Existence of Positive Solution for a Higher Order Fractional Integral Boundary Value Problem*

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***Abstract*—In this article, we investigate the existence of a positive solution to an integral boundary value problem for a higher order fractional differential equation**

$$
\begin{aligned}
&-D_{0+}^{\alpha-2}((p(\chi)z'(\chi))'-q(\chi)z(\chi))+f(\chi,z(\chi),z'(\chi))=0,\\
&\qquad 0<\chi<1,\\
&(p(0)z'(0))'-q(0)z(0)=[(p(0)z'(0))'-q(0)z(0)]'=\cdots\\
&=[(p(0)z'(0))'-q(0)z(0)]^{(n-4)}=0,\\
&(p(1)z'(1))'-q(1)z(1)=0,\\
&\alpha_1 z(0)-\beta_1 z'(0)=\int_0^1 z(\varrho)dX(\varrho),\\
&\alpha_1 z(1)+\beta_1 z'(1)=\int_0^1 z(\varrho)dY(\varrho),
\end{aligned}
$$

here, $D_{0+}^{\alpha-2}$ denotes the standard Riemann-Liouville fractional derivative of order $\alpha-2$, where $n-1\le\alpha\le n$ and $\alpha\ge 4$. The constants $\alpha_1,\beta_1,\alpha_2$ and β_2 are positive, and $p,q\in C([0,1],(0,\infty))$. The integrals $\int_0^1 z(\varrho)dX(\varrho)$ and $\int_0^1 z(\varrho)dY(\varrho)$ are defined in the Riemann-Stieltjes sense regarding the nondecreasing functions $X(\chi)$ and $Y(\chi)$, respectively. $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous. The first derivative of z is involved into the nonlinear term, and the existence of a positive solution is demonstrated by transforming the differential equation into an integral equation through Green's function and applying Krasnosel'skii conclusion.

***Index Terms*—Fractional differential equation, Integral boundary, Positive solution, Fixed point.**

I. Introduction

FRACTIONAL calculus is the quantitative analysis of non-integer-order integrals and derivatives, where the order can be real numbers, complex numbers, or even functions of variables. Compared to classical calculus, fractional differential equations exhibit broader applicability and richer physical interpretations. With advancements in science and technology, fractional differential equations have become essential in modeling diverse phenomena across multiple disciplines, including physics, aerodynamics, viscoelasticity, electromagnetism, control theory, chemistry, biology, and economics. The study of positive solutions to such equations is particularly significant, as it enhances understanding of their physical meaning and practical utility.

For a thorough review of fractional integral equations with boundary constraints, readers may consult [1], [2], [3], [10], [17], [18], [19]. Specifically, Asaduzzaman [4] established the existence of positive solutions for Caputo-type fractional boundary value problems employing fixed-point theorems. For Sturm-Liouville problems, refer to [5], [6]. There have been some papers concerning fixed point theory applied to existence proofs for fractional differential equations, see [7], [8], [9].

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In [10], Xinan Hao examines the subsequent Riemann-Liouville fractional DE with integral boundary restrictions and a parameter:

$$
\begin{aligned}
&-D_{0+}^{\eta-2}\left(u''(t)\right)+\lambda f\left(t,u(t)\right)=0,\quad t\in(0,1),\\
&u''(0)=u'''(0)=\cdots=u^{(n-2)}(0)=0,\\
&D_{0+}^{\kappa-2}\left(u''(t)\right)|_{t=1}=0,\\
&\alpha u(0)-\beta u'(0)=\int_0^1 u(s)\mathrm{d}A(s),\\
&\gamma u(1)+\delta u'(1)=\int_0^1 u(s)\mathrm{d}B(s),
\end{aligned}
$$

by employing the Guo-Krasnoselskii fixed point theorem of cone, the positive solutions exist in different intervals of parameters under different nonlinear conditions is determined.

In [11], PAUL W. ELOE studied the following set of analytical solutions of two-point BVPs for linear fractional DE, he found that the Green's function of fractional derivative of orders α can be expressed as the convolution of Green's functions of order 2 and $\alpha-2$.

