Applications of Generalized Contractions in non-Archimedean Fuzzy Metric Spaces

Geeta Jatav, Gopal Meena

Abstract— **In the present research paper, we have introduced some generalized contractions called $(\psi - \xi)$ contraction and $(\psi - \xi)$ weak contraction and as the applications of these contractions, we established some fixed point results in non-Archimedean fuzzy metric spaces. In the final phase of our study, we used the obtained results to verify that a solution to the equation of motion exists.**

Index Terms— **Weak contraction, Fuzzy metric space, Cauchy sequence, Fixed point.**

I. Introduction

THE conception of fuzzy sets was launched initially by Zadeh [29] in 1965 and initiated important applications in the various fields of science. At that time, this concept was operated in topology and analysis by many authors and gave new applications. In particular, [1] to [7], [12] and [13] have acquainted the concept of fuzzy metric space with different strategies. In the present era, many directions of research have opened in view of fuzzy metric spaces. It has been seen in the research of [28], how tripolar fuzzy interior ideals work in ordered semigroups. Also, some more studies and some basic properties of tripolar fuzzy ideals in semigroups have been presented in [16]. Some applications of fixed point problems in the emerging fields have been given in [10]. Authors like [8] commenced the view of fixed point theory in fuzzy metric spaces, which is aligned to fixed point theory in probabilistic metric spaces. Many researchers have used this concept to initiate and investigate the various types of fuzzy contractive mappings. For example, after the commencement of fuzzy contractive mapping, the Banach contraction theorem has been proved by utilizing a strong condition for completeness in [9]. Researchers in [14], [15] have released this strong condition and introduced a new fuzzy contraction called ψ-contraction; also refer to [22], [26]. Recently, Vetro [25] defined the concept of a weak non-Archimedean fuzzy metric space and established common fixed point results for a pair of generalized contractive-type mappings. For a better approach to the generalization of inequalities and the uses of simple fixed-point iterations, we refer to [17] and [23].

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II. Preliminaries

Definition 2.1. [20] $* : [0,1] \times [0,1] \to [0,1]$, a binary operation is called a continuous triangular norm, if it satisfies the following conditions:

(i) $*$ is commutative and associative,
(ii) $*$ is continuous,
(iii) $*(j,1) = j$, for every $j \in [0,1]$,
(iv) $*(j, k) \leq *(l, m)$, whenever $j \leq l, k \leq m$ and $j,k,l,m \in [0,1]$.

Definition 2.2. [19] A fuzzy metric space is an ordered triple $(\mathcal{M}, \rho, *)$, such that $\mathcal{M}$ is non-empty set, $*$ is a continuous t-norm and ρ is a fuzzy set on $\mathcal{M} \times \mathcal{M} \times (0,\infty)$, satisfying the following conditions:

for all $\mu, v, w \in \mathcal{M}; k,t > 0$:
(ρ1) $\rho(\mu,v,t) > 0$,
(ρ2) $\rho(\mu,v,t) = 1$ iff $\mu = v$,
(ρ3) $\rho(\mu,v,t) = \rho(v,\mu,t)$,
(ρ4) $\rho(\mu,v,.) : (0,\infty) \to [0,1]$ is continuous,
(ρ5) $\rho(\mu,w,t+k) \geq \rho(\mu,v,t) * \rho(v,w,k)$.

If, in the above definition, the triangular inequality (ρ5) is replaced by

$$\rho(\mu,w,\max\{t,k\}) \geq \rho(\mu,v,t) * \rho(v,w,k)$$

for all $\mu,v,w \in \mathcal{M}; k,t > 0$, or equivalently

$$\rho(\mu,w,t) \geq \rho(\mu,v,t) * \rho(v,w,t).$$

Then, the triple $(\mathcal{M}, \rho, *)$ is called a non-Archimedean fuzzy metric space [27].

Definition 2.3. [18] Let $(\mathcal{M}, \rho, *)$ be a fuzzy metric space. A sequence $\{\mu_n\}$ in $\mathcal{M}$ is said to be converges to μ in $\mathcal{M}$, denoted by $\mu_n \to \mu$ if and only if $\lim_{n\to\infty} \rho(\mu_n, \mu, t) = 1$ for all $t > 0$; i.e. for each $z \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\rho(\mu_n, \mu, t) > 1 - z$ for all $n \geq n_0$.

Definition 2.4 [24] Let $(\mathcal{M}, \rho, *)$ be a fuzzy metric space. A sequence $\{\mu_n\}$ is a ρ -Cauchy sequence if and only if for all $\epsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\rho(\mu_n, \mu_m, t) > 1 - \epsilon$ for all $m > n \geq n_0$. A sequence $\{\mu_n\}$ is a G-Cauchy sequence if and only if $\lim_{n\to\infty} \rho(\mu_n, \mu_{n+p}, t) = 1$ for any $p > 0$ and $t > 0$.

Definition 2.5. [18] The fuzzy metric space $(\mathcal{M}, \rho, *)$ is called ρ-complete (G-complete) if every ρ –Cauchy (G - Cauchy) sequence is convergent.

