

A Class of Equations of the Basic Coupling and Norm

Zhifeng Zhu, Junchao Zhou

Abstract—In this paper, we focus on a class of equations of the basic coupling and norm. By means of the coupling method, we introduce an equation of the basic coupling and total variation norm. Further, by improving the equation of the basic coupling, an equation of the basic coupling and f -norm is derived. This class of equations establishes a link between the basic coupling and norm, which has wide applications in the derivation and proof of inequalities and in judging the random order relationship between two probabilities.

Index Terms—Markov process, coupling, the basic coupling, f -norm.

I. INTRODUCTION

RECENTLY the coupling method is widely used in stochastic process. Investigated by many scholars, this method has become an important tool to study the stability of the Markov process [1-6]. The coupling method is a powerful tool for analyzing the properties of two or more related Markov chains by constructing their synchronous evolution. Transforming complex probability problems into path level comparisons to simplify analysis. The core value of the coupling method lies in constructing a path of "synchronous evolution" to transform abstract probability distribution relationships into concrete events (such as encounters, order relationships), thereby simplifying the analysis of key properties such as convergence, traversal, and mixing time of Markov processes. It is not only an important tool in theoretical research (such as proving convergence), but also provides quantifiable boundaries and estimation methods for practical problems (such as algorithm design, system optimization).

Norm and f -norm, as fundamental tools in mathematics and engineering, have a wide range of applications covering measurement, optimization, analysis, and other scenarios, and has attracted many scholars to study them. The convergence of Markov processes essentially solves the problem of whether the long-term behavior of a system is predictable and stable, and its applications run through multiple levels such as predictive analysis, algorithm design, risk assessment, and model simplification. Whether it is natural sciences, engineering technology, or social sciences, convergence is a key tool for judging system laws and optimizing decisions as long as it involves long-term evolution analysis of dynamic systems [7-10].

Manuscript received January 22, 2025; revised September 9, 2025. This work was supported by the National Natural Science Foundation of China (12301667) and the Natural Science Foundation of Hubei Province (2021CFB275).

Zhifeng Zhu is an associate professor in the School of Mathematics and Statistics, Hubei Engineering University, Xiaogan, 432000, China (e-mail: 376574200@qq.com).

Junchao Zhou is an associate professor in the School of Mathematics and Statistics, Hubei Engineering University, Xiaogan, 432000, China (corresponding author to provide e-mail: zhoujunchao@hbeu.edu.cn).

In order to study the convergence rate of ergodicity, Nummelin and Tweedie focused on geometric ergodicity (also known as exponential ergodicity) and f -geometric ergodicity (also known as f -exponential ergodicity) in the sense of total variation norm of general state space. Geometric ergodicity is a strong ergodicity of Markov processes, which refers to the exponential decay of the convergence speed to a steady distribution (i.e., the convergence speed is controlled by a certain geometric series). The core value of geometric ergodicity lies in its strong guarantee of convergence speed (exponential decay), which makes it a key condition in fields such as statistical inference, algorithm design, and stochastic process analysis. It not only ensures the stability and predictability of the long-term behavior of Markov processes, but also provides a solid theoretical foundation for quantitative analysis and practical applications, especially in scenarios that require efficient convergence and accurate estimation (such as MCMC, reinforcement learning) [11-20]. In [21], Meyn and Tweedie focused on the more general form of f -geometric ergodicity. Using the coupling method, they investigated the f -geometric ergodicity of discrete-time Markov chains. The ergodicity of Markov processes has important and wide applications involving Markov processes derived from queueing theory and birth and death processes in q processes.

Zhu [4] studied the f -geometric ergodicity of Markov chains in discrete-time general state space. Zhang [2, 3, 23] proposed the concepts of the φ -optimal coupling and the optimal coupling operator, where φ is a nonnegative lower semicontinuous function. Zhang also proved the existence of the optimal measurable coupling of transition probability and the existence of optimal coupling operator of jump process. Both Chen [22] and Zhang [24] used the coupling method to investigate the ergodicity of homogeneous Markov processes. Zhu [5] applied the coupling method to study the ergodicity of Markov processes.

