Soft Intersection Bi-quasi-interior Ideals of Semigroups

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***Abstract*—In this paper, we introduce the concept of the soft intersection (S-int) bi-quasi-interior (ɃQĪ) ideal of semigroups and provide an equivalent definition. The relationships between S-int ideals and S-int ɃQĪ ideals are established. We prove that every S-int bi-ideal, left ideal, right ideal, interior ideal, quasi-ideal, bi-interior ideal, left/right bi-quasi-ideal, and left/right quasi-interior ideal is also an S-int ɃQĪ ideal. Counterexamples are given to show that the converses do not hold, and we demonstrate that additional conditions, such as regularity or right/left simplicity, are required for the converses. We also show that if a subsemigroup of a semigroup is a ɃQĪ ideal, then its soft characteristic function is an S-int ɃQĪ ideal, and the converse holds as well. Thus, this work establishes an important connection between classical semigroup theory and soft set theory. Furthermore, we show that finite soft AND-products, Cartesian products, and intersections of S-int ɃQĪ ideals remain S-int ɃQĪ ideals, whereas finite soft OR-products and unions do not. This study provides a broad conceptual characterization and analysis of S-int ɃQĪ ideals.**

***Index Terms*—(soft intersection) bi-quasi-interior ideals, (regular) semigroup, soft set**

I. Introduction

Semigroups play a crucial role in various areas of mathematics. In applied mathematics, semigroups, first studied in the early 20th century, serve as essential tools for analyzing linear time-invariant processes. Moreover, because finite semigroups are closely related to finite automata, their study is fundamental in theoretical computer science. In probability theory, semigroups are also connected to Markov processes. The concept of ideals plays a key role in understanding mathematical structures and their applications. Consequently, many mathematicians have focused their research on generalizing ideals within algebraic structures. Indeed, further exploration of algebraic structures necessitates the generalization of ideals. By employing the notions and properties of these generalized ideals, many mathematicians have made significant contributions, providing new insights and characterizations of algebraic structures. Dedekind introduced the concept of ideals in the context of algebraic number theory, and Noether later extended this idea to associative rings. The notion of a one-sided ideal in any algebraic structure can be seen as a generalization of the original ideal concept, and both one-sided and two-sided ideals continue to be central to the study of ring theory.

In 1952, Good and Hughes [1] introduced the concept of bi-ideals in semigroups. Steinfeld [2] was the first to define the concept of quasi-ideals in semigroups and later extended this notion to rings Quasi-ideals generalize Ɍ and Ł ideals, while bi-ideals represent a further generalization of quasi-ideals. The notion of interior ideals was first introduced by Lajos [3] and later studied in more detail by Szasz [4,5]. The concept of interior ideals arose from the generalization of the ideal concept. Rao [6-11] introduced several novel types of semigroup ideals, which expand upon existing ones, such as bi-interior ideals, ɃQĪ ideals, bi-quasi ideals, quasi-interior ideals, weak-interior ideals, tri-ideals and tri-quasi ideals. Rao [6–11] introduced several novel types of semigroup ideals, extending existing ones, including bi-interior ideals, ɃQĪ ideals, bi-quasi-ideals, quasi-interior ideals, weak-interior ideals, tri-ideals, and tri-quasi ideals. Baupradist et al. [12] also introduced the concept of essential ideals in semigroups. As a broader extension of various types of ideals, the notion of "almost" ideals was proposed, and their characteristics, together with their relationships to other ideals, were thoroughly investigated. In this context, the concept of almost ideals in semigroups was first introduced in [13]. Subsequently, [14] expanded the notion of bi-ideals to almost bi-ideals in semigroups. The notion of almost interior ideals was introduced in [15], while almost quasi-ideals in semigroups were first studied in [16]. The authors in [17–19] proposed certain types of almost ideals. Moreover, in [15, 17–22], various fuzzy almost ideal types for semigroups were examined. Many of these new structures have been employed to further characterize semigroups and their generalizations.

Molodtsov [23] introduced the "Soft Set Theory" (Ƨƨ Theory) to address problems involving uncertainty and to find appropriate solutions for them. Since then, numerous important studies have focused on various concepts related to Ƨƨ, particularly on the operations performed on them. Maji et al. [24] proposed several definitions related to Ƨƨ and established specific operations for them. Pei and Mia [25] as well as Ali et al. [26] introduced different operations on Ƨƨ. Sezgin and Atagün [27] also contributed to the study of Ƨƨ operations. For further insights into Ƨƨ operations, which have gained popularity since their inception, we refer to [28-39]. The concept and operations of Ƨƨ were further modified by Çağman and Enginoğlu [40]. Later, Çağman et

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al. [41] developed the idea of S-int groups, which led to the exploration of various soft algebraic systems. Sezer et al. [42-44], by applying Ss to semigroup theory, introduced soft intersection (S-int) semigroups, ideals, interior ideals, quasi-ideal ideals, (generalized) bi-ideals of semigroups, providing an in-depth analysis of their fundamental properties. In the context of S-int substructures of semigroups, Sezgin and Orbay [44] defined and classified several types of semigroups. Soft intersection (int) almost ideals, as a generalization of various types of S-int ideals, were introduced and examined in [45-56]. The soft versions of different algebraic structures were explored in [57-68].

Rao [7] introduced the concept of the ɃQĪ ideal as a generalization of the bi-ideal, quasi-ideal, interior ideal, bi-quasi ideal, and bi-interior ideal of a semigroup, and examined the properties of these ideals and their relationships. In Rao [69], the concept of ɃQĪ ideal is studied as a generalization of bi-ideal, quasi-ideal, interior ideal, bi-quasi ideal, and bi-interior ideal in Γ-semirings. The properties of these ideals and their relationships with other ideals are explored. The regular ($\mathscr{R}$) and simple Γ-semirings are characterized, and the conditions necessary for a Γ-semiring to be $\mathscr{R}$ or simple are examined. Additionally, Rao [70,71] investigates the properties of ɃQĪ ideals in Γ-semigroups and semirings and explores their relationships with other ideals. The regularity and simplicity properties of Γ-semigroups and semirings are characterized, and the conditions for Γ-semigroups and semirings to be $\mathscr{R}$ or simple are identified.

The motivation for this study stems from the fact that generalizing classical ideals within the framework of soft set theory opens new directions for algebraic research. In this study, the concept of "Soft intersection bi-quasi ideal" (S-int ɃQĪ Ideal) of a semigroup is defined for Ss theory, illustrated with examples, and its properties and relationships with other S-int ideals are examined in detail. It is concluded that if a subsemigroup of a semigroup is a ɃQĪ ideal, then its soft characteristic function is also an S-int ɃQĪ ideal, and the converse is also true. This is an crucial theorem establishing a significant connection between classical semigroup theory and Ss theory. Furthermore, by investigating the relationships of S-int ɃQĪ ideal of a semigroup with other S-int ideals, it is observed that every S-int bi-ideal, S-int ideal, S-int interior ideal, S-int quasi-ideal, S-int bi-interior ideal, S-int bi-quasi ideal, and S-int quasi-interior ideal of a semigroup is an S-int ɃQĪ ideal, and counterexample sare provided to show that the converses do not hold. The conditions for the converses to hold are also obtained. Additionally, the relationships between S-int ɃQĪ ideals and soft set operations as well as the concepts like soft image and soft inverse image are investigated. The paper is structured into four sections. Section 1 offers a general introduction to the topic, while Section 2 explores the fundamental concepts of semigroups and Ss ideals, along with their associated definitions and implications. We introduce the concept of S-int ɃQĪ ideals in Section 3 and examine their properties, as well as how they relate to other types of S-int ideals, through concrete examples. Section 4 provides a summary of our findings and considers potential directions for future research.

II. Preliminaries

In this section, we recall some basic definitions and results that will be used throughout this paper. S denotes a semigroup throughout this paper. A subsemigroup Ꝁ of S is called a bi-quasi-interior(*ɃQĪ*) ideal of S if ꝀSꝀSꝀ ⊆ Ꝁ [7]. If there exists an element $y \in S$ such that $x = xyx$ for all $x \in S$, then S is called a regular ($\mathscr{R}$) semigroup,

Theorem 1[72] *Let S be a semigroup. Then*

(1) *S is Ł (Ɽ) simple if and only if (iff)* $Sa = S$ $(aS = S)$ *for all* $a \in S$. *That is, for every* $a, b \in S$, *there exists* $c \in S$ *such that* $b = ca$ $(b = ac)$

(2) *S is simple iff S is a group. (both Ł and Ɽ simple)*

Definition 1 *[23,40] Let* E *be the parameter set,* U *be the universal set,* $P(U)$ *be the power set of* U, *and* Ꝁ ⊆ E. *The soft set (Ss)*$\mathfrak{f}_Ꝁ$ *over* U *is a function such that* $\mathfrak{f}_Ꝁ: E \to P(U)$, *where for all* ɏ ∉ Ꝁ, $\mathfrak{f}_Ꝁ$(ɏ) = ∅. *That is,*

$$\mathfrak{f}_Ꝁ = \{(ɏ, \mathfrak{f}_Ꝁ(ɏ)) : ɏ \in E, \mathfrak{f}_Ꝁ(ɏ) \in P(U)\}$$

The set of all Ss over U is designated by $S_E(U)$.

Definition 2 *[40] Let* $\mathfrak{f}_Ꝁ \in S_E(U)$. *If* $\mathfrak{f}_Ꝁ$(ɏ) = ∅ *for all* ɏ ∈ E, *then* $\mathfrak{f}_Ꝁ$ *is called a null Ss and indicated by* $\emptyset_E$.

Definition 3*[40] Let* $\mathfrak{f}_A, \mathfrak{f}_B \in S_E(U)$. *If* $\mathfrak{f}_A$(ɏ) ⊆ $\mathfrak{f}_B$(ɏ), *for all* ɏ ∈ E, *then* $\mathfrak{f}_A$ *is a soft subset of* $\mathfrak{f}_ℵ$ *and indicated by* $\mathfrak{f}_A \,\tilde{\subseteq}\, \mathfrak{f}_B$. *If* $\mathfrak{f}_A$(ɏ) = $\mathfrak{f}_B$(ɏ), *for all* ɏ ∈ E, *then* $\mathfrak{f}_A$ *is called soft equal to* $\mathfrak{f}_B$ *and denoted by* $\mathfrak{f}_A = \mathfrak{f}_B$.

Definition 4 *[40] Let* $\mathfrak{f}_A, \mathfrak{f}_B \in S_E(U)$. *The union (intersection) of* $\mathfrak{f}_A$ *and* $\mathfrak{f}_B$ *is the Ss* $\mathfrak{f}_A \,\tilde{\cup}\, \mathfrak{f}_B (\mathfrak{f}_A \,\tilde{\cap}\, \mathfrak{f}_B)$, *where* $(\mathfrak{f}_A \,\tilde{\cup}\, \mathfrak{f}_B)$(ƀ) = $\mathfrak{f}_A$(ƀ) ∪ $\mathfrak{f}_B$(ƀ)(($\mathfrak{f}_A \,\tilde{\cap}\, \mathfrak{f}_B$)(ƀ) = $\mathfrak{f}_A$(ƀ) ∩ $\mathfrak{f}_B$(ƀ)), *for all* ƀ ∈ E, *respectively.*

Definition 5*[40] Let* $\mathfrak{f}_A, \mathfrak{f}_B \in S_E(U)$. *Then,* ∧*-product* (∨*-product) of* $\mathfrak{f}_A$ *and* $\mathfrak{f}_B$, *denoted by* $\mathfrak{f}_A \wedge \mathfrak{f}_B (\mathfrak{f}_A \vee \mathfrak{f}_B)$ *is defined by*$(\mathfrak{f}_A \wedge \mathfrak{f}_B)(x, y) = \mathfrak{f}_A(x) \cap \mathfrak{f}_B(y)\big((\mathfrak{f}_A \vee \mathfrak{f}_B)(x, y) = \mathfrak{f}_A(x) \cup \mathfrak{f}_B(y)\big)$ *for all* $(x, y) \in E \times E$, *respectively .*

Definition 6*[41] Let* $\mathfrak{f}_A, \mathfrak{f}_B \in S_E(U)$ *and* φ *be a function from A to B. Then, the soft image of* $\mathfrak{f}_A$ *under* φ, *and the soft pre-image (soft inverse image) of* $\mathfrak{f}_B$ *under* φ *are the Ss* $\varphi(\mathfrak{f}_A)$ *and* $\varphi^{-1}(\mathfrak{f}_B)$ *such that*

$$\big(\varphi(\mathfrak{f}_A)\big)(x) = \begin{cases} \bigcup\{\mathfrak{f}_A(k) \mid k \in A \text{ and } \varphi(k) = x\}, & if\ \varphi^{-1}(x) \neq \emptyset \\ \emptyset, & otherwise \end{cases}$$

for all $b \in B$ *and* $\big(\varphi^{-1}(\mathfrak{f}_B)\big)(k) = \mathfrak{f}_B\big(\varphi(k)\big)$ *for all* $k \in A$.

Definition 7 *[41] Let* $\mathfrak{f}_A \in S_E(U)$ *and* $\alpha \subseteq U$. *Then, the upper* α*-inclusion of* $\mathfrak{f}_A$, *denoted by* $\mathcal{U}(\mathfrak{f}_A; \alpha)$, *is defined as*$\mathcal{U}(\mathfrak{f}_A; \alpha) = \{x \in A \mid \mathfrak{f}_A(x) \supseteq \alpha\}$.

