

Lexicographic Ordering in n -dimensional Space: Exploring Completeness Properties

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Abstract—The arrangement of abstract objects is a fundamental topic in mathematics. While the ordering of real numbers is well known, ordering elements in $\mathbb{R}^n$ is considerably more complex. This paper explores the completeness of $\mathbb{R}^n$ when ordered lexicographically—a method analogous to how words are arranged in a dictionary, a process studied in lexicography. We establish the presence of a lexicographic order in $\mathbb{R}^n$ and introduce new definitions for inequalities, the Archimedean property, and the completeness property, along with several basic characteristics of these concepts in $\mathbb{R}^n$. In practical applications, lexicographic order is critical for databases that sort records based on string values, as well as for programming algorithms and data structures—such as binary search trees—that require efficient sorting. Consequently, a thorough understanding of lexicographic ordering is essential for professionals in language processing, data management, and software development.

Index Terms — Data management, Lexicography, Programming, Software development.

I. INTRODUCTION

LEXICOGRAPHY is both the theory and practice behind creating dictionaries [1]. It entails systematically gathering, analyzing and presenting words along with their meanings, thereby reflecting not only the language but also the cultural and contextual background in which it is used. The term "lexicography" is derived from the Greek words "lexis" (word) and "grapho" (to write). This emphasis on both the words themselves and their definitions highlights the discipline's complexity and depth.

Lexicography divides into two primary categories: theoretical and practical. Theoretical lexicography examines the underlying principles and frameworks of dictionary-making, while practical lexicography deals with the actual creation of dictionaries. This practical aspect includes tasks

such as selecting words, crafting definitions, and providing usage examples. A central concern in lexicography is the ordering of words. Although dictionaries usually arrange words alphabetically for easy access, lexicographers also consider methods for grouping related terms and addressing synonyms, antonyms, and subtle differences in meaning. In certain cases, specialized dictionaries may organize entries by themes or concepts. For further reading on lexicography, see [2]–[4].

A key aspect of practical lexicography is the ordering of words using lexicographic order. Also called dictionary order, it is a method that arranges words or entries based on the alphabetical sequence of their letters. This ordering method is widely used in dictionaries, glossaries, and indexes. The following principles guide lexicographic order:

1. **Alphabetical Sequence:** Words are arranged from A to Z. For example, "apple" comes before "banana."
2. **Character Comparison:** When comparing words, lexicographic order evaluates them character by character:
 - Compare the first letter of each word. The word with the earlier letter comes first.
 - If the first letters are the same, compare the second letters, and so on.
3. **Handling Ties:** If two words share the same prefix, the longer word will come later. For instance, "bat" comes before "batter."
4. **Case Sensitivity:** In some systems, case may matter. Typically, uppercase letters are treated as coming before lowercase letters, so "Apple" would come before "apple."
5. **Special Characters and Digits:** When special characters (like hyphens or apostrophes) or numbers are involved, their positioning in the ASCII or Unicode table can influence order. For example, "apple" would come before "apple-pie" because the hyphen is considered.

In addition to its application in dictionaries for easy word lookup, lexicographic order is also essential in databases for sorting records based on string values, and in programming for algorithms and data structures that require sorting, such as binary search trees. This may be found in some references such as [5]–[8]. Thus, understanding lexicographic order is crucial for anyone involved in language processing, data management, or software development.

On the other hand, ordering abstract objects is also one of the interests of mathematics. The well-known ordering mathematical objects is ordering of real numbers. This confers ordering properties, which constitute one of the most fundamental attributes of the real number system. An elegant discussion about this, yet easily followed by beginners, may be found in [9]–[11]. The ordering properties characteristic of the real numbers can be formally stated as follows [9], [12],

Manuscript received March 8, 2025; revised August 14, 2025.

This research was supported in part by Academic Field Group Strengthening Research Grant and Professor Expertise Research Grant 2025, Universitas Pendidikan Indonesia.

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[13], [14]:

Ordering Properties of Real Numbers [9]

Let P denote a nonempty subset of $\mathbb{R}$, referred to as the set of positive real numbers, which satisfies the following axioms:

- (i) Closure under Addition:
If $a, b \in P$, then $a + b \in P$.
- (ii) Closure under Multiplication:
If $a, b \in P$, then $ab \in P$.
- (iii) Trichotomy Property:
For any $a \in \mathbb{R}$, exactly one of the following statements holds:

$$a \in P, \quad a = 0, \quad -a \in P.$$

Completeness Axiom

An ordered field F is said to be complete if every nonempty subset of F that is bounded above in F possesses a least upper bound (supremum) within F . This axiom is satisfied by the real number field $\mathbb{R}$ but not by the rational number field $\mathbb{Q}$. Essentially, it asserts that there are no "gaps" in the set $\mathbb{R}$. Consequently, $\mathbb{R}$ can be characterized as a complete ordered field. Moreover, this completeness property establishes a foundation that links the monotonicity properties of functions with extensions to the monotonicity of real matrix functions [15]-[17].

In this article we will generalize such properties in $\mathbb{R}^n$ by applying lexicographic order, then discuss the consequences of these properties in the space $\mathbb{R}^n$. We will start with the discussion of lexicographic order in $\mathbb{R}^2$ and the consequences in this two-dimensional space. Next, we extend the discussion for $\mathbb{R}^n$. The potential applications of the proposed idea.

II. LEXICOGRAPHIC ORDER IN $\mathbb{R}^2$

A commonly order used in $\mathbb{R} \times \mathbb{R}$ is a partial order. Ghorpade and Limaye [18] introduced the pointwise order for $\mathbb{R}^n$:

$$\text{If } x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n \text{ then} \\ x \leq y \Leftrightarrow x_i \leq y_i, \text{ for all } i = 1, 2, \dots, n. \quad (1)$$

Ghorpade and Limaye's pointwise ordering is an example of a partial order. This means that some elements cannot be compared with each other using this order. In contrast, a total order allows any two elements to be compared. In $\mathbb{R}^n$, the pointwise order allows for vectors to be compared component-wise, but this does not guarantee that all vectors can be compared, making it a partial order rather than a total order. For example, we cannot compare which one is greater between (1,2) and (2,1). Also (1,1,2) cannot be compared to (2,2,1). However, pointwise orders are commonly used in mathematics and have produced so many theories that are developing today both in $\mathbb{R}^2$ and in general in $\mathbb{R}^n$.

In this article we will discuss a concept of such an order in $\mathbb{R}^n$, particularly for $\mathbb{R} \times \mathbb{R}$ is totally ordered. This order is called the lexicographic order:

$$\text{If } x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n \text{ then} \\ x \leq y \Leftrightarrow x_1 < y_1, \text{ or } x_i = y_i \text{ for } i = 1, 2, 3, \dots, k < n \\ \text{and } x_{k+1} \leq y_{k+1} \quad (2)$$

Now, applying this definition, we can compare of two vectors, such as which one is greater between (1,2) and (2,1). Is (1,2) less than (2,1)? Also (1,1,2) can be compared to (2,2,1) which is (1,1,2) less than (2,2,1). Even (1,1,2) can be compared with (1,1,3).

The completeness property in $\mathbb{R} \times \mathbb{R}$ by using lexicographic order has been presented by Cahya in [12], the result properties $\mathbb{R}^2$ motivated by the existing of properties in $\mathbb{R}$. Some properties in $\mathbb{R}$ were examined their enforceability in $\mathbb{R}^n$ under the total order, particularly in $\mathbb{R} \times \mathbb{R}$. In $\mathbb{R}^2$ it has been shown the properties that are fulfilled by the lexicographic order, such as the existing of lexicographic order, inequality, modified Archimedean, completeness in $\mathbb{R} \times \mathbb{R}$, and their influence to the properties of multivariable functions, such as monotonicity property.

Cahya [12] in his researched for $\mathbb{R} \times \mathbb{R}$ starting the prove by dividing $\mathbb{R} \times \mathbb{R}$ in such a way so that $\mathbb{R} \times \mathbb{R}$ splits into three disjoint subsets. Suppose K is a subset of $\mathbb{R} \times \mathbb{R}$, given by

$$K = \{(x, y) \in \mathbb{R}^2: x > 0\} \cup \{(0, y) \in \mathbb{R}^2: y > 0\}. \quad (3)$$

Through the set K we will divide the set $\mathbb{R} \times \mathbb{R}$ into three disjoint sub sets, i.e.

$$K = \{(x, y) \in \mathbb{R}^2: x > 0\} \cup \{(0, y) \in \mathbb{R}^2: y > 0\}, \\ O = (0,0) \text{ and } (K \cup O)^c.$$

Using the above definitions and the usual addition, subtraction and scalar multiplication operations that apply in $\mathbb{R}^2$, then the set K possesses the following properties.