In this article, we focus on the basic coupling and f -norm and obtain an equation of the basic coupling and total variation norm. Also we improve the equation of the basic coupling, and thus an equation of the basic coupling and f -norm is derived.

The remainder of this paper is organized as follows. Section 2 introduces some notations and necessary background. Section 3 obtains an equation of the basic coupling and total variation norm. Section 4 derives an equation of the basic coupling and f -norm. Section 5 presents the applications of our results. Section 6 concludes this study.

Let the state space E is a polish space (i.e., a complete and separable metric space), and $\mathcal{E}$ is a σ algebra generated by a countable subset of E .

II. PRELIMINARIES

Definition 2.1: Let μ be a sign measure on $\mathcal{E}$, and g and f be measurable functions on $\mathcal{E}$. The total variation norm of μ is defined as

$$\|\mu\| := \sup\{|v(g)| : |g| \leq 1\}.$$

Define the f -norm of μ as

$$\|\mu\|_f := \sup\{|v(g)| : |g| \leq f\}, f \geq 1.$$

Remark 1: μ can be a measure, a function, or a point.

Remark 2: When $f \equiv 1$, the f -norm becomes the total variation norm.

Definition 2.2: Given two probability measures μ_1 and μ_2 on $\mathcal{E}$, and a probability measure $\tilde{\mu}$ on $\mathcal{E} \times \mathcal{E}$, $\tilde{\mu}$ is called the coupling of μ_1 and μ_2 if it satisfies the marginality

$$\tilde{\mu}(A_1 \times E) = \mu_1(A_1) \text{ for all } A_1 \in \mathcal{E}$$

and

$$\tilde{\mu}(E \times A_2) = \mu_2(A_2) \text{ for all } A_2 \in \mathcal{E}.$$

The set of all coupling of μ_1 and μ_2 is denoted by $K(\mu_1, \mu_2)$.

Definition 2.3: For two probability measures μ_1 and μ_2 in the Polish space $(X, \rho, \mathcal{E})$, we call

$$W_\varphi(\mu_1, \mu_2) = \inf\left\{\int \mu(dx, dy)\varphi(x, y) : \mu \in K(\mu_1, \mu_2)\right\}$$

the minimum probability distance of μ_1 and μ_2 . If there exists $\tilde{\mu} \in K(\mu_1, \mu_2)$ satisfying

$$W_\varphi(\mu_1, \mu_2) = \int \tilde{\mu}(dx, dy)\varphi(x, y),$$

then $\tilde{\mu}$ is called the φ -optimal coupling of μ_1 and μ_2 .

It is known that the φ -optimal coupling always exists [1].

Lemma 2.1: Let μ_1 and μ_2 be probability measures on $\mathcal{E}$, and set $\mu' = \mu_1 + \mu_2$. Define the derivatives

$$g_1 = \frac{d\mu_1}{d\mu'}, \quad g_2 = \frac{d\mu_2}{d\mu'},$$

let

$$g = \min\{g_1, g_2\}, \quad \gamma = \int g d\mu',$$

and introduce the signed measures

$$v_1(A) = \int_A (g_1 - g) d\mu', \quad A \in \mathcal{E},$$

$$v_2(A) = \int_A (g_2 - g) d\mu', \quad A \in \mathcal{E}.$$

Finally, define

$$Q(B) = \int_{B \cap \{(x, y) : x=y\}} g(x) \mu' d(x), \quad B \in \mathcal{E} \times \mathcal{E}.$$

Then, we have the following:

1. $0 \leq \gamma \leq 1$;
2. each of v_1 and v_2 is a probability measure on $\mathcal{E}$;
3. Q is a probability measure on $\mathcal{E} \times \mathcal{E}$.