Definition 8 *[42] Let* $\mathfrak{f}_Ꝁ, \mathfrak{t}_Ꝁ \in S_S(U)$, *where* Ꝁ *is a semigroup. S-int product* $\mathfrak{f}_Ꝁ \circ \mathfrak{t}_Ꝁ$ *is defined by*

$$(\mathfrak{f}_Ꝁ \circ \mathfrak{t}_Ꝁ)(ɏ) = \begin{cases} \bigcup_{ɏ = yz}\{\mathfrak{f}_Ꝁ(x) \cap \mathfrak{t}_Ꝁ(y)\}, & if\ \exists x, y \in Ꝁ \text{ such that } ɏ = xy \\ \emptyset, & otherwise \end{cases}$$

Theorem 2 *[42] Let* $ꞙ_Ꝁ, ŧ_Ꝁ, ꝑ_Ꝁ \in S_Ꝁ(U)$*, where* ₭ *is a semigroup. Then,*
i) $(ꞙ_Ꝁ \circ ŧ_Ꝁ) \circ ꝑ_Ꝁ = ꞙ_Ꝁ \circ (ŧ_Ꝁ \circ ꝑ_Ꝁ)$
ii) $ꞙ_Ꝁ \circ ŧ_Ꝁ \neq ꞙ_Ꝁ \circ ŧ_Ꝁ$
iii) $ꞙ_Ꝁ \circ (ŧ_Ꝁ \tilde{\cup}\, ꝑ_Ꝁ) = (ꞙ_Ꝁ \circ ŧ_Ꝁ) \tilde{\cup} (ꞙ_Ꝁ \circ ꝑ_Ꝁ)$ *and* $(ꞙ_Ꝁ \tilde{\cup}\, ŧ_Ꝁ) \circ ꝑ_Ꝁ = (ꞙ_Ꝁ \circ ꝑ_Ꝁ) \tilde{\cup} (ŧ_Ꝁ \circ ꝑ_Ꝁ)$
iv) $ꞙ_Ꝁ \circ (ŧ_Ꝁ \tilde{\cap}\, ꝑ_Ꝁ) = (ꞙ_Ꝁ \circ ŧ_Ꝁ) \tilde{\cap} (ꞙ_Ꝁ \circ ꝑ_Ꝁ)$ *and* $(ꞙ_Ꝁ \tilde{\cap}\, ŧ_Ꝁ) \circ ꝑ_Ꝁ = (ꞙ_Ꝁ \circ ꝑ_Ꝁ) \tilde{\cap} (ŧ_Ꝁ \circ ꝑ_Ꝁ)$
v) If $ꞙ_Ꝁ \tilde{\subseteq} ŧ_Ꝁ$*, then* $ꞙ_Ꝁ \circ ꝑ_Ꝁ \tilde{\subseteq} ŧ_Ꝁ \circ ꝑ_Ꝁ$ *and* $ꝑ_Ꝁ \circ ꞙ_Ꝁ \tilde{\subseteq} ꝑ_Ꝁ \circ ŧ_Ꝁ$
vi) If $ƶ_Ꝁ, ꞩ_Ꝁ \in S_Ꝁ(U)$ *such that* $ƶ_Ꝁ \tilde{\subseteq} ꞙ_Ꝁ$ *and* $ꞩ_Ꝁ \tilde{\subseteq} ŧ_Ꝁ$*, then* $ƶ_Ꝁ \circ ꞩ_Ꝁ \tilde{\subseteq} ꞙ_Ꝁ \circ ŧ_Ꝁ$.

Definition 9 *[42] Let* $\emptyset \neq K \subseteq S$*. The soft characteristic function (Ꞩ¢ƒŋ) of* K*, denoted by* S_K*, is defined as*

$$S_K(x) = \begin{cases} U, & if\ x \in K \\ \emptyset, & if\ x \in S \backslash K \end{cases}$$

Theorem 3*[42,51] Let* $Ǥ, Ƀ \subseteq S$*. Then*
i) $Ǥ \subseteq Ƀ$ *iff* $S_Ǥ \tilde{\subseteq} S_Ƀ$
ii) $S_Ǥ \tilde{\cap} S_Ƀ = S_{Ǥ \cap Ƀ}$ *and* $S_Ǥ \tilde{\cup} S_Ƀ = S_{Ǥ \cup Ƀ}$
iii) $S_Ǥ \circ S_Ƀ = S_{ǤɃ}$

From now on, ₭ denotes a semigroup like S.

Definition 10 *[42] An Ꞩꞩ* $ꞙ_Ꝁ$ *over U is called an S-int subsemigroup of S if* $ꞙ_Ꝁ(ᶅᶇ) \supseteq ꞙ_Ꝁ(ᶅ) \cap ꞙ_Ꝁ(ᶇ)$ *for all* $ᶅ, ᶇ \in S$.

Note that in [42], the definition of "S-int subsemigroup of S" is given as "S-int semigroup of S"; however in this paper, without loss of generality, we prefer to use "S-int subsemigroup of S".

From now on, "ideal" is abbreviated by "id".

Definition 11*[42,43] An Ꞩꞩ* $ꞙ_Ꝁ$ *over U is called an S-int Ł (Ꞧ) id of S if* $ꞙ_Ꝁ(ᶅᶇ) \supseteq ꞙ_Ꝁ(ᶇ)\big(ꞙ_Ꝁ(ᶅᶇ) \supseteq ꞙ_Ꝁ(ᶅ)\big)$ *for all* $ᶅ, ᶇ \in S$*, and is called an S-int two-sidedid (S-intid) of S if it is both S-int Ł id of S over U and S-int Ꞧ id of S over U. An S-int subsemigroup* $ꞙ_Ꝁ$ *is called an S-int bi-id of S if* $ꞙ_Ꝁ(ᶅᶇƫ) \supseteq ꞙ_Ꝁ(ᶅ) \cap ꞙ_Ꝁ(ƫ)$ *for all* $ᶅ, ᶇ, ƫ \in S$*. An Ꞩꞩ* $ꞙ_Ꝁ$ *over U is called an S-int interior id of S if* $ꞙ_Ꝁ(ᶅᶇƫ) \supseteq ꞙ_Ꝁ(ᶇ)$ *for all* $ᶅ, ᶇ, ƫ \in S$.

An Ꞩꞩ $ꞙ_Ꝁ$ *over U is called an S-int Ł weak-interior (Ꞧ) id of S if* $ꞙ_Ꝁ(ᶅᶇƫ) \supseteq ꞙ_Ꝁ(ᶇ) \cap ꞙ_Ꝁ(ƫ)$ *(*$ꞙ_Ꝁ(ᶅᶇƫ) \supseteq ꞙ_Ꝁ(ᶅ) \cap ꞙ_Ꝁ(ᶇ)$*) for all* $ᶅ, ᶇ, ƫ \in S$*, and is called an S-int weak-interior id of S if it is both S-int Ł weak-interior id of S over Uand S-int Ꞧ weak-interior id of S over U. An Ꞩꞩ* $ꞙ_Ꝁ$ *over U is called an S-int Ł quasi-interior (Ꞧ) id of S if* $ꞙ_Ꝁ(ᶅᶇƫɚ) \supseteq ꞙ_Ꝁ(ᶇ) \cap ꞙ_Ꝁ(ɚ)$ *(*$ꞙ_Ꝁ(ᶅᶇƫɚ) \supseteq ꞙ_Ꝁ(ᶅ) \cap ꞙ_Ꝁ(ƫ)$*) for all* $ᶅ, ᶇ, ƫ, ɚ \in S$*, and is called an S-int quasi-interior id of S if it is both S-int Ł quasi-interior id of S over U and S-int Ꞧ quasi-interior id of S over U[73,74].*

If $ꞙ_Ꝁ(x) = U$ for all $x \in S$, then $ꞙ_Ꝁ$ is an S-int subsemigroup (id, bi-id, interior id, weak-interior id, quasi-interior id). We denote such a kind of S-int subsemigroup (id, bi-id, interior id, weak-interior id, quasi-interior id) by $\widetilde{\mathbb{S}}$ [42,43,73-75]. Moreover, $\widetilde{\mathbb{S}} = S_S$, that is, $\widetilde{\mathbb{S}}(x) = U$ for all $x \in S$[42].

Definition 12*[73-75] An Ꞩꞩ* $ꞙ_Ꝁ$ *over U is called an S-int quasi-id of S over U if* $\left(\widetilde{\mathcal{S}} \circ ꞙ_Ꝁ\right) \tilde{\cap} \left(ꞙ_Ꝁ \circ \widetilde{\mathcal{S}}\right) \tilde{\subseteq} ꞙ_Ꝁ$*. An Ꞩꞩ* $ꞙ_Ꝁ$ *over U is called an S-int bi-interior id of S over U if* $(\widetilde{\mathcal{S}} \circ ꞙ_Ꝁ \circ \widetilde{\mathcal{S}}) \tilde{\cap} \left(ꞙ_Ꝁ \circ \widetilde{\mathcal{S}} \circ ꞙ_Ꝁ\right) \tilde{\subseteq} ꞙ_Ꝁ$*. An Ꞩꞩ* $ꞙ_Ꝁ$ *over U is called an S-int Ł bi-quasi (Ꞧ) id of S if* $(\widetilde{\mathcal{S}} \circ ꞙ_Ꝁ) \tilde{\cap} \left(ꞙ_Ꝁ \circ \widetilde{\mathcal{S}} \circ ꞙ_Ꝁ\right) \tilde{\subseteq} ꞙ_Ꝁ$ *(*$(ꞙ_Ꝁ \circ \widetilde{\mathcal{S}}) \tilde{\cap} \left(ꞙ_Ꝁ \circ \widetilde{\mathcal{S}} \circ ꞙ_Ꝁ\right) \tilde{\subseteq} ꞙ_Ꝁ$*), and is called an S-intbi-quasi id of S if it is both S-int Ł bi-quasi id of S over U and S-int Ꞧ bi-quasi id of S over U.*

Theorem 4 *[42]Let* $ḟ_S \in S_S(U)$*. Then,*

i) $\widetilde{\mathcal{S}} \circ \widetilde{\mathcal{S}} \tilde{\subseteq} \widetilde{\mathcal{S}}$
ii) $\widetilde{\mathcal{S}} \circ ḟ_S \tilde{\subseteq} \widetilde{\mathcal{S}}$ *and* $ḟ_S \circ \widetilde{\mathcal{S}} \tilde{\subseteq} \widetilde{\mathcal{S}}$
iii) $ḟ_S \tilde{\cup} \widetilde{\mathcal{S}} = \widetilde{\mathcal{S}}$ *and* $ḟ_S \tilde{\cap} \widetilde{\mathcal{S}} = ḟ_S$.

Theorem 5 *[42,43] Let* ₭ *be a nonempty subset of a semigroup S. Then,* ₭ *is a subsemigroup of S iff* $S_Ꝁ$ *is an S-int subsemigroup.*

Theorem 6 *[42,43,73-75] Let* $ꞙ_S \in S_S(U)$*. Then,*

(1) $ꞙ_S$ *is an S-int subsemigroup iff* $(ꞙ_S \circ ꞙ_S) \tilde{\subseteq} ꞙ_S$,
(2) $ꞙ_S$ *is an S-int Ł (Ꞧ) id iff* $\left(\widetilde{\mathcal{S}} \circ ꞙ_S\right) \tilde{\subseteq} ꞙ_S$ *and* $\left(ꞙ_S \circ \widetilde{\mathcal{S}}\right) \tilde{\subseteq} ꞙ_S$,
(3) $ꞙ_S$ *is an S-int bi-id iff* $(ꞙ_S \circ ꞙ_S) \tilde{\subseteq} ꞙ_S$ *and* $\left(ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S\right) \tilde{\subseteq} ꞙ_S$,
(4) $ꞙ_S$ *is an S-int interior id iff* $\left(\widetilde{\mathcal{S}} \circ ꞙ_S \circ \widetilde{\mathcal{S}}\right) \tilde{\subseteq} ꞙ_S$,
(5) $ꞙ_S$ *is an S-int Ł (Ꞧ) weak-interior id iff* $(\widetilde{\mathcal{S}} \circ ꞙ_S \circ ꞙ_S) \tilde{\subseteq} ꞙ_S$ $((ꞙ_S \circ ꞙ_S \circ \widetilde{\mathcal{S}}) \tilde{\subseteq} ꞙ_S)$,
(6) $ꞙ_S$ *is an S-int Ł (Ꞧ) quasi-interior id iff* $\left(\widetilde{\mathcal{S}} \circ ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S\right) \tilde{\subseteq} ꞙ_S \left(\left(ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S \circ \widetilde{\mathcal{S}}\right) \tilde{\subseteq} ꞙ_S\right)$,

Theorem 7*[42,43] The following assertions hold:*

(1) *Every S-int Ł (Ꞧ/two-sided) id is an S-int subsemigroup (S-int bi-id/S-int quasi-id).*
(2) *Every S-int id is an S-int interior id (S-int quasi-id).*
(3) *Every S-int quasi id is an S-int subsemigroup (S-int bi-id).*

Theorem 8*[42] Let* $ꞙ_S \in S_S(U)$*,* α *be a subset of U,* $Im(ꞙ_S)$ *be the image of* $ꞙ_S$ *such that* $\alpha \in Im(ꞙ_S)$*. If* $ꞙ_S$ *is an S-int subsemigroup of S, then* $\mathcal{U}(ꞙ_S; \alpha)$ *is a subsemigroup of S.*

III. SOFT INTERSECTION BI-QUASI-INTERIOR IDEALS OF SEMIGROUPS

In this section, we present the concept of soft intersection *(S-int)* bi-quasi-interior ids in semigroups, provide its examples, thoroughly examine its relationships with other soft intersection ids, and analyze the concept in terms of certain SS concepts and operations.

Definition 13 *A soft set* f_S *over U is called a soft intersection(S-int)* $ɃQ\bar{I}$ *ideal of S over U iff* $f_S(ƀđꞙꝳꞥ) \supseteq f_S(ƀ) \cap f_S(ꞙ) \cap f_S(ꞥ)$ *for all* $ƀ, đ, ꞙ, ꝳ, ꞥ \in S$.