In [12], Xiping Liu examines the fixed point approaches to positive solutions in singular Sturm-Liouville boundary value problems with integral-type boundary restrictions

$$
\begin{aligned}
&-(p(u)z'(u))'+q(u)z(u)=f(u,z(u)),0<u<1,\\
&\phi_1 z(0)-\psi_1 z'(0)=\int_0^1 z(\tau)\mathrm{d}A(\tau),\\
&\phi_2 z(1)+\psi_2 z'(1)=\int_0^1 z(\tau)\mathrm{d}B(\tau).
\end{aligned}
$$

The parameters $\phi_1,\phi_2,\psi_1,\psi_2$ are all non-negative real numbers such that $\phi_1\phi_2+\phi_1\psi_2+\psi_1\phi_2$ is positive. $A(t),B(t)$ are nondecreasing on $[0,1]$, let u be integrable with respect to A and B in the Riemann-Stieltjes sense, where the integrals are given by: $\int_0^1 z(\tau)\mathrm{d}A(\tau)$ and $\int_0^1 z(\tau)\mathrm{d}B(\tau)$. $p\in C^1([0,1],(0,+\infty)),q\in C([0,1],[0,+\infty))$, $f\in C((0,1)\times(0,+\infty)),[0,+\infty))$ may be singular at $u=0,u=1$ and $z=0$.

Inspired by the above three papers, we aim to explore the fixed point approaches to positive solutions in fractional boundary value problems with integral-type boundary restrictions. This study combins the two equations to achieve this goal.

Currently, significant progress has been developed in the research of integral boundary value problems. Furthermore, for research on the existence of positive solutions, fixed point theorems serve as crucial tools. Depending on the characteristics of the equations, appropriate theorems are selected to construct function spaces and operators, and through complex derivations, existence conditions are obtained.

In the future, research should focus on these directions for breakthroughs and innovations, propelling the study of integral boundary value problems of fractional differential equations to new heights and providing stronger theoretical and technical support for the development of numerous disciplines.

In this study, we take into account the positive solution of the subsequent boundary value problem for a differential equation related to the Riemann-Liouville fractional order derivative:

$$\begin{cases} -D_{0^+}^{\alpha-2}((p(\chi)z'(\chi))' - q(\chi)z(\chi)) + f(\chi, z(\chi), z'(\chi)) = 0, \\ \qquad 0 < \chi < 1, \\ (p(0)z'(0))' - q(0)z(0) = [(p(0)z'(0))' - q(0)z(0)]' = \cdots \\ = [(p(0)z'(0))' - q(0)z(0)]^{(n-4)} = 0, \\ (p(1)z'(1))' - q(1)z(1) = 0, \\ \alpha_1 z(0) - \beta_1 z'(0) = \int_0^1 z(\varrho)dX(\varrho), \\ \alpha_1 z(1) + \beta_1 z'(1) = \int_0^1 z(\varrho)dY(\varrho). \end{cases} \tag{1}$$

where $D_{0^+}^{\alpha-2}$ is the standard Riemann-Liouville fractional derivative of orders $\alpha - 2$. $n-1 \le \alpha \le n$, $\alpha \ge 4$, $\alpha_1, \beta_1, \alpha_2, \beta_2 > 0$, $p, q \in C([0,1], (0,\infty))$, $\int_0^1 z(\varrho)dX(\varrho)$ and $\int_0^1 z(\varrho)dY(\varrho)$ denote the integrals of z in the Riemann-Stieltjes sense regarding X and Y, respectively. $X(\chi)$ and $Y(\chi)$ are nondecreasing on [0,1], $f : [0,1]\times[0,\infty) \to [0,\infty)$ is continuous.

II. The preliminary lemmas

In this part, we are going to provide some definitions and lemmas.

Definition 2.1 ([13], [14]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $h : (0,+\infty) \to R$ is given by

$$I_{0^+}^{\alpha} h(i) = \frac{1}{\Gamma(\alpha)} \int_0^i (i-k)^{\alpha-1} h(k)\, \mathrm{d}k, \quad i > 0,$$

on condition that the right-hand side is pointwise defined on $(0,+\infty)$.

Definition 2.2 ([13], [14]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $h : (0,+\infty) \to R$ is given by

$$D_{0^+}^{\alpha} h(i) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{di}\right)^n \int_0^i \frac{h(k)}{(i-k)^{\alpha-n+1}} dk,$$

where n is the smallest integer which is no less than α, on condition that the right-hand side is pointwise defined on $(0,+\infty)$.

