{c}_b}\left(2\omega + \frac{d\mu}{d\xi_2}\right). \tag{261}$$

The correction functionals for (260)-(261) are:

$$\mu_{n+1}(\xi_2) = \mu_n(\xi_2) + \int_0^{\xi_2}(\Xi_2 - \xi_2)\left(\frac{d^2\mu_n}{d\Xi_2^2} + \frac{2\mathfrak{v}_r}{\mathfrak{v} + \mathfrak{v}_r}\frac{d\omega_n}{d\Xi_2}\right)(\Xi_2)d\Xi_2, \tag{262}$$

$$\omega_{n+1}(\xi_2) = \omega_n(\xi_2) + \int_0^{\xi_2} \frac{\sqrt{\mathfrak{c}_a + \mathfrak{c}_\mathfrak{d}}}{2\sqrt{\mathfrak{v}_r}} \sinh\left(\frac{2\sqrt{\mathfrak{v}_r}}{\sqrt{\mathfrak{c}_a + \mathfrak{c}_\mathfrak{d}}}(\Xi_2 - \xi_2)\right)\left(\frac{d^2\omega_n}{d\Xi_2^2} - \frac{2\mathfrak{v}_r}{\mathfrak{c}_a + \mathfrak{c}_\mathfrak{d}}\left(2\omega_n + \frac{d\omega_n}{d\Xi_2}\right)\right)(\Xi_2)d\Xi_2, \tag{263}$$

Initial approximations are chosen as:

$$\mu_0 = \mathfrak{A}\xi_2, \tag{264}$$

$$\omega_0 = \mathfrak{B}, \tag{265}$$

$$\rho_0 = \mathfrak{C}, \tag{266}$$

where the boundary conditions determine the constants $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$. Substituting these into (262) and (263) yields the first iterates:

$$\mu_1 = \mathfrak{A}\xi_2, \tag{267}$$

$$\omega_1 = \mathfrak{B} + (\mathfrak{A} + 2\mathfrak{B})\sinh^2\left(\sqrt{\frac{\mathfrak{v}_r}{c_a + c_d}}\xi_2\right), \tag{268}$$

$$\rho_1 = \mathfrak{C}. \tag{269}$$

Enforcing the microrotation boundary condition (259.4) implies $\mathfrak{B} = -\frac{1}{2}\mathfrak{A}$. Employing (259.3) gives $\mathfrak{A} = \frac{\mathfrak{v}}{\mathfrak{h}}$ and accordingly $\mathfrak{B} = -\frac{\mathfrak{v}}{2\mathfrak{h}}$. This results in the exact solution:

$$\mu(\xi_2) = \frac{\mathfrak{v}}{\mathfrak{h}}\xi_2, \tag{270}$$

$$\omega(\xi_2) = -\frac{\mathfrak{v}}{2\mathfrak{h}}, \tag{271}$$

$$\rho_1 = \mathfrak{C} \text{ (arbitrary constant)}. \tag{272}$$

Example 21 ([33, 38]). Look at the flow of micropolar fluids in a narrow film with the velocity field $\boldsymbol{\mu} = (\mu_1(\xi_1, \xi_2), \mu_2(\xi_1, \xi_2), 0)$, the microrotation field $\boldsymbol{\omega} = \big(0,0, \omega(\xi_1, \xi_2)\big)$, and the pressure $\rho = \rho(\xi_1, \xi_2)$, within the domain:

$$\mho = \{(\xi_1, \xi_2, \xi_3) \in \mathcal{R}^3 : -a < \xi_1 < a, \xi_2 = (i-1)\mathfrak{h}(\tau), i = 1,2\}.$$

The governing equations simplify to the following system within $\mho$:

$$\begin{cases} (\mathfrak{v} + \mathfrak{v}_r)\dfrac{\partial^2\mu_1}{\partial\xi_2^2} + 2\mathfrak{v}_r\dfrac{\partial\omega_3}{\partial\xi_2} - \dfrac{\partial\rho}{\partial\xi_1} = 0 & \text{in } \mho, \\ \dfrac{\partial\rho}{\partial\xi_2} = 0 & \text{in } \mho, \\ (\mathfrak{c}_a + \mathfrak{c}_\mathfrak{d})\dfrac{\partial^2\omega_3}{\partial\xi_2^2} - 2\mathfrak{v}_r\left(2\omega_3 + \dfrac{\partial\mu_1}{\partial\xi_2}\right) = 0 & \text{in } \mho, \\ \dfrac{\partial\mu_1}{\partial\xi_1} + \dfrac{\partial\mu_2}{\partial\xi_2} = 0 & \text{in } \mho, \end{cases} \tag{273}$$