Definition 2.6. [28] A fuzzy set ρ of a universal set $\mathcal{M}$ is said to be tripolar fuzzy set, if $\rho \coloneqq \{\mu; \rho^+(\mu),$

$\rho^*(\mu), \rho^-(\mu) | \mu \in \mathcal{M}$ and
$$0 \le \rho^+(\mu)\rho^*(\mu) \le 1 \},$$
where $\rho^+: \mathcal{M} \to [0,1]$, $\rho^*: \mathcal{M} \to [0,1]$, $\rho^-: \mathcal{M} \to [-1,0]$.

For simplicity $\rho := (\rho^+,\ \rho^*,\ \rho^-)$ uses for the tripolar fuzzy set.

The research in paper [21] led to the conclusion that the concepts of "contraction" and "weak contraction" can be broadened to improve upon the existing findings. This likely means the author found ways to extend or refine these concepts, making them more broadly applicable or more powerful in a given context.

III. MAIN RESULTS

First, we define the contraction, which will be further utilized in fixed point theory:

Definition 3.1. Let $\psi : [0,1) \to R$ and $\xi : [0,1) \to [0,1)$ are strictly increasing, continuous mappings and for each sequence $\{q_n\}_{n \in \mathbb{N}}$ of positive numbers, $\lim_{n\to\infty} q_n = 1$ if and only if $\lim_{n\to\infty} \psi(\xi(q_n)) = +\infty$. A mapping $S : \mathcal{M} \to \mathcal{M}$ is said to be $(\psi - \xi)$ contraction, if there exists $\zeta \in (0,1)$ such that
$$\rho(S\mu, Sv, t) < 1 \Rightarrow$$
$$\psi(\xi(\rho(S\mu, Sv, t))) \ge \psi(\xi(\rho(\mu, v, t))) + \zeta, \qquad (1)$$
for all $\mu, v \in \mathcal{M}$.

Remark 3.2. For $\psi(\mu) = \frac{1}{1-\mu}$ and (1), it is easy to conclude that every contraction S is contractive mapping, that is, $\rho(S\mu, Sv, t) > \rho(\mu, v, t)$ for all $\mu, v \in \mathcal{M}$, such that $S\mu \ne Sv$. Thus every $(\psi - \xi)$ contraction is continuous mapping.

Theorem 3.3. Let $(\mathcal{M},\ \rho,\ *)$ be a non-Archimedean fuzzy metric space and a mapping $S : \mathcal{M} \to \mathcal{M}$ be a $(\psi - \xi)$ contraction. Then, S has a unique fixed point in $\mathcal{M}$.

Proof. Let $\mu_0 \in \mathcal{M}$, be arbitrary and fixed. Define sequence $\{\mu_n\}$, by
$$S\mu_n = \mu_{n+1} \text{ for all } n \in \mathbb{N}.$$
If $\mu_n = \mu_{n+1}$, then μ_n is a fixed point of S; then the proof is finished.
Suppose that $\mu_n \ne \mu_{n+1}$ for all $n \in \mathbb{N}$.
Therefore, by (1), we get
$$\psi(\xi(\rho(S\mu_{n-1}, S\mu_n, t))) \ge \psi(\xi(\rho(\mu_{n-1}, \mu_n, t))) + \zeta.$$
Repeating this process, we get
$$\psi(\xi(\rho(S\mu_{n-1}, S\mu_n, t))) \ge \psi(\xi(\rho(\mu_{n-2}, \mu_{n-1}, t))) + 2\zeta,$$
$$\cdots$$
$$\ge \psi(\xi(\rho(\mu_0, \mu_1, t))) + n\zeta. \qquad (2)$$
Letting $n \to \infty$, from (2), we get
$$\lim_{n\to\infty} \psi(\xi(\rho(S\mu_{n-1}, S\mu_n, t))) = +\infty.$$
Then, we have
$$\lim_{n\to\infty} \rho(S\mu_n, \mu_n, t) = 1. \qquad (3)$$
Now, we want to show that the sequence $\{\mu_n\}$ is a Cauchy sequence.
Suppose to the contrary that $\{\mu_n\}$ is not a Cauchy sequence. Then there are $\epsilon \in (0,1)$ and $t_0 > 0$, such that for all $h \in \mathbb{N}$, there exists $n(h),\ m(h) \in \mathbb{N}$ with $n(h) > m(h) > h$ and
$$\rho\left(\mu_{n(h)}, \mu_{m(h)}, t_0\right) \le 1 - \epsilon. \qquad (4)$$
Assume that, $m(h)$ is the least integer exceeding $n(h)$, and satisfying inequality (4). Then, we have
$$\rho\left(\mu_{m(h)-1}, \mu_{n(h)}, t_0\right) > 1 - \epsilon.$$
So, for all $h \in \mathbb{N}$, we get
$$1 - \epsilon \ge \rho\left(\mu_{n(h)}, \mu_{m(h)}, t_0\right)$$
$$\ge \rho\left(\mu_{m(h)-1}, \mu_{m(h)}, t_0\right) * \rho\left(\mu_{m(h)-1}, \mu_{n(h)}, t_0\right)$$
$$\ge \rho\left(\mu_{m(h)-1}, \mu_{m(h)}, t_0\right) * (1 - \epsilon). \qquad (5)$$
By taking $h \to \infty$ in (5) and using (3), we obtain
$$\lim_{h\to\infty} \rho\left(\mu_{n(h)}, \mu_{m(h)}, t_0\right) = 1 - \epsilon.$$
From(ρ5), we get
$$\rho\left(\mu_{m(h)+1}, \mu_{n(h)+1}, t_0\right) \ge \rho\left(\mu_{m(h)+1}, \mu_{n(h)}, t_0\right)$$
$$* \rho\left(\mu_{m(h)}, \mu_{n(h)}, t_0\right) * \rho\left(\mu_{m(h)}, \mu_{n(h)+1}, t_0\right).$$
Taking the limit as $h \to \infty$, we obtain
$$\lim_{n\to\infty} \rho\left(\mu_{n(h)+1}, \mu_{m(h)+1}, t_0\right) = 1 - \epsilon.$$
By applying inequality (1), with $\mu = \mu_{m(h)}$ and $v = v_{n(h)}$, we have
$$\psi(\xi\left(\rho\left(\mu_{n(h)+1}, \mu_{m(h)+1}, t\right)\right))$$
$$\ge \psi(\xi\left(\rho\left(\mu_{n(h)}, \mu_{m(h)}, t\right)\right)) + \zeta.$$
Taking the limit as $h \to \infty$ with the continuity of $(\psi - \xi)$, we obtain
$$\psi(\xi(1 - \epsilon)) \ge \psi(\xi(1 - \epsilon)) + \zeta,$$
which is a contradiction. Thus, $\{\mu_n\}$ is a Cauchy sequence in $\mathcal{M}$. From the completeness of $(\mathcal{M},\ \rho,\ *)$, there exists $r \in \mathcal{M}$, such that
$$\lim_{n\to\infty} \mu_n = r.$$
Finally, the continuity of S yields
$$\rho(Sr, r, t) = \lim_{n\to\infty} \rho(S\mu_n, \mu_n, t)$$
$$= \lim_{n\to\infty} \rho(S\mu_n, \mu_n, t) = 1.$$
Thus r is a fixed point of S.
Now, we show that S has a unique fixed point. Suppose r_1 and r_2 are two fixed points of S.
Indeed, if for $r_1,\ r_2 \in \mathcal{M},\ Sr_1 = r_1 \ne r_2 = Sr_2$, then we get
$$\psi(\xi(\rho(r_1, r_2, t))) \ge \psi(\xi(\rho(r_1, r_2, t))) + \zeta,$$
which is a contradiction. Thus, S has a unique fixed point.