Let $-(p(\chi)z'(\chi))' + q(\chi)z(\chi) = \Phi(\chi)$, then the BVP

$$\begin{cases} -D_{0^+}^{\alpha-2}((p(\chi)z'(\chi))' - q(\chi)z(\chi)) + f(\chi, z(\chi), z'(\chi)) = 0 \\ \qquad 0 < \chi < 1, \\ (p(0)z'(0))' - q(0)z(0) = [(p(0)z'(0))' - q(0)z(0)]' = \\ = [(p(0)z'(0))' - q(0)z(0)]^{(\alpha-4)} = 0, \\ (p(1)z'(1))' - q(1)z(1) = 0, \end{cases}$$

becomes

$$\begin{cases} D_{0^+}^{\alpha-2}\Phi(\chi) + f(\chi, z(\chi), z'(\chi)) = 0, \quad \chi \in (0,1), \\ \Phi(0) = \Phi'(0) = \cdots = \Phi^{(n-4)}(0) = 0, \quad \Phi(1) = 0. \end{cases} \tag{2}$$

The following result is in our possession.

Lemma 2.3 ([15]) The BVP (2) has a unique solution

$$\Phi(\chi) = \int_0^1 R(\chi,\varrho) f(\varrho, z(\varrho), z'(\varrho)) d\varrho, \quad \chi \in [0,1],$$

where

$$R(\chi,\varrho) = \begin{cases} \frac{(1-\varrho)^{\alpha-3}\chi^{\alpha-3} - (\chi-\varrho)^{\alpha-3}}{\Gamma(\alpha-2)}, & 0 \le \varrho \le \chi < 1, \\ \frac{(1-\varrho)^{\alpha-3}\chi^{\alpha-3}}{\Gamma(\alpha-2)}, & 0 < \chi \le \varrho \le 1. \end{cases}$$

By direction computations we obtain the properties of $R(\chi,\varrho)$.

Lemma 2.4

$$0 \le R(\chi,\varrho) \le R(\varrho,\varrho) = \frac{(1-\varrho)^{\alpha-3}\varrho^{\alpha-3}}{\Gamma(\alpha-2)}, \quad \chi, \varrho \in [0,1].$$

Now we take into account the following integral BVP:

$$\begin{cases} -(p(\chi)z'(\chi))' + q(\chi)z(\chi) = \Phi(\chi), \\ \alpha_1 z(0) - \beta_1 z'(0) = \int_0^1 z(\upsilon)d\lambda(\upsilon), \\ \alpha_2 z(1) + \beta_2 z'(1) = \int_0^1 z(\upsilon)d\epsilon(\upsilon). \end{cases} \tag{3}$$

Lemma 2.5 ([12])Suppose μ and ν be the solutions of the linear problems

$$-(p(\chi)\mu'(\chi))' + q(\chi)\mu(\chi) = 0, 0 < \chi < 1,$$
$$\mu(0) = \beta_1, \mu'(0) = \alpha_1,$$

and

$$-(p(\chi)\nu'(\chi))' + q(\chi)\nu(\chi) = 0, 0 < \chi < 1,$$
$$\nu(1) = \beta_2, \nu'(1) = -\alpha_2,$$

respectively. Then
(i) μ is strictly increasing on the interval from 0 to 1,and $\mu(\chi)$ is positive on $(0,1]$,
(ii) ν is strictly decreasing on the interval from 0 to 1, and $\nu(\chi)$ is positive on $[0,1)$,
(iii) $\sigma = p(\chi)(\mu'(\chi)\nu(\chi) - \mu(\chi)\nu'(\chi))$ is a positive constant, μ and ν are linearly independent. Let

$$G(t,u) = \frac{1}{\sigma}\begin{cases} \mu(\chi)\nu(\varrho), & 0 \le \chi \le \varrho \le 1, \\ \mu(\varrho)\nu(\chi), & 0 \le \varrho \le \chi \le 1. \end{cases}$$