Based on:

$$\begin{cases} \mu_1(\xi_1, 0) = \mathrm{M}_0, \mu_1(\xi_1, \mathfrak{h}) = 0, \\ \mu_2(\xi_1, 0) = 0, \mu_2(\xi_1, \mathfrak{h}) = 0, \\ \omega(\xi_1, 0) = \dfrac{\mathrm{M}_0}{\mathfrak{h}}, \omega(\xi_1, \mathfrak{h}) = 0, \end{cases}$$

The rewriting of equations (273.1) and (273.2) is as follows:

$$\frac{\partial^2\mu_1}{\partial\xi_2^2} = -\frac{1}{\mathfrak{v} + \mathfrak{v}_r}\left(2\mathfrak{v}_r\frac{\partial\omega_3}{\partial\xi_2} - \frac{\partial\rho}{\partial\xi_1}\right) \tag{274}$$

$$\frac{\partial^2\omega_3}{\partial\xi_2^2} = \frac{2\mathfrak{v}_r}{\mathfrak{c}_a + \mathfrak{c}_\mathfrak{d}}\left(2\omega_3 + \frac{\partial\mu_1}{\partial\xi_2}\right) \tag{275}$$

The pressure ρ satisfies the Neumann problem:

$$\begin{cases} \Delta\rho = 0 & \text{in } \mho, \\ \dfrac{\partial\rho}{\partial n} = 0 & \text{in } \partial\mho. \end{cases} \tag{276}$$

The correction functionals for (274), (275), and (276.1) are:

$$\mu_{1,n+1}(\xi_2) = \mu_{1,n}(\xi_2) + \int_0^{\xi_2} (\Xi_2 - \xi_2)\left(\frac{\partial^2\mu_{1,n}}{\partial\Xi_2^2} + \frac{1}{\mathfrak{v} + \mathfrak{v}_r}\left(2\mathfrak{v}_r\frac{\partial\omega_{3,n}}{\partial\Xi_2} - \frac{\partial\rho_n}{\partial\xi_1}\right)\right)(\Xi_2)d\Xi_2, \tag{277}$$

$$\omega_{3,n+1}(\xi_2) = \omega_{3,n}(\xi_2) + \int_0^{\xi_2} \frac{\sqrt{\mathfrak{c}_a + \mathfrak{c}_\mathfrak{d}}}{2\sqrt{\mathfrak{v}_r}} \sinh\left(\frac{2\sqrt{\mathfrak{v}_r}}{\sqrt{\mathfrak{c}_a + \mathfrak{c}_\mathfrak{d}}}(\Xi_2 - \xi_2)\right)\left(\frac{\partial^2\omega_{3,n}}{\partial\Xi_2^2} - \frac{2\mathfrak{v}_r}{\mathfrak{c}_a + \mathfrak{c}_\mathfrak{d}}\left(2\omega_{3,n} + \frac{\partial\omega_{3,n}}{\partial\Xi_2}\right)\right)(\Xi_2)d\Xi_2, \tag{278}$$

$$\rho_{n+1}(\xi_1, \xi_2) = \rho_n(\xi_1, \xi_2) + \int_0^{\xi_1}\int_0^{\xi_2} (\Xi_1 - \xi_1)\Delta\rho_n(\Xi_1, \Xi_2)d\Xi_1 d\Xi_2. \tag{279}$$

Starting with the Initial approximations:

$$\mu_{1,0} = \mathfrak{A}(\xi_2 - \mathfrak{h})^2, \tag{280}$$

$$\omega_{3,0} = \mathfrak{B}(\xi_2 - \mathfrak{h}), \tag{281}$$

$$\rho_0 = \mathfrak{C}\xi_1, \tag{282}$$

where $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ are arbitrary constants, the first iterative solutions are obtained:

$$\mu_{1,1} = \mathfrak{A}(\xi_2 - \mathfrak{h})^2 - \frac{1}{2}\left(2\mathfrak{A} - \frac{\mathfrak{C} - 2\mathfrak{B}\mathfrak{v}_r}{\mathfrak{v} + \mathfrak{v}_r}\right)\xi_1\xi_2^2, \tag{283}$$

$$\omega_{3,1} = \mathfrak{B}(\xi_2 - \mathfrak{h}) + \left(\frac{1}{2}\sqrt{\frac{\mathfrak{c}_a + \mathfrak{c}_b}{\mathfrak{v}_r}}\sinh\left(2\sqrt{\frac{\mathfrak{v}_r}{\mathfrak{c}_a + \mathfrak{c}_b}}\xi_2\right) + (\mathfrak{h} - \xi_2) - \mathfrak{h}\cosh\left(2\sqrt{\frac{\mathfrak{v}_r}{\mathfrak{c}_a + \mathfrak{c}_b}}\xi_2\right)\right)(\mathfrak{A} + \mathfrak{B})\xi_1, \tag{284}$$

$$\rho_1 = \mathfrak{C}\xi_1. \tag{285}$$

By setting $\mathfrak{B} = -\mathfrak{A}, \mathfrak{C} = 2\mathfrak{A}\mathfrak{v}$, the first iterative solutions (268)-(270) become the exact solution that satisfy the system and boundary conditions:

$$\mu_{1,1} = \mathfrak{A}(\xi_2 - \mathfrak{h})^2, \tag{286}$$

$$\omega_{3,1} = -\mathfrak{A}(\xi_2 - \mathfrak{h}), \tag{287}$$

$$\rho_1 = 2\mathfrak{v}a\xi_1. \tag{288}$$

Applying (273.5) determines the constants:

$$\mathfrak{A} = \frac{M_0}{\mathfrak{h}^2}, \quad \mathfrak{B} = -\frac{M_0}{\mathfrak{h}^2}, \quad \mathfrak{C} = \frac{2\mathfrak{v}M_0}{\mathfrak{h}^2}.$$

Therefore, the exact solution for the flow is:

$$\mu_{1,1} = \frac{M_0}{\mathfrak{h}^2}(\xi_2 - \mathfrak{h})^2, \tag{289}$$

$$\omega_{3,1} = \frac{M_0}{\mathfrak{h}^2}(\xi_2 - \mathfrak{h}), \tag{290}$$

$$\rho_1 = \frac{2\mathfrak{v}M_0}{\mathfrak{h}^2}\xi_1. \tag{291}$$

Example 22 ([5, 6, 33]). Let us examine the two-dimensional lubrication problem defined on the domain:

$$\mho = \{(\xi_1, \xi_2) \in \mathcal{R}^2 : 0 < \xi_1 < 2\pi; 0 < \xi_2 < \mathfrak{h}(\xi_1)\}$$

assuming the velocity field $\boldsymbol{\mu} = (\mu_1(\xi_1, \xi_2), \mu_2(\xi_1, \xi_2), 0)$, the microrotation field $\boldsymbol{\omega} = (0, 0, \omega(\xi_1, \xi_2))$, and the pressure $\rho = \rho(\xi_1, \xi_2)$, the governing system is:

$$\begin{cases} \dfrac{\partial\rho}{d\xi_1} = \dfrac{\partial^2\mu_1}{\partial\xi_2^2} + 2\left(\dfrac{\mathfrak{v}_r}{\mathfrak{v} + \mathfrak{v}_r}\right)^2\dfrac{\partial\omega_3}{\partial\xi_2} & \text{in } \mho, \\ \dfrac{\partial\rho}{d\xi_2} = 0 & \text{in } \mho, \\ \dfrac{\partial^2\omega_3}{\partial\xi_2^2} = 2\left(\dfrac{\mathfrak{v}_r}{\mathfrak{c}_a + \mathfrak{c}_b}\right)^2\left(2\omega_3 + \dfrac{\partial\mu_1}{\partial\xi_2}\right) & \text{in } \mho, \end{cases} \tag{292}$$

bound by:

$$\begin{cases} \mu_1(\xi_1, 0) = 0, \mu_1(\xi_1, \mathfrak{h}(\xi_1)) = \mathfrak{h}^2(\xi_1) \\ \omega_3(\xi_1, 0) = 0, \omega_3(\xi_1, \mathfrak{h}(\xi_1)) = -\mathfrak{h}(\xi_1) \end{cases}$$

Rewriting equations (292) as:

$$\frac{\partial^2\mu_1}{\partial\xi_2^2} = -\left(2\left(\frac{\mathfrak{v}_r}{\mathfrak{v} + \mathfrak{v}_r}\right)^2\frac{\partial\omega_3}{\partial\xi_2} - \frac{\partial\rho}{\partial\xi_1}\right), \tag{293}$$

$$\frac{\partial^2\omega_3}{\partial\xi_2^2} = 2\left(\frac{\mathfrak{v}_r}{\mathfrak{c}_a + \mathfrak{c}_b}\right)^2\left(2\omega_3 + \frac{\partial\mu_1}{\partial\xi_2}\right), \tag{294}$$

$$\Delta\rho = \frac{\partial^3\mu_1}{\partial\xi_2^2\partial\xi_1} + 2\left(\frac{\mathfrak{v}_r}{\mathfrak{v} + \mathfrak{v}_r}\right)^2\frac{\partial^2\omega_3}{\partial\xi_2\partial\xi_1}. \tag{295}$$