Example 3.4. Let $\mathcal{M} = [0,1),\ a * b = min\{a, b\}$,
$$\rho(\mu, v, t) = \begin{cases} \dfrac{1}{1 + \max\{\mu, v\}}, & \mu \ne v \\ 1, & \mu = v, \end{cases}$$
for all $t > 0$. Let $\psi : [0,1) \to R$ and $\xi : [0,1) \to [0,1)$ such that $\psi(\mu) = \frac{4}{3}\mu$ and $\xi(\mu) = \frac{1}{3}\mu$ for all $\mu \in [0,1)$ and define $S : \mathcal{M} \to \mathcal{M}$, by $S(\mu) = \frac{2\mu}{3}$ for all $\mu \in \mathcal{M}$.
Clearly, $(\mathcal{M},\ \rho,\ *)$ is a non-Archimedean fuzzy metric space.

Case-1. We assume that $\mu < v$ for all $\mu, v \in (0,1)$. Since $\mu^2 < \mu$ and $v^2 < v$, then $\max\{\mu, v\} > \max\{S\mu, Sv\}$. So, there exists $\zeta \in (0,1)$, such that
$$\frac{1}{3}\left(\frac{1}{1 + \max\{S\mu, Sv\}}\right) \ge \frac{1}{3}\left(\frac{1}{1 + \max\{\mu, v\}}\right) + \zeta$$
$$\frac{4}{3}\left(\frac{1}{3}\left(\frac{1}{1 + \max\{S\mu, Sv\}}\right)\right) \ge \frac{4}{3}\left(\frac{1}{3}\left(\frac{1}{1 + \max\{\mu, v\}}\right)\right) + \zeta$$
$$\psi(\xi(\rho(S\mu, Sv, t))) \ge \psi(\xi(\rho(\mu, v, t))) + \zeta.$$

Case-2. We assume that $\mu = 0$ and $v \in (0,1)$. Since $\mu^2 =$

0, $v^2 < v$, then $\max\{\mu, v\} = v > v^2 = \max\{S\mu, Sv\}$. Hence, we have

$$\frac{1}{3}\left(\frac{1}{1+\max\{S\mu, Sv\}}\right) \geq \frac{1}{3}\left(\frac{1}{1+\max\{\mu, v\}}\right) + \zeta$$

$$\frac{4}{3}\left(\frac{1}{3}\left(\frac{1}{1+\max\{S\mu,Sv\}}\right)\right) \geq \frac{4}{3}\left(\frac{1}{3}\left(\frac{1}{1+\max\{\mu,v\}}\right)\right) + \zeta,$$

that is,

$$\psi(\xi(\rho(S\mu, Sv, t))) \geq \psi(\xi(\rho(\mu, v, t))) + \zeta.$$

Therefore, S is a $(\psi - \xi)$ contraction. Thus, all the conditions of theorem 3.3 holds and S has the unique fixed point $\mu = 0$.