Further, let

$$\bar{\mu} = \begin{cases} Q, & \gamma = 1, \\ \frac{v_1 \times v_2}{1-\gamma} + Q, & \gamma \neq 1. \end{cases}$$

Then $\bar{\mu}$ is the coupling of μ_1 and μ_2 . Here $\bar{\mu}$ is called the basic coupling of μ_1 and μ_2 .

Remark: The basic coupling between the point measure δ_x and the probability measure μ is denoted as $\bar{\mu}$.

III. AN EQUATION OF THE BASIC COUPLING AND TOTAL VARIATION NORM

Let $\varphi(x, y) := d(x, y)[f(x) + f(y)]$, where

$$d(x, y) := \begin{cases} 1, & x \neq y, \\ 0, & x = y. \end{cases}$$

Lemma 3.1: For two probability measures μ_1, μ_2 , we have

$$\frac{1}{2} \|\mu_1 - \mu_2\| = \inf_{\tilde{\mu} \in K(\mu_1, \mu_2)} \int d(x, y) \tilde{\mu}(dx, dy).$$

Now we can get our first main result as follows.

Theorem 3.1: Let μ_1, μ_2 be probability measures. Then

$$\|\mu_1 - \mu_2\| = 2 \int d(x, y) \bar{\mu}(dx, dy), \quad (1)$$

where $\bar{\mu}$ is the basic coupling of μ_1 and μ_2 .

Proof. Let h be a measurable function on $\mathcal{E}$. Then

$$\begin{aligned} & \|\mu_1 - \mu_2\| \\ &= \sup_{|h| \leq 1} \left| \int h d\mu_1 - \int h d\mu_2 \right| \\ &= \sup_{|h| \leq 1} \left| \int h g_1 d\mu' - \int h g_2 d\mu' \right| \\ &= \sup_{|h| \leq 1} \left\{ \int_{\{g_1 \geq g_2\}} h(g_1 - g_2) d\mu' - \int_{\{g_1 < g_2\}} h(g_1 - g_2) d\mu' \right\} \\ &= \int_{\{g_1 \geq g_2\}} (g_1 - g_2) d\mu' - \int_{\{g_1 < g_2\}} (g_1 - g_2) d\mu' \\ &= \int |g_1 - g_2| d\mu' \\ &= \int (g_1 - g) d\mu' + \int (g_2 - g) d\mu' \\ &= \left(\int g_1 d\mu' - \int g d\mu' \right) + \left(\int g_2 d\mu' - \int g d\mu' \right) \\ &= (1 - \gamma) + (1 - \gamma) \\ &= 2(1 - \gamma), \end{aligned}$$

where the fifth equality holds due to

$$|g_1 - g_2| \equiv (g_1 - g) + (g_2 - g).$$

To finish the proof, it suffices to prove

$$\int d(x, y) \bar{\mu}(dx, dy) = 1 - \gamma,$$

which is proven in the following two cases.

(i) When $\gamma = 1$, we get

$$\begin{aligned} & \int d(x, y) \bar{\mu}(dx, dy) \\ &= \int d(x, y) Q(dx, dy) \\ &= \int d(x, y) \int_{\{(x, y) : x=y\}} g(x) \mu' d(x) \\ &= \int \int_{\{(x, y) : x=y\}} d(x, y) g(x) \mu' d(x) \\ &= 0. \end{aligned}$$

Thus

$$\int d(x, y) \bar{\mu}(dx, dy) = 0 = 1 - \gamma.$$

(ii) When $\gamma < 1$, if $x = y$, then

$$\int d(x, y) \bar{\mu}(dx, dy) = 0.$$

Hence,

$$\begin{aligned} & \int d(x, y) \bar{\mu}(dx, dy) \\ &= \int_{\{(x, y): x \neq y\}} d(x, y) \frac{v_1(dx) \times v_2(dy)}{1 - \gamma} \\ & \quad + \int_{\{(x, y): x = y\}} d(x, y) Q(dx, dy) \\ &= \int_{\{(x, y): x \neq y\}} d(x, y) \frac{v_1(dx) \times v_2(dy)}{1 - \gamma} + 0 \\ &= \int_{\{(x, y): x \neq y\}} d(x, y) \frac{(1 - \gamma) \times (1 - \gamma)}{1 - \gamma} \\ &= \int_{\{(x, y): x \neq y\}} d(x, y) (1 - \gamma) \\ &= 1 - \gamma. \end{aligned}$$

Therefore (1) is concluded, and the proof is completed.