The correction functionals for (293), (294), and (295) are:

$$\mu_{1,n+1}(\xi_2) = \mu_{1,n}(\xi_2) + \int_0^{\xi_2}(\Xi_2 - \xi_2)\left(\frac{\partial^2\mu_{1,n}}{\partial\Xi_2^2} + 2\left(\frac{\mathfrak{v}_r}{\mathfrak{v} + \mathfrak{v}_r}\right)^2\frac{\partial w_{3,n}}{d\Xi_2} - \frac{\partial\rho_n}{\partial\Xi_1}\right)(\Xi_2)d\Xi_2, \tag{296}$$

$$\omega_{3,n+1}(\xi_2) = \omega_{3,n}(\xi_2) + \int_0^{\xi_2}\frac{\mathfrak{c}_a + \mathfrak{c}_b}{2\mathfrak{v}_r}\sinh\left(\frac{2\mathfrak{v}_r}{\mathfrak{c}_a + \mathfrak{c}_b}(\Xi_2 - \xi_2)\right)\left(\frac{\partial^2\omega_{3,n}}{\partial\Xi_2^2} - 2\left(\frac{\mathfrak{v}_r}{\mathfrak{c}_a + \mathfrak{c}_b}\right)^2\left(2\omega_{3,n} + \frac{\partial\mu_{1,n}}{\partial\Xi_2}\right)\right)(\Xi_2)d\Xi_2, \tag{297}$$

$$\rho_{n+1}(\xi_1) = \rho_n(\xi_1) + \int_0^{\xi_2}\int_0^{\xi_1}\frac{\gamma_{\lambda_\rho}(\Xi_1, \Xi_2) - \gamma_{\lambda_\rho}(\xi_1, \xi_2)}{a_{\gamma_{\lambda_\rho}}}\left(\Delta\rho_n - \left(\frac{\partial^3\mu_{1,n}}{\partial\xi_2^2\partial\xi_1} + 2\left(\frac{\mathfrak{v}_r}{\mathfrak{v} + \mathfrak{v}_r}\right)^2\frac{\partial^2\omega_{3,n}}{\partial\xi_2\partial\xi_1}\right)\right)(\Xi_1, \Xi_2)d\Xi_1\, d\Xi_2, \tag{298}$$

Starting with arbitrary quadratic initial guesses:

$$\mu_{1,0} = \mathfrak{A}_1\xi_2^2 + \mathfrak{B}_1\xi_2 + \mathfrak{C}_1, \tag{299}$$

$$\omega_{3,0} = \mathfrak{A}_2\xi_2^2 + \mathfrak{B}_2\xi_2 + \mathfrak{C}_2, \quad (300)$$

$$\rho_0 = \mathfrak{A}_3\xi_1 + \mathfrak{B}_3, \quad (301)$$

where $\mathfrak{A}_i, \mathfrak{B}_i, \mathfrak{C}_i (i = 1,2,3)$ are unconstrained constants, the first iterative solutions are:

$$\mu_{1,1} = \left(\frac{1}{2}\mathfrak{B}_3 - \left(\frac{\mathfrak{v}_\mathrm{r}}{\mathfrak{v} + \mathfrak{v}_\mathrm{r}}\right)^2 \mathfrak{B}_2 + \mathfrak{A}_3\xi_1 - \frac{2}{3}\left(\frac{\mathfrak{v}_\mathrm{r}}{\mathfrak{v} + \mathfrak{v}_\mathrm{r}}\right)^2 \mathfrak{A}_2\xi_2\right)\xi_2^2 + \mathfrak{B}_1\xi_2 + \mathfrak{C}_1, \quad (302)$$

$$\omega_{3,1} = -\mathfrak{A}_1\xi_2 - \frac{1}{2}\mathfrak{B}_1 + \left(\frac{1}{2}\mathfrak{B}_1 + \mathfrak{C}_2\right)\cosh\left(\frac{2\mathfrak{v}_\mathrm{r}}{\mathfrak{c}_\mathrm{a} + \mathfrak{c}_\mathfrak{d}}\xi_2\right) + (\mathfrak{A}_1 + \mathfrak{B}_2)\frac{\mathfrak{c}_\mathrm{a} + \mathfrak{c}_\mathfrak{d}}{2\mathfrak{v}_\mathrm{r}}\sinh\left(\frac{2\mathfrak{v}_\mathrm{r}}{\mathfrak{c}_\mathrm{a} + \mathfrak{c}_\mathfrak{d}}\xi_2\right), \quad (303)$$

$$\rho_1 = \mathfrak{A}_3\xi_1^2 + \mathfrak{B}_3\xi_1 + \mathfrak{C}_3 - \mathfrak{A}_3\xi_1^2\xi_2. \quad (304)$$

Imposing the boundary conditions requires setting $\mathfrak{A}_2 = 0, \mathfrak{B}_2 = -\mathfrak{A}_1, \mathfrak{C}_2 = -\frac{\mathfrak{B}_1}{2}, \mathfrak{A}_3 = 0,$ and $\mathfrak{B}_3 = 2\mathfrak{A}_1\left(1 - \left(\frac{\mathfrak{v}_\mathrm{r}}{\mathfrak{v}+\mathfrak{v}_\mathrm{r}}\right)^2\right)$ which simplify (294)-(296) to:

$$\mu_{1,1} = \mathfrak{A}_1\xi_2^2 + \mathfrak{B}_1\xi_2 + \mathfrak{C}_1, \quad (305)$$

$$\omega_{3,1} = -\mathfrak{A}_1\xi_2 - \frac{1}{2}\mathfrak{B}_1, \quad (306)$$

$$\rho_1 = 2\mathfrak{A}_1\left(1 - \left(\frac{\mathfrak{v}_\mathrm{r}}{\mathfrak{v} + \mathfrak{v}_\mathrm{r}}\right)^2\right)\xi_1 + \mathfrak{C}_3. \quad (307)$$