Lemma 2.6 ([12])For any $\Phi \in L[0,1]$, z is the solution of the BVP

$$-(p(\chi)z'(\chi))' + q(\chi)z(\chi) = \Phi(\chi), 0 < \chi < 1,$$
$$\alpha_1 z(0) - \beta_1 z'(0) = 0, \alpha_2 z(1) + \beta_2 z'(1) = 0,$$

if and only if z can be expressed by

$$z(\chi) = \int_0^1 G(\chi,\varrho)\Phi(\varrho)d\varrho. \tag{4}$$

Let $M(\chi) = \frac{\nu(\chi)}{\alpha_1\nu(0) - \beta_1\nu'(0)} = \frac{p(0)\nu(\chi)}{\sigma}$, $N(\chi) = \frac{\mu(\chi)}{\alpha_2\mu(1) + \beta_2\mu'(1)} = \frac{p(1)\mu(\chi)}{\sigma}$. Then the $M(\chi)$ and $N(\chi)$ are the solution of

$$-(p(\chi)M'(\chi))' + q(\chi)M(\chi) = 0, 0 < \chi < 1,$$
$$\alpha_1 M(0) - \beta_1 M'(0) = 1, \alpha_2 M(1) + \beta_2 M'(1) = 0,$$

and

$$-(p(\chi)N'(\chi))' + q(\chi)N(\chi) = 0, 0 < \chi < 1,$$
$$\alpha_1 N(0) - \beta_1 N'(0) = 1, \alpha_2 N(1) + \beta_2 N'(1) = 0.$$

Denote

$$k_1 = 1 - \int_0^1 M(\upsilon)d\lambda(\upsilon), \quad k_2 = 1 - \int_0^1 N(\upsilon)d\epsilon(\upsilon),$$
$$k_3 = \int_0^1 M(\upsilon)d\epsilon(\upsilon), \quad k_4 = \int_0^1 N(\upsilon)d\lambda(\upsilon),$$
$$A(\varrho) = \frac{k_2\int_0^1 G(\upsilon,\varrho)d\lambda(\upsilon) + k_4\int_0^1 G(\upsilon,\varrho)d\epsilon(\upsilon)}{k_1k_2 - k_3k_4},$$

$$B(\varrho)=\frac{k_1\int_0^1 G(\upsilon,\varrho)d\epsilon(\upsilon)+k_3\int_0^1 G(\upsilon,\varrho)d\lambda(\upsilon)}{k_1k_2-k_3k_4}.$$

The following hypothesis will be utilized by us:

($\mathbf{H1}$) $k_1>0$, $k_1k_2-k_3k_4>0$.

Obviously, k_3, $k_4\geq 0$. And $k_2>0$ if (H1) holds.

Lemma 2.7 ([12])Suppose ($\mathbf{H1}$) holds. For any $\Phi\in L[0,1]$, z is the solution of the nonlinear BVP(3) precisely when z can be expressed by

$$z(\chi)=\int_0^1\left(G(\chi,\varrho)+M(\chi)A(\varrho)+N(\chi)B(\varrho)\right)\Phi(\varrho)d\varrho. \quad (5)$$

Therefore, the solution of BVP(1) can be expressed by

$$\begin{aligned}z(\chi)=&\int_0^1(G(\chi,\varrho)+M(\chi)A(\varrho)+N(\chi)B(\varrho))\\&\int_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho.\end{aligned} \quad (6)$$

Let $W=\max_{t\in[0,1]}\{\|\mu\|,\|\nu\|\}$. We denote

$$\gamma_0=\frac{1}{W}\min\{\beta_1,\beta_2\}.$$

Lemma 2.8 ([12])(1) $G(\chi,\varrho)=G(\varrho,\chi)\leq G(\varrho,\varrho)\leq\frac{M^2}{\sigma}$ for all $(\chi,\varrho)\in[0,1]\times[0,1]$,

(2) $0<\gamma_0G(\varrho,\varrho)\leq G(\chi,\varrho)$, for $\chi,\varrho\in[0,1]$.

Lemma 2.9 ([12])(1) $B(\varrho)$ and $A(\varrho)$ are bounded and nonnegative on the interval from 0 to 1,

(2) $M(\chi)$ is strictly decreasing on the interval from 0 to 1, and $M(\chi)$ is positive on the interval from 0 to 1,

(3) $N(\chi)$ is strictly increasing on the interval from 0 to 1, and $N(\chi)>0$ on the interval from 0 to 1.

Write $c(\chi)=\min\left\{\gamma_0,\frac{\mu(\chi)}{\mu(1)},\frac{\nu(\chi)}{\nu(0)}\right\}$ and $\Theta(\varrho)=G(\varrho,\varrho)+M(0)A(\varrho)+N(1)B(\varrho)$,the following Lemma can be easily obtained by us.