The following result is a straightforward consequence of Lemma 3.1 and Theorem 3.1.

Theorem 3.2: Given two probability measures μ_1, μ_2 , and the basic coupling $\bar{\mu}$ of μ_1 and μ_2 , we get

$$\begin{aligned} & \|\mu_1 - \mu_2\| \\ &= 2 \inf_{\bar{\mu} \in K(\mu_1, \mu_2)} \int d(x, y) \bar{\mu}(dx, dy) \\ &= 2 \int d(x, y) \bar{\mu}(dx, dy). \end{aligned} \quad (2)$$

IV. AN EQUATION OF THE BASIC COUPLING AND f -NORM

Theorem 4.1: Let μ_1, μ_2 be probability measures. Then

$$\|\mu_1 - \mu_2\|_f = \int \varphi(x, y) \bar{\mu}(dx, dy), \quad (3)$$

where $\bar{\mu}$ is the basic coupling of μ_1 and μ_2 .

Proof. We have

$$\begin{aligned} & \|\mu_1 - \mu_2\|_f \\ &= \sup_{|v| \leq f} \left| \int v d\mu_1 - \int v d\mu_2 \right| \\ &= \sup_{|v| \leq f} \left| \int v g_1 d\mu' - \int v g_2 d\mu' \right| \\ &= \int_{\{g_1 \geq g_2\}} f(g_1 - g_2) d\mu' - \int_{\{g_1 < g_2\}} f(g_1 - g_2) d\mu' \\ &= \int f|g_1 - g_2| d\mu' \\ &= \int f(g_1 - g) d\mu' + \int f(g_2 - g) d\mu', \end{aligned}$$

where the fourth equality holds since

$$|g_1 - g_2| \equiv (g_1 - g) + (g_2 - g).$$

We discuss it in terms of the following two cases.

(i) When $\gamma = 1$, one gets $g = g_1 = g_2$ and μ' -a.s.. Then it can be readily verified that

$$\int f(g_1 - g) d\mu' + \int f(g_2 - g) d\mu' = 0.$$

Moreover,

$$\begin{aligned} & \int \varphi(x, y) \bar{\mu}(dx, dy) \\ &= \int \varphi(x, y) Q(dx, dy) \\ &= \int \varphi(x, y) \int_{\{(x, y): x = y\}} g(x) \mu' d(x) \\ &= \int \int_{\{(x, y): x = y\}} \varphi(x, y) g(x) \mu' d(x) \\ &= 0, \end{aligned}$$

where the last equality holds due to the fact

$$\varphi(x, y) = 0, \text{ if } x = y.$$

Therefore,

$$\int \varphi(x, y) \bar{\mu}(dx, dy) = \int f(g_1 - g) d\mu' + \int f(g_2 - g) d\mu'.$$