S-int bi-quasi-interior of S over U is abbreviated by S-int ɃQĪ-id in what follows.

Example 1 Let the semigroup $\mathrm{S} = \{ȥ, Ꮭ, պ\}$ be defined as follows:

::	ȥ	Ꮭ	պ
ȥ	պ	պ	պ
Ꮭ	ȥ	Ꮭ	Ղ
պ	պ	պ	Ղ

Let $ŧ_S$ and $ɓ_S$ be Ꞩss over $U = Z_8^*$ as follows:

$ŧ_S = \{(ȥ, \{\bar{1}, \bar{3}, \bar{5}\}), (Ꮭ, \{\bar{1}, \bar{3}, \bar{7}\}), (պ, \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\})\}, ɓ_S = \{(ȥ, \{\bar{1}, \bar{3}, \bar{5}\}), (Ꮭ, \{\bar{1}, \bar{3}, \bar{7}\}), (պ, \{\bar{1}, \bar{3}, \bar{5}\})\}$

$ŧ_S$ is an S-int ɃQĪ-id of S. Here, we find it appropriate to give a few concrete examples of elements for ease of illustration in order to be more understandable. In fact,

$ŧ_S(ȥȥᏝᏝպ) = ŧ_S(պ) \supseteq ŧ_S(ȥ) \cap ŧ_S(Ꮭ) \cap ŧ_S(պ)$, $ŧ_S(ᏝᏝᏝᏝպ) = ŧ_S(պ) \supseteq ŧ_S(Ꮭ) \cap ŧ_S(Ꮭ) \cap ŧ_S(պ)$, $ŧ_S(ȥȥȥᏝȥ) = ŧ_S(պ) \supseteq ŧ_S(ȥ) \cap ŧ_S(ȥ) \cap ŧ_S(ȥ)$

It can be easily shown that the Ꞩs set $ŧ_S$ satisfies the S-int ɃQĪ-id condition for all other element combinations of the set S. However, since $ɓ_S(ᏝᏝᏝպᏝ) = ɓ_S(պ) \not\supseteq ɓ_S(Ꮭ) \cap ɓ_S(Ꮭ) \cap ɓ_S(Ꮭ)$ $ɓ_S$ is not an S-int ɃQĪ-id.

Theorem 9 *Let $ꞙ_S \in S_S(U)$. Then, $ꞙ_S$ is an S-int ɃQĪ-id iff $ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S \mathrel{\widetilde{\subseteq}} ꞙ_S$.*

Proof: Suppose that $ꞙ_S$ is an S-int ɃQĪ-id and $ȿ \in S$. If $(ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S)(ȿ) = \emptyset$, then $ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S \mathrel{\widetilde{\subseteq}} ꞙ_S$. Otherwise, there exist elements $ƚ, ꞑ, ŧ, ɚ, ȥ, Ꮭ, պ, ƅ \in S$ such that $ȿ = ƚꞑ$, $ƚ = ŧɚ$, $ŧ = ȥᏝ$ and $ȥ = պƅ$, for $ȿ \in S$. Since $ꞙ_S$ is an S-int ɃQĪ-id, $ꞙ_S(ȿ) = ꞙ_S(ƚꞑ) = ꞙ_S(պƅᏝɚꞑ) \supseteq ꞙ_S(պ) \cap ꞙ_S(Ꮭ) \cap ꞙ_S(ꞑ)$ Therefore,

$$\begin{aligned}
\left(ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S\right)(ȿ) &= \left[(ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S \circ \widetilde{\mathcal{S}}) \circ ꞙ_S\right](ȿ) \\
&= \bigcup_{ȿ=ƚꞑ} \left\{(ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S \circ \widetilde{\mathcal{S}})(ƚ) \cap (ꞙ_S)(ꞑ)\right\} \\
&= \bigcup_{ȿ=ƚꞑ} \left\{\bigcup_{ƚ=ŧɚ} \left\{(ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S)(ŧ) \cap \widetilde{\mathcal{S}}(ɚ)\right\} \cap ꞙ_S(ꞑ)\right\} \\
&= \bigcup_{ȿ=ƚꞑ} \left\{\bigcup_{ƚ=ŧɚ} \left\{\bigcup_{ŧ=ȥᏝ} \left\{(ꞙ_S \circ \widetilde{\mathcal{S}})(ȥ) \cap (ꞙ_S)(Ꮭ)\right\} \cap \widetilde{\mathcal{S}}(ɚ) \cap ꞙ_S(ꞑ)\right\}\right\} \\
&= \bigcup_{ȿ=ƚꞑ} \left\{\bigcup_{ƚ=ŧɚ} \left\{\bigcup_{ŧ=ȥᏝ} \left\{\bigcup_{ȥ=պƅ} \left\{(ꞙ_S)(պ) \cap \widetilde{\mathcal{S}}(ƅ)\right\} \cap (ꞙ_S)(Ꮭ) \cap \widetilde{\mathcal{S}}(ɚ) \cap ꞙ_S(ꞑ)\right\}\right\}\right\} \\
&= \bigcup_{ȿ=պƅᏝɚꞑ} \{ꞙ_S(պ) \cap ꞙ_S(Ꮭ) \cap ꞙ_S(ꞑ)\} \\
&\subseteq \bigcup_{ȿ=պƅᏝɚꞑ} \{ꞙ_S(պƅᏝɚꞑ)\} \\
&= ꞙ_S(ƚꞑ) \\
&= ꞙ_S(ȿ)
\end{aligned}$$

Thus, we have $ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S \mathrel{\widetilde{\subseteq}} ꞙ_S$. Moreover, in the case where $ȿ = ƚꞑ$ and $ƚ \neq պƅᏝɚꞑ$ for $ȿ \in S$, since $\left(ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S \circ \widetilde{\mathcal{S}}\right)(ƚ) = \emptyset$, $ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S \mathrel{\widetilde{\subseteq}} ꞙ_S$ is satisfied.

Conversely, assume that $ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S \mathrel{\widetilde{\subseteq}} ꞙ_S$. Let $ȿ = ƚʉŧɚȥ$ for $ƚ, ʉ, ŧ, ɚ, ȥ \in S$. Then, we have

$$\begin{aligned}
ꞙ_S(ƚʉŧɚȥ) &= ꞙ_S(ȿ) \\
&\supseteq \left(ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S\right)(ȿ) \\
&= \left[\left(ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S \circ \widetilde{\mathcal{S}}\right) \circ ꞙ_S\right](ȿ) \\
&= \bigcup_{ȿ=ƅȥ} \left\{\left(ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S \circ \widetilde{\mathcal{S}}\right)(ƅ) \cap ꞙ_S(ȥ)\right\} \\
&= \bigcup_{ȿ=ƅȥ} \left\{\bigcup_{ƅ=պɚ} \left\{(ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S)(պ) \cap \widetilde{\mathcal{S}}(ɚ)\right\} \cap ꞙ_S(ȥ)\right\} \\
&= \bigcup_{ȿ=ƅȥ} \left\{\bigcup_{ƅ=պɚ} \left\{\bigcup_{պ=ꞑŧ} \left\{(ꞙ_S \circ \widetilde{\mathcal{S}})(ꞑ) \cap ꞙ_S(ŧ)\right\} \cap \widetilde{\mathcal{S}}(ɚ) \cap ꞙ_S(ȥ)\right\}\right\} \\
&= \bigcup_{ȿ=ƅȥ} \left\{\bigcup_{ƅ=պɚ} \left\{\bigcup_{պ=ꞑŧ} \left\{\bigcup_{ꞑ=ƚʉ} ꞙ_S(ƚ) \cap \widetilde{\mathcal{S}}(ʉ) \cap ꞙ_S(ŧ) \cap \widetilde{\mathcal{S}}(ɚ) \cap ꞙ_S(ȥ)\right\}\right\}\right\} \\
&\supseteq \left(ꞙ_S \circ \widetilde{\mathcal{S}} \circ ꞙ_S \circ \widetilde{\mathcal{S}}\right)(ƚʉŧɚ) \cap ꞙ_S(ȥ) \\
&= \bigcup_{ȿ=ƚʉŧɚȥ} \left\{ꞙ_S(ƚ) \cap \widetilde{\mathcal{S}}(ʉ) \cap ꞙ_S(ŧ) \cap \widetilde{\mathcal{S}}(ɚ) \cap ꞙ_S(ȥ)\right\} \\
&\supseteq ꞙ_S(ƚ) \cap \widetilde{\mathcal{S}}(ʉ) \cap ꞙ_S(ŧ) \cap \widetilde{\mathcal{S}}(ɚ) \cap ꞙ_S(ȥ) \\
&= ꞙ_S(ƚ) \cap U \cap ꞙ_S(ŧ) \cap U \cap ꞙ_S(ȥ) \\
&= ꞙ_S(ƚ) \cap ꞙ_S(ŧ) \cap ꞙ_S(ȥ)
\end{aligned}$$

Hence, $ꞙ_S(ƚʉŧɚȥ) \supseteq ꞙ_S(ƚ) \cap ꞙ_S(ŧ) \cap ꞙ_S(ȥ)$ implying that $ꞙ_S$ is an S-int ɃQĪ-id.

Corollary 1 $\widetilde{\mathcal{S}}$ and $\emptyset_S$ are S-int ɃQĪ-ids.

Theorem 10 *Let Ꝁ be a subsemigroup of S. Then, Ꝁ is a ɃQĪ id of S iff $S_Ꝁ$, the Ꞩ¢ƒ of Ꝁ, is an S-int ɃQĪ-id.*

Proof: Let Ꝁ be a ɃQĪ id of S. Then, $ꝀSꝀSꝀ \subseteq Ꝁ$. By Theorem 3, $S_Ꝁ \circ \widetilde{\mathcal{S}} \circ S_Ꝁ \circ \widetilde{\mathcal{S}} \circ S_Ꝁ = S_Ꝁ \circ S_S \circ S_Ꝁ \circ S_S \circ S_Ꝁ = S_{ꝀSꝀSꝀ} \mathrel{\widetilde{\subseteq}} S_Ꝁ$ Hence, $S_Ꝁ$ is an S-int ɃQĪ-id. Conversely, let $S_Ꝁ$ be an S-int ɃQĪ-id and Ꝁ be a subsemigroup of S. Then, $S_Ꝁ \circ$

$\widetilde{S} \circ S_{Ꝁ} \circ \widetilde{S} \circ S_{Ꝁ} \,\widetilde{\subseteq}\, S_{Ꝁ}$. Let $x \in ꝀSꝀSꝀ$. Then, $S_{Ꝁ}(x) \supseteq \left(S_{Ꝁ} \circ \widetilde{S} \circ S_{Ꝁ} \circ \widetilde{S} \circ S_{Ꝁ}\right)(x) = (S_{Ꝁ} \circ S_S \circ S_{Ꝁ} \circ S_S \circ S_{Ꝁ})(x) = S_{ꝀSꝀSꝀ}(x) = U$ Thus, $S_{Ꝁ}(x) = U$ and so $x \in Ꝁ$, implying that $ꝀSꝀSꝀ \subseteq Ꝁ$. Hence, Ꝁ is a ₿QĪ id of S.

Example 2 Consider the semigroup in Example 1. $A = \{ʑ, ɰ\}$ is a ₿QĪ id of S. By the definition of S¢fŋ, $S_A = \{(ʑ, U), (ʟ, \emptyset)(ɰ, U)\}$. S_A is an S-int ₿QĪ-id. Conversely, by choosing the S-int ₿QĪ-id as $ŧ_S = \{(ʑ, U), (ʟ, \emptyset)(ɰ, U)\}$, which is the S¢fŋ of $X = \{ʑ, ɰ\}$, X is a ₿QĪ idof S.

Now, we continue with the relationships between S-int ₿QĪ-ids and other types of S-int ids of S.

Theorem 11 *Every S-int bi-id is an S-int ₿QĪ-id.*

Proof: Let $ŧ_S$ be an S-int bi-id of S. Then, $ŧ_S \circ \widetilde{S} \circ ŧ_S \,\widetilde{\subseteq}\, ŧ_S$. Thus, $(ŧ_S \circ \widetilde{S} \circ ŧ_S) \circ \widetilde{S} \circ ŧ_S \,\widetilde{\subseteq}\, ŧ_S \circ \widetilde{S} \circ ŧ_S \,\widetilde{\subseteq}\, ŧ_S$ Hence, $ŧ_S$ is an S-int ₿QĪ-id of S.

We show with a counterexample that the converse of Theorem 11 is not true:

Example 3 Let the semigroup $S = \{ч, ℘, ъ, ϖ̈\}$ be defined as follows:

⊗	ч	℘	ъ	ϖ̈
ч	ч	ч	Ч	ч
℘	ч	ч	Ч	ч
ъ	ч	ч	Ч	℘
ϖ̈	ч	ч	℘	ъ

Let $ŧ_S$ be Ss over $U = \mathbb{Z}$ as follows: $ŧ_S = \{(ч, \{1,2,3,4,5\}), (℘, \{2,3\}), (ъ, \{4,5\})(ϖ̈, \{1\})\}$. Here, $ŧ_S$ is an S-int ₿QĪ-id. In fact,
$\left(ŧ_S \circ \widetilde{S} \circ ŧ_S \circ \widetilde{S} \circ ŧ_S\right)(ч) = ŧ_S(ч) \cup ŧ_S(℘) \cup ŧ_S(ъ) \cup ŧ_S(ϖ̈) \subseteq ŧ_S(ч)$, $\left(ŧ_S \circ \widetilde{S} \circ ŧ_S \circ \widetilde{S} \circ ŧ_S\right)(℘) = \emptyset \subseteq ŧ_S(℘)$, $\left(ŧ_S \circ \widetilde{S} \circ ŧ_S \circ \widetilde{S} \circ ŧ_S\right)(ъ) = \emptyset \subseteq ŧ_S(ъ)$, $\left(ŧ_S \circ \widetilde{S} \circ ŧ_S \circ \widetilde{S} \circ ŧ_S\right)(ϖ̈) = \emptyset \subseteq ŧ_S(ϖ̈)$
Thus, $ŧ_S$ is an S-int ₿QĪ-id of S. However, since $(ŧ_S \circ \widetilde{S} \circ ŧ_S)(℘) = ŧ_S(ϖ̈) \not\subseteq ŧ_S(℘)$ $ŧ_S$ is not an S-int bi-id.