Definition 3.5. Let $(\mathcal{M}, \rho, *)$ be a non-Archimedean fuzzy metric space and a mapping $S : \mathcal{M} \to \mathcal{M}$ is said to be $(\psi - \xi)$ weak contraction, if there exists $\zeta \in (0, 1)$ such that $\rho(S\mu, Sv, t) < 1 \Rightarrow$

$$\psi(\xi(\rho(S\mu, Sv, t))) \geq \psi(\xi(P(\mu, v, t))) + \zeta, \tag{6}$$

for all $\mu, v \in \mathcal{M}$, where

$$P(\mu, v, t) = \min\{\rho(\mu, v, t), \rho(\mu, S\mu, t), \rho(v, Sv, t)\}$$

Remark 3.6. Every $(\psi - \xi)$ contraction is a $(\psi - \xi)$ weak contraction. But the converse is not true.

Example 3.7. Let $\mathcal{M} = A \cup B$, where $A = \left\{\frac{1}{20}, \frac{1}{4}, 1, 2, 3\right\}$, $B = [5,7]$. Let $j * k = min\{j, k\}$ and $\rho(\mu, v, t) = \frac{min\{\mu,v\}}{max\{\mu,v\}}$ for all $t > 0$.

Clearly, $(\mathcal{M}, \rho, *)$ is a complete non-Archimedean fuzzy metric space. Let $\psi: [0,1) \to R$ such that $\psi(\mu) = \frac{1}{\sqrt{(1+\mu)}}$ for all $\mu \in [0,1)$ and define $S: X \to X$, by

$$S(\mu) = \begin{cases} \frac{1}{20}, \mu \in A \\ \frac{1}{4}, v \in B \end{cases},$$

$$\xi(\mu) = \frac{1}{1+\mu}$$

for all $u \in [0,1)$. Since S is not continuous, so by remark (3.2), S is not a $(\psi - \xi)$ contraction.

Now, we show that, for all $\mu \in A$ and $v \in B$, S is a $(\psi - \xi)$ weak contraction.

Case-1. Let $\mu = 1$ and $v \in B$, then

$$\rho(S\mu, Sv, t) = \rho(\frac{1}{20}, \frac{1}{4}, t)$$

$$= \frac{min\{\mu, v\}}{max\{\mu, v\}}$$

$$= \frac{min\{\frac{1}{20}, \frac{1}{4}\}}{max\{\frac{1}{20}, \frac{1}{4}\}}$$

$$= \frac{\frac{1}{20}}{\frac{1}{4}}$$

$$= \frac{1}{5}.$$

Also,

$$\rho(\mu, v, t) = \frac{\mu}{v},$$

$$\rho(\mu, S\mu, t) = \frac{1}{20\mu},$$

$$\rho(v, Sv, t) = \frac{1}{4v}.$$

So,

$$P(\mu, v, t) = min\{\rho(\mu, v, t), \rho(\mu, S\mu, t), \rho(v, Sv, t)\},$$

$$= min\left\{\frac{\mu}{v}, \frac{1}{20\mu}, \frac{1}{4v}\right\}$$

$$= \frac{1}{4v}.$$

Thus

$$\rho(S\mu, Sv, t) = \frac{1}{5} > \frac{1}{4v} = min\left\{\frac{\mu}{v}, \frac{1}{20\mu}, \frac{1}{4v}\right\}.$$

Then, we have

$$\xi(\rho(S\mu, Sv, t)) = \frac{1}{1+\frac{1}{5}} = \frac{5}{6},$$

and

$$\psi\left(\xi(\rho(S\mu, Sv, t))\right) = \psi\left(\frac{5}{6}\right) = \frac{1}{\sqrt{(1+\frac{5}{6})}}$$

and

$$\xi(P(\mu, v, t)) = \xi\left(\frac{1}{4v}\right) = \frac{1}{1+\frac{1}{4v}} = \frac{4v}{4v+1},$$

$$\psi\left(\xi(P(\mu, v, t))\right) = \frac{1}{\sqrt{\left(1+\frac{4v}{4v+1}\right)}}.$$

So, there exists $\zeta \in (0, 1)$, such that

$$\psi\left(\xi(\rho(S\mu, Sv, t))\right) \geq \psi\left(\xi(P(\mu, v, t))\right) + \zeta.$$

Case-2. Let $\mu \in \{2, 3\}$ and $v \in B$, then

$$\rho(S\mu, Sv, t) = \frac{1}{5}$$

and

$$P(\mu, v, t) = min\left\{\frac{\mu}{v}, \frac{1}{20\mu}, \frac{1}{4v}\right\}$$

$$= \frac{1}{20\mu}.$$

Thus

$$\psi(\rho(S\mu, Sv, t)) = \frac{1}{5} > \frac{1}{20\mu} = min\left\{\frac{\mu}{v}, \frac{1}{20\mu}, \frac{1}{4v}\right\}.$$

Then, we have

$$\xi(P(\mu, v, t)) = \xi\left(\frac{1}{20\mu}\right) = \frac{1}{1+\frac{1}{20\mu}} = \frac{20\mu}{20\mu+1},$$

and

$$\psi\left(\xi(P(\mu, v, t))\right) = \frac{1}{\sqrt{\left(1+\frac{20\mu}{20\mu+1}\right)}},$$

and

$$\psi\left(\frac{1}{1+\frac{1}{5}}\right) \geq \psi\left(\frac{20\mu}{20\mu+1}\right).$$

So, there exists $\zeta \in (0, 1)$, such that

$$\psi\left(\xi(\rho(S\mu, Sv, t))\right) \geq \psi\left(\xi(P(\mu, v, t))\right) + \zeta.$$