(ii) When $\gamma < 1$, if $x = y$, then

$$\varphi(x, y) = d(x, y) [f(x) + f(y)] = 0.$$

Note that

$$\begin{aligned} & \int \varphi(x, y) \bar{\mu}(dx, dy) \\ &= \int \varphi(x, y) \frac{v_1(dx) \times v_2(dy)}{1 - \gamma} \\ & \quad + \int \varphi(x, y) Q(dx, dy) \\ &= \int_{B \cap \{(x, y): x \neq y\}} [f(x) + f(y)] \frac{v_1(dx) \times v_2(dy)}{1 - \gamma} \\ & \quad + 0 \\ &= \int_{B \cap \{(x, y): x \neq y\}} f(x) \frac{v_1(dx) \times v_2(dy)}{1 - \gamma} \\ & \quad + \int_{B \cap \{(x, y): x \neq y\}} f(y) \frac{v_1(dx) \times v_2(dy)}{1 - \gamma} \\ &= \int_{B \cap \{(x, y): x \neq y\}} f(x) \frac{v_1(dx) \times (1 - \gamma)}{1 - \gamma} \\ & \quad + \int_{B \cap \{(x, y): x \neq y\}} f(y) \frac{(1 - \gamma) \times v_2(dy)}{1 - \gamma} \\ &= \int f(x) v_1(dx) + \int f(y) v_2(dy) \\ &= \int f(g_1 - g) d\mu' + \int f(g_2 - g) d\mu', \end{aligned}$$

where the second equality holds due to that

$$\begin{aligned} & \int \varphi(x, y) Q(dx, dy) \\ &= \int \varphi(x, y) \int_{B \cap \{(x, y): x = y\}} g(x) \mu' d(x) \\ &= \int \int_{B \cap \{(x, y): x = y\}} \varphi(x, y) g(x) \mu' d(x) \\ &= 0. \end{aligned}$$

Then we obtain

$$\begin{aligned} & \int \varphi(x, y) \bar{\mu}(dx, dy) \\ &= \int f(g_1 - g) d\mu' + \int f(g_2 - g) d\mu'. \end{aligned}$$

Thus, (3) holds. This finishes the proof.

Theorem 4.2: Let μ_1 and μ_2 be probability measures and $f \geq 1$ be a measurable function on $\mathcal{E}$. Then

$$\inf_{\tilde{\mu} \in K(\mu_1, \mu_2)} \int \varphi(x, y) \tilde{\mu}(dx, dy) \geq \|\mu_1 - \mu_2\|_f. \quad (4)$$

Proof. For any measurable function g , if it satisfies

$$0 < g \leq f,$$

then

$$\begin{aligned} & \varphi(x, y) \\ &= d(x, y)[f(x) + f(y)] \\ &\geq d(x, y)[g(x) + g(y)] \\ &\geq |g(x) - g(y)|, \end{aligned}$$

from which we can derive that

$$\begin{aligned} & \int \varphi(x, y) \tilde{\mu}(dx, dy) \\ &\geq \int |g(x) - g(y)| \tilde{\mu}(dx, dy) \\ &\geq \left| \int [g(x) - g(y)] \tilde{\mu}(dx, dy) \right| \\ &= \left| \int g(x) \tilde{\mu}(dx, dy) - \int g(y) \tilde{\mu}(dx, dy) \right| \\ &= \left| \int g(x) \mu_1(dx) - \int g(y) \mu_2(dy) \right|. \end{aligned}$$

This means that

$$\int \varphi(x, y) \tilde{\mu}(dx, dy) \geq \left| \int g(x) \mu_1(dx) - \int g(y) \mu_2(dy) \right|.$$

Based on the above inequality, we can obtain the following result

$$\begin{aligned} & \int \varphi(x, y) \tilde{\mu}(dx, dy) \\ &\geq \sup_{|g| \leq f} \left| \int g(x) \mu_1(dx) - \int g(y) \mu_2(dy) \right| \\ &= \|\mu_1 - \mu_2\|_f \end{aligned}$$

by calculating the supremum of $|g| \leq f$ on both sides. Similarly, from the above inequality, we can derive the following result

$$\inf_{\tilde{\mu} \in K(\mu_1, \mu_2)} \int \varphi(x, y) \tilde{\mu}(dx, dy) \geq \|\mu_1 - \mu_2\|_f$$

by computing the infimum of $\tilde{\mu} \in K(\mu_1, \mu_2)$ on both sides. Therefore, (4) holds.

Based on Theorems 4.1 and 4.2, the following result can be immediately obtained, which is our second main result.