Theorem 12 *Let* $ŧ_S \in S_S(U)$ *and* S *be an* ℛ *semigroup. Then, the following conditions are equivalent:*
1. $ŧ_S$ *is an S-int bi-id.*
2. $ŧ_S$ *is an S-int ₿QĪ-id.*

Proof: (1) implies (2) is by Theorem 11. Let $ŧ_S$ is an S-int ₿QĪ-id and $ɭ, ɳ, ƭ \in S$. By assumption, there exists $ᵯ, ᵰ \in S$ such that $ɭ = ɭᵯɭ$ and $ɳ = ɳᵰɳ$ Thus,

$$ŧ_S(ɭɳƭ) = ŧ_S\big((ɭᵯɭ)ɳƭ\big) = ŧ_S(ɭᵯɭɳƭ) \supseteq ŧ_S(ɭ) \cap ŧ_S(ɭ) \cap ŧ_S(ƭ) = ŧ_S(ɭ) \cap ŧ_S(ƭ)$$

$$ŧ_S(ɭɳ) = ŧ_S(ɭᵯɭ(ɳy)ɳ) \supseteq ŧ_S(ɭ) \cap ŧ_S(ɭ) \cap ŧ_S(ɳ) = ŧ_S(ɭ) \cap ŧ_S(ɳ)$$

Thus, $ŧ_S$ is an S-int bi-id.

Proposition 1 *Every S-int Ł id is an S-int ₿QĪ-id.*

Proof: Let $ŧ_S$ be an S-int Ł id of S. Then, by Theorem 7, $ŧ_S$ is an S-int bi-id. The rest of the proof is obvious by Theorem 11. Hence, $ŧ_S$ is an S-int ₿QĪ-id of S.

We show with a counterexample that the converse of Proposition 1 is not true:

Example 4 Let the semigroup $ℜ = \{ɖ, ƙ\}$ be defined as follows:

⋆	ɖ	ƙ
ɖ	ɖ	ɖ
ƙ	ƙ	ƙ

Let $ŧ_ℜ$ be Ss over $U = \left\{\begin{bmatrix}x\\0\end{bmatrix} \middle| x \in \mathbb{Z}\right\}$ as follows: $ŧ_ℜ = \left\{\left(ɖ, \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}4\\0\end{bmatrix}\right\}\right), \left(ƙ, \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}\right\}\right)\right\}$
Here, $ŧ_ℜ$ is an S-int ₿QĪ-id. In fact,

$\left(ŧ_ℜ \circ \widetilde{S} \circ ŧ_ℜ \circ \widetilde{S} \circ ŧ_ℜ\right)(ɖ) = ŧ_ℜ(ɖ) \subseteq ŧ_ℜ(ɖ)$, $\left(ŧ_ℜ \circ \widetilde{S} \circ ŧ_ℜ \circ \widetilde{S} \circ ŧ_ℜ\right)(ƙ) = ŧ_ℜ(ƙ) \subseteq ŧ_ℜ(ƙ)$

Thus, $ŧ_ℜ$ is an S-int ₿QĪ-id of ℜ. However, since $ŧ_ℜ(ƙɖ) = ŧ_ℜ(ƙ) \not\supseteq ŧ_ℜ(ɖ)$ $ŧ_ℜ$ is not an S-int Ł id.

Proposition 2 *Let* $ŧ_S \in S_S(U)$ *and* S *be an* ℛ *semigroup and* Ɍ *simple semigroup. Then, the following conditions are equivalent:*
1. $ŧ_S$ is an S-int Ł id.
2. $ŧ_S$ is an S-int ₿QĪ-id.

Proof: (1) implies (2) is by Proposition 1. Let $ŧ_S$ is an S-int ₿QĪ-id and $ɭ, ɳ, ƭ \in S$. By assumption, there exists $ᵯ, ᵰ \in S$ such that $ɭ = ɳᵯ$ and $ɳ = ɳᵰɳ$. Thus, $ŧ_S(ɭɳ) = ŧ_S\big((ɳᵯ)(ɳᵰɳ)\big) = ŧ_S(ɳᵯɳᵰɳ) \supseteq ŧ_S(ɳ) \cap ŧ_S(ɳ) \cap ŧ_S(ɳ) = ŧ_S(ɳ)$. Thus, $ŧ_S$ is an S-int Ł id.

Proposition 3 *Every S-int Ɍ id is an S-int ₿QĪ-id.*

Proof: Let $ŧ_S$ be an S-int Ɍ id of S. Then, by Theorem 7, $ŧ_S$ is an S-int bi-id. The rest of the proof is obvious by Theorem 11. Hence, $ŧ_S$ is an S-int ₿QĪ-id of S.

We show with a counterexample that the converse of Proposition 3 is not true:

Example 5 Consider the Ss $ŧ_S$ in Example 1. It was shown in Example 1 that $ŧ_S$ is an S-int ₿QĪ-id. Since, $ŧ_S(ʟʑ) = ŧ_S(ʑ) \not\supseteq ŧ_S(ʟ)$ $ŧ_S$ is not an S-int Ɍ id.

Proposition 4 *Let* $ŧ_S \in S_S(U)$ *and* S *be an* ℛ *semigroup and* Ł *simple semigroup. Then, the following conditions are equivalent:*
1. $ŧ_S$ *is an S-int Ɍ id.*
2. $ŧ_S$ *is an S-int ₿QĪ-id.*

Proof: (1) implies (2) is by Proposition 3. Let $ŧ_S$ is an S-int ₿QĪ-id and $ɭ, ɳ \in S$. By assumption, there exists $ᵯ, ᵰ \in S$ such that $ɳ = ᵯɭ$ and $ɭ = ɭᵰɭ$. Thus, $ŧ_S(ɭɳ) =$

$ŧ_S\big((ƚꞥƚ)(ꝳƚ)\big) = ŧ_S(ƚꞥƚꝳƚ) \supseteq ŧ_S(ƚ) \cap ŧ_S(ƚ) \cap ŧ_S(ƚ) = ŧ_S(ƚ)$Thus, $ŧ_S$ is an S-int Ɍ id.

Theorem 13 *Every S-int id is an S-int ɃQĪ-id.*

Proof: It is followed by Proposition 1 and Proposition 3. Here note that the converse of Theorem 13 is not true follows from Example 4 and Example 5.

Theorem 14 shows that the converse of Theorem 13 holds for $\mathscr{R}$ groups.

Theorem 14 *Let $ŧ_S \in S_S(U)$ andS be an $\mathscr{R}$ group. Then, the following conditions are equivalent:*
1. $ŧ_S$ is an S-intid.
2. $ŧ_S$ is an S-int ɃQĪ-id.

Proof: (1) implies (2) is by Proposition 1 and Proposition 3. Let$ŧ_S$ is an S-int ɃQĪ-id of a group S. Then, by Theorem 1, S is both an Ł simple and an Ɍ simple semigroup. The rest of the proof follows from Proposition 2 and Proposition 4.

Theorem 15 *Every S-int interior id is an S-int ɃQĪ-id.*

Proof: Let $ŧ_S$ be an S-int interior id of S. Then, $\widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}} \,\widetilde{\subseteq}\, ŧ_S$.Thus,$ŧ_S \circ \left(\widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}}\right) \circ ŧ_S \,\widetilde{\subseteq}\, ŧ_S \circ ŧ_S \circ ŧ_S \,\widetilde{\subseteq}\, \widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}} \,\widetilde{\subseteq}\, ŧ_S$. Hence, $ŧ_S$ is an S-int ɃQĪ-id of S.

We show with a counterexample that the converse of Theorem 15 is not true:

Example 6 Consider the Ss $ŧ_S$ in Example 1. It was shown in Example 1 that $ŧ_S$ is an S-int ɃQĪ id. Since$ŧ_S(ꝡꝡȥ) = ŧ_S(ȥ) \not\supseteq ŧ_S(ꝡ)$, $ŧ_S$ is not an S-int interior id.

Theorem 16 shows that the converse of Theorem 15 holds for the groups.

Theorem 16 *Let $ŧ_S \in S_S(U)$ and S be a group. Then, the following conditions are equivalent:*
1. *$ŧ_S$ is an S-int interior id.*
2. *$ŧ_S$ is an S-int ɃQĪ-id.*

Proof: (1) implies (2) is by Theorem 15. Let$ŧ_S$ is an S-int ɃQĪ-id and $ƚ, ɳ, ŧ \in S$. By assumption, there exists $ꝳ \in S$ such that $ƚ = ɳꝳ$ and $ŧ = ꞥɳ$. Thus, $ŧ_S(ƚɳŧ) = ŧ_S\big((ɳꝳ)ɳ(ꞥɳ)\big) = ŧ_S(ɳꝳɳꞥɳ) \supseteq ŧ_S(ɳ) \cap ŧ_S(ɳ) \cap ŧ_S(ɳ) = ŧ_S(ɳ)$. Thus, $ŧ_S$ is an S-int interior id.

Theorem 17 *Every S-int quasi-id is an S-int ɃQĪ-id.*

Proof: Let $ŧ_S$ be an S-int quasi-id of S. Then, by Theorem 7, $ŧ_S$ is an S-int bi-id. The rest of the proof is obvious by Theorem 11. Hence, $ŧ_S$ is an S-int ɃQĪ-id of S.

We show with a counterexample that the converse of Theorem 17 is not true:

Example 7 Consider the Ss $ŧ_S$ in Example 3. It was shown in Example 3 that $ŧ_S$ is an S-int ɃQĪ-id. Since,$\left(ŧ_S \circ \widetilde{\mathbb{S}}\right)(\wp) \cap \left(\widetilde{\mathbb{S}} \circ ŧ_S\right)(\wp) = ŧ_S(\bar{ѿ}) \cup ŧ_S(ъ) \not\subseteq ŧ_S(\wp)$.$ŧ_S$ is not an S-int quasi-id.

Theorem 18 *Let $ŧ_S \in S_S(U)$ and S be an $\mathscr{R}$ group. Then, the following conditions are equivalent:*
1. *$ŧ_S$ is an S-int quasi-id.*
2. *$ŧ_S$ is an S-int ɃQĪ-id.*

Proof: (1) implies (2) is by Theorem 17. Let$ŧ_S$ is an S-int ɃQĪ-id. Since S is an $\mathscr{R}$ group, then, by Theorem 14, $ŧ_S$ is an S-int id. The rest of the proof is obvious by Theorem 7 $ŧ_S$ is an S-int quasi-id of S.

Theorem 19 *Every S-int bi-interior id is an S-int ɃQĪ-id.*

Proof: Let $ŧ_S$ be an S-int bi-interior id of S. Then, $\left(\widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left(ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S\right) \widetilde{\subseteq} ŧ_S$.Since, $\left(ŧ_S \circ \widetilde{\mathbb{S}}\right) \circ ŧ_S \circ \left(\widetilde{\mathbb{S}} \circ ŧ_S\right) \widetilde{\subseteq} \widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}}$and $ŧ_S \circ \left(\widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}}\right) \circ ŧ_S \widetilde{\subseteq} ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S$ it is obtained $ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \widetilde{\subseteq} \left(\widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left(ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S\right) \widetilde{\subseteq} ŧ_S$. Hence, $ŧ_S$ is an S-int ɃQĪ-id of S.

Conversely, an S-int ɃQĪ ideal is not necessarily an S-int bi-interior id.

Example 8 Consider the Ss $ŧ_S$ in Example 3. It was shown in Example 3 that $ŧ_S$ is an S-int ɃQĪ-id. Since,$\left(\widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}}\right)(\wp) \widetilde{\cap} \left(ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S\right)(\wp) = ŧ_S(\bar{ѿ}) \not\subseteq ŧ_S(\wp)$$ŧ_S$ is not an S-int bi-interior id.

The following theorem offers another characterization of S-int ɃQĪ ideals.

Theorem 20 *Le t$ŧ_S \in S_S(U)$ and S be an $\mathscr{R}$ group. Then, the following conditions are equivalent:*
1. *$ŧ_S$ is an S-int bi-interior id.*
2. *$ŧ_S$ is an S-int ɃQĪ-id.*

Proof: (1) implies (2) is by Theorem 19. Let $ŧ_S$ is an S-int ɃQĪ-id. Since S is an $\mathscr{R}$ group, then, by Theorem 12, $ŧ_S$ is an S-int bi-id. The rest of the proof is obvious by Theorem 7 $ŧ_S$ is an S-int bi-interior id of S.

This characterization will be useful for proving subsequent results.

Proposition 5 *Every S-int Ł bi-quasi id is an S-int ɃQĪ-id.*

Proof: Let $ŧ_S$ be an S-int ɃQĪ-id of S. Then, $\left(\widetilde{\mathbb{S}} \circ ŧ_S\right) \widetilde{\cap} \left(ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S\right) \widetilde{\subseteq} ŧ_S$. Since, $\left(ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}}\right) \circ ŧ_S \widetilde{\subseteq} \widetilde{\mathbb{S}} \circ ŧ_S$and $ŧ_S \circ \left(\widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}}\right) \circ ŧ_S \widetilde{\subseteq} ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S$ it is obtained $ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \widetilde{\subseteq} \left(\widetilde{\mathbb{S}} \circ ŧ_S\right) \widetilde{\cap} \left(ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S\right) \widetilde{\subseteq} ŧ_S$. Hence, $ŧ_S$ is an S-int Ł bi-quasi id of S.