Lemma 2.10 ([12])Let us suppose that condition ($\mathbf{H1}$) is fulfilled. Then for $\chi,\varrho\in[0,1]$,

$$c(\chi)\Theta(\varrho)\leq G(\chi,\varrho)+M(\chi)A(\varrho)+N(\chi)B(\varrho)\leq\Theta(\varrho).$$

Theorem 2.11 ([16]) Let C be a Banach space and $I\subset C$ be a cone in C. Assume Λ_1 and Λ_2 are open subsets of C with $0\in\Lambda_1\subset\overline{\Lambda}_1\subset\Lambda_2$, and let $J:I\cap(\overline{\Lambda}_2\setminus\Lambda_1)\to I$ be completely continuous , if it is satisfied:

•$\|Js\|\leq\|s\|$, for all $s\in I\cap\partial\Lambda_1$, $\|Js\|\geq\|s\|$, for all $s\in I\cap\partial\Lambda_2$, or

•$\|Js\|\leq\|s\|$, for all $s\in I\cap\partial\Lambda_2$, $\|Js\|\geq\|s\|$, for all $s\in I\cap\partial\Lambda_1$.

Then, J has a fixed point at least in $I\cap(\overline{\Lambda}_2\setminus\Lambda_1)$.

III. Main results

We consider the Banach space $C=C^1[0,1]$. Let C be endowed with the norm

$$\|z\|=\max\left\{\max_{\chi\in[0,1]}|z(\chi)|,\max_{\chi\in[0,1]}|z'(\chi)|\right\},$$

and define a cone $I\subset C$ by

$$I=\{z\in C:z(\chi)\geq 0,\min_{0\leq\chi\leq 1}z(\chi)\geq\xi\|z\|\},$$

where

$$\xi=\min\{\delta_0,\delta_1\},\quad\delta_0=\min\{\gamma_0,\frac{\beta_1}{\mu(1)},\frac{\beta_2}{\nu(0)}\},$$

$$\delta_1=\min\{\frac{\beta_1}{m_2},\frac{\beta_2}{n_2}\}.$$

Because of Lemma 2.5, we know $\mu''(\chi)$ and $\nu''(\chi)$ exsit, then $\mu'(\chi)$ and $\nu'(\chi)$ are continuous and have maximum and minimum value. We suppose $m_1,m_2,n_1,n_2>0$ are constants, such that $\mu'(\chi)$ and $\nu'(\chi)$ are always satisfied $m_1\leq\mu'(\chi)\leq m_2$, $n_1\leq|\nu'(\chi)|\leq n_2$, $0\leq\chi\leq 1$ and $m_2>\beta_1$, $n_2>\beta_2$.

Now, we define an operator J maps I to C as follows:

$$\begin{aligned}Jz(\chi)&=\int_0^1(G(\chi,\varrho)+M(\chi)A(\varrho)+N(\chi)B(\varrho))\\&\int_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho,\end{aligned} \quad (7)$$

therefore, the BVP(1) has a solution $z=z(\chi)$ precisely when z is a fixed point of J.

For convenient we denote

$$K(\chi,\varrho)=\int_0^1\left(G(\chi,\upsilon)+M(\chi)A(\upsilon)+N(\chi)B(\upsilon)\right)R(\upsilon,\varrho)d\upsilon,$$

$$\overline{K}(\varrho)=\int_0^1\Theta(\upsilon)R(\upsilon,\varrho)d\upsilon,$$

then the operator J defined by (7) becomes

$$Jz(\chi)=\int_0^1 K(\chi,\varrho)f(\varrho,z(\varrho),z'(\varrho))d\varrho.$$

As $\delta_0=\min\{\gamma_0,\frac{\beta_1}{\mu(1)},\frac{\beta_2}{\nu(0)}\}$, then we have $0<\delta_0\leq c(\chi)$. We can effortlessly get the following Lemma 3.1 based on Lemma 2.10.

Lemma 3.1 Suppose ($\mathbf{H1}$) holds. Then

$$\delta_0\overline{K}(\varrho)\leq K(\chi,\varrho)\leq\overline{K}(\varrho).$$

Lemma 3.2 The operator J maps I to I is continuous and compact.