Proposition 1

Let $K \subset \mathbb{R}^2$, be the set defined in (3). For every $a, b \in K$, and $c > 0, c \in \mathbb{R}$ we have:

- (i). $a + b \in K$
- (ii). $ca \in K$

However, the multiplication of two elements in $\mathbb{R}^2$ that satisfy the closed property has not been specifically defined, although the ordinary two-point multiplication operation in $\mathbb{R}^2$ and the Sets of Complex number already exists.

The point wise product in $\mathbb{R}^2$ is not closed in K , because $(0,1)(1,0) = (0,0) \notin K$. The multiplication of Complex number, i.e. $(a,b)(c,d) = (ac - bd, ad + bc)$ is also not closed in K , for example $(0,1)(1,0) = (-1,0) \notin K$. Therefore, it is not true that $a, b \in K$ implies $ab \in K$.

Under the lexicographic order in (2), the Trichotomy property is valid in $\mathbb{R}^2$, i.e. for every $a \in \mathbb{R}^2$ then either

$$a \in K, \text{ or } a = O, \text{ or } -a \in K.$$

When $a \in K$, we write $a > 0$ and a is called positive. When $a \in K \cup O$, we denote this by writing $a \geq 0$, and say that a is nonnegative. Conversely, if $-a \in K$, we denote this by $a < 0$ and refer to a as negative. When $-a \in K \cup O$, we write $a \leq 0$, and a is called non positive. This is different with the concept of positive element in $\mathbb{R}$, the positive element in $\mathbb{R}^2$ is $a = (x, y), x > 0$ or $x = 0$ and $y > 0$. This means $a = (x, y) > 0$ not only for $x > 0$ and $y > 0$. This concept, implies to the definition of positive element in a bigger space, which is $\mathbb{R}^n$. Thus, lexicographic order has an important role in defining inequality in $\mathbb{R}^n, n \geq 2, n \in \mathbb{N}$. And then, the inequality of two points in $\mathbb{R}^2$ is defined as follow.

Definition 2

Suppose $p, q \in \mathbb{R}^2$.

- (i). If $p - q \in K$, then we write $p > q$ or $q < p$.
- (ii). If $p - q \in K \cup O$, then we write $p \geq q$ or $q \leq p$.

For all p, q in $\mathbb{R}^2$, the Trichotomy Law ensures that precisely one of these relations is true: $p > q$, $p = q$ or $p < q$.

Therefore, if we have $a \geq b$ and $a \leq b$ then $a = b$. Hence forward, $a \geq b$ is meant as $a > b$ or $a = b$. Just like in $\mathbb{R}$, the inequality $a < b < c$ means $a < b$ and $b < c$. The inequalities $a \leq b < c$, $a < b \leq c$, and $a \leq b \leq c$ can be interpreted similarly.

We will give some interpretation of the inequalities. Let $a = (a_1, a_2), b = (b_1, b_2)$ be in $\mathbb{R}^2$. $a - b \in K$ means $(a_1 - b_1, a_2 - b_2) \in K$. Furthermore, $(a_1 - b_1, a_2 - b_2) \in K$ means $(a_1 - b_1 > 0)$ or $(a_1 - b_1 = 0 \text{ and } a_2 - b_2 > 0)$ or $(a_1 > b_1)$ or $(a_1 = b_1 \text{ and } a_2 > b_2)$. Consider the following example, $(1, 3) > (1, 2)$ because $(1, 3) - (1, 2) = (0, 1) \in K$.

Finally, we can define the existing of lexicographic order in $\mathbb{R} \times \mathbb{R}$ as follow.

Definition 3

Suppose $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$. We define

$$a < b \Leftrightarrow a_1 < b_1 \text{ or } a_1 = b_1 \text{ and } a_2 < b_2 \quad (4)$$

Using the above definition, we have the following proposition.

Proposition 4

Suppose $a, b, c \in \mathbb{R}^2$ and $k \in \mathbb{R}$.

- (i) If $a > b$ and $b > c$ then $a > c$.
- (ii) If $a > b$ then $a + c > b + c$.
- (iii) If $a > b$ and $k > 0$ then $ka > kb$.
If $a > b$ and $k < 0$ then $ka < kb$.

Furthermore, the modified Archimedean properties in $\mathbb{R} \times \mathbb{R}$ in [5], are also satisfied for $\mathbb{R} \times \mathbb{R}$ in the order (4).

Proposition 5 (Modified Archimedean property, [18])

Suppose $a, b \in \mathbb{R} \times \mathbb{R}$ with $a = (x, y), x > 0$, then there is $k \in \mathbb{N}$ such that $ka > b$.

Proof. To prove the proposition, is enough to consider the case $s > 0$, and show that there is $k \in \mathbb{N}$ such that $kx > s$. It can be shown that the relation on (4), is a total order lexicographic order in $\mathbb{R} \times \mathbb{R}$ [18].

Consider the following example, lets $a = (2, 1)$ and $b = (7, 5)$, then there is $k = 4 \in \mathbb{N}$ such that $4(2, 1) > (7, 5)$. Similarly, for $a = (\frac{1}{2}, 1)$ and $b = (5, 6)$ there is $k = 10 \in \mathbb{N}$ such that $10(\frac{1}{2}, 1) > (5, 6)$.

Next, we will utilize the concept of lexicographic order in solving inequalities in $\mathbb{R}^2$. Suppose that inequality $(x, y) > (a, b)$, for some $(a, b) \in \mathbb{R}^2$. We will look for a set of solutions to these inequalities. Write the inequality as $(x, y) - (a, b) > 0$ its mean $(x - a, y - b) \in K$. Then we have (i). $x - a > 0$ or (ii). $x - a = 0$ and $y - b > 0$. Or (i). $x > a$ or (ii). $x = a$ and $y > b$. From (i) and (ii) we have a set of solutions is

$$\{(x, y) \in \mathbb{R}^2: x > a \text{ for some } a \in \mathbb{R}\} \cup \{(x, y) \in \mathbb{R}^2: x = a \text{ and } y > b \text{ for some } a, b \in \mathbb{R}\}.$$

Example 1

We seek to determine the solution set of the given inequality $(x, y) > (2, 4)$. As a first step, we reformulate the inequality

in the following form $(x, y) - (2, 4) > 0$ or equivalently $(x - 2, y - 4) \in K$. This implies two possibilities: (i). $x - 2 > 0$ or (ii). $x - 2 = 0$ and $y - 4 > 0$. In other words, we obtain the conditions: (i). $x > 2$ or (ii). $x = 2$ and $y > 4$. Thus, the solution set is: $\{(x, y) \in \mathbb{R}^2: x > 2\} \cup \{(x, y) \in \mathbb{R}^2: x = 2 \text{ and } y > 4\}$. Suppose that $A = \{(x, y) \in \mathbb{R}^2: x = 2 \text{ and } y > 4\}$ and $B = \{(x, y) \in \mathbb{R}^2: x > 2\}$, then the set of solution is $A \cup B$, and illustrated as follows.

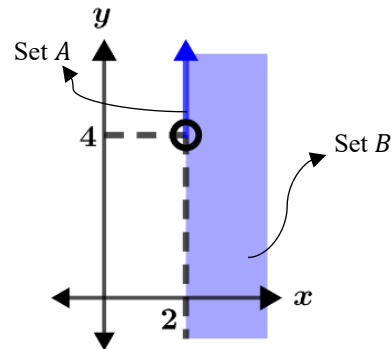


Figure 1. $A \cup B$ is the set of solutions in Example 1

Next, consider the inequality $(p, q) < (x, y) < (a, b)$, for some fixed $(p, q), (a, b) \in \mathbb{R}^2$. We can break this compound inequality into two separate parts: $(p, q) < (x, y)$ and $(x, y) < (a, b)$. For the inequality $(x, y) < (a, b)$ we write it as $(a - x, b - y) \in K$. Which implies either $a - x > 0$ or $a - x = 0$ and $b - y > 0$ (i.e., $x = a$ and $y < b$). Thus, the solution set for $(x, y) < (a, b)$ is $\{(x, y) \in \mathbb{R}^2: x < a \text{ for some } a \in \mathbb{R}\} \cup \{(x, y) \in \mathbb{R}^2: x = a \text{ and } y < b \text{ for some } a, b \in \mathbb{R}\}$. Similarly, for the inequality $(p, q) < (x, y)$, we write it as $(x - p, y - q) \in K$, which implies either $x - p > 0$ or $x = p$ and $y - q > 0$. Or $x > p$ or $x = p$ and $y > q$. So the solution set for case 2 is $\{(x, y) \in \mathbb{R}^2: x < a \text{ for some } a \in \mathbb{R}\} \cup \{(x, y) \in \mathbb{R}^2: x = a \text{ and } y < b \text{ for some } a, b \in \mathbb{R}\}$. The overall solution set for the inequality $(p, q) < (x, y) < (a, b)$ is the intersection of the two solution sets above—namely, those (x, y) that satisfy both conditions simultaneously.