We show with a counterexample that the converse of Proposition 5 is not true:

Example 9 Consider the Ss $ŧ_S$ in Example 3. It was shown in Example 3 that $ŧ_S$ is an S-int ɃQĪ-id. Since,$\big(\widetilde{\mathbb{S}} \circ$

$ŧ_S\big)(\wp) \,\widetilde{\cap}\, \big(ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S\big)(\wp) = ŧ_S(\ddot{\varpi}) \nsubseteq ŧ_S(\wp)$. $ŧ_S$ is not an S-int Ł bi-quasi id.

Proposition 6 *Let $ŧ_S \in S_S(U)$ and S bean Ɍ simple ℛ semigroup. Then, the following conditions are equivalent:*
1. *$ŧ_S$ is an S-int Ł bi-quasi id.*
2. *$ŧ_S$ is an S-int ɃQĪ-id.*

Proof: (1) implies (2) is by Proposition 5. Let$ŧ_S$ is an S-int ɃQĪ-id. Since S isɌ simple ℛ semigroup, then, by Theorem 12, $ŧ_S$ is an S-int bi-id and by Proposition 2, $ŧ_S$ is an S-int Ł id. Since, $\big(\widetilde{\mathbb{S}} \circ ŧ_S\big) \widetilde{\subseteq} ŧ_S$and $\big(ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S\big) \widetilde{\subseteq} ŧ_S$ it is obtained $\big(\widetilde{\mathbb{S}} \circ ŧ_S\big) \,\widetilde{\cap}\, \big(ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S\big) \widetilde{\subseteq} ŧ_S$.

Proposition 7 *Every S-int Ɍ bi-quasi id is also an S-int ɃQĪ-id.*

Proof: Let $ŧ_S$ be an S-int ɃQĪ-id of S. Then, $\big(ŧ_S \circ \widetilde{\mathbb{S}}\big) \,\widetilde{\cap}\, \big(ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S\big) \widetilde{\subseteq} ŧ_S$. Since, $ŧ_S \circ \big(\widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S\big) \widetilde{\subseteq} ŧ_S \circ \widetilde{\mathbb{S}}$and $ŧ_S \circ \big(\widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}}\big) \circ ŧ_S \widetilde{\subseteq} ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S$ it is obtained $ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \widetilde{\subseteq} \big(ŧ_S \circ \widetilde{\mathbb{S}}\big) \,\widetilde{\cap}\, \big(ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S\big) \widetilde{\subseteq} ŧ_S$. Hence, $ŧ_S$ is an S-int Ɍ bi-quasi id of S.

We show with a counterexample that the converse of Proposition 7 is not true:

Example 10 Consider the Ꞩꞩ $ŧ_S$ in Example 3. It was shown in Example 3 that $ŧ_S$ is an S-int ɃQĪ-id. Since,$\big(ŧ_S \circ \widetilde{\mathbb{S}}\big)(\wp) \,\widetilde{\cap}\, \big(ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S\big)(\wp) = ŧ_S(\ddot{\varpi}) \nsubseteq ŧ_S(\wp)$$ŧ_S$ is not an S-int Ɍ bi-quasi id.

Proposition 8 *Let$ŧ_S \in S_S(U)$ and$S$ bean Ł simple ℛ semigroup. Then, the following conditions are equivalent:*
1. *$ŧ_S$ is an S-int Ɍ bi-quasi id.*
2. *$ŧ_S$ is an S-int ɃQĪ-id.*

Proof: (1) implies (2) is by Proposition 7. Let$ŧ_S$ is an S-int ɃQĪ-id. Since S isan Ł simple ℛ semigroup, then, by Theorem 12, $ŧ_S$ is an S-int bi-id and by Proposition 4, $ŧ_S$ is an S-int Ɍ id. Since, $\big(ŧ_S \circ \widetilde{\mathbb{S}}\big) \widetilde{\subseteq} ŧ_S$and $\big(ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S\big) \widetilde{\subseteq} ŧ_S$ it is obtained $\big(ŧ_S \circ \widetilde{\mathbb{S}}\big) \,\widetilde{\cap}\, \big(ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S\big) \widetilde{\subseteq} ŧ_S$.

Theorem 21 *Every S-int bi-quasi id is also an S-int ɃQĪ-id.*

Proof: It follows by Proposition 5 and Proposition 7.

Theorem 22 *Let $ŧ_S \in S_S(U)$ and S be an ℛ group. Then, the following conditions are equivalent:*
1. *$ŧ_S$ is an S-int bi-quasi id.*
2. *$ŧ_S$ is an S-int ɃQĪ-id.*

Note that the converse of Theorem 21 is not true, following from Example 9 and Example 10. Theorem 22 shows that the converse of Theorem 21 holds for ℛ groups.

Proposition 9 *Every S-int Ł quasi-interior id is also an S-int ɃQĪ-id.*

Proof: Let $ŧ_S$ be an S-int Ł quasi-interior id of S. Then, $\widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \widetilde{\subseteq} ŧ_S$. Thus,$\big(ŧ_S \circ \widetilde{\mathbb{S}}\big) \circ ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \widetilde{\subseteq} \widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \widetilde{\subseteq} ŧ_S$.Hence, $ŧ_S$ is an S-int ɃQĪ-id of S.

Proposition 10 *Let $ŧ_S \in S_S(U)$ and S be an Ɍ simple semigroup. Then, the following conditions are equivalent:*
1. $ŧ_S$ is an S-int Ł quasi-interior id.
2. $ŧ_S$ is an S-int ɃQĪ-id.

Proof: (1) implies (2) is by Proposition 9. Let $ŧ_S$ is an S-int ɃQĪ-id and ᶅ, ᶇ, ƫ, đ $\in S$. By assumption, there exists ᵯ $\in S$ such that ᶅ $=$ ᶇᵯ. Thus,$ŧ_S$(ᶅᶇƫđ) $=$ $ŧ_S$((ᶇᵯ)ᶇƫđ) $=$ $ŧ_S$(ᶇᵯᶇƫđ) $\supseteq$ $ŧ_S$(ᶇ) $\cap$ $ŧ_S$(ᶇ) $\cap$ $ŧ_S$(đ) $=$ $ŧ_S$(ᶇ) $\cap$ $ŧ_S$(đ). Thus, $ŧ_S$ is an S-int Ł quasi-interior id.

Proposition 11 *Every S-int Ɍ quasi-interior id is also an S-int ɃQĪ-id.*

Proof: Let $ŧ_S$ be an S-int Ɍ quasi-interior id of S. Then, $ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} ŧ_S$. Thus, $ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \circ \big(\widetilde{\mathbb{S}} \circ ŧ_S\big) \widetilde{\subseteq} ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} ŧ_S$. Hence, $ŧ_S$ is an S-int ɃQĪ-id of S.

Example 11 Let $S = \{$ℵ, ɘ, ɍ, 3, ư$\}$ be defined as follows:

⁂	ℵ	ɘ	ɍ	3	ư
ℵ	ℵ	3	ℵ	3	3
ɘ	ℵ	ɘ	ℵ	3	3
ɍ	ℵ	3	ɍ	3	ư
3	ℵ	3	ℵ	3	3
ư	ℵ	3	ɍ	3	ư

Let f_S be Ꞩꞩ over Uas follows: $f_S = \{$(ℵ, U), (ɘ, $\emptyset$), (ɍ, U), (3, $\emptyset$), (ư, $\emptyset$)$\}$. Here, f_S is an S-int ɃQĪ-id. However, sincef_S(ɍɘɍɘ) $= f_S(3) \nsupseteq f_S$(ɍ), f_S is not an S-int Ɍ quasi-interior id.

Proposition 12 *Let $f_S \in S_S(U)$ and S be an Ł simple semigroup. Then, the following conditions are equivalent:*
1. $ŧ_S$ is an S-int Ɍ quasi-interior id.
2. $ŧ_S$ is an S-int ɃQĪ-id.

Proof: (1) implies (2) is by Proposition 11. Let$ŧ_S$ is an S-int ɃQĪ-id and ᶅ, ᶇ, ƫ, đ $\in S$. By assumption, there exists ᵯ $\in S$ such that đ $=$ ᵯƫ. Thus,$ŧ_S$(ᶅᶇƫđ) $=$ $ŧ_S$(ᶅᶇƫ(ᵯƫ)) $=$ $ŧ_S$(ᶅᶇƫᵯƫ) $\supseteq$ $ŧ_S$(ᶅ) $\cap$ $ŧ_S$(ƫ) $\cap$ $ŧ_S$(ƫ) $=$ $ŧ_S$(ᶅ) $\cap$ $ŧ_S$(ƫ).Thus, $ŧ_S$ is an S-int Ɍ quasi-interior id.

Theorem 23 *Every S-int quasi-interior id is also an S-int ɃQĪ-id.*

Proof: It follows from Proposition 9 and Proposition 11.

Theorem 24 *Let $ŧ_S \in S_S(U)$ and S be a group. Then, the following conditions are equivalent:*
1. *$ŧ_S$ is an S-int quasi-interior id.*

2. $ŧ_S$ *is an S-int ɃQĪ-id.*

Proof: (1) implies (2) is by Proposition 9 and Proposition 11. Let S be a group. The rest of the proof is obvious by Proposition 10 and Proposition 12. Note here that the converse of Theorem 23 is not true, following from Example 11. Theorem 24 shows that the converse of Theorem 23 holds for groups.

The following theorem analyzes the behavior of S-int ɃQĪ ideals under soft set operations. The following theorem analyzes the behavior of S-int ɃQĪ ideals under soft set operations. We prove that finite soft AND-products, Cartesian products, and intersections of S-int ɃQĪ ideals remain S-int ɃQĪ ideals. In contrast, finite soft OR-products and unions of S-int ɃQĪ ideals are not S-int ɃQĪ ideals.

Theorem 25 *Let* $ʒ_S$ *and* $ʒ_T$ *be S-int ɃQĪ-ids of* S *and* T*, respectively. Then,* $ʒ_S \wedge ʒ_T$ *is an S-int ɃQĪ-ids of* $S \times T$*.*

Proof: Let $(ɵ_1, ƃ_1)$, $(ɵ_2, ƃ_2)$, $(ɵ_3, ƃ_3)$, $(ɵ_4, ƃ_4)$, $(ɵ_5, ƃ_5) \in S \times T$Then,

$$\begin{aligned} &ʒ_{S\wedge T}\big((ɵ_1, ƃ_1)(ɵ_2, ƃ_2)(ɵ_3, ƃ_3)(ɵ_4, ƃ_4)(ɵ_5, ƃ_5)\big) \\ &= ʒ_{S\wedge T}(ɵ_1ɵ_2ɵ_3ɵ_4ɵ_5, ƃ_1ƃ_2ƃ_3ƃ_4ƃ_5) \\ &= ʒ_S(ɵ_1ɵ_2ɵ_3ɵ_4ɵ_5) \cap ʒ_T(ƃ_1ƃ_2ƃ_3ƃ_4ƃ_5) \\ &\supseteq \big(ʒ_S(ɵ_1) \cap ʒ_S(ɵ_3) \cap ʒ_S(ɵ_5)\big) \cap \big(ʒ_T(ƃ_1) \cap ʒ_T(ƃ_3) \cap ʒ_T(ƃ_5)\big) \\ &= \big(ʒ_S(ɵ_1) \cap ʒ_T(ƃ_1)\big) \cap \big(ʒ_S(ɵ_3) \cap ʒ_T(ƃ_3)\big) \cap \big(ʒ_S(ɵ_5) \cap ʒ_T(ƃ_5)\big) \\ &= ʒ_{S\wedge T}(ɵ_1, ƃ_1) \cap ʒ_{S\wedge T}(ɵ_3, ƃ_3) \cap ʒ_{S\wedge T}(ɵ_5, ƃ_5) \end{aligned}$$

Thus, $ʒ_S \wedge ʒ_T$ is an S-int ɃQĪ-id of $S \times T$.

Note her that $ʒ_S \vee ʒ_T$ is not always an S-int ɃQĪ-id with Example 12.

Example 12 Let's consider the subsemigroups in Example 1 and Example 4. Let $ŧ_S$ and $ŧ_{\mathfrak{K}}$ be Ꞩꞩ over $U = Z_6 = \{0,1,2,3,4,5\}$ as follows:

$ŧ_S = \{(ȥ, \{1,3\}), (ʯ, \{2,4,5\}), (ɰ, \{1,2,3,4,5\})\}$,$ŧ_{\mathfrak{K}} = \{(ȡ, \{1,3,4,5\}), (ƙ, \{2\})\}$

$ŧ_S$ and $ŧ_{\mathfrak{K}}$are S-int ɃQĪ-ids. Here, since

$$ŧ_{S\vee\mathfrak{K}}\big((ʯ, ȡ)(ʯ, ȡ)(ʯ, ȡ)(ʯ, ȡ)(ȥ, ƙ)\big) \not\supseteq ŧ_{S\vee\mathfrak{K}}(ʯ, ȡ) \cap ŧ_{S\vee\mathfrak{K}}(ʯ, ȡ) \cap ŧ_{S\vee\mathfrak{K}}(ȥ, ƙ)$$

$ŧ_S \vee ŧ_{\mathfrak{K}}$ is not an S-int ɃQĪ-id.