Proof:

$$\begin{aligned}Jz\left(\chi\right)&=\int_0^1 K\left(\chi,\varrho\right)f\left(\varrho,z(\varrho),z'(\varrho)\right)d\varrho\\&\geq\delta_0\int_0^1\overline{K}(\chi,\varrho)f(\varrho,z(\varrho),z'(\varrho))d\varrho\\&\geq\delta_0\max_{0\leq\chi\leq 1}\left|Jz(\chi)\right|.\end{aligned} \quad (8)$$

$$\begin{aligned}&|(Jz)'(\chi)|\\&\leq\int_0^\chi\left(\frac{\mu(\varrho)}{\sigma}|\nu'(\chi)|+A(\varrho)\frac{p(0)}{\sigma}|\nu'(\chi)|+B(\varrho)\frac{p(1)}{\sigma}\mu'(\chi)\right)\\&\int_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho\\&+\int_\chi^1\left(\frac{\nu(\varrho)}{\sigma}\mu'(\chi)+A(\varrho)\frac{p(0)}{\sigma}|\nu'(\chi)|+B(\varrho)\frac{p(1)}{\sigma}\mu'(\chi)\right)\\&\int_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho\\&=\int_0^\chi\left(\frac{\mu(\varrho)+p(0)A(\varrho)}{\sigma}|\nu'(\chi)|+\frac{p(1)B(\varrho)}{\sigma}\mu'(\chi)\right)\\&\int_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho\\&+\int_\chi^1\left(\frac{\nu(\varrho)+p(1)B(\varrho)}{\sigma}\mu'(\chi)+\frac{p(0)A(\varrho)}{\sigma}|\nu'(\chi)|\right)\\&\int_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho.\end{aligned}$$

So we have

$$\begin{aligned}&Jz(\chi)\\&=\int_0^1(G(\chi,\varrho)+M(\chi)A(\varrho)+N(\chi)B(\varrho))\\&\int_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho\\&=\int_0^\chi\left(\tfrac{\mu(\varrho)+p(0)A(\varrho)}{\sigma}\nu(\chi)+\tfrac{p(1)B(\varrho)}{\sigma}\mu(\chi)\right)\\&\int_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho\\&+\int_\chi^1\left(\tfrac{\nu(\varrho)+p(1)B(\varrho)}{\sigma}\mu(\chi)+\tfrac{p(0)A(\varrho)}{\sigma}\nu(\chi)\right)\\&\int_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho\\&\geq\int_0^\chi\left(\tfrac{\mu(\varrho)+p(0)A(\varrho)}{\sigma}\delta_1|\nu'(\chi)|+\tfrac{p(1)B(\varrho)}{\sigma}\delta_1\mu'(\chi)\right)\\&\in\chi_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho\\&+\int_\chi^1\left(\tfrac{\nu(\varrho)+p(1)B(\varrho)}{\sigma}\delta_1\mu'(\chi)+\tfrac{p(0)A(\varrho)}{\sigma}\delta_1|\nu'(\chi)|\right)\\&\int_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho\\&\geq\delta_1|Jz'(\chi)|.\end{aligned}$$

Then we have

$$Jz(\chi)\geq\delta_1\max_{0\leq\chi\leq1}|Jz'(\chi)|. \tag{9}$$

Given (8)(9) and $\xi=\min\{\delta_0,\delta_1\}$ we have

$$Jz(\chi)\geq\xi\|Jz(\chi)\|,\quad\text{then}\quad\min_{0\leq\chi\leq1}Jz(\chi)\geq\xi\|Jz\|.$$

So that J maps I to I.

Clearly, J is continuous. Now, consider $\Lambda\subset I$ as a bounded set. It's evident that $J\Lambda$ is both bounded and equicontinuous. By applying the Arzela-Ascoli theorem, we conclude that $J\Lambda$ is relatively compact. Consequently, J is compact. In summary, we can assert that $J:I\to I$ is a completely continuous operator. ■

Now, for the purpose of convenience, let's introduce the notations below:

$$f_1=\lim_{|z|+|z'|\to0}\inf\min_{\chi\in[0,1]}\frac{f(\chi,z,z')}{|z|+|z'|},$$

$$f_2=\lim_{|z|+|z'|\to\infty}\inf\min_{\chi\in[0,1]}\frac{f(\chi,z,z')}{|z|+|z'|},$$

$$f_3=\lim_{|z|+|z'|\to0}\sup\max_{\chi\in[0,1]}\frac{f(\chi,z,z')}{|z|+|z'|},$$

$$f_4=\lim_{|z|+|z'|\to\infty}\sup\max_{\chi\in[0,1]}\frac{f(\chi,z,z')}{|z|+|z'|}.$$

The following theorems constitute the principal results of this paper.