Example 2

Our goal is to determine all values that fulfill the inequality $(1, 2) < (x, y) < (3, 4)$. This means we need to find the intersection of the solution sets for $(1, 2) < (x, y)$ and $(x, y) < (3, 4)$. For $(1, 2) < (x, y)$, we write as $(x - 1, y - 2) \in K$, which implies either: $x > 1$ that is $x = 1$ and $y > 2$. So the first solution is $P = \{(x, y): x > 1\} \cup \{(x, y): x = 1 \text{ and } y > 2\}$. Similarly, for $(x, y) < (3, 4)$ then $(3 - x, 4 - y) \in K$. So we have $x < 3$ or $x = 3$ and $y < 4$. So the second solution is $Q = \{(x, y): x < 3\} \cup \{(x, y): x = 3 \text{ and } y < 4\}$. So, the solution set is

$$P \cap Q = \{(x, y): 1 < x < 3\} \cup \{(x, y): x = 1 \text{ and } y > 2\} \cup \{(x, y): x = 3 \text{ and } y < 4\}$$

$$= A \cup B \cup C,$$

which

$$A = \{(x, y): 1 < x < 3\},$$

$$B = \{(x, y): x = 1 \text{ and } y > 2\},$$

$$C = \{(x, y): x = 3 \text{ and } y < 4\}.$$

See figure 2.

Completeness Property in $\mathbb{R} \times \mathbb{R}$

In this section, we discuss the general concept of boundedness in $\mathbb{R} \times \mathbb{R}$, as well as the definitions of a modified supremum and infimum for sets within $\mathbb{R} \times \mathbb{R}$. These definitions are adaptations of the ones presented in [18]. We have revised the existing notions because bounded subsets in $\mathbb{R}$ differ from those in $\mathbb{R} \times \mathbb{R}$, particularly when comparing the partially ordered structure described in [18] with the total order defined in (4).

For instance, consider the set $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : 0 < x < 1\}$. Under the partial order defined in [18], S is not bounded because there is no element $(a, b) \in \mathbb{R} \times \mathbb{R}$ that serves as an upper bound for every $(x, y) \in S$. In contrast, when we use the lexicographic order defined in (4), S does have both an upper and a lower bound, even though it may lack a supremum and infimum. Thus, although S is an infinite set, it is considered bounded according to the lexicographic order.

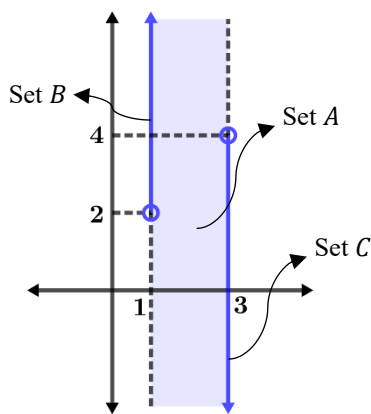


Figure 2. $A \cup B \cup C$ is the set of solutions in Example 2

This distinction affects the definitions of supremum and infimum. Under the partial order described in [18], the supremum and infimum of a set can lie far from the set's elements. For instance, consider the set $S = \{(x, y) : y = 1 - x, 0 < x < 1\}$. According to [18], $\text{Sup}(S) = (1, 1)$ and $\text{Inf}(S) = (0, 0)$. However, with the total order defined in (4), the supremum and infimum of S become $(1, 0)$ and $(0, 1)$, respectively. In fact, many open sets under the total order in $\mathbb{R} \times \mathbb{R}$ lack a true supremum or infimum, even if they have upper and lower bounds. This example shows that the completeness property in $\mathbb{R} \times \mathbb{R}$, when considered under the total order, is rather weak. Therefore, it is necessary to redefine the concepts of bounded sets, as well as the supremum and infimum, in $\mathbb{R} \times \mathbb{R}$.

In the following discussion, we propose new definitions for bounded sets, supremum, and infimum in $\mathbb{R} \times \mathbb{R}$.

Definition 6

Suppose $S \neq \emptyset$, $S \subseteq \mathbb{R} \times \mathbb{R}$. S is right-bounded in $\mathbb{R} \times \mathbb{R}$ if there exists an interval I that is bounded above in $\mathbb{R}$ and a bounded interval J in $\mathbb{R}$ such that $S \subseteq I \times J$. The point (a, b) in $\mathbb{R} \times \mathbb{R}$ that satisfies $(x, y) \leq (a, b)$ for every $(x, y) \in S$ is called a right bound of S .

Definition 7

Suppose $S \subseteq \mathbb{R} \times \mathbb{R}$, $S \neq \emptyset$. We say that S is bounded on the left if there exists an interval I , bounded below in $\mathbb{R}$, and a bounded interval J in $\mathbb{R}$ such that $S \subseteq I \times J$. The point

(a, b) in $\mathbb{R} \times \mathbb{R}$ that satisfies $(x, y) \geq (a, b)$ for every $(x, y) \in S$ is referred to as a left bound of S .

Definition 8

Let $S \subseteq \mathbb{R} \times \mathbb{R}$, $S \neq \emptyset$. We say that S is totally bounded if S is bounded in the left and the right in $\mathbb{R} \times \mathbb{R}$.

Consequently, a set S is called totally bounded in $\mathbb{R} \times \mathbb{R}$ if there are bounded intervals I and J in $\mathbb{R}$ such that $S \subseteq I \times J$.

Example 3

The set

$$S = \{(x, x) \mid 0 < x < 1\}$$

is totally bounded in $\mathbb{R} \times \mathbb{R}$, because there are bounded intervals $I = (-1, 2)$ and $J = (-1, 2)$ such that $S \subseteq I \times J$. For any (x, y) in S , we have $x \in (0, 1) \subseteq (-1, 2) = I$, $y \in (0, 1) \subseteq (-1, 2) = J$.

Moreover, since S is contained within the closed intervals $[-1, 2]$ and $[-1, 2]$ such that $S \subseteq [-1, 2] \times [-1, 2]$ and $(x, y) \leq (2, 2)$ for every (x, y) in S , then $(2, 2)$ is a right bound for S . But is $(1, 1) \in \bar{S}$ a right bound as well? Yes—it turns out that there exist intervals $[-1, 1]$ such that $S \subseteq [-1, 1] \times [-1, 1]$ and $(x, y) \leq (1, 1)$ for every (x, y) in S .

According to Definition 8, assume that S is totally bounded in $\mathbb{R} \times \mathbb{R}$. A point $(p, q) \in \mathbb{R} \times \mathbb{R}$ that satisfies $(p, q) \leq (x, y)$ for every $(x, y) \in S$ is called a left bound of S . Conversely, a point $(a, b) \in \mathbb{R} \times \mathbb{R}$ for which $(x, y) \leq (a, b)$ holds for every $(x, y) \in S$ is called a right bound of S .

In the following section, we will introduce the concepts of the least right bound and the greatest left bound of a set S in $\mathbb{R} \times \mathbb{R}$, which emerge from our definitions of bounded sets and the total order on $\mathbb{R} \times \mathbb{R}$. For any $S \subseteq \mathbb{R} \times \mathbb{R}$, we denote $\bar{S}$ (closure of S) as the smallest closed set that contains S . If S is closed and totally bounded, then we called S as compact set. Heine-Borel Theorem ensures that S is compact set if and only if S is closed and totally bounded.

Definition 9

Suppose $S \subseteq \mathbb{R} \times \mathbb{R}$, $S \neq \emptyset$.

- If S is right-bounded, then a point $(a, b) \in \bar{S}$ is called the optimal supremum of S if it satisfies the following conditions:
 - (a, b) is a right bound for S , and
 - If $(\alpha, \beta) \in \bar{S}$ is a right bound of S , then $(a, b) \leq (\alpha, \beta)$.
- If S is left-bounded, then a point $(s, t) \in \bar{S}$ is called the optimal infimum of S if it satisfies:
 - (s, t) is a left bound for S , and
 - If $(\alpha, \beta) \in \bar{S}$ is a left bound of S , then $(s, t) \geq (\alpha, \beta)$.

Lemma 10

The optimal supremum of a set S is unique.