Thus, the exact solution is:

$$\mu_{1,1} = \xi_2^2, \quad (308)$$

$$\omega_{3,1} = -\xi_2, \quad (309)$$

$$\rho_1 = 2\left(1 - \left(\frac{\mathfrak{v}_\mathrm{r}}{\mathfrak{v} + \mathfrak{v}_\mathrm{r}}\right)^2\right)\xi_1 + \mathfrak{C}_3. \quad (310)$$

Example 23 ([5, 6, 33]). Suppose the three-dimensional lubrication problem defined in the domain:

$$\mho = \{(\xi_1, \xi_2, \xi_3) \in \mathcal{R}^3 : 0 < \xi_1 < 2\pi; 0 < \xi_2 < 2\pi; 0 < \xi_3 < \mathfrak{h}(\xi_1, \xi_2)\}.$$

Assuming the velocity field $\boldsymbol{\mu} = (\mu_1(\xi_1, \xi_2, \xi_3), \mu_2(\xi_1, \xi_2, \xi_3), 0)$, the microrotation field $\boldsymbol{\omega} = (0,0,\omega(\xi_1, \xi_2, \xi_3))$, and the pressure $\rho = \rho(\xi_1, \xi_2, \xi_3)$, the governing system is:

$$\begin{cases} \dfrac{\partial\rho}{d\xi_1} = \dfrac{\partial^2\mu_1}{\partial\xi_3^2} - 2\left(\dfrac{\mathfrak{v}_\mathrm{r}}{\mathfrak{v} + \mathfrak{v}_\mathrm{r}}\right)^2 \dfrac{\partial\omega_2}{\partial\xi_3} & \text{in } \mho, \\ \dfrac{\partial\rho}{d\xi_2} = \dfrac{\partial^2\mu_2}{\partial\xi_3^2} + 2\left(\dfrac{\mathfrak{v}_\mathrm{r}}{\mathfrak{v} + \mathfrak{v}_\mathrm{r}}\right)^2 \dfrac{\partial\omega_1}{\partial\xi_3} & \text{in } \mho, \\ \dfrac{\partial\rho}{d\xi_3} = 0 & \text{in } \mho, \\ \dfrac{\partial^2\omega_1}{\partial\xi_3^2} = 2\left(\dfrac{\mathfrak{v}_\mathrm{r}}{\mathfrak{c}_\mathrm{a} + \mathfrak{c}_\mathfrak{d}}\right)^2\left(2\omega_1 + \dfrac{\partial\mu_2}{\partial\xi_3}\right) & \text{in } \mho, \\ \dfrac{\partial^2\omega_2}{\partial\xi_3^2} = 2\left(\dfrac{\mathfrak{v}_\mathrm{r}}{\mathfrak{c}_\mathrm{a} + \mathfrak{c}_\mathfrak{d}}\right)^2\left(2\omega_2 - \dfrac{\partial\mu_1}{\partial\xi_3}\right) & \text{in } \mho, \end{cases} \quad (311)$$

conditioned by:

$$\begin{cases} \mu_1(\xi_1, \xi_2, 0) = 0, \mu_1(\xi_1, \xi_2, \mathfrak{h}) = U_1, \\ \mu_2(\xi_1, \xi_2, 0) = 0, \mu_2(\xi_1, \xi_2, \mathfrak{h}) = U_2, \\ \omega_1(\xi_1, \xi_2, 0) = -\dfrac{1}{2}\dfrac{\partial\mu_2}{\partial\xi_3}(\xi_1, \xi_2, 0), \\ \omega_1(\xi_1, \xi_2, \mathfrak{h}) = -\dfrac{1}{2}\dfrac{\partial\mu_2}{\partial\xi_3}(\xi_1, \xi_2, \mathfrak{h}), \\ \omega_2(\xi_1, \xi_2, 0) = \dfrac{1}{2}\dfrac{\partial\mu_1}{\partial\xi_3}(\xi_1, \xi_2, 0), \\ \omega_2(\xi_1, \xi_2, \mathfrak{h}) = \dfrac{1}{2}\dfrac{\partial\mu_1}{\partial\xi_3}(\xi_1, \xi_2, \mathfrak{h}). \end{cases}$$