Theorem 3.1 If hypothesis (**H1**) holds true and $f_2=\lim_{|z|+|z'|\to\infty}\inf\min_{\chi\in[0,1]}\frac{f(\chi,z,z')}{|z|+|z'|}\in[B,+\infty)$, $f_3=\lim_{|z|+|z'|\to0}\sup\max_{\chi\in[0,1]}\frac{f(\chi,z,z')}{|z|+|z'|}\in(0,A]$. Then BVP(1)has a nonnegaitve solution, where

$$A=\min\Big\{\frac{\sigma}{\int_0^1 2R(\varrho,\varrho)(\theta(\varrho)\nu(0)+\eta(\varrho)\mu(1))d\varrho},\\ \frac{\sigma}{\int_0^1 2R(\varrho,\varrho)(\theta(\varrho)n_2+\eta(\varrho)m_2)d\varrho}\Big\},$$

$$B=\frac{1}{\delta_0\xi\int_0^1\overline{K}(\varrho)d\varrho},$$

$$\theta(\varrho)=\mu(\varrho)+p(0)A(\varrho),\qquad\eta(\varrho)=\nu(\varrho)+p(1)B(\varrho).$$

Proof:

On the one side, $f_3\in(0,A]$, then there is a $r_1>0$, such that when $|z|+|z'|\leq2r_1$,we have

$$f(\chi,z,z')\leq A(|z|+|z'|).$$

Because$\|z\|=\max\left\{\max_{\chi\in[0,1]}|z(\chi)|,\max_{\chi\in[0,1]}|z'(\chi)|\right\}$, so $|z(\chi)|+|z'(\chi)|\leq2\|z\|$.Define an open subset of C, $\Lambda_1=\{z\in I:\|z(\chi)\|<r_1\}$, then when $z\in I\cap\partial\Lambda_1$, we have $z(\chi)\leq\|z\|=r_1$, $z'(\chi)\leq\|z\|=r_1$, that is $|z(\chi)|+|z'(\chi)|\leq2r_1$.Sequentially,

$$\begin{aligned}&\max_{\chi\in[0,1]}|Jz(\chi)|\\&=\int_0^\chi\left(\tfrac{\mu(\varrho)+p(0)A(\varrho)}{\sigma}\nu(\chi)+\tfrac{p(1)B(\varrho)}{\sigma}\mu(\chi)\right)\\&\in\chi_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho\\&+\int_\chi^1\left(\tfrac{\nu(\varrho)+p(1)B(\varrho)}{\sigma}\mu(\chi)+\tfrac{p(0)A(\varrho)}{\sigma}\nu(\chi)\right)\\&\int_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho\\&\leq\int_0^\chi\left(\tfrac{\mu(\varrho)+p(0)A(\varrho)}{\sigma}\nu(0)+\tfrac{p(1)B(\varrho)}{\sigma}\mu(1)\right)\\&\int_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho\\&+\int_\chi^1\left(\tfrac{\nu(\varrho)+p(1)B(\varrho)}{\sigma}\mu(1)+\tfrac{p(0)A(\varrho)}{\sigma}\nu(0)\right)\\&\int_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho\\&\leq\int_0^1\tfrac{[\mu(\varrho)+p(0)A(\varrho)]\nu(0)+[\nu(\varrho)+p(1)B(\varrho)]\mu(1)}{\sigma}\\&\int_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho\\&\leq\int_0^1\tfrac{[\mu(\varrho)+p(0)A(\varrho)]\nu(0)+[\nu(\varrho)+p(1)B(\varrho)]\mu(1)}{\sigma}\\&A(|z|+|z'|)\int_0^1 R(\varrho,\upsilon)d\upsilon d\varrho\\&\leq\int_0^1\tfrac{[\mu(\varrho)+p(0)A(\varrho)]\nu(0)+[\nu(\varrho)+p(1)B(\varrho)]\mu(1)}{\sigma}\\&2A\|z\|R(\varrho,\varrho)d\varrho\leq\|z\|,\end{aligned}$$