Theorem 4.3: Let μ_1 and μ_2 be probability measures, $f \geq 1$, and $\bar{\mu}$ be the basic coupling of μ_1 and μ_2 . Then

$$\begin{aligned} & \|\mu_1 - \mu_2\|_f \\ &= \inf_{\tilde{\mu} \in K(\mu_1, \mu_2)} \int \varphi(x, y) \tilde{\mu}(dx, dy) \\ &= \int \varphi(x, y) \bar{\mu}(dx, dy). \end{aligned} \quad (5)$$

Remark 4.1: Take $f = 1$, then the equation (5) is equivalent to equation (2), from which we can deduce that Theorem 4.3 is a generalization of Theorem 3.3.

V. APPLICATIONS OF OUR CONCLUSION

Define $\mathcal{P}(X)$ as the set of all probability measures on E , and let x_0 be any given point on E . Set

$$\mathcal{M} = \left\{ \mu \in \mathcal{P}(X) : \int \varphi(x_0, x) \mu(dx) < \infty \right\}.$$

Proposition 5.1: Let μ_1 and μ_2 be probability measures, and $\mu_1, \mu_2 \in \mathcal{M}$. Then $\|\mu_1 - \mu_2\|_f \in \mathcal{M}$.

Proof. Since $\mu_1, \mu_2 \in \mathcal{M}$, we have

$$\begin{aligned} & \|\mu_1 - \mu_2\|_f \\ &= \int \varphi(x, y) \bar{\mu}(dx, dy) \\ &\leq \int \varphi(x, x_0) \bar{\mu}(dx, dy) + \int \varphi(x_0, y) \bar{\mu}(dx, dy) \\ &= \int \varphi(x, x_0) \mu_1(dx) + \int \varphi(x_0, y) \mu_2(dy) \\ &< \infty. \end{aligned}$$

Thus, $\|\mu_1 - \mu_2\|_f \in \mathcal{M}$.

Proposition 5.2: Let P be a probability kernel on Polish space $(X, \rho, \mathcal{E})$, and μ_1, μ_2 be probability measures such that

$$\|\mu_1 P - \mu_2 P\|_f \leq c \|\mu_1 - \mu_2\|_f.$$

If any $P \in \mathcal{M}$, and any probability measure $\mu \in \mathcal{M}$, then $\mu P \in \mathcal{M}$.

Proof. Assume that $\bar{\mu}$ is the basic coupling between the point measure δ_x and the probability measure μ . According to Theorem 4.1, we can get

$$\|\delta_x - \mu\|_f = \int \bar{\mu}(du, dv) \varphi(u, v) = \int \mu(dv) \varphi(x, v).$$

Since $\mu \in \mathcal{M}$ and $P \in \mathcal{M}$, it follows that

$$\|\delta_{x_0} - \mu\|_f = \int \mu(dy) \varphi(x_0, y) < \infty$$

and

$$\|\delta_{x_0} - \delta_{x_0} P\|_f = \int P(x_0, dy) \varphi(x_0, y) < \infty.$$

From the condition

$$\|\mu_1 P - \mu_2 P\|_f \leq c \|\mu_1 - \mu_2\|_f$$

we obtain that

$$\|\delta_{x_0} P - \mu P\|_f \leq c \|\delta_{x_0} - \mu\|_f.$$

Then

$$\begin{aligned} & \int \mu P(x_0, dy) \varphi(x_0, y) \\ &= \|\delta_{x_0} - \mu P\|_f \\ &\leq \|\delta_{x_0} - \delta_{x_0} P\|_f + \|\delta_{x_0} P - \mu P\|_f \\ &\leq \|\delta_{x_0} - \delta_{x_0} P\|_f + c \|\delta_{x_0} - \mu\|_f \\ &< \infty. \end{aligned}$$

Hence, $\mu P \in \mathcal{M}$. The proof is completed.