The explicit form of (311) is:

$$\frac{\partial^2\mu_1}{\partial\xi_3^2} = 2\left(\frac{\mathfrak{v}_\mathrm{r}}{\mathfrak{v} + \mathfrak{v}_\mathrm{r}}\right)^2 \frac{\partial\omega_2}{\partial\xi_3} + \frac{\partial\rho}{d\xi_1}, \quad (312)$$

$$\frac{\partial^2\mu_2}{\partial\xi_3^2} = -2\left(\frac{\mathfrak{v}_\mathrm{r}}{\mathfrak{v} + \mathfrak{v}_\mathrm{r}}\right)^2 \frac{\partial\omega_1}{\partial\xi_3} + \frac{\partial\rho}{d\xi_2}, \quad (313)$$

$$\frac{\partial^2\omega_1}{\partial\xi_3^2} = 2\left(\frac{\mathfrak{v}_\mathrm{r}}{\mathfrak{c}_\mathrm{a} + \mathfrak{c}_\mathfrak{d}}\right)^2\left(2\omega_1 + \frac{\partial\mu_2}{\partial\xi_3}\right), \quad (314)$$

$$\frac{\partial^2\omega_2}{\partial\xi_3^2} = 2\left(\frac{\mathfrak{v}_\mathrm{r}}{\mathfrak{c}_\mathrm{a} + \mathfrak{c}_\mathfrak{d}}\right)^2\left(2\omega_2 - \frac{\partial\mu_1}{\partial\xi_3}\right), \quad (315)$$

$$\Delta\rho = \frac{\partial^3\mu_1}{\partial\xi_1\partial\xi_3^2} + \frac{\partial^3\mu_2}{\partial\xi_2\partial\xi_3^2} + 2\left(\frac{\mathfrak{v}_\mathrm{r}}{\mathfrak{v} + \mathfrak{v}_\mathrm{r}}\right)^2\left(\frac{\partial^2\omega_1}{\partial\xi_2\partial\xi_3} - \frac{\partial^2\omega_2}{\partial\xi_1\partial\xi_3}\right). \quad (316)$$

The correction functionals for solving (312)-(316) iteratively over $\mho$ are:

$$\mu_{1,n+1}(\xi_3) = \mu_{1,n}(\xi_3) + \int_0^{\xi_3}(\Xi_3 - \xi_3)\left(\frac{\partial^2\mu_{1,n}}{\partial\xi_3^2} - 2\left(\frac{v_r}{v + v_r}\right)^2\frac{\partial\omega_{2,n}}{\partial\xi_3} - \frac{\partial\rho_n}{d\xi_1}\right)(\Xi_3)d\Xi_3, \quad (317)$$

$$\mu_{2,n+1}(\xi_3) = \mu_{2,n}(\xi_3) + \int_0^{\xi_3} (\Xi_3 - \xi_3)\left(\frac{\partial^2 \mu_{2,n}}{\partial \xi_3^2} + 2\left(\frac{v_r}{v+v_r}\right)^2 \frac{\partial \omega_{1,n}}{\partial \xi_3} - \frac{\partial \rho_n}{d\xi_2}\right)(\Xi_3)\, d\Xi_3, \quad (318)$$

$$\omega_{1,n+1}(\xi_3) = \omega_{3,n}(\xi_3) + \int_0^{\xi_3} \frac{\mathfrak{c}_{\mathfrak{a}} + \mathfrak{c}_{\mathfrak{d}}}{2\mathfrak{v}_{\mathrm{r}}} \sinh\left(\frac{2\mathfrak{v}_{\mathrm{r}}}{\mathfrak{c}_{\mathfrak{a}} + \mathfrak{c}_{\mathfrak{d}}}(\Xi_3 - \xi_3)\right)\left(\frac{\partial^2 \omega_{1,n}}{\partial \xi_3^2} - 2\left(\frac{v_r}{c_a + c_d}\right)^2 \left(2\omega_{1,n} + \frac{\partial \mu_{2,n}}{\partial \xi_3}\right)\right)(\Xi_3)\, d\Xi_3, \quad (319)$$

$$\omega_{2,n+1}(\xi_3) = \omega_{3,n}(\xi_3) + \int_0^{\xi_3} \frac{\mathfrak{c}_{\mathfrak{a}} + \mathfrak{c}_{\mathfrak{d}}}{2\mathfrak{v}_{\mathrm{r}}} \sinh\left(\frac{2\mathfrak{v}_{\mathrm{r}}}{\mathfrak{c}_{\mathfrak{a}} + \mathfrak{c}_{\mathfrak{d}}}(\Xi_3 - \xi_3)\right)\left(\frac{\partial^2 \omega_{2,n}}{\partial \xi_3^2} - 2\left(\frac{v_r}{c_a + c_d}\right)^2 \left(2\omega_{2,n} - \frac{\partial \mu_{1,n}}{\partial \xi_3}\right)\right)(\Xi_3)\, d\Xi_3, \quad (320)$$