Case-3. Let $\mu \in \left\{\frac{1}{20}, \frac{1}{4}\right\}$ and $v \in B$, then

$$\rho(S\mu, Sv, t) = \frac{1}{5},$$

and

$$P(\mu, v, t) = min\left\{\frac{\mu}{v}, \frac{1}{20\mu}, \frac{1}{4v}\right\} = \frac{\mu}{v}.$$

Thus

$$\rho(S\mu, Sv, t) = \frac{1}{5} > \frac{\mu}{v} = min\left\{\frac{\mu}{v}, \frac{1}{20\mu}, \frac{1}{4v}\right\}.$$

Then, we have

$$\xi\big(P(\mu,v,t)\big) = \xi\left(\frac{\mu}{v}\right) = \frac{1}{1+\frac{\mu}{v}} = \frac{v}{v+\mu}$$

and

$$\psi\left(\xi\big(P(\mu,v,t)\big)\right) = \frac{1}{\sqrt{\left(1+\frac{v}{v+\mu}\right)}}$$

and

$$\psi\left(\frac{1}{1+\frac{1}{5}}\right) \geq \psi\left(\frac{v}{v+\mu}\right).$$

So, there exists $\zeta \in (0,1)$, such that

$$\psi\left(\xi\big(\rho(S\mu, Sv, t)\big)\right) \geq \psi\left(\xi\big(P(\mu,v,t)\big)\right) + \zeta.$$

Therefore, S is a $(\psi - \xi)$ weak contraction.

Theorem 3.8. Let $(\mathcal{M}, \rho, *)$ be non-Archimedean fuzzy metric space and let $S: \mathcal{M} \to \mathcal{M}$ be a $(\psi - \xi)$ weak contraction. Then, S has a unique fixed point in $\mathcal{M}$.

Proof. Let $\mu_0 \in \mathcal{M}$ be an arbitrary element. Define sequence $\{\mu_n\}$, by $S\mu_n = \mu_{n+1}$ for all $n \in \mathbb{N}$.
If $\mu_n = \mu_{n+1}$, then μ_{n+1} is a fixed point of S, then the proof is finished.
Suppose that, $\mu_n \neq \mu_{n+1}$ for all $n \in \mathbb{N}$, therefore, by (6), we get

$$\psi\left(\xi\left(\rho(S\mu_{n-1}, S\mu_n, t)\right)\right)$$
$$\geq \psi\big(\xi\big(\min\{\rho(\mu_{n-1},\mu_n,t), \rho(\mu_{n-1}, S\mu_{n-1}, t), \rho(\mu_n, S\mu_n, t)\}\big)\big) + \zeta$$
$$= \psi\big(\xi\big(\min\{\rho(\mu_{n-1},\mu_n,t), \rho(\mu_{n-1},\mu_n,t), \rho(\mu_n,\mu_{n+1},t)\}\big)\big) + \zeta$$

$$= \psi\left(\xi\big(min\{\rho(\mu_{n-1},\mu_n,t), \rho(\mu_n,\mu_{n+1},t)\}\big)\right) + \zeta. \quad (7)$$

If there exists $n \in \mathbb{N}$, such that

$$\min\{\rho(\mu_{n-1},\mu_n,t), \rho(\mu_n,\mu_{n+1},t)\} = \rho(\mu_n,\mu_{n+1},t).$$

Thus, from (7), it becomes

$$\psi\left(\xi\left(\rho(S\mu_{n-1}, S\mu_n, t)\right)\right) = \psi\left(\xi\left(\rho(\mu_n,\mu_{n+1},t)\right)\right)$$
$$\geq \psi\left(\rho(\mu_n,\mu_{n+1},t)\right) + \zeta,$$

which is a contradiction, therefore,

$$\min\{\rho(\mu_{n-1},\mu_n,t), \rho(\mu_n,\mu_{n+1},t)\} = \rho(\mu_{n-1},\mu_n,t), \quad (8)$$

for all $n \in \mathbb{N}$.
Thus, from (7), we have

$$\psi\left(\xi\left(\rho(\mu_n,\mu_{n+1},t)\right)\right) \geq \psi\left(\xi\left(\rho(\mu_{n-1},\mu_n,t)\right)\right) + \zeta,$$

for all $n \in \mathbb{N}$. It implies that

$$\psi\left(\xi\left(\rho(\mu_n,\mu_{n+1},t)\right)\right) \geq \psi\left(\xi\left(\rho(\mu_0,\mu_1,t)\right)\right) + n\zeta.$$

Taking, as $n \to \infty$, we get

$$\lim_{n\to\infty} \psi\left(\xi\left(\rho(\mu_n,\mu_{n+1},t)\right)\right) = +\infty.$$

Then, we have

$$\lim_{n\to\infty} \rho(S\mu_{n-1}, S\mu_n, t) = 1.$$

The proof, that $\{\mu_n\}$ is a Cauchy sequence can be shown as in Theorem (3.3). From the completeness of $(\mathcal{M}, \rho, *)$, there exists $r \in \mathcal{M}$, such that

$$\lim_{n\to\infty} \mu_n = r.$$

Now, we show that r is a fixed point of S . Since ψ and ξ are continuous, then there are two cases arise.