Proof:

Assume for contradiction that S has two distinct optimal suprema, say (a, b) and (s, t) . If $(a, b) < (s, t)$, then (a, b) cannot be a right bound for S , contradicting its optimality. Conversely, if $(s, t) < (a, b)$, then (s, t) fails to serve as a right bound for S . Therefore, we must have $(a, b) = (s, t)$. A similar argument establishes the uniqueness of the optimal infimum.

In $\mathbb{R}$, the following statement is true.

“A point $u \in S$ is the supremum of a non-empty S if and only if the following conditions are met:

(1). $s \leq u$ for all $s \in S$.

(2). If $v < u$, then there is $s' \in S$ such that $v < s'$.”

But it is not true for optimal supremum in $\mathbb{R}^2$, for example $S = \{(x, 1) \mid 0 < x < 2\}$. Choose $u = (2, 1)$ and $v = (2, 0)$, but $\{v < (x, y) < u\} = \emptyset$.

Remark. Consider set $A = \{(x, 1) \in \mathbb{R}^2 \mid 0 < x < 1\}$, $u = (2, 0)$ is optimal supremum of A and $v = (2, 0)$ is right bound of A , but there is no $(x, y) \in A$ such that $v < (x, y) < u$. Also, let $B = \{x \in \mathbb{R} \mid 1 < x < 3 \text{ or } x = 4\}$, $\sup B = 4$, and $3\frac{1}{2}$ is upper bound of B , but there is no $x \in B$ such that $3\frac{1}{2} < x < 4$.

Example 4

Consider the set $S = \{(x, y) : 0 < x < 1, y = 1 - x\}$. This set is a totally bounded because it is contained within the rectangle $[0, 1] \times [0, 1]$. Notice that $(1, 0) \in \bar{S}$, and every $(x, y) \in S$ satisfies $(x, y) \leq (1, 0)$, which makes $(1, 0)$ a right bound for S . Similarly, every $(x, y) \in S$ satisfies $(x, y) \geq (0, 1)$, so $(0, 1)$ is a left bound for S and $(0, 1) \in \bar{S}$. Therefore $(1, 0)$ is the optimal supremum of S , and $(0, 1)$ is the optimal infimum of S .

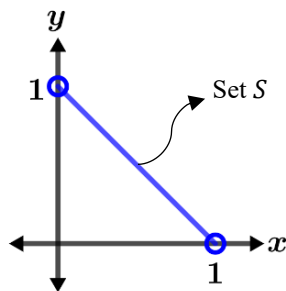


Figure 3 Illustrative representation of set S in Example 4.

Example 5

Consider a set $T = \{(x, y) \in \mathbb{R}^2 \mid x \in (0, 1), y \in \{1, 2\}\}$. Since it is contained within $[0, 1] \times [1, 2]$ then it is a totally bounded set. Points $(1, 1)$ and $(1, 2)$ are right bounds for T , and elements of $\bar{T} = \{(x, y) \in \mathbb{R}^2 \mid x \in [0, 1], y \in \{1, 2\}\}$. The set of all right bounds of T in $\bar{T}$ is $A = \{(1, 1), (1, 2)\}$. So, the optimal supremum is the smallest element in A , that is $(1, 1)$.

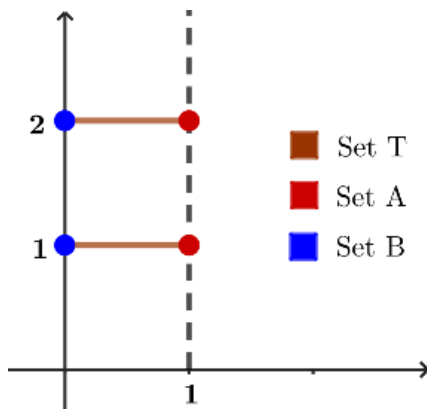


Figure 4. A, B are the sets of all right and left bounds of T in $\bar{T}$

On the other hand, the set of all left bound of T in $\bar{T}$ is $B = \{(0, 1), (0, 2)\}$, so the optimal infimum is the largest element in B , that is $(0, 2)$. Although, for any $t < 1$, $(1, t) < (1, 1)$ and $(1, t)$ is a right bound of T , we should notice that it is not an element of $\bar{T}$, so it is not in our consideration. Similarly, for any $s > 2$, $(0, s) > (0, 2)$ and a left bound of S , but $(0, s) \notin \bar{T}$.

Example 6

Consider set $U = \{(x, y) \mid 0 \leq x \leq 1, y \in \{1, 2\}\}$. Notice that $\bar{U} = U$, and the only right bound of U in $\bar{U}$ is $(1, 2)$, so $(1, 2)$ is the optimal supremum.

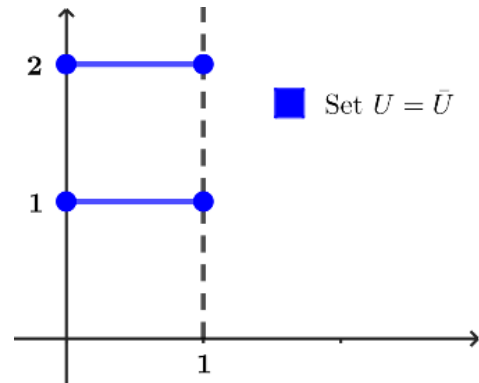


Figure 5. $U = \bar{U}$. $(1, 2)$ is the only right bound of U in $\bar{U}$

Again, although for any $t > 2$, $(1, t)$ is a right bound of U , we will not consider it since it is not in $\bar{U}$. Similarly, $(0, 1)$ is the optimal infimum of U .

Example 7

Consider a set

$$U = \{(x, y) \mid 0 < x < 1, y \in \{1, 2, 3\}\} \cup \{(0, 2), (1, 2)\}.$$

We have

$$\bar{U} = \{(x, y) \mid 0 \leq x \leq 1, y \in \{1, 2, 3\}\}$$

and the set of all right bounds of U is $P = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \geq (1, 2)\}$, therefore the set of all right bounds of S in $\bar{U}$ is $C = P \cap \bar{U} = \{(1, 2), (2, 3)\}$. So, the optimal supremum is the smallest element in C , that is $(1, 2)$.

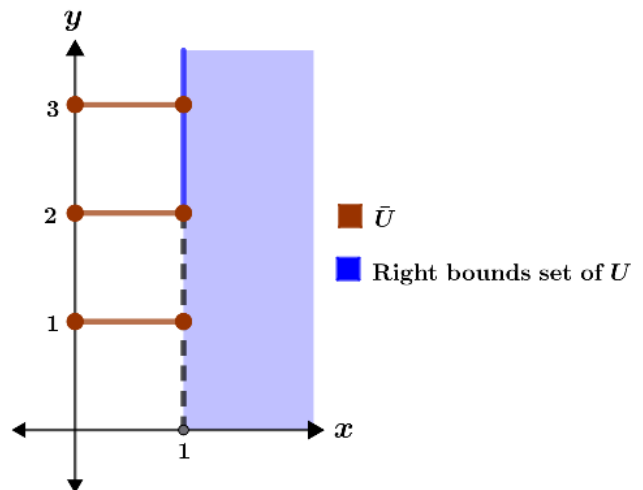


Figure 6. $\bar{U}$ and right bound set of U .

On the other hand, the set of all left bounds of U is $Q = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \leq (0, 2)\}$, therefore the set of all left bounds of S in $\bar{U}$ is $D = Q \cup \bar{U} = \{(0, 1), (0, 2)\}$. So, the optimal infimum is the largest element in D , that is $(0, 2)$.

Let $S \subseteq \mathbb{R} \times \mathbb{R}$ is compact set. Since S is closed and bounded, then the existence of optimal supremum and optimal infimum should be guaranteed. Here (a^*, b^*) is the candidate of the optimal supremum of compact set S ,

$$a^* = \sup \{x \in \mathbb{R} \mid (x, y) \in \bar{S}\}$$

and

$$b^* = \sup \{y \in \mathbb{R} \mid (a^*, y) \in \bar{S}\}.$$

Similarly, the candidate for optimal infimum is (a_*, b_*) which

$$a_* = \inf \{x \in \mathbb{R} \mid (x, y) \in \bar{S}\}$$

and

$$b_* = \inf \{y \in \mathbb{R} \mid (a^*, y) \in \bar{S}\}.$$

Proposition 11

Let $S \subset \mathbb{R} \times \mathbb{R}$ is compact set, then $(a^*, b^*) \in S$.