$$\rho_{n+1}(\xi_1, \xi_2, \xi_3) = \rho_n(\xi_1, \xi_2, \xi_3) + \int_0^{\xi_3}\int_0^{\xi_2}\int_0^{\xi_1} \frac{\gamma_{\lambda_\rho}(\Xi_1, \Xi_2, \Xi_3) - \gamma_{\lambda_\rho}(\xi_1, \xi_2, \xi_3)}{a_{\gamma_{\lambda_\rho}}}\left(\Delta\rho_n - \frac{\partial^3 \mu_{1,n}}{\partial \xi_1 \partial \xi_3^2} - \frac{\partial^3 \mu_{2,n}}{\partial \xi_2 \partial \xi_3^2} - 2\left(\frac{\mathfrak{v}_{\mathrm{r}}}{\mathfrak{v} + \mathfrak{v}_{\mathrm{r}}}\right)^2 \left(\frac{\partial^2 \omega_{1,n}}{\partial \xi_2 \partial \xi_3} - \frac{\partial^2 \omega_{2,n}}{\partial \xi_1 \partial \xi_3}\right)\right)(\Xi_1, \Xi_2, \Xi_3)\, d\Xi_1\, d\Xi_2\, d\Xi_3, \quad (321)$$

Using the initial guesses:

$$\mu_{1,0} = \mathfrak{A}_1 \xi_3^2 + \mathfrak{B}_1 \xi_3 + \mathfrak{C}_1, \quad (322)$$

$$\mu_{2,0} = \mathfrak{A}_2 \xi_3^2 + \mathfrak{B}_2 \xi_3 + \mathfrak{C}_2, \quad (323)$$

$$\omega_{1,1} = \mathfrak{A}_3 \xi_3^2 + \mathfrak{B}_3 \xi_3 + \mathfrak{C}_3, \quad (324)$$

$$\omega_{2,1} = \mathfrak{A}_4 \xi_3^2 + \mathfrak{B}_4 \xi_3 + \mathfrak{C}_4, \quad (325)$$

$$\rho_0 = \mathfrak{A}_5 \xi_1^2 + \mathfrak{B}_5 \xi_1 + \mathfrak{C}_5 + \mathfrak{A}_6 \xi_2^2 + \mathfrak{B}_6 \xi_2 + \mathfrak{C}_6, \quad (326)$$

where $\mathfrak{A}_i, \mathfrak{B}_i, \mathfrak{C}_i, (i = 1, \ldots, 6)$ are constants, the first iterates are:

$$\mu_{1,1} = \mathfrak{C}_1 + \mathfrak{B}_1 \xi_3 + \left(\frac{1}{2}\mathfrak{B}_5 + \left(\frac{\mathfrak{v}_{\mathrm{r}}}{\mathfrak{v} + \mathfrak{v}_{\mathrm{r}}}\right)^2 \mathfrak{B}_4\right)\xi_3^2 + \frac{2}{3}\left(\frac{\mathfrak{v}_{\mathrm{r}}}{\mathfrak{v} + \mathfrak{v}_{\mathrm{r}}}\right)^2 \mathfrak{A}_4 \xi_3^3 + \mathfrak{A}_5 \xi_1 \xi_3^2, \quad (327)$$

$$\mu_{2,1} = \mathfrak{C}_2 + \mathfrak{B}_2 \xi_3 + \left(\frac{1}{2}\mathfrak{B}_6 - \left(\frac{\mathfrak{v}_{\mathrm{r}}}{\mathfrak{v} + \mathfrak{v}_{\mathrm{r}}}\right)^2 \mathfrak{B}_3\right)\xi_3^2 + \mathfrak{A}_6 \xi_2 \xi_3^2 - \frac{2}{3}\left(\frac{\mathfrak{v}_{\mathrm{r}}}{\mathfrak{v} + \mathfrak{v}_{\mathrm{r}}}\right)^2 \mathfrak{A}_3 \xi_3^3, \quad (328)$$

$$\omega_{1,1} = \mathfrak{C}_3 + \mathfrak{B}_3 \xi_3 + \left(\frac{\mathfrak{v}_{\mathrm{r}}}{\mathfrak{c}_{\mathfrak{a}} + \mathfrak{c}_{\mathfrak{d}}}\right)^2 (\mathfrak{B}_2 + 2\mathfrak{C}_3)\xi_3^2 + \frac{2}{3}\left(\frac{\mathfrak{v}_{\mathrm{r}}}{\mathfrak{c}_{\mathfrak{a}} + \mathfrak{c}_{\mathfrak{d}}}\right)^2 (\mathfrak{A}_2 + \mathfrak{B}_3)\xi_3^3 + \frac{1}{3}\left(\frac{\mathfrak{v}_{\mathrm{r}}}{\mathfrak{c}_{\mathfrak{a}} + \mathfrak{c}_{\mathfrak{d}}}\right)^2 \mathfrak{A}_3 \xi_3^4, \quad (329)$$