VI. CONCLUSIONS

In this paper, we investigate the basic coupling of Markov chains. Firstly, we obtain an equation of the basic coupling and total variation norm. Then we improve the equation of the basic coupling and thus an equation of the basic coupling and f -norm is obtained. These two equations have extensive applications in studying the Convergence of Markov Chains and f -empirical ergodicity of Markov Chains. Since Chen [22] proved Lemma 3.1, many scholars have used it to study the ergodicity and geometric ergodicity of Markov processes [5,6,7]. This article proves a relation between the basic coupling and the total variation norm (see Theorem 3.1) based on Lemma 3.1. Theorem 3.1 is more general than Lemma 3.1. Furthermore, we have demonstrated another equation involving the basic coupling, and consequently derived a relation between the basic coupling and the f -norm (see Theorem 4.1). Researchers can use Theorem 4.1 to further investigate the f -ergodicity and f -geometric ergodicity of Markov processes. For example, Zhu [5] utilized the ideas of the equation of basic coupling and total variational norm to study the convergence of nonhomogeneous Markov chains. Zhu [4] utilized the ideas of the equation of the basic coupling and f -norm to study f -exponential ergodicity of Markov chains by coupling method.

REFERENCES

- [1] Lindvall T. Lectures on the coupling method, New York: Wiley, 1992.
- [2] S.Y. Zhang, K. Xu, "The Existence of Optimal Measurable Coupling of Transition Probability", *Acta Mathematica Sinica*, vol. 40, no. 1, pp. 5-13, 1997.
- [3] S.Y. Zhang, "The Measurable Coupling and Probability Distance of Transition Probability", *Chinese Annals of Mathematics*, vol. 6, no. 16, pp. 769-775, 1995.
- [4] Z. F. Zhu, S.Y. Zhang, "Study on f -exponential ergodicity of Markov chains by coupling method", *Acta Mathematica Sinica*, vol. 62, no. 2, pp. 287-292, 2019.
- [5] Z. F. Zhu, S.Y. Zhang, "Study on the Convergence of Nonhomogeneous Markov Chains with Probability Distance", *Acta Mathematica Scientia*, vol. 38A, no. 4, pp. 963-969, 2018.
- [6] Z. F. Zhu, Zhang S.Y. and Tian F.J., "Study the Convergence of Non-homogeneous Markov Chains in General State Spaces by Coupling Method", *Acta Mathematica Scientia*, vol. 41B, no. 5, pp. 1777-1787, 2021.
- [7] Yong Hua, "Strong Ergodicity for Markov Processes by Coupling Methods", *Journal of Applied Probability*, vol. 39, no. 4, pp. 839-852, 2002.
- [8] Holden, Lars, "Convergence of Markov Chains in the Relative Supremum Norm", *Journal of Applied Probability*, vol. 37, no. 4, pp. 1074-1083, 2000.
- [9] Ioannis Kontoyiannis; Sean P. Meyn, "On the f -norm ergodicity of Markov processes in continuous time", *Electronic Communications in Probability*, vol. 21, no. 77, pp. 1-10, 2016.
- [10] Iryna Rozora; Yurii Mlavets; Olga Vasylyk; Volodymyr Polishchuk, "On Convergence of the Uniform Norm and Approximation for Stochastic Processes from the Space $\mathbf{F}_\psi(\Omega)$ ", *Journal of Theoretical Probability*, vol. 37, no. 2, pp. 1627-1653, 2024.
- [11] Borovkov, A. A.; Hordijk, A., "Characterization and Sufficient Conditions for Normed Ergodicity of Markov Chains", *Advances in Applied Probability*, vol. 36, no. 1, pp. 227-242, 2004.
- [12] Farrukh Mukhamedov, "The Dobrushin ergodicity coefficient and the ergodicity of noncommutative Markov chains", *Journal of Mathematical Analysis and Applications*, vol. 408, no. 1, pp. 364-373, 2013.
- [13] Marco A. Gallegos-Herrada; David Ledvinka; Jeffrey S., "Rosenthal. Equivalences of Geometric Ergodicity of Markov Chains", *Journal of Theoretical Probability*, vol. 37, no. 2, pp. 1230-1256, 2024.