Proof:

Since $S = \bar{S}$, then we just need to prove $(a^*, b^*) \in \bar{S}$, that is by verifying $B((a^*, b^*), r) \cap \bar{S} \neq \emptyset$ for any $r > 0$. By contradiction, suppose that there exists a $r_0 > 0$ such that $B((a^*, b^*), r_0) \cap \bar{S} = \emptyset$. By definition of b^* , we have

$$(a^*, b^*) \geq (a^*, y) \quad \forall (a^*, y) \in \bar{S}.$$

Since

$$B((a^*, b^*), r_0) \cap \bar{S} = \emptyset,$$

then $b^* - r_0 \geq y$ for any $(a^*, y) \in \bar{S}$, implies $b^* - \frac{r_0}{2} > y$. Therefore

$$(a^*, b^* - \frac{r_0}{2}) > (a^*, y) \quad \forall (a^*, y) \in \bar{S}.$$

Contradict with the definition of b^* . The proof is complete.

The following theorem shows that (a^*, b^*) is the optimal supremum of a compact set S .

Proposition 12

Let $S \subset \mathbb{R} \times \mathbb{R}$ is a compact set, then (a^*, b^*) , which

$$a^* = \sup \{x \in \mathbb{R} \mid (x, y) \in \bar{S}\}$$

and

$$b^* = \sup \{y \in \mathbb{R} \mid (a^*, y) \in \bar{S}\},$$

is the optimal supremum of S .

Proof. Since S is compact, then $S = \bar{S}$. Since $a^* \geq x$ for any $(x, y) \in \bar{S}$, then $(a^*, y) \geq (x, y)$ for any $(x, y) \in \bar{S}$. Since $b^* \geq y$ for any $(a^*, y) \in \bar{S}$, then $(a^*, b^*) \geq (a^*, y)$ for any $(a, y) \in \bar{S}$. Therefore, $(a^*, b^*) \geq (x, y)$ for any $(x, y) \in \bar{S}$. This shows that (a, b) is right bound for $\bar{S}$, which implies a right bound for S . Since (a^*, b^*) is right bound of S and $(a^*, b^*) \in S$ (previous proposition), then (a, b) is smallest right bound for $\bar{S} = S$. So, (a, b) is optimal supremum of S .

Definition 13

For any set $S \subseteq \mathbb{R}^2$ and any point $a \in \mathbb{R}^2$, we define the translated set as:

$$a + S = \{a + x : x \in S\}$$

Example 8

Suppose that $S = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$, $a = (2, 2)$. Then $a + S = \{(2 + x, 2 + y) : 0 \leq x \leq 1, 0 \leq y \leq 1\} = \{(x, y) : 2 \leq x \leq 3, 2 \leq y \leq 3\}$.

The optimal supremum of S is $(1, 1)$ and its optimal infimum is $(0, 0)$. Consequently, the optimal supremum of $a + S = (3, 3) = (2, 2) + (1, 1) = a + \text{optimal supremum of } S$ and the optimal infimum of $a + S = (2, 2) = (2, 2) + (0, 0) = a + \text{optimal infimum of } S$.

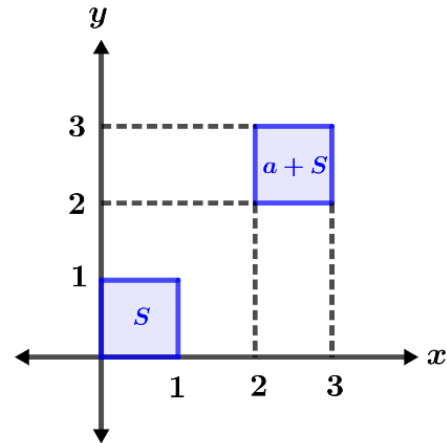


Figure 7. S and $a + S$

Proposition 14

Suppose $S \subseteq \mathbb{R}^2$ is totally bounded, and $a \in \mathbb{R}^2$.

Then $\sup op(a + S) = a + \sup op(S)$ and $\inf op(S) = a + \inf op(S)$.

Note: $\sup op(S)$ = the optimal supremum of S and $\inf op(S)$ = the optimal infimum of S .

Example 9

Consider that $S = \{(x, y) : 0 < x < 1, y = 1 - x\}$ and $a = (2, 1)$. See Fig. 5.

Consider that

$$\begin{aligned} a + S &= \{a + s : s \in S\} \\ &= \{(2, 1) + (x, y) : 0 < x < 1, y = 1 - x\} \\ &= \{(2 + x, 1 + y) : 2 < x + 2 < 3, y = 1 - x\} \\ &= \{(x', y') : 2 < x' < 3, y' - 1 = 1 - (x' - 2)\} \\ &= \{(x, y) : 2 < x < 3, y = 4 - x\} \end{aligned}$$

Then $\sup Op(S) = (1, 0)$ dan $\inf op(S) = (0, 1)$.

$\sup op(a + S) = (3, 1) = (2, 1) + (1, 0) = a + \sup op(S)$

$\inf op(a + S) = (2, 2) = (2, 1) + (0, 1) = a + \inf op(S)$

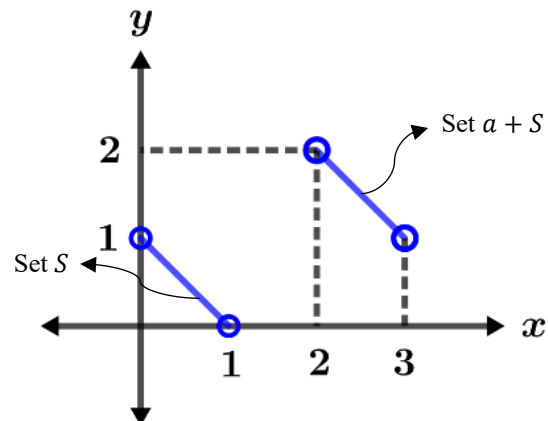


Figure 8. Another example of S and $a + S$

MONOTONIC FUNCTIONS

The study of monotone real functions has inspired research into monotone matrix functions [15]-[17]. Naturally, extending the notion of single-variable monotonicity to multivariable functions is a logical step. This article presents preliminary findings from our project aimed at generalizing monotone matrix functions to their multivariable counterparts.

In this section, we discuss the concept of monotonicity for real functions of two variables—a definition that differs from that in [18] and aligns with the total order we are employing.

Definition 15

Suppose $D \subseteq \mathbb{R}^2$ and $f: D \rightarrow \mathbb{R}$ be any function, and let I and J be intervals in $\mathbb{R}$ such that $I \times J \subseteq D$. We say

- (i) f is strictly increasing on $I \times J$ if for every (x_1, y_1) and (x_2, y_2) in $I \times J$, we have if $(x_1, y_1) < (x_2, y_2)$ then
 - (1) $f(x_1, y_1) < f(x_2, y_1)$ or $f(x_1, y_2) < f(x_2, y_2)$ if $x_1 < x_2$ or
 - (2) $f(x_1, y_1) < f(x_2, y_2)$ if $x_1 = x_2, y_1 < y_2$
- (ii) f is strictly decreasing on $I \times J$ if for every (x_1, y_1) and (x_2, y_2) in $I \times J$, we have if $(x_1, y_1) < (x_2, y_2)$ then
 - (1) $f(x_1, y_1) > f(x_2, y_1)$ or $f(x_1, y_2) > f(x_2, y_2)$ if $x_1 < x_2$ or
 - (2) $f(x_1, y_1) > f(x_2, y_2)$ if $x_1 = x_2, y_1 < y_2$
- (iii) f is strictly monotone on $I \times J$ if f is strictly increasing or strictly decreasing on $I \times J$.
- (iv) f is increasing on $I \times J$ if for every (x_1, y_1) and (x_2, y_2) in $I \times J$, we have if $(x_1, y_1) \leq (x_2, y_2)$ then
 - (1) $f(x_1, y_1) \leq f(x_2, y_1)$ or $f(x_1, y_2) \leq f(x_2, y_2)$ if $x_1 < x_2$ or
 - (2) $f(x_1, y_1) \leq f(x_2, y_2)$ if $x_1 = x_2, y_1 < y_2$
- (v) f is decreasing on $I \times J$ if for every (x_1, y_1) and (x_2, y_2) in $I \times J$, we have if $(x_1, y_1) \leq (x_2, y_2)$ then
 - (1) $f(x_1, y_1) \geq f(x_2, y_1)$ or $f(x_1, y_2) \geq f(x_2, y_2)$ if $x_1 < x_2$ or
 - (2) $f(x_1, y_1) \geq f(x_2, y_2)$ if $x_1 = x_2, y_1 < y_2$
- (vi) f is monotone on $I \times J$ if f is increasing or decreasing on $I \times J$.

Example 10

The functions $f(x, y) = x^3$ and $f(x, y) = y^3$ are increasing on $\mathbb{R} \times \mathbb{R}$, meanwhile the function $f(x, y) = y^3 + x^3$ is strictly increasing on $\mathbb{R} \times \mathbb{R}$.

Next, we consider a special class of two variables functions, and we discuss their relation with monotonicity of functions.