$$\omega_{2,1} = \mathfrak{C}_4 + \mathfrak{B}_4 \xi_3 + \left(\frac{\mathfrak{v}_{\mathrm{r}}}{\mathfrak{c}_{\mathfrak{a}} + \mathfrak{c}_{\mathfrak{d}}}\right)^2 (2\mathfrak{C}_4 - \mathfrak{B}_1)\xi_3^2 + \frac{2}{3}\left(\frac{\mathfrak{v}_{\mathrm{r}}}{\mathfrak{c}_{\mathfrak{a}} + \mathfrak{c}_{\mathfrak{d}}}\right)^2 (\mathfrak{B}_4 - \mathfrak{A}_1)\xi_3^3 + \frac{1}{3}\left(\frac{\mathfrak{v}_{\mathrm{r}}}{\mathfrak{c}_{\mathfrak{a}} + \mathfrak{c}_{\mathfrak{d}}}\right)^2 \mathfrak{A}_4 \xi_3^4, \quad (330)$$

$$\rho_1 = \mathfrak{A}_5 \xi_1^2 - (\mathfrak{A}_5 + \mathfrak{A}_6)\xi_1 \xi_2 \xi_3^2 + \mathfrak{B}_5 \xi_1 + \mathfrak{A}_6 \xi_2^2 + \mathfrak{B}_6 \xi_2 + \mathfrak{C}_5 + \mathfrak{C}_6. \quad (331)$$

The exact solution to system (311) is attained when the constants satisfy:

$$\mathfrak{A}_3 = \mathfrak{A}_4 = \mathfrak{A}_5 = \mathfrak{A}_6 = 0, \mathfrak{B}_3 = -\mathfrak{A}_2, \mathfrak{C}_3 = -\frac{1}{2}\mathfrak{B}_2,$$

$$\mathfrak{B}_4 = \mathfrak{A}_1, \mathfrak{C}_4 = \frac{1}{2}\mathfrak{B}_1, \mathfrak{B}_5 = 2\mathfrak{A}_1\left(1 - \left(\frac{\mathfrak{v}_{\mathrm{r}}}{\mathfrak{v} + \mathfrak{v}_{\mathrm{r}}}\right)^2\right),$$

$$\mathfrak{B}_6 = 2\mathfrak{A}_2\left(1 - \left(\frac{\mathfrak{v}_{\mathrm{r}}}{\mathfrak{v} + \mathfrak{v}_{\mathrm{r}}}\right)^2\right).$$

This yields the simplified solution:

$$\mu_{1,1} = \mathfrak{A}_1 \xi_3^2 + \mathfrak{B}_1 \xi_3 + \mathfrak{C}_1, \quad (324)$$

$$\mu_{2,1} = \mathfrak{A}_2 \xi_3^2 + \mathfrak{B}_2 \xi_3 + \mathfrak{C}_2, \quad (325)$$

$$\omega_{1,1} = -\mathfrak{A}_2 \xi_3 - \frac{1}{2}\mathfrak{B}_2, \quad (326)$$

$$\omega_{2,1} = \mathfrak{A}_1 \xi_3 + \frac{1}{2}\mathfrak{B}_1, \quad (327)$$

$$\rho_1 = 2\left(1 - \left(\frac{\mathfrak{v}_{\mathrm{r}}}{\mathfrak{v} + \mathfrak{v}_{\mathrm{r}}}\right)^2\right)(\mathfrak{A}_1 \xi_1 + \mathfrak{A}_2 \xi_2) + \mathfrak{C}_5 + \mathfrak{C}_6. \quad (328)$$

Enforcing the boundary condition requires $\mathfrak{A}_1 = \mathfrak{A}_2 = 0$ (due to the periodicity of ρ in $\xi_1$and ξ_2), $\mathfrak{C}_1 = \mathfrak{C}_2 = 0, \mathfrak{B}_1 = \frac{\mathrm{M}_1}{\mathfrak{h}}, \mathfrak{B}_2 = \frac{\mathrm{M}_2}{\mathfrak{h}}$. The exact solution then simplifies to:

$$\mu_{1,1} = \frac{\mathrm{M}_1}{\mathfrak{h}}\xi_3, \quad (329)$$

$$\mu_{2,1} = \frac{\mathrm{M}_2}{\mathfrak{h}}\xi_3, \quad (330)$$

$$\omega_{1,1} = -\frac{1}{2}\frac{\mathrm{M}_2}{\mathfrak{h}}, \quad (331)$$

$$\omega_{2,1} = \frac{1}{2}\frac{\mathrm{M}_1}{\mathfrak{h}}, \quad (332)$$

$$\rho_1 = \mathfrak{C}, \quad (333)$$

where $\mathfrak{C} = \mathfrak{C}_5 + \mathfrak{C}_6$.

VI. CONCLUSION

In this work, an iterative method for analytically solving time-asymmetric partial differential equation systems is useꝺ. Using this approach, two symmetric systems in the spatial variable are built and the associated correction functionals are produced. For instance, the incompressible fluid flow and incompressible micropolar fluid flow governing equations need to be solved.

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