Case-1. For each $n \in \mathbb{N}$, there exists $l_n \in \mathbb{N}$ such that $\mu_{l_{n+1}} = Sr$ and $l_n > l_{n-1}$, where $l_0 = 1$. Then, we get

$$r = \lim_{n\to\infty} \mu_{l_{n+1}} = Sr.$$

This proves that r is a fixed point of S.

Case-2. If there exists $n_0 \in \mathbb{N}$ such that $\mu_{n+1} \neq Sr$ for all $n \geq n_0$.
That is, $\rho(S\mu_n, Sr, t) < 1$ for all $n \geq n_0$. It follows from (6) with the properties of ψ and ξ,

$$\psi\left(\xi\left(\rho(\mu_{n+1}, Sr, t)\right)\right) = \psi\left(\xi\left(\rho(S\mu_n, Sr, t)\right)\right)$$
$$\geq \psi\big(\xi\big(\min\{\rho(\mu_n, r, t), \rho(\mu_n, S\mu_n, t), \rho(r, Sr, t)\}\big)\big) + \zeta$$
$$\geq \psi\big(\xi\big(\min\{\rho(\mu_n, r, t), \rho(\mu_n,\mu_{n+1},t), \rho(r, Sr, t)\}\big)\big) + \zeta. \quad (9)$$

If $\rho(r, Sr, t) < 1$, then, we have

$$\lim_{n\to\infty} \rho(\mu_n, r, t) = 1,$$

and there exits $n_1 \in \mathbb{N}$, such that, for all $n \geq n_1$ we get

$$\min\{\rho(\mu_n, r, t), \rho(\mu_n,\mu_{n+1},t), \rho(r, Sr, t)\} = \rho(r, Sr, t).$$

From (9), we have

$$\psi\left(\xi\left(\rho(\mu_{n+1}, Sr, t)\right)\right) \geq \psi\left(\xi\big(\rho(r, Sr, t)\big)\right) + \zeta. \quad (10)$$

For all $n \geq \max\{n_0, n_1\}$. Since, ψ and ξ are continuous, taking the limit as $n \to \infty$ in (10), we obtain

$$\psi\left(\xi\big(\rho(r, Sr, t)\big)\right) \geq \psi\left(\xi\big(\rho(r, Sr, t)\big)\right) + \zeta,$$

which is a contradiction. Therefore, $\rho(r, Sr, t) = 1$; that is, r is a fixed point of S.
Now, we prove that the fixed point of S is unique. Let r_1, r_2 be two fixed points of S. Suppose that $r_1 \neq r_2$; then we have $Sr_1 \neq Sr_2$. It follows from (6), that we have

$$\psi\left(\xi\big(\rho(r_1, r_2, t)\big)\right) = \psi\left(\xi\big(P(Sr_1, Sr_2, t)\big)\right)$$
$$\geq \psi\big(\xi\big(\min\{\rho(r_1, r_2, t), \rho(r_1, Sr_1, t), \rho(r_2, Sr_2, t)\}\big)\big) + \zeta$$
$$= \psi\big(\xi\big(\min\{\rho(r_1, r_2, t), \rho(r_1, r_1, t), \rho(r_2, r_2, t)\}\big)\big) + \zeta$$
$$= \psi\left(\xi\big(\rho(r_1, r_2, t)\big)\right) + \zeta,$$

which is a contradiction. Then $\rho(r_1, r_2, t) = 1$, that is, $r_1 = r_2$.Therefore, the fixed point of S is unique.

IV. STABILITY RESULTS

Lemma 4.1. Let $(\mathcal{M}, \rho, *)$ be a non-Archimedean fuzzy metric space and $\{P_n: \mathcal{M} \to \mathcal{M}\}$ be a sequence of $(\psi - \xi)$ contractions, which is uniformly convergent to $P: \mathcal{M} \to \mathcal{M}$, then P is also a $(\psi - \xi)$ contraction.

Proof. Since P_n for all $n \geq 1$ be a sequence of $(\psi - \xi)$ contractions, thus

$$\rho(P_n\mu, P_n v, t)) < 1 \Rightarrow$$

$\psi(\xi(\rho(P_n\mu, P_n v, t))) \geq \psi(\xi(\rho(\mu, v, t))) + \zeta$, for all $u, v \in W$.

Letting $n \to \infty$ and $\{P_n\}$ converges uniformly to P, then

$$\rho(P\mu, Pv, t)) < 1 \Rightarrow$$

$\psi(\xi(\rho(P\mu, Pv, t))) \geq \psi(\xi(\rho(\mu, v, t))) + \zeta$, for all $\mu, v \in W$.

Hence P is a $(\psi - \xi)$ contraction mapping.

Theorem 4.2. Let $(\mathcal{M}, \rho, *)$ be a non-Archimedean fuzzy metric space and $\{P_n: \mathcal{M} \to \mathcal{M}\}$ be a sequence of $(\psi - \xi)$ contractions, which is uniformly convergent to $P: \mathcal{M} \to \mathcal{M}$, then $\lim_{n\to\infty} H_p\big(F(P_n), F(P)\big), t\big) = 1$. Hence the fixed point sets of the sequence $\{P_n\}$ are stable.

Proof. Since $F(P_n)$ is non-empty, so $\mu_n \in F(P_n)$ such that $\rho(P_n\mu_n, \mu_n, t) = 1$.