$$\begin{aligned}&\max_{\chi\in[0,1]}|Jz'(\chi)|\\&\leq\int_0^\chi\left(\tfrac{\mu(\varrho)+p(0)A(\varrho)}{\sigma}|\nu'(\chi)|+\tfrac{p(1)B(\varrho)}{\sigma}\mu'(\chi)\right)\\&\int_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho\\&+\int_\chi^1\left(\tfrac{\nu(\chi)+p(1)B(\varrho)}{\sigma}\mu'(\chi)+\tfrac{p(0)A(\varrho)}{\sigma}|\nu'(\chi)|\right)\\&\int_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho\\&\leq\int_0^\chi\left(\tfrac{\mu(\varrho)+p(0)A(\varrho)}{\sigma}n_2+\tfrac{p(1)B(\varrho)}{\sigma}m_2\right)\\&\int_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho\\&+\int_\chi^1\left(\tfrac{\nu(\chi)+p(1)B(\varrho)}{\sigma}m_2+\tfrac{p(0)A(\varrho)}{\sigma}n_2\right)\\&\int_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho\\&\leq\int_0^1\tfrac{[\mu(\varrho)+p(0)A(\varrho)]n_2+[\nu(\varrho)+p(1)B(\varrho)]m_2}{\sigma}\\&\int_0^1 R(\varrho,\upsilon)f(\upsilon,z(\upsilon),z'(\upsilon))d\upsilon d\varrho\\&\leq\int_0^1\tfrac{[\mu(\varrho)+p(0)A(\varrho)]n_2+[\nu(\varrho)+p(1)B(\varrho)]m_2}{\sigma}\\&A(|z|+|z'|)\int_0^1 R(\varrho,\varrho)d\upsilon d\varrho\\&\leq\int_0^1\tfrac{[\mu(\varrho)+p(0)A(\varrho)]n_2+[\nu(\varrho)+p(1)B(\varrho)]m_2}{\sigma}\\&2A\|z\|R(\varrho,\varrho)d\varrho\leq\|z\|.\end{aligned}$$

Thus,

$$\|Jz\|\leq\|z\|,\quad\forall z\in I\cap\partial\Lambda_1. \tag{10}$$

On the other side, by condition $f_2\in[B,+\infty)$, then there is a $r_3>0$, such that when $|z|+|z'|\geq r_3$,we have

$$f(\chi,z,z')\geq B(|z|+|z'|).$$

Let $r_2=\max\left\{\frac{r_3}{\xi},r_1\right\}$, $\Lambda_2=\{z\in C,\|z\|<r_2\}$, when $z\in I\cap\partial\Lambda_2$, we get

$$\xi\|z\|\leq|z|\leq\|z\|,\quad|z|+|z'|\geq\xi\|z\|=\xi r_2.$$

Hence

$$\begin{aligned} Jz(\chi) &= \int_0^1 K(\chi,\varrho) f(\varrho, z(\varrho), z'(\varrho))\,d\varrho \\ &\geq \delta_0 \int_0^1 \overline{K}(\varrho) f(\varrho, z(\varrho), z'(\varrho))d\varrho \\ &\geq \delta_0 B(|z| + |z'|) \int_0^1 \overline{K}(\varrho)d\varrho \\ &\geq \delta_0 B\xi \|z\| \int_0^1 \overline{K}(\varrho)d\varrho \\ &\geq \|z\|. \end{aligned}$$

$$\begin{aligned} \|Jz\| &= \max\left\{\max_{\chi\in[0,1]} |Jz(\chi)|, \max_{\chi\in[0,1]} |Jz'(\chi)|\right\} \\ &\geq \max_{\chi\in[0,1]} |Jz(\chi)| \geq \|z\|. \end{aligned}$$

Thus,

$$\|Jz\| \geq \|z\|, \quad \forall z \in I \cap \partial\Lambda_2. \tag{11}$$

Applying Theorem 2.11 along with the inequalities (10) and (11), we can conclude that the operator J possesses a fixed point, denoted as z^*, belonging to $I \cap \overline{\Lambda}_2 \backslash \Lambda_1$, satisfying $r_1 < \|z^*\| < r_2$. Notably, it is evident that z^* constitutes a nonnegative solution to the BVP(1). ■

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