Theorem 26 *Let* $ŧ_S$ *and* $ŧ_T$ *be S-int ɃQĪ-ids of* S *and* T *over* U*, respectively. Then,* $ŧ_S \times ŧ_T$ *is an S-int ɃQĪ-id of* $S \times T$ *over* $U \times U$*.*

Proof: Let $ŧ_S \times ŧ_T = ŧ_{S\times T}$, where $ŧ_{S\times T}(ɵ, ƃ) = ŧ_S(ɵ) \times ŧ_T(ƃ)$ for all $(ɵ, ƃ) \in S \times T$. Then, for all $(ɵ_1, ƃ_1)$, $(ɵ_2, ƃ_2)$, $(ɵ_3, ƃ_3)$, $(ɵ_4, ƃ_4)$, $(ɵ_5, ƃ_5) \in S \times T$,

$$\begin{aligned} &ŧ_{S\times T}\big((ɵ_1, ƃ_1)(ɵ_2, ƃ_2)(ɵ_3, ƃ_3)(ɵ_4, ƃ_4)(ɵ_5, ƃ_5)\big) \\ &= ŧ_{S\times T}(ɵ_1ɵ_2ɵ_3ɵ_4ɵ_5, ƃ_1ƃ_2ƃ_3ƃ_4ƃ_5) \\ &= ŧ_S(ɵ_1ɵ_2ɵ_3ɵ_4ɵ_5) \times ŧ_T(ƃ_1ƃ_2ƃ_3ƃ_4ƃ_5) \\ &\supseteq \big(ŧ_S(ɵ_1) \cap ŧ_S(ɵ_3) \cap ŧ_S(ɵ_5)\big) \times \big(ŧ_T(ƃ_1) \cap ŧ_T(ƃ_3) \cap ŧ_T(ƃ_5)\big) \\ &= \big(ŧ_S(ɵ_1) \times ŧ_T(ƃ_1)\big) \cap \big(ŧ_S(ɵ_3) \times ŧ_T(ƃ_3)\big) \cap \big(ŧ_S(ɵ_5) \times ŧ_T(ƃ_5)\big) \\ &= ŧ_{S\times T}(ɵ_1, ƃ_1) \cap ŧ_{S\times T}(ɵ_3, ƃ_3) \cap ŧ_{S\times T}(ɵ_5, ƃ_5) \end{aligned}$$

Hence, $ŧ_S \times ŧ_T = ŧ_{S\times T}$ is an S-int ɃQĪ-id of $S \times T$ over $U \times U$.

Theorem 27 *Let* $ɷ_S$ *and* $ƅ_S$ *be S-int ɃQĪ-ids. Then,* $ɷ_S \widetilde{\cap} ƅ_S$ *is an S-int ɃQĪ-id.*

Proof: Let $ɷ_S$ and $ƅ_S$ be S-int ɃQĪ-ids of S. Then, $ɷ_S \circ \widetilde{\mathbb{S}} \circ ɷ_S \circ \widetilde{\mathbb{S}} \circ ɷ_S \,\widetilde{\subseteq}\, ɷ_S$ and $ƅ_S \circ \widetilde{\mathbb{S}} \circ ƅ_S \circ \widetilde{\mathbb{S}} \circ ƅ_S \,\widetilde{\subseteq}\, ƅ_S$. Thus,$(ɷ_S \widetilde{\cap} ƅ_S) \circ \widetilde{\mathbb{S}} \circ (ɷ_S \widetilde{\cap} ƅ_S) \circ \widetilde{\mathbb{S}} \circ (ɷ_S \widetilde{\cap} ƅ_S) \,\widetilde{\subseteq}\, ɷ_S \circ \widetilde{\mathbb{S}} \circ ɷ_S \circ \widetilde{\mathbb{S}} \circ ɷ_S \,\widetilde{\subseteq}\, ɷ_Sand(ɷ_S \widetilde{\cap} ƅ_S) \circ \widetilde{\mathbb{S}} \circ (ɷ_S \widetilde{\cap} ƅ_S) \circ \widetilde{\mathbb{S}} \circ (ɷ_S \widetilde{\cap} ƅ_S) \,\widetilde{\subseteq}\, ƅ_S \circ \widetilde{\mathbb{S}} \circ ƅ_S \circ \widetilde{\mathbb{S}} \circ ƅ_S \,\widetilde{\subseteq}\, ƅ_S$.Hence,$(ɷ_S \widetilde{\cap} ƅ_S) \circ \widetilde{\mathbb{S}} \circ (ɷ_S \widetilde{\cap} ƅ_S) \circ \widetilde{\mathbb{S}} \circ (ɷ_S \widetilde{\cap} ƅ_S) \,\widetilde{\subseteq}\, ɷ_S \widetilde{\cap} ƅ_S$. Thus, $ɷ_S \widetilde{\cap} ƅ_S$ is an S-int ɃQĪ-id.

Corollary 2 Let $ɷ_S$ be an S-int Ɽ id (Ł id/id/bi-id/interior-id/quasi-id/bi-interior/left bi-quasi/right bi-quasi/bi-quasi/left quasi-interior/right quasi-interior/quasi-interior)and $ƅ_S$ be an S-int Ɽ id (Ł id/id/bi-id/interior-id/quasi-id/bi-interior/left bi-quasi/right bi-quasi/bi-quasi/left quasi-interior/right quasi-interior/quasi-interior). Then, $ɷ_S \widetilde{\cap} ƅ_S$ is an S-int ɃQĪ-id.

Proposition 13 *Let* $ɷ_S$ *be an S-int Ł id and* $ƅ_S$ *be an Ꞩꞩ. Then,* $ɷ_S \circ ƅ_S$ *is an S-int ɃQĪ-id.*

Proof: Let $ɷ_S$ be S-int Ł id of S. Then, $\widetilde{\mathbb{S}} \circ ɷ_S \,\widetilde{\subseteq}\, ɷ_S$ and $ŧ_S \circ ŧ_S \,\widetilde{\subseteq}\, ŧ_S$. Thus,$(ɷ_S \circ ƅ_S) \circ \widetilde{\mathbb{S}} \circ (ɷ_S \circ ƅ_S) \circ \widetilde{\mathbb{S}} \circ (ɷ_S \circ ƅ_S) = ɷ_S \circ ƅ_S \circ \left(\widetilde{\mathbb{S}} \circ ɷ_S\right) \circ ƅ_S \circ \left(\widetilde{\mathbb{S}} \circ ɷ_S\right) \circ ƅ_S \,\widetilde{\subseteq}\, ɷ_S \circ ƅ_S \circ ɷ_S \circ ƅ_S \circ ɷ_S \circ ƅ_S \,\widetilde{\subseteq}\, ɷ_S \circ \left(\widetilde{\mathbb{S}} \circ ɷ_S\right) \circ \left(\widetilde{\mathbb{S}} \circ ɷ_S\right) \circ ƅ_S \,\widetilde{\subseteq}\, (ɷ_S \circ ɷ_S) \circ ɷ_S \circ ƅ_S \,\widetilde{\subseteq}\, (ɷ_S \circ ɷ_S) \circ ƅ_S \,\widetilde{\subseteq}\, ɷ_S \circ ƅ_S$. Thus, $ŧ_S \circ ƅ_S$ is an S-int ɃQĪ-id.

Proposition 14 *Let* $ɷ_S$ *be an S-int Ɽ id and* $ƅ_S$ *be an Ꞩꞩ. Then,* $ɷ_S \circ ƅ_S$ *is an S-int ɃQĪ-id.*

Proof: Let $ɷ_S$ be S-int Ɽ id of S. Then, $ɷ_S \circ \widetilde{\mathbb{S}} \,\widetilde{\subseteq}\, ɷ_S$ and $ɷ_S \circ ɷ_S \,\widetilde{\subseteq}\, ŧ_S$. Thus,$(ɷ_S \circ ƅ_S) \circ \widetilde{\mathbb{S}} \circ (ɷ_S \circ ƅ_S) \circ \widetilde{\mathbb{S}} \circ (ɷ_S \circ ƅ_S) \,\widetilde{\subseteq}\, ɷ_S \circ \left(\widetilde{\mathbb{S}} \circ \widetilde{\mathbb{S}}\right) \circ ɷ_S \circ \left(\widetilde{\mathbb{S}} \circ \widetilde{\mathbb{S}}\right) \circ ɷ_S \circ ƅ_S \,\widetilde{\subseteq}\, \left(ɷ_S \circ \widetilde{\mathbb{S}}\right) \circ \left(ɷ_S \circ \widetilde{\mathbb{S}}\right) \circ ɷ_S \circ ƅ_S \,\widetilde{\subseteq}\, (ɷ_S \circ ɷ_S) \circ ɷ_S \circ ƅ_S \,\widetilde{\subseteq}\, (ɷ_S \circ ɷ_S) \circ ƅ_S \,\widetilde{\subseteq}\, ɷ_S \circ ƅ_S$.Thus, $ɷ_S \circ ƅ_S$ is an S-int ɃQĪ-id.

Theorem 28 *Let* $ɷ_S$ *be an S-int id and* $ƅ_S$ *be an Ꞩꞩ. Then,* $ɷ_S \circ ƅ_S$ *is an S-int ɃQĪ-id.*

Proposition 15 *Let* $ƅ_S$ *be an S-int Ł id and* $ɷ_S$ *be an Ꞩꞩ. Then,* $ɷ_S \circ ƅ_S$ *is an S-int ɃQĪ-id.*

Proof: Let $ƅ_S$be S-int Ł id of S. Then, $\widetilde{\mathbb{S}} \circ ƅ_S \,\widetilde{\subseteq}\, ƅ_S$ and $ƅ_S \circ ƅ_S \,\widetilde{\subseteq}\, ƅ_S$.

Then,$(ω_S ∘ ƃ_S) ∘ \widetilde{\mathbb{S}} ∘ (ω_S ∘ ƃ_S) ∘ \widetilde{\mathbb{S}} ∘ (ω_S ∘ ƃ_S) \tilde{⊆} ω_S ∘ ƃ_S ∘ \left(\widetilde{\mathbb{S}} ∘ \widetilde{\mathbb{S}}\right) ∘ ƃ_S ∘ \left(\widetilde{\mathbb{S}} ∘ \widetilde{\mathbb{S}}\right) ∘ ƃ_S \tilde{⊆} ω_S ∘ ƃ_S ∘ \left(\widetilde{\mathbb{S}} ∘ ƃ_S\right) ∘ \left(\widetilde{\mathbb{S}} ∘ ƃ_S\right) \tilde{⊆} ω_S ∘ (ƃ_S ∘ ƃ_S) ∘ ƃ_S \tilde{⊆} ω_S ∘ (ƃ_S ∘ ƃ_S) \tilde{⊆} ω_S ∘ ƃ_S$. Thus, $ω_S ∘ ƃ_S$ is an S-int ꞖQĪ-id.

Proposition 16 *Let $ƃ_S$ be an S-int Ɍ id and $ω_S$ be an Ꞩs. Then, $ω_S ∘ ƃ_S$ is an S-int ꞖQĪ-id.*

Proof: Let $ƃ_S$ be S-int Ɍ id of S. Then, $ƃ_S ∘ \widetilde{\mathbb{S}} \tilde{⊆} ƃ_S$ and $ƃ_S ∘ ƃ_S \tilde{⊆} ƃ_S$. Thus, $(ω_S ∘ ƃ_S) ∘ \widetilde{\mathbb{S}} ∘ (ω_S ∘ ƃ_S) ∘ \widetilde{\mathbb{S}} ∘ (ω_S ∘ ƃ_S) \tilde{⊆} ω_S ∘ \left(ƃ_S ∘ \widetilde{\mathbb{S}}\right) ∘ ω_S ∘ \left(ƃ_S ∘ \widetilde{\mathbb{S}}\right) ∘ ω_S ∘ ƃ_S \tilde{⊆} ω_S ∘ ƃ_S ∘ ω_S ∘ ƃ_S ∘ ω_S ∘ ƃ_S \tilde{⊆} ω_S ∘ \left(ƃ_S ∘ \widetilde{\mathbb{S}}\right) ∘ \left(ƃ_S ∘ \widetilde{\mathbb{S}}\right) ∘ ƃ_S \tilde{⊆} ω_S ∘ (ƃ_S ∘ ƃ_S) ∘ ƃ_S \tilde{⊆} ω_S ∘ (ƃ_S ∘ ƃ_S) \tilde{⊆} ω_S ∘ ƃ_S$. Thus, $ŧ_S ∘ ƃ_S$ is an S-int ꞖQĪ-id.

Theorem 29 *Let $ƃ_S$ be an S-int id and $ω_S$ be an Ꞩs. Then, $ω_S ∘ ƃ_S$ is an S-int ꞖQĪ-id.*

Proposition 17 *Let $ω_S$ and $ƀ_S$ be Ꞩss over U. If $ω_S ∘ ƀ_S$ is an* S-int Ł id*, then it is an S-int ꞖQĪ-id.*

Proof: Let $ŧ_S ∘ ƀ_S$ is an S-int Ł id. Then,$(ω_S ∘ ƀ_S) ∘ \left[\widetilde{\mathbb{S}} ∘ (ω_S ∘ ƀ_S)\right] ∘ \left[\widetilde{\mathbb{S}} ∘ (ω_S ∘ ƀ_S)\right] \tilde{⊆} [(ω_S ∘ ƀ_S) ∘ (ω_S ∘ ƀ_S)] ∘ (ω_S ∘ ƀ_S) \tilde{⊆} (ω_S ∘ ƀ_S) ∘ (ω_S ∘ ƀ_S) \tilde{⊆} (ω_S ∘ ƀ_S)$, implying that $ŧ_S ∘ ƀ_S$ is an S-int ꞖQĪ-id.