Since $\{P_n\}$, the sequence of $(\psi - \xi)$ contractions is uniformly convergent to P, so P is also $(\psi - \xi)$ contraction. It implies that there exists $\mu \in F(P)$ such that

$$\rho(P\mu, \mu, t) = 1.$$

Now we know that Housdorff fuzzy metric is defined as $H_p(F(P_n), F(P), t)$

$$= min\left\{\inf_{\mu_n \in F(P_n)} \sup_{\mu \in F(P)} \rho(\mu_n, \mu, t), \inf_{v \in F(P)} \sup_{v_n \in F(P_n)} \rho(v_n, v, t)\right\}$$

and implies that $\lim_{n\to\infty} H_p(F(P_n),\ F(P)), t) = 1$.

V. APPLICATION TO PHYSICAL PROBLEM

Let $C_f([0,1], R)$ be the collection of all continuous functions $f: [0,1] \to R$ and ρ be a fuzzy set function on $C_f([0,1], R) \times C_f([0,1], R) \times (0, \infty)$ with $\rho(\mu(p), v(p), t) = \frac{1}{1+\sup_{p\in[0,1]} |\mu(p)-v(p)|}$, and triangular norm $*$ is defined by $a * b = \min(a, b)$.
Thus $(C_f([0,1], R),\ \rho,\ *)$ be a non-Archimedean fuzzy metric space.

Now, we consider the equation of motion

$$m\frac{d^2\alpha}{dp^2} = f(p,\ \alpha(p)) \qquad (11)$$

and $\alpha(0) = 0, \alpha'(0) = 1$, where $f: [0,1] \times R \to R$ is continuous.
Green's function associated with (11) is defined by

$$\mathrm{F}(p, l) = \begin{cases} p, & if\ 0 \le p \le l \le 1, \\ 2p - l, & if\ 0 \le l \le p \le 1. \end{cases}$$

It is clear that the solution of (11) is equivalent to $\alpha \in C_f([0,1], R)$, a solution of the integral equation

$$\alpha(p) = \int_0^1 \mathrm{F}(p, l) f(l, x(l))\, dl, \qquad p \in [0,1].$$

Let $S: C_f([0,1], R) \to C_f([0,1], R)$, be defined by $S(\alpha(p)) = \int_0^1 \mathrm{F}(p, l) f(l, \alpha(l))\, dl$.

Theorem 5.1. Under the assumptions
(i) $|f(p, \alpha(p)) - f(p, \gamma(p))| \le \sup\{|\alpha(p) - \gamma(p)| - 1\}$, for all $p \in [0,1]$ and $\alpha, \gamma \in C_f([0,1], R)$,
(ii) Let $\psi : [0,1) \to R$ and $\xi : [0,1) \to [0,1)$ such that $\psi(\mu) = \frac{4}{3}\mu$ and $\xi(\mu) = \frac{1}{3}\mu$.
The equation (11) has a solution.

Proof. Consider

$$1 + \sup_{p\in[0,1]} |S\mu(p) - Sv(p)| = 1 + \sup_{p\in[0,1]} \left|\int_0^1 \mathrm{F}(p, l) f(l, \mu(l))\, dl - \int_0^1 \mathrm{F}(p, l) f(l, v(l))\, dl\right|,$$

$$\le 1 + \sup_{p\in[0,1]} \int_0^1 \mathrm{F}(p, l) |f(l, \mu(l)) - f(l, v(l))|\, dl,$$

$$\le 1 + \sup_{l\in[0,1]} \{|\mu(l) - v(l)| - 1\} \sup_{p\in[0,1]} \int_0^1 \mathrm{F}(p, l)\, dl.$$

Since $\int_0^1 \mathrm{F}(p, l)\, dl = -\frac{p^2}{2} + \frac{1}{2}$ for all $p \in [0,1]$, then $\sup_{p\in[0,1]} \int_0^1 \mathrm{F}(p, l)\, dl = \frac{1}{2}$.

$$\frac{1}{\sup_{p\in[0,1]} |S\mu(p) - Sv(p)|} \ge \frac{2}{2 + \sup_{l\in[0,1]} \{|\mu(l) - v(l)| - 1\}}$$

$$\frac{1}{1 + \max_{p\in[0,1]} |S\mu(p) - Sv(p)|} \ge \frac{2}{1 + \sup_{l\in[0,1]} |\mu(l) - v(l)|}$$

$$\frac{4}{9}\left(\frac{1}{1 + \max_{p\in[0,1]} |S\mu(p) - Sv(p)|}\right) \ge \frac{4}{9}\left(\frac{2}{1 + \sup_{l\in[0,1]} |\mu(l) - v(l)|}\right).$$

Thus,

$$\psi(\xi(\rho(S\mu(p), Sv(p), t))) \ge \psi(\xi(\rho(\mu(p), v(p), t))) + \zeta$$

for all $\mu(p), v(p) \in C_f([0,1], R)$.

Therefore, the mapping S is a $(\psi - \xi)$ contraction. Thus, by theorem 3.3, S has the unique fixed point in $C_f([0,1], R)$, which in turns the solution of equation (11).

VI. CONCLUSION

Fixed point results have been given by applying the newly defined contractions, and comparative studies in the form of examples for the properties of the contractions are also presented. In the fourth section of the study, the obtained results were used to verify that a solution to an equation of motion exists. This application highlights the theoretical framework that was built throughout the study. This research can be extended to find the best proximity point for non-self-mappings as the applications of these contractions.