Proposition 16

Suppose $I \neq \emptyset, J \neq \emptyset$, I, J are intervals in $\mathbb{R}$. For any functions

$\mu: I \rightarrow \mathbb{R}$, and $\vartheta: J \rightarrow \mathbb{R}$, define $f: I \times J \rightarrow \mathbb{R}$ and $h: I \times J \rightarrow \mathbb{R}$ by $f(x, y) = \mu(x) + \vartheta(y)$ and $h(x, y) = \mu(x)\vartheta(y)$ for every $(x, y) \in I \times J$. Then:

- (1) f is strictly increasing on $I \times J$ and only if μ is strictly increasing on I and ϑ is strictly increasing on J .
- (2) If $\mu(x) > 0$ and $\vartheta(y) > 0$ for all $x \in I$ and $y \in J$. Then h is strictly increasing on $I \times J$ if only if μ is strictly increasing on I and ϑ is strictly increasing on J .

Proof.

Both parts follow directly from the definitions.

A similar statement holds for decreasing functions. These propositions, along with the next one, provide a foundation for constructing various examples of monotonic functions.

Proposition 17

Suppose $I \neq \emptyset, J \neq \emptyset, I, J$ are intervals in $\mathbb{R}$. The set $I + J := \{x + y: x \in I \text{ and } y \in J\}$ is interval in $\mathbb{R}$. Further, let $\varphi: I + J \rightarrow \mathbb{R}$ be any function and consider $f: I \times J \rightarrow \mathbb{R}$ defined by $f(x, y) := \varphi(x + y)$ for $(x, y) \in I \times J$. Then:

1. φ is increasing on $I + J \Rightarrow f$ is increasing on $I \times J$
2. φ is decreasing on $I + J \Rightarrow f$ is decreasing on $I \times J$

Proof. First, one shows that $I + J$ is an interval in $\mathbb{R}$ (see [18, p. 16]). Then the result follows from the definition (14).

The properties we have established in $\mathbb{R}^2$ have inspired this study. We examine certain properties in $\mathbb{R}^2$ —such as various inequality relations, the modified Archimedean property, and the modified completeness property—and assess their validity in $\mathbb{R}^n$ under the total order. In this research, we employ analytical methods to extend the concept of lexicographic order from $\mathbb{R}^2$ to $\mathbb{R}^n$, and we have obtained several promising results.

IV. EXTENSION TO LEXICOGRAPHIC ORDER IN $\mathbb{R}^n$

In this section, we will expand on the foundational concept of lexicographic order in $\mathbb{R}^n$, building upon the ideas developed in [12] to clearly demonstrate its extension to higher dimensions. We begin with the following definition of lexicographic order in $\mathbb{R}^n$ (another results about Lexicographic in $\mathbb{R}^n$ can be found in [19]-[23]).

Definition 18

If $x = (x_1, x_2, x_3, \dots, x_n), y = (y_1, y_2, y_3, \dots, y_n) \in \mathbb{R}^n$ then $x < y \Leftrightarrow x_1 < y_1$, or $x_i = y_i$, for $i = 1, 2, 3, \dots, k < n$ and $x_{k+1} < y_{k+1}$, (5)

Next, we partition $\mathbb{R}^n$ into three distinct subsets whose union is $\mathbb{R}^n$, similar to how $\mathbb{R}^2$ is divided into three disjoint subsets.

Specifically, for any $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, and K_i is a set of all element in $\mathbb{R}^n$ such that first elements which is not zero is the i^{th} element and is a real positive. For examples of set $K_3 = \{(0, 0, x_3, x_4, \dots, x_n) \in \mathbb{R}^n: x_3 > 0\}$. Next, define $-K_j = \{(0, 0, \dots, 0, -x_j, -x_{j+1}, \dots, -x_n) \in \mathbb{R}^n: x_j > 0\}$, and let $K = \bigcup_1^n K_i \subset \mathbb{R}^n$. (6)

Using the set K , we can partition $\mathbb{R}^n$ into three disjoint subsets, i.e $K, O, -K$ with $O = \{(0, 0, \dots, 0)\}$ such $\mathbb{R}^n = K \cup O \cup -K$.

Under the standard operations of addition and scalar

multiplication in $\mathbb{R}^n$ the set K satisfies the following properties.

Proposition 19

Let $K \subset \mathbb{R}^n$ be the set defined in (6). For every $a, b \in K$, and $c > 0, c \in \mathbb{R}$, the following properties hold:

- 1) $a + b \in K$
- 2) $ca \in K$

Note that the pointwise product in $\mathbb{R}^n$ is not closed in K . for example $(0,1,0, \dots, 0)(1,0,0, \dots, 0) = (0,0, \dots, 0) \notin K$. Thus, it is not true that $a, b \in K$ implies that $ab \in K$.

Under the lexicographic order defined in (1), the Trichotomy property holds in $\mathbb{R}^n$, i.e. for any vector $a \in \mathbb{R}^n$ exactly one of the following is true:

$$a \in K, \text{ or } a = 0, \text{ or } -a \in K$$

When $a \in K$, we denote this by writing $a > 0$ and call a positive. If $a \in K \cup 0$, we write $a \geq 0$, and say that a is nonnegative. Similarly, when $-a \in K$, we write $a < 0$, and say that a is negative, and if $-a \in K \cup 0$, we write $a \leq 0$, and say that a is non positive.

With these definitions, the ordering of two points in $\mathbb{R}^n$ is established.

Definition 20

Let $a, b \in \mathbb{R}^n$,

- (i). If $a - b \in K$, then we write $a > b$ (or equivalently, $b < a$).
- (ii). If $a - b \in K \cup 0$, then we write $a \geq b$ (or $b \leq a$).

By the trichotomy property, for any two vectors a and b in $\mathbb{R}^n$, exactly one of the following holds: $a > b$, or $a = b$ or $a < b$. Consequently, if both $a \geq b$ and $a \leq b$ then $a = b$. Henceforth, $a \geq b$ is understood to mean either $a > b$ or $a = b$. Just like in $\mathbb{R}$, the notation $a < b < c$ means $a < b$ and $b < c$, and similarly, the compound inequalities $a \leq b < c, a < b \leq c$, and $a \leq b \leq c$ are interpreted in the same way.

To illustrate these inequalities, consider vectors a and b in $\mathbb{R}^n$. The condition $a, b \in \mathbb{R}^n$, then $a - b \in K$ means $a_1 - b_1 > 0$ or $a_i - b_i = 0, i = 1, 2, 3, \dots, k < n$ and $a_{k+1} - b_{k+1} > 0$.

Example 11

$$\begin{aligned} (2, 3, \dots, 6) &> (1, 2, \dots, 9) \text{ because } (2, 3, \dots, 6) \\ &\quad - (1, 2, \dots, 9) = (1, 1, \dots, -3) \in K. \\ (2, 3, \dots, 1) &> (2, 2, \dots, 0) \text{ because } (2, 3, \dots, 1) - (2, 2, \dots, 0) \\ &= (0, 1, \dots, 1) \in K, \end{aligned}$$

The order relation " $\leq$ " as defined in (5) for $\mathbb{R}^n$ satisfies the properties of a total order.

Proposition 21

The relation of $\leq$ in (5) is the totally order.

Different with properties of the order that we know, which is for a and b in $\mathbb{R}^n, a \leq b \Leftrightarrow a_i \leq b_i$ (7)

Relation $\leq$ in (7) is the partial order of $\mathbb{R}^n$, because the three condition of partial order is satisfied. But $\leq$ is not the total

order in $\mathbb{R}^n$, because not every element a, b in $\mathbb{R}^n$ satisfies $a \leq b$ or $a \geq b$. To illustrate the basic properties of our order and their implications for inequalities, we conclude the result as follow.

Proposition 22

Suppose $a, b, c \in \mathbb{R}^n$ and $k \in \mathbb{R}$.

- (i) If $a > b$ and $b > c$ then $a > c$.
- (ii) If $a > b$ then $a + c > b + c$.
- (iii) If $a > b$ and $k > 0$ then $ka > kb$.
If $a > b$ and $k < 0$ then $ka < kb$.

Proof

- (i). If $a - b \in K$ and $b - c \in K$ then $a - c = (a - b) + (b - c) \in K$. We conclude that $a > c$.

- (ii). If $a - b \in K$ then $a - b + 0 = (a - b) + (c - c) = (a + c) + (-b - c) = (a + c) - (b + c) \in K$ (because of if $a \in K$ then $a + 0 = a \in K$). We conclude that $a + c > b + c$.

- (iii). If $a - b \in K$, then $a_1 > b_1$ or $a_i = b_i, i = 1, 2, 3, \dots, p < n$ and $a_{p+1} > b_{p+1}$.