Theorem 30 *Let $ŧ_S$ be an S-int subsemigroup over U, $α$ be a subset of U, $Iƃ(ŧ_S)$ be the image of $ŧ_S$ such that $α ∈ Im(ŧ_S)$. If $ŧ_S$ is an S-int ꞖQĪ-id of S, then $\mathcal{U}(ŧ_S; α)$ is a ꞖQĪ id of S.*

Proof: Since, $ŧ_S(x) = α$ for some$x ∈ S, ∅ ≠ \mathcal{U}(ŧ_S; α) ⊆ S$. Let$k ∈ \mathcal{U}(ŧ_S; α) · S · \mathcal{U}(ŧ_S; α) · S · \mathcal{U}(ŧ_S; α)$. Then, there exist $x, y, z ∈ \mathcal{U}(ŧ_S; α)$ and $b ∈ S$ such that $k = xbybz$. Thus, $ŧ_S(x) ⊇ α$, $ŧ_S(y) ⊇ α$ and $ŧ_S(z) ⊇ α$. Since $ŧ_S$ is an S-int ꞖQĪ-id,$ŧ_S(k) = ŧ_S(xbybz) ⊇ ŧ_S(x) ∩ ŧ_S(y) ∩ ŧ_S(z) ⊇ α ∩ α ∩ α ⊇ α$. Hence, $ŧ_S(k) ⊇ α$, implying that $k ∈ \mathcal{U}(ŧ_S; α)$. Therefore, $\mathcal{U}(ŧ_S; α) · S · \mathcal{U}(ŧ_S; α) · S · \mathcal{U}(ŧ_S; α) ⊆ \mathcal{U}(ŧ_S; α)$. Moreover, since $ŧ_S$ is an S-int subsemigroup over U, by Theorem 8, $\mathcal{U}(ŧ_S; α)$ is a subsemigroup of S. Thus,$\mathcal{U}(ŧ_S; α)$ is a ꞖQĪ id.

We illustrate Theorem 30 with Example 13.

Example 13 Consider Example 1. It was shown in Example 1 that $ŧ_{\mathrm{S}}$ is an S-int ꞖQĪ-id.

By considering the image set of $ŧ_{\mathrm{S}}$, that is, $Im(ŧ_{\mathrm{S}}) = \left\{\{\bar{1}, \bar{3}, \bar{5}\}, \{\bar{1}, \bar{3}, \bar{7}\}, \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\}\right\}$, we obtain the following:

$$\mathcal{U}(ŧ_{\mathrm{S}}; α)=\begin{cases} \{ȥ, ɰ\}, & α = \{\bar{1}, \bar{3}, \bar{5}\} \\ \{Ꞁ, ɰ\}, & α = \{\bar{1}, \bar{3}, \bar{7}\} \\ \{ɰ\}, & α = \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\} \end{cases}$$

Here,$\{ȥ, ɰ\}, \{Ꞁ, ɰ\}$and $\{ɰ\}$are all ꞖQĪ ids of S. In fact, since$\{ȥ, ɰ\} · \{ȥ, ɰ\} ⊆ \{ȥ, ɰ\}, \{Ꞁ, ɰ\} · \{Ꞁ, ɰ\} ⊆ \{Ꞁ, ɰ\},\{ɰ\} · \{ɰ\} ⊆ \{ɰ\}$each $\mathcal{U}(ŧ_{\mathrm{S}}; α)$ is a subsemigroup of S. Similarly, since$\{ȥ, ɰ\} · \mathrm{S} · \{ȥ, ɰ\} · \mathrm{S} · \{ȥ, ɰ\} = \{ɰ\} ⊆ \{ȥ, ɰ\}, \{Ꞁ, ɰ\} · S · \{Ꞁ, ɰ\} · S · \{Ꞁ, ɰ\} = \{Ꞁ, ɰ\} ⊆ \{Ꞁ, ɰ\}, \{ɰ\} · S · \{ɰ\} · S · \{ɰ\} = \{ɰ\} ⊆ \{ɰ\}$each $\mathcal{U}(ŧ_{\mathrm{S}}; α)$ is a ꞖQĪ id of S. Now, consider the Ꞩs $ɓ_{\mathrm{S}}$ in Example 1. By taking into account $Im(ɓ_{\mathrm{S}}) = \left\{\{\bar{1}, \bar{3}, \bar{5}\}, \{\bar{1}, \bar{3}, \bar{7}\}\right\}$ we obtain the following:

$$\mathcal{U}(ɓ_{\mathrm{S}}; α)=\begin{cases} \{ȥ, ɰ\} & α = \{\bar{1}, \bar{3}, \bar{5}\} \\ \{Ꞁ\} & α = \{\bar{1}, \bar{3}, \bar{7}\} \end{cases}$$

Here, $\{Ꞁ\}$is not a ꞖQĪ id of S. In fact, since$\{Ꞁ\} · \mathrm{S} · \{Ꞁ\} · \mathrm{S} · \{Ꞁ\} = \{ȥ, Ꞁ, ɰ\} ⊈ \{Ꞁ\}$one of the $\mathcal{U}(ɓ_{\mathrm{S}}; α)$ is not a ꞖQĪ id of S, hence it is not a ꞖQĪ id of S, It is seen that each of $\mathcal{U}(ɓ_{\mathrm{S}}; α)$ is not a ꞖQĪ id of S. On the other hand, in Example 1 it was shown that $ɓ_{\mathrm{S}}$ is not an S-int ꞖQĪ-id of S.

Definition 14 *Let $ŧ_S$ be an S-int subsemigroup and S-int ꞖQĪ-id of S. Then, the ꞖQĪ ids $\mathcal{U}(ŧ_S; α)$ are called upper α-ꞖQĪ ids of $ŧ_S$.*

Proposition 18 *Let $ŧ_S$ be an Ꞩs over $U, \mathcal{U}(ŧ_S; α)$ be upper α-ꞖQĪ of $ŧ_S$ for each $α ⊆ U$ and $Im(ŧ_S)$ be an ordered set by inclusion. Then, $ŧ_S$ is an S-int ꞖQĪ-id.*

Proof: Let $x, y, z, b, c ∈ S$ and $ŧ_S(x) = α_1$, $ŧ_S(y) = α_2$ and $ŧ_S(z) = α_3$. Suppose that $α_1 ⊆ α_2$. It is obvious that $x ∈ \mathcal{U}(ŧ_S; α_1)$ and $y ∈ \mathcal{U}(ŧ_S; α_2)$. Since $α_1 ⊆ α_2 ⊆ α_3$, $x, y, z ∈ \mathcal{U}(ŧ_S; α_1)$ and since $\mathcal{U}(ŧ_S; α)$ is a ꞖQĪ of S for all $α ⊆ U$, it follows that $xbycz ∈ \mathcal{U}(ŧ_S; α_1)$. Hence, $ŧ_S(xbycz) ⊇ α_1 = α_1 ∩ α_2 ∩ α_3$. Thus, $ŧ_S$ is an S-int ꞖQĪ-id.

Proposition 19 *Let $ŧ_S$ and $ŧ_T$ be Ꞩs over U, and Ψ be a semigroup isomorphism from S to T. If $ŧ_S$ is an S-int ꞖQĪ-id of S, then $Ψ(ŧ_S)$ is an S-int ꞖQĪ-id of T.*

Proof: Let $ƀ_1, ƀ_2, ƀ_3, ƀ_4, ƀ_5 ∈ T$. Since Ψ is surjective, there exist $ɵ_1, ɵ_2, ɵ_3, ɵ_4, ɵ_5 ∈ S$ such that $Ψ(ɵ_1) = ƀ_1$, $Ψ(ɵ_2) = ƀ_2$, $Ψ(ɵ_3) = ƀ_3$, $Ψ(ɵ_4) = ƀ_4$ and $Ψ(ɵ_5) = ƀ_5$. Then,

$$\begin{aligned}
&\left(Ψ(ŧ_S)\right)(ƀ_1ƀ_2ƀ_3ƀ_4ƀ_5) \\
&= \bigcup\{ŧ_S(ɵ): ɵ ∈ S, Ψ(ŧ_S) = ƀ_1ƀ_2ƀ_3ƀ_4ƀ_5\} \\
&= \bigcup\{ŧ_S(ɵ): ɵ ∈ S, ɵ = Ψ^{-1}(ƀ_1ƀ_2ƀ_3ƀ_4ƀ_5)\} \\
&= \bigcup\{ŧ_S(ɵ): ɵ ∈ S, ɵ = Ψ^{-1}\left(Ψ(ɵ_1ɵ_2ɵ_3ɵ_4ɵ_5)\right) = ɵ_1ɵ_2ɵ_3ɵ_4ɵ_5\} \\
&= \bigcup\{ŧ_S(ɵ_1ɵ_2ɵ_3ɵ_4ɵ_5): ɵ_i ∈ S, Ψ(ɵ_i) = ƀ_i, i = 1,2,3,4,5\} \\
&⊇ \bigcup\{ŧ_S(ɵ_1) ∩ ŧ_S(ɵ_3) ∩ ŧ_S(ɵ_5): ɵ_1, ɵ_3, ɵ_5 ∈ S, Ψ(ɵ_1) = ƀ_1, Ψ(ɵ_3) = ƀ_3 \text{ and } Ψ(ɵ_5) = ƀ_5\} \\
&= \left(Ψ(ŧ_S)\right)(ɵ_1) ∩ \left(Ψ(ŧ_S)\right)(ɵ_3) ∩ \left(Ψ(ŧ_S)\right)(ɵ_5)
\end{aligned}$$

Hence, $Ψ(ŧ_S)$ is an S-int ꞖQĪ-id of T.

Proposition 20 *Let $ŧ_S$ and $ŧ_T$ be Ꞩs over U, and Ψ be a semigroup isomorphism from S to T. If $ŧ_T$ is an S-int ꞖQĪ-id of S, then $Ψ^{-1}(ŧ_T)$ is an S-int ꞖQĪ-id of T.*

Proof: Let $ɵ_1, ɵ_2, ɵ_3, ɵ_4, ɵ_5 ∈ S$. Then,

$$\left(Ψ^{-1}(ŧ_T)\right)(ɵ_1ɵ_2ɵ_3ɵ_4ɵ_5) = ŧ_T\left(Ψ(ɵ_1ɵ_2ɵ_3ɵ_4ɵ_5)\right)$$

$$= ŧ_T(\Psi(\vartheta_1)\Psi(\vartheta_2)\Psi(\vartheta_3)\Psi(\vartheta_4)\Psi(\vartheta_5))$$
$$\supseteq ŧ_T\,(\Psi(\vartheta_1)) \cap ŧ_T(\Psi(\vartheta_3)) \cap ŧ_T(\Psi(\vartheta_5)))$$
$$= \left(\Psi^{-1}(ŧ_T)\right)(\vartheta_1) \cap \left(\Psi^{-1}(ŧ_T)\right)(\vartheta_3) \cap \left(\Psi^{-1}(ŧ_T)\right)(\vartheta_5)$$

Thus, $\Psi^{-1}(ŧ_T)$ is an S-int ɃQĪ-id of T.

Theorem 31 *For a semigroup S, the following conditions are equivalent:*
1. S is a $\mathscr{R}$ semigroup.
2. $ŧ_S = ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S$ for every S-int ɃQĪ-id of S.

Proof: First assume that (1) holds. Let S be an $\mathscr{R}$ semigroup, $ŧ_S$ be an S-int ɃQĪ-id of S and $ᵯ \in S$. Then, $ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \,\widetilde{\subseteq}\, ŧ_S$ and there exists an element $ᵰ \in S$ such that $ᵯ = ᵯᵰᵯ$. Thus,

$$\left(ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S\right)(ᵯ)$$
$$= \bigcup_{ᵯ = ƀᵰ} \left\{\left(ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}}\right)(ƀ) \cap ŧ_S(ᵰ)\right\}$$
$$\supseteq \left(ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}}\right)(ᵯᵰ) \cap ŧ_S(ᵯ)$$
$$= \bigcup_{ᵯᵰ = pq} \left\{\left(ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S\right)(p) \cap \widetilde{\mathbb{S}}(q)\right\} \cap ŧ_S(ᵯ)$$
$$\supseteq \left(ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S\right)(ᵯ) \cap \widetilde{\mathbb{S}}(ᵰᵯᵰ) \cap ŧ_S(ᵯ)$$
$$= \bigcup_{ᵯ = ƀᵰ} \left\{\left(ŧ_S \circ \widetilde{\mathbb{S}}\right)(ƀ) \cap ŧ_S(ᵰ)\right\} \cap ŧ_S(ᵯ)$$
$$\supseteq \left(ŧ_S \circ \widetilde{\mathbb{S}}\right)(ᵯᵰ) \cap ŧ_S(ᵯ) \cap ŧ_S(ᵯ)$$
$$= \bigcup_{ᵯᵰ = pq} \left\{ŧ_S(p) \cap \widetilde{\mathbb{S}}(q)\right\} \cap ŧ_S(ᵯ)$$
$$\supseteq ŧ_S(ᵯ) \cap \widetilde{\mathbb{S}}(ᵰᵯᵰ) \cap ŧ_S(ᵯ)$$
$$= ŧ_S(ᵯ)$$

Therefore, $ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \,\widetilde{\supseteq}\, ŧ_S$ implying that $ŧ_S = ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S$.

Conversely, let $ŧ_S = ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S \circ \widetilde{\mathbb{S}} \circ ŧ_S$ where $ŧ_S$ is an S-int ɃQĪ-id of S. To show that S is an $\mathscr{R}$ semigroup, we need to show that $A = ASASA$ for every ɃQĪ id A of S. It is obvious that $ASASA \subseteq A$. Thus, it is enough to show that $A \subseteq ASASA$. Let $ƀ \in A$, and A be any ɃQĪ id of S. Then, S_A is an S-int ɃQĪ of S. Hence, $S_A(ƀ) = (S_A \circ \widetilde{\mathbb{S}} \circ S_A \circ \widetilde{\mathbb{S}} \circ S_A)(ƀ) = (S_A \circ S_S \circ S_A \circ S_S \circ S_A)(ƀ) = S_{ASASA}(ƀ) = U$ implying that $ƀ \in ASASA$. Hence, $A = ASASA$, so S is an $\mathscr{R}$ semigroup.