REFERENCES

[1] K. Atanassov, "Intuitionistic fuzzy sets," Fuzzy Sets and Systems, Vol. 20, pp. 87-96, 1986.
[2] S. Banach, "Sur les operations dans les ensembles abstraits et leur application aux equations integrals," Fundamenta Mathematicae, vol.3, pp. 133–181, 1922.
[3] B.S. Choudhury, C. Bandyopadhyay, "Stability of fixed point sets of a class of multi-valued non-linear contractions," Journal of Mathematics, p. 4 Article ID 302012, 2015.
[4] Z. Deng, "Fuzzy pseudo-metric spaces," Journal of Mathematical Analysis and Applications, Vol.86, pp. 74-95, 1982.
[5] P.N. Dutta, B.S. Choudhury, "A generalisation of contraction principle in metric spaces," Fixed Point Theory and Applications, Article ID 406368, 2008.
[6] M.A. Erceg, "Metric spaces in fuzzy set theory," Journal of Mathematical Analysis and Applications, Vol. 69,pp 205-230, 1979.
[7] A. George, P. Veeramani, "On some results in fuzzy metric spaces," Fuzzy Sets and Systems, Vol. 64, pp. 395-399, 1994.
[8] M. Grabiec, "Fixed points in fuzzy metric spaces," Fuzzy Sets and Systems, Vol. 27, pp. 385-389, 1988.
[9] V. Gregori, A. Sapena, "On fixed point theorems in fuzzy metric spaces," Fuzzy Sets and Systems, Vol. 125, pp. 245-252, 2002.
[10] S.K. Jain, G. Meena and J.K. Maitra, "Coupled fixed point results and application to integral equations," Vol. 55, no. 4, pp. 149-159, 2023.
[11] O. Kaleva, S. Seikkala, "On fuzzy metric spaces," Fuzzy Sets and Systems, Vol. 12, pp. 215-229, 1984.
[12] I. Kramosil, J. Michalek, "Fuzzy metric and statistical metric spaces," Ky-bernetica, Vol. 11, pp. 326-334, 1975.
[13] J.T. Markin, "A fixed point stability theorem for non-expansive set valued mappings,"Journal ofMathematical Analysis and Applications, Vol. 54, pp. 441-443, 1976.
[14] D. Mihet, "Fuzzy ψ-contractive mappings in non-Archimedean fuzzy metric spaces," Fuzzy Sets and Systems, Vol. 159, pp.739-744, 2008.
[15] D. Mihet, "A class of contractions in fuzzy metric spaces," Fuzzy Sets and Systems, Vol. 161, pp.1131-1137, 2010.
[16] T. Promai, A. Iampan and T, Gaketem, "Tripolar fuzzy ideals in semi groups," IAENG International Journal of Applied Mathematics, Vol. 54, no.12, pp. 2275-2782, 2024.
[17] S. Purwani and R.A. Ibrahim, "Using Simple Fixed-Point Iterations to Estimate Generalized Pareto Distribution Parameters," IAENG International Journal of Applied Mathematics, Vol.54, No. 2, pp. 194-204, 2024.
[18] S. Radenovic, F. Vetro and J. Vujakovic, "An alternative and easy

approach to fixed point results via simulation functions," Demonstratio Mathematica,Vol. 50, pp. 223-230, 2017.
[19] B.E. Rhoades, "Some theorems on weakly contractive maps," Nonlinear Analysis, TMA,Vol. 47, No.4, pp.2683–2693, 2001.
[20] B. Schweizer, A. Sklar, "Statistical metric space," Pac J Math, Vol.10, pp.314-334, 1960.
[21] M.S. Sezen, "Fixed Point Theorem for New Type Contractive Mappings,"Journal of Function Spaces, Vol. 2019, p. 6, Article ID 2153563.
[22] W. Shatanawi, "Fixed point theorem for weakly C- contractive mappings in metric space," Mathematical and Computer Modelling, Vol. 54, pp. 1944-1956, 2011.
[23] M.S. Singh, B. Chanam, "Extension and Generalization of Polynomial Inequalities,"IAENG International Journal of Applied Mathematics, Vol.54, No. 4, pp. 651-656, 2024.
[24] R. Vasuki and P. Veeramani, "Fixed point theorems and Cauchysequences in fuzzy metric spaces," Fuzzy Sets and Systems, Vol. 135,No. 3, pp. 415–417, 2003.
[25] C. Vetro, "Fixed points in weak non-Archimedean Fuzzy metric space," Fuzzy Sets and Systems,Vol.162, pp.84-90, 2013.
[26] S. Wang, "Answers to some open questions on fuzzy ψ-contractions in fuzzy metric spaces," Fuzzy Sets and Systems, Vol. 222, pp.115-119, 2013.
[27] D. Wardowski, "Fixed points of a new type of contractive mappings incomplete metric spaces," Fixed Point Theory and Applications, Vol. 94, 2012.
[28] N. Wattanasiripong, N.Lekkoksung and S.Lekkoksung, "On Tripolar fuzzy interior ideals in ordered semi groups," International Journal of Innovative Computing, Information and Control, Vol. 28, no. 4, pp. 1291-1304, 2022.
[29] L.A. Zadeh, "Fuzzy sets," Information and Control, Vol. 8,pp. 338-353, 1965.