- [14] Roberts, Gareth O; Rosenthal, Jeffrey S; Segers, Johan; Sousa, Bruno, "Extremal indices, geometric ergodicity of Markov chains, and MCMC", *Extremes*, vol. 9, no. 3, pp. 213-230, 2006.
- [15] Peter H., "Renewal Theory and Computable Convergence Rates for Geometrically Ergodic Markov Chains", *Annals of Applied Probability*, vol. 15B, no. 1, pp. 700-738, 2005.
- [16] Kontoyiannis, I.; Meyn, S. P., "Spectral Theory and Limit Theorems for Geometrically Ergodic Markov Processes", *Annals of Applied Probability*, vol. 13, no. 1, pp. 304-362, 2003.
- [17] Alain Durmus; Eric Moulines; Alexey Naumov; Sergey Samsonov, "Probability and Moment Inequalities for Additive Functionals of Geometrically Ergodic Markov Chains", *Journal of Theoretical Probability*, vol. 37, no. 3, pp. 2184-2233, 2024.
- [18] Antoine Havet; Matthieu Lerasle; Eric Moulines; Elodie Vernet, "A quantitative McDiarmids inequality for geometrically ergodic Markov chains", *Electronic Communications in Probability*, vol. 25, no. 15, pp. 1-11, 2020.
- [19] Kushal Kr. Dey; Sourabh Bhattacharya, "On geometric ergodicity of additive and multiplicative transformation-based Markov Chain Monte Carlo in high dimensions", *Brazilian Journal of Probability and Statistics*, vol. 30, no. 4, pp. 570-613, 2016.
- [20] Ioannis Kontoyiannis; Sean P. Meyn, "On the f -norm ergodicity of Markov processes in continuous time", *Electronic Communications in Probability*, vol. 21, no. 77, pp. 1-10, 2016.
- [21] Meyn S. P, Tweedie R. L, Markov chains and stochastic stability. London: Springer-Verlag, 2008.
- [22] M. F. Chen, From markov chains to non-equilibrium particle systems (second edition). Singapore: World Scientific, 2004.
- [23] S.Y. Zhang, "The existence of optimal coupling operator for jump processes", *Acta Mathematica Sinica*, vol. 41, no. 2, pp. 393-398, 1998.
- [24] S.Y. Zhang, "Existence of optimal measurable coupling and ergodicity of Markov processes", *Chinese Science (Series A)*, vol. 28, no. 11, pp. 999-1008, 1998.
- [25] Zhang Shuili. Stochastic stability of jump processes in general state space. Wuhan: Hubei University, 2014.
- [26] Cinlar E. Introduction to stochastic processes. Chicago: Courier Corporation, 2013.
- [27] Zhang S. Y., "Regularity and existence of invariant measures for jump processes", *Acta Mathematica Sinica*, vol. 48, no. 4, pp. 785-788, 2005.
- [28] Mohamed Boumazgour, "Norm inequalities for sums of two basic elementary operators", *Journal of Mathematical Analysis and Applications*, vol. 342, no. 1, pp. 386-393, 2008.
- [29] Mohamed Barraa; Mohamed Boumazgour, "Norm equality for a basic elementary operator", *Journal of Mathematical Analysis and Applications*, vol. 286, no. 1, pp. 359-362, 2003.
- [30] Elizaveta Rebrova, "Constructive Regularization of the Random Matrix Norm", *Journal of Theoretical Probability*, vol. 33, no. 3, pp. 1768-1790, 2020.

Zhifeng Zhu received his Doctor's degree from the School of Mathematics and Statistics, Hubei University, in 2019. He is currently an Associate Professor at the School of Mathematics and Statistics, Hubei Engineering University. The main research directions are Markov Process, Statistics, and Random Series.

Junchao Zhou received her Doctor's degree from the School of Mathematics and Statistics, Hubei University, in 2021. She is currently an Associate Professor at the School of Mathematics and Statistics, Hubei Engineering University. The main research directions are Probability and cryptography.