IV. CONCLUSION

In this study, the concept of the soft intersection (S-int) bi-quasi-interior (ɃQĪ) ideal in semigroups is proposed and the relations of several types of S-int ideals with S-int ɃQĪ ideals are provided. We established equivalent definitions and investigated the fundamental properties of this ideal. It is obtained that every S-int bi-ideal, S-int ideal, S-int interior ideal, S-int quasi-ideal, S-int bi-interior ideal, S-int bi-quasi ideal, and S-int quasi-interior ideal of a semigroup is an S-int ɃQĪ ideal. We also showed, by means of counterexamples, that the converses do not hold in general. The conditions for the converses to hold are also explored. In addition, we proved that the class of S-int ɃQĪ ideals is closed under finite soft AND-products, Cartesian products, and intersections, but not under finite soft OR-products or unions. Moreover, it is shown that if a subsemigroup of a semigroup is a ɃQĪ ideal, then its soft characteristic function is also an S-int ɃQĪ ideal, and the converse is also true. The results presented here build a bridge between classical semigroup theory and soft set theory, offering new directions for future research. Future work could explore more characterizations of an S-int ɃQĪ ideal with certain types of semigroups like intra-regular, weakly-regular, quasi-regular, semisimple and duo semigroups.

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REFERENCES

[1] R. A. Good and D. R. Hughes, "Associated groups for a semigroup, Bulletin of the American Mathematical Society, vol. 58, pp.624–625, 1952.
[2] O. Steinfeld, "Uher die quasi ideals," Von halbgruppend Publication Mathematical Debrecen, vol. 4, pp. 262-275, 1956.
[3] S. Lajos, " (m;k;n)-ideals in semigroups. Notes on semigroups II."Karl Marx University of Economics Department of Mathematics Budapest, vol. 1, pp. 12–19, 1976.
[4] G. Szasz, "Interior ideals in semigroups. Notes on semigroups IV," Karl Marx University of Economics Department of MathematicsBudapest, vol. 5, pp. 1-7, 1977.
[5] G. Szasz, "Remark on interior ideals of semigroup," Studia Scientiarum Mathematicarum Hungarica, vol. 16, pp. 61-63, 1981.
[6] M. M. K. Rao, "Bi-interior ideals of semigroups," Discussiones Mathematica e-general Algebra and Applications, vol. 38, pp. 69-78, 2018.
[7] M. M. K. Rao, "A study of a generalization of bi-ideal, quasi ideal and interior ideal of semigroup," Mathematica Moravica, vol. 22, pp. 103-115, 2018.
[8] M. M. K. Rao, "Left Bi-quasi ideals of semigroups," Southeast Asian Bulletin of Mathematics, vol. 44, pp. 369–376, 2020.
[9] M. M. K. Rao, "Quasi-interior ideals and weak-interior ideals," Asia Pacific Journal of Mathematics, vol. 7, no. 21, pp. 1-20, 2020.
[10] M. M. K. Rao, "Tri-ideals of semigroups," Bulletin of International Mathematical Virtual Institute, vol. 14, no. 2, pp. 213-223, 2024.
[11] M. M. K. Rao, R. K. Kona, N. Rafi, "Tri-quasi ideals and fuzzy tri-quasi ideals of semigroups," Annals of Communications in Mathematics, vol. 7, no. 3, pp. 281-295, 2024.
[12] S. Baupradist, B. Chemat, K. Palanivel and R. Chinram "Essential ideals and essential fuzzy ideals in semigroups," Journal of Discrete Mathematical Sciences and Cryptography, Vol. 24, no. 1, pp. 223-233, 2021.
[13] O. Grosek and L. Satko, "A new notion in the theory of semigroup,"Semigroup Forum, vol. 20, pp. 233–240, 1980.
[14] S. Bogdanovic, "Semigroups in which some bi-idealis a group,"Univerzitet u Novom Sadu, Zbornik Radova Prirodno-Matematičkog Fakulteta, Serija Matematika, vol. 11, pp. 261–266, 1981.
[15] K. Wattanatripop, R. Chinram and T. Changphas, "Quasi-A-ideals and fuzzy A-ideals in semigroups," Journal of Discrete Mathematical Sciences and Cryptography, vol. 21, pp. 1131–1138, 2018.
[16] N. Kaopusek, T. Kaewnoi and R. Chinram, "On almost interior ideals and weakly almost interior ideals of semigroups," Journal of Discrete Mathematical Sciences and Cryptograpyh, vol. 23, pp. 773–778, 2020.
[17] A. Iampan, R. Chinram and P. Petchka, "A note on almost subsemigroups of semigroups," International Journal of Mathematics and Computer Science, vol. 16, no. 4, pp. 1623–1629, 2021.
[18] R. Chinram and W. Nakkhasen, "Almost bi-quasi-interior ideals and fuzzy almost bi-quasi-interior ideals of semigroups," Journal of Mathematics and Computer Science, vol. 26, pp. 128–136, 2022.

[19] T. Gaketem, “Almost bi-interior ideal in semigroups and their fuzzifications,” European Journal of Pure and Applied Mathematics, vol. 15, no. 1, pp. 281-289, 2022.
[20] T. Gaketem and R. Chinram, “Almost bi-quasi-ideal ideals and their fuzzifications in semigroups,” Annals of the University of Craiova Mathematicsa and Computer Science Series, vol. 50, pp. 342–352, 2023.
[21] K. Wattanatripop, R. Chinram and T. Changphas, “Fuzzy almost bi-ideals in semigroups,” International Journal of Mathematics and Computer Science, vol. 13, pp. 51–58, 2018.
[22] W. Krailoet, A. Simuen, R. Chinram and P. Petchkaew , “A note on fuzzy almost interior ideals in semigroups,” International Journal of Mathematics and Computer Science, vol. 16, pp. 803–808, 2021.
[23] D. Molodtsov, “Soft set theory-first results,” Computers and Mathematics with Applications, vol. 37 no. 4-5, pp. 19–31, 1999.
[24] P. K. Maji, R. Biswas and A. R. Roy, “Soft set theory,” Computers and Mathematics with Applications, vol. 45, pp. 555–562, 2003.
[25] D. Pei and D. Miao, “From soft sets to information systems,” IEEE International Conference on Granular Computing, vol. 2, pp.617-621, 2005.
[26] M. I. Ali, F. Feng, X. Liu,W. K. Min and M. Shabir, “On some new operations in soft set theory,” Computers Mathematics with Applications. vol. 57, no. 9, pp. 1547-1553, 2009.
[27] A. Sezgin and A. O. Atagün , “On operations of soft sets,” Computers and Mathematics with Applications, vol. 61, no. 5, pp. 1457-1467, 2011.
[28] A. Sezgin, M. Sarıalioğlu, “Complementary extended gamma operation: A new soft set operation”, Natural and Applied Sciences Journal, vol. 7, no.1, pp.15-44, 2024.
[29] A. Sezgin and K. Dagtoros, “Complementary soft binary piecewise symmetric difference operation: A novel soft set operation,” Scientific Journal of Mehmet Akif Ersoy University, vol. 6, no. 2, pp.31-45, 2023.
[30] A. Sezgin and H. Çalışıcı, “A comprehensive study on soft binary piecewise difference operation,” Eskişehir Technical University Journal of Science and Technology B- Theoretical Sciences, vol. 12, no. 1, pp. 32-54, 2024.
[31] A. Sezgin, N. Çağman, A. O. Atagün and F. N. Aybek, “Complemental operations of sets and their application to group theory,” Matrix Science Mathematic, vol. 7, no. 2, pp.114-121, 2023.
[32] A. Sezgin and E. Yavuz, “Soft binary piecewise plus operation: A new type of operation for soft sets,” Uncertainty Discourse and Applications, vol. 1, no. 1, pp. 79-100, 2024.
[33] A. Sezgin, A. O. Atagün and N. Çağman, “A complete study on and-product of soft sets,” Sigma Journal of Engineering and Natural Sciences, vol. 14, no. 1, pp 1-14, 2025.
[34] A. Sezgin, N. Çağman and F. Çıtak, “α-inclusions applied to group theory via soft set and logic,” Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, vol. 68, no. 1, pp. 334-352, 2019.
[35] A. Sezgin, F. N. Aybek and N. B. Güngör, “A new soft set operation: Complementary soft binary piecewise union operation,” Acta Informatica Malaysia, vol. 7, no. 1, pp. 38-53, 2023.
[36] A. Sezgin and A. M. Demirci, “A new soft set operation: Complementary soft binary piecewise star operation,” Ikonion Journal of Mathematics, vol. 5, no. 2, pp. 24-52, 2023.
[37] A. Sezgin, E. Yavuz and Ş. Özlü, “Insight into soft binary piecewise lambda operation: a new operation for soft sets,” Journal of Umm Al-Qura University for Applied Sciences, pp. 1-15, 2024.
[38] A. Sezgin and E. Yavuz, “A new soft set operation: Soft binary piecewise symmetric difference operation,” Necmettin Erbakan University Journal of Science and Engineering, vol. 5, no. 2, pp. 189-208, 2023.
[39] A. Sezgin, A. and E. Şenyiğit, “A new product for soft sets with its decision-making: Soft star-product. Big data and computing visions,” vol. 5, no. 1, pp.52-73, 2025.
[40] N. Çağman and S. Enginoğlu, “Soft set theory and uni-int decision making,” European Journal of Operational Research,vol. 207, no, 2, pp.848-855, 2010.
[41] N. Çağman, F. Çıtak and H. Aktaş, “Soft int-group and its applications to group theory,” Neural Computing and Applications, vol. 2, pp. 151–158, 2012.
[42] A. S. Sezer, N. Çağman, A. O. Atagün, M. İ. Ali and E. Türkmen, “Soft intersection semigroups, ideals and bi-ideals; A new application on semigroup theory I” Filomat, vol. 29, no. 5, pp. 917-946, 2015.
[43] A. S. Sezer, N. Çağman, A. O. Atagün, “Soft intersection interior ideals, quasi-ideals and generalized bi-ideals: A New Approach to SemigroupTheory II.” Journal of Multiple-Value Logicand Soft Computing, vol. 29, no. 5, pp. 917-946, 2015.
[44] A. Sezgin and M. Orbay, “Analysis of semigroups with soft intersection ideals,” Acta Universitatis Sapientiae Mathematica, vol. 14, no. 1, pp.166-210, 2022.
[45] A. Sezgin and A. İlgin, “Soft intersection almost ideals of semigroups,” Journal of Innovative Engineering and Natural Science, vol. 4, no. 2, pp.466-481, 2024.
[46] A. Sezgin and A. İlgin, “Soft intersection almost bi-interior ideals of semigroups,” Journal of Natural and Applied Sciences Pakistan, vol. 6, no. 1, pp.1619-1638, 2024.
[47] A. Sezgin and A. İlgin, “Soft intersection almost bi-quasi ideals of semigroups,” Soft Computing Fusion with Applications, vol. 1, no.1, pp.27-42, 2024.
[48] A. Sezgin and A. İlgin, “Soft intersection almost weak-interior ideals of semigroups,” Journal of Natural Sciences and Mathematics of UT, vol. 9, no.17-18, pp.372-385, 2024.
[49] A. Sezgin and B. Onur, “Soft intersection almost bi-ideals of semigroups,” Systemic Analytics, vol. 2, no.1, pp.95-105, 2024.
[50] A. Sezgin, F. Z. Kocakaya and A. İlgin, “Soft intersection almost quasi-interior ideals of semigroups,” Eskişehir Teknik Üniversitesi Bilim ve Teknoloji Dergisi B – Teorik Bilimler, vol. 12, no.2, pp.81-99, 2024.
[51] A. Sezgin and A. İlgin, “Soft intersection almost subsemigroups of semigroups,” International Journal of Mathematics and Physics, vol. 15, no. 1, pp.13-20, 2024.
[52] A. Sezgin, Z. H. Baş, and A. İlgin, “Soft intersection almost bi-quasi-interior ideals of semigroups,” Journal of Fuzzy Extension and Application, vol. 6, no. 1, pp. 43-58, 2024.
[53] A. Sezgin and B. Onur and A. İlgin, “Soft intersection almost tri-ideals of semigroups,” SciNexuses, vol. 1, pp. 126-138, 2024.
[54] A. Sezgin, A. İlgin and A. O. Atagün, “Soft intersection almost tri-bi-ideals of semigroups,” Science&Technology Asia, vol. 29, no. 4, pp. 1-13, 2024.
[55] A. Sezgin and F. Z. Kocakaya, “Soft intersection almost quasi-ideals of semigroups,” Songklanakarin Journal of Science and Technology. in press
[56] A. Sezgin and Z. H. Baş, “Soft-int almost interior ideals for semigroups,” Information Sciences with Applications, vol. 4, pp. 25-36, 2024.
[57] A. O. Atagün, H. Kamacı, İ. Taştekin and A. Sezgin, “P-properties in near-rings,” Journal of Mathematical and Fundamental Sciences, vol. 51, no. 2, pp. 152-167, 2019.
[58] C. Jana, M. Pal, F. Karaaslan and A. Sezgin, “(α, β)-soft Intersectional rings and ideals with their applications,” New Mathematics and Natural Computation. vol. 15, no. 2, pp. 333–350, 2019.
[59] T. Manikantan, P. Ramasany and A. Sezgin, “Soft quasi-ideals of soft near-rings,” Sigma Journal of Engineering and Natural Science, vol. 41, no. 3, pp. 565-574, 2023.
[60] A. Khan, I. Izhar and A. Sezgin, “Characterizations of Abel Grassmann's groupoids by the properties of their double-framed soft ideals,” International Journal of Analysis and Applications, vol. 15, no. 1, pp. 62-74, 2017.
[61] A. O. Atagün and A. S. Sezer, “Soft Sets, Soft semimodules and soft substructures of semimodules,” Mathematical Sciences Letters, vol. 4, no. 3, pp. 235-242, 2015.
[62] M. Gulistan, F. Feng, M. Khan