There are two cases. The first, if $a_1 - b_1 > 0$ and $k > 0, k \in \mathbb{R}$ then $ka_1 - kb_1 = k(a_1 - b_1)$. We conclude, $ka - kb \in K$. The second, if $a_i = b_i, i = 1, 2, 3, \dots, p < n$ and $a_{p+1} > b_{p+1}$ then $k(a_{p+1} - b_{p+1}) > 0$ for $k > 0$ such that $ka_{p+1} - kb_{p+1} > 0$ with $a_i = b_i, i = 1, 2, 3, \dots, p < n$. We conclude, $ka - kb \in K$ or $ka > kb$. The similar proof, can be used for case $k < 0$.

COMPLETENESS PROPERTY IN $\mathbb{R}^n$

In this section, we introduce the definition of a bounded set in $\mathbb{R}^n$ and extend the concepts of supremum and infimum for sets in $\mathbb{R}^2$, building on the definitions presented in [12].

Definition 23

Suppose $S \neq \emptyset, S \subseteq \mathbb{R}^n$. S is right-bounded in $\mathbb{R}^n$ if there exists an interval I_1 in $\mathbb{R}$ that is bounded above, along with bounded intervals $I_k, k = 2, \dots, n$ in $\mathbb{R}$ such that $S \subseteq I_1 \times I_2 \times \dots \times I_n$. A point $s \in \mathbb{R}^n$ that satisfies $s \geq x$ for every $x \in S$ is called a right bound of S .

Definition 24

Suppose $S \neq \emptyset, S \subseteq \mathbb{R}^n$. S is left-bounded in $\mathbb{R}^n$ if there is an interval I_1 in $\mathbb{R}$ that is bounded below, along with bounded intervals $I_k, k = 2, \dots, n$ in $\mathbb{R}$ such that $S \subseteq I_1 \times I_2 \times \dots \times I_n$. A point $s \in \mathbb{R}^n$ that satisfies $s \leq x$ for every $x \in S$ is called a left bound of S .

Definition 25

Let $S \subseteq \mathbb{R}^n, S \neq \emptyset$. We say that S is totally bounded if S bounded in the left and the right in $\mathbb{R}^n$.

Equivalently, S is totally bounded if there exist bounded intervals $I_k, k = 1, 2, \dots, n$ in $\mathbb{R}$ such that $S \subseteq I_1 \times I_2 \times \dots \times I_n$.

Next, we will discuss the concept of the smallest right bound and the largest left bound of a set S in $\mathbb{R}^n$ as a

consequence of our definitions of bounded sets and the lexicographic order on $\mathbb{R}^n$. The definitions of supremum and infimum will be presented as extensions of the corresponding definitions in [12].

Definition 26

Suppose $S \neq \emptyset$, $S \subseteq \mathbb{R}^n$, and $\bar{S}$ is its closure.

- If S is bounded right, a point $a \in \bar{S}$ is called the optimal supremum of S if it satisfies the conditions:
 - a is a right bound for S (that is, $a \geq x$ for every $x \in S$), and
 - a is the smallest right bound of S , meaning that for any right bound $t \in \bar{S}$ of S , we have $a \leq t$.
- If S is left bounded, a point $u \in \bar{S}$ is called the optimal infimum of S if:
 - u is a left bound for S ,
 - u is the greatest left bound for S , meaning that for any left bound $w \in \bar{S}$ of S , we have $u \geq w$.

Part (a)(ii) ensures that a is the minimal right bound, while part (b)(ii) guarantees that u is the maximal left bound. In what follows, the optimal supremum of S is denoted by $\sup op(S)$ and optimal infimum by $\inf op(S)$. Let $S \subseteq \mathbb{R}^n$ is totally compact set. Since S is closed and bounded, then the existence of optimal supremum and optimal infimum should be guaranteed. Following previous studies, the candidate for optimal supremum is $(a_1^*, a_2^*, \dots, a_n^*)$, which

$$\begin{aligned} a_1^* &= \sup \{x_1 \in \mathbb{R} \mid (x_1, x_2, x_3, \dots, x_n) \in \bar{S}\}, \\ a_2^* &= \sup \{x_2 \in \mathbb{R} \mid (a_1^*, x_2, x_3, \dots, x_n) \in \bar{S}\}, \\ a_3^* &= \sup \{x_3 \in \mathbb{R} \mid (a_1^*, a_2^*, x_3, \dots, x_n) \in \bar{S}\}, \\ &\vdots \\ a_n^* &= \sup \{x_n \in \mathbb{R} \mid (a_1^*, a_2^*, a_3^*, \dots, a_{n-1}^*, x_n) \in \bar{S}\}. \end{aligned}$$

Similarly, the candidate for optimal infimum is $(a_{1*}, a_{2*}, \dots, a_{n*})$, which

$$\begin{aligned} a_{1*} &= \sup \{x_1 \in \mathbb{R} \mid (x_1, x_2, x_3, \dots, x_n) \in \bar{S}\}, \\ a_{2*} &= \sup \{x_2 \in \mathbb{R} \mid (a_{1*}, x_2, x_3, \dots, x_n) \in \bar{S}\}, \\ a_{3*} &= \sup \{x_3 \in \mathbb{R} \mid (a_{1*}, a_{2*}, x_3, \dots, x_n) \in \bar{S}\}, \\ &\vdots \\ a_{n*} &= \sup \{x_n \in \mathbb{R} \mid (a_{1*}, a_{2*}, a_{3*}, \dots, a_{n-1*}, x_n) \in \bar{S}\}. \end{aligned}$$

Proposition 27

Let $S \subset \mathbb{R}^n$ is a compact set, then

$$(a_1^*, a_2^*, \dots, a_n^*) \text{ and } (a_{1*}, a_{2*}, \dots, a_{n*}) \in S.$$

Proof:

Mimic the proof of Proposition 11.

Proposition 28

Let $S \subset \mathbb{R} \times \mathbb{R}$ is a compact set, then (a^*, b^*) is the optimal supremum of S .

Proof:

Mimic the proof of Proposition 12.

An analogous condition holds for the optimal infimum.

Example 12

Consider the set $S = \{(x, y, z) \in \mathbb{R}^3 : x + y + 2z \leq 4, x > 0, y > 0, z > 0\}$ (as illustrated in Fig. 9).

This set is totally bounded in $\mathbb{R}^3$ because there exist bounded intervals $I_1 = [0, 5]$, $I_2 = [0, 5]$, and $I_3 = [0, 3]$, such that $S \subseteq I_1 \times I_2 \times I_3$. In this case, for any $(x, y, z) \in S$, then $x \in (0, 4] \subseteq [0, 5] = I_1$, $y \in (0, 4] \subseteq [0, 5] = I_2$, $z \in (0, 2] \subseteq [0, 3] = I_3$.

The point $(4, 0, 0) \in \bar{S}$, and serves as a right bound of S , since for every $(x, y, z) \in S$ satisfies $(x, y, z) \leq (4, 0, 0)$. To see that $(4, 0, 0)$ is the smallest right bound, assume there exists another right bound (p, q, r) such that $(p, q, r) < (4, 0, 0)$. If $p = 4$ then necessarily $q \leq 0$, which contradicts the condition $y > 0$. If $y > 0$ then $(p, q, r) \in S$. Thus, $(4, 0, 0)$ is the optimal supremum of S . Similarly, one can show that the optimal infimum of S is $(0, 0, 0)$.

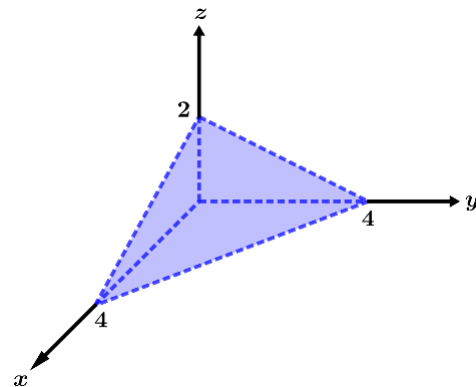


Fig. 9. Illustrative representation of set S in Example 12

Lemma 29

The optimal supremum of any nonempty set S is unique.

Proof:

Assume that S has two optimal suprema, say x and y . If $x < y$, then, given that y is the optimal supremum, x cannot serve as a right bound for S . Conversely, if $y < x$, then x cannot be the optimal supremum. Therefore, we must have $x = y$. A similar argument establishes the uniqueness of the optimal infimum.

An analogous condition holds for the optimal infimum.

Proposition 30

For any $S \neq \emptyset$, $S \subseteq \mathbb{R}^2$ that is right bounded has an optimal supremum in $\mathbb{R}^2$. Similarly, for any $S \neq \emptyset$, $S \subseteq \mathbb{R}^2$ that is left bounded has an optimal infimum in $\mathbb{R}^2$.

Proof:

Let D be a nonempty, right-bounded subset of $\mathbb{R}^2$. Then there exists an interval I_1 in $\mathbb{R}$ that is bounded above and an interval I_2 in $\mathbb{R}$ such that $D \subseteq I_1 \times I_2$. Define $M_i = \{x_i / (x_1, x_2) \in D\}$. $M_1 \subseteq I_1$ is bounded above, M_1 has a supremum, say α , in $\mathbb{R}$; similarly, M_2 is bounded. Three cases rise:

Case (1), if $\alpha \in M_1$ then $(\alpha, y) = \sup op D$ with $y = \sup \{x_2 / (\alpha, x_2) \in D\}$ or y is the only element in $di M_2$ then $(\alpha, y) \in \bar{D}$. Case (2), if $\alpha \notin M_1$ then $(\alpha, y) = \sup op D$ with $y = \inf \{x_2 / (\alpha, x_2) \in D\}$ or y is the only element in M_2 then $(\alpha, y) \in \bar{D}$. Case (3), if $\alpha \in M_1$ at once $\alpha \notin M_1$ (D is combination of several subsets in $\mathbb{R}^2$), a similar analysis applies as in the previous cases.

A similar line of reasoning shows that every left-bounded subset D of $\mathbb{R}^2$ has an optimal infimum. This framework can be extended to subsets of $\mathbb{R}^n$.

Proposition 31

For any $S \neq \emptyset$, $S \subseteq \mathbb{R}^n$ that is right bounded has an optimal supremum in $\mathbb{R}^n$. Likewise, for any $S \neq \emptyset$, $S \subseteq \mathbb{R}^n$ that is left bounded has an optimal infimum in $\mathbb{R}^n$. In conclusion, any totally bounded set in $\mathbb{R}^n$ possesses both an optimal supremum and an optimal infimum.

In what follows, we extend the translation operation defined in Definition 12 to $\mathbb{R}^n$.

Definition 32

For any subset $S \subseteq \mathbb{R}^n$, and any point $a \in \mathbb{R}^n$, define

$$a + S = \{a + x : x \in S\}$$

Example 13

Let $S = \{(x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$ and let $a = (2, 2, 2)$. Then $a + S = \{(2 + x, 2 + y, 2 + z) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\} = \{(x, y, z) : 2 \leq x \leq 3, 2 \leq y \leq 3, 2 \leq z \leq 3\}$.

Given that the optimal supremum of S is $\text{Sup } \text{Op}(S) = (1, 1, 1)$ and the optimal infimum is $\text{Inf } \text{Op}(S) = (0, 0, 0)$, it follows that $\text{Sup } \text{Op}(a + S) = (3, 3, 3) = (2, 2, 2) + (1, 1, 1) = a + \text{Sup } \text{Op}(S)$, and $\text{Inf } \text{Op}(a + S) = (2, 2, 2) = (2, 2, 2) + (0, 0, 0) = a + \text{Inf } \text{Op}(S)$.

Example 14

Consider the set $S = \{(x, y, z) \in \mathbb{R}^3 : 3x + 2y + 2z \leq 6, x > 0, y > 0, z > 0\}$ and let $a = (2, 2, 2)$ (see Fig. 2). S is totally bounded in $\mathbb{R}^3$ since there exist bounded intervals $I_1 = [0, 3]$, $I_2 = [0, 4]$, and $I_3 = [0, 4]$, such that $S \subseteq I_1 \times I_2 \times I_3$. In particular, for every $(x, y, z) \in S$, we have $x \in (0, 2] \subseteq [0, 3] = I_1$, $y \in (0, 3] \subseteq [0, 4] = I_2$, $z \in (0, 3] \subseteq [0, 4] = I_3$. The point $(2, 0, 0) \in \bar{S}$, and is a right bound of S because every $(x, y, z) \in S$ satisfies $(x, y, z) \leq (2, 0, 0)$. To show that $(2, 0, 0)$ is the smallest right bound, assume there exists another right bound (p, q, r) such that $(p, q, r) < (2, 0, 0)$. If $p = 2$ then $q \leq 0$, this is the opposite of $y > 0$ hypothesis. If $y > 0$ then $(p, q, r) \in S$. Thus, $(2, 0, 0)$ is the optimal supremum of S . Then $\text{Sup } \text{Op}(a + S) = a + \text{Sup } \text{Op}(S) = (2, 2, 2) + (2, 0, 0) = (4, 2, 2)$. Or by direct calculation $a + S = \{(2, 2, 2) + (x, y, z) \in \mathbb{R}^3 : 3x + 2y + 2z \leq 6, x > 0, y > 0, z > 0\} = \{(2 + x, 2 + y, 2 + z) \in \mathbb{R}^3 : 3x + 2y + 2z \leq 6, x > 0, y > 0, z > 0\} = \{(x', y', z') \in \mathbb{R}^3 : 3(x' - 2) + 2(y' - 2) + 2(z' - 2) \leq 6, x' - 2 > 0, y' - 2 > 0, z' - 2 > 0\} = \{(x, y, z) \in \mathbb{R}^3 : 3x + 2y + 2z \leq 20, x > 2, y > 2, z > 2\}$.

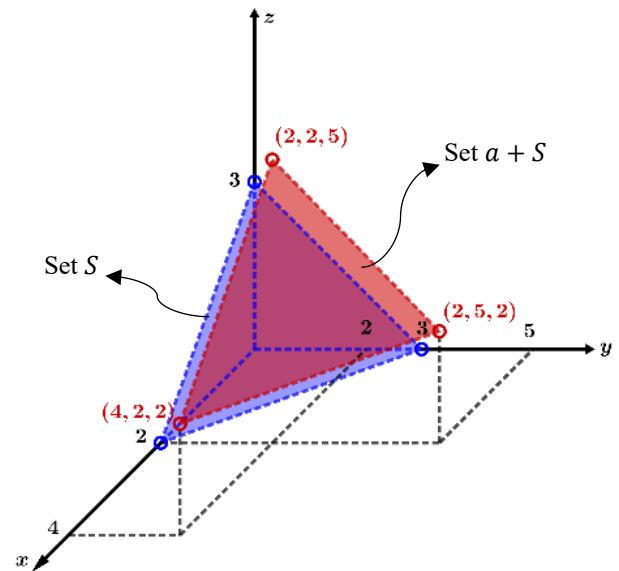


Figure 10. S and $a + S$ in $\mathbb{R}^3$

Proposition 33

Suppose $S \subseteq \mathbb{R}^n$ is totally bounded, and $a \in \mathbb{R}^n$.

Then $\text{sup } \text{Op}(a + S) = a + \text{sup } \text{Op}(S)$ and $\text{inf } \text{Op}(S) = a + \text{inf } \text{Op}(S)$.

Proof.

Let $m = \text{sup } \text{Op}(S)$, so that for every $\forall x \in S, x \leq m$. Then, for every $x \in S$, we have $a + x \leq a + m$, implying that $a + m$ is the right bound for $a + S$. Hence, $\text{sup } \text{Op}(a + S) \leq a + m = a + \text{sup } \text{Op}(S)$. Conversely, if v is any right bound of $a + S$, then $a + x \leq v, \forall x \in S$. So that $v - a$ is the right bound of S . Hence $w = \text{sup } \text{Op}(S) \leq v - a$, which implies that $a + w \leq v$. Since v is an arbitrary right bound of $a + S$ it follows that $a + \text{sup } \text{Op}(S) = a + w \leq \text{sup } \text{Op}(a + S)$.

V. DISCUSSION AND CONCLUSION

We have presented the properties of lexicographic order in $\mathbb{R}^n$. In general, there exist no conventional order of elements in $\mathbb{R}^n$. We introduced a lexicographic order, motivated by ordering of words in dictionary, which is non-quantitative data. Moreover, we generate mathematical theory to the elements of $\mathbb{R}^n$ regarding the order and the properties. This mathematical theory might be a potential to be applied in non-quantitative data management and programming such as [24] – [27].

In paper, we have started by presenting the properties of lexicographic order in $\mathbb{R}^2$. Applying ordering properties and completeness axiom of real number $\mathbb{R}$ we have constructed theory of lexicographic order of $\mathbb{R}^2$. Starting from this, we have continued to generalize the properties of lexicographic order established in $\mathbb{R}^2$ to higher order space $\mathbb{R}^n$. These extended properties encompass the nature of inequality, the modified Archimedean property, and the modified completeness property in $\mathbb{R}^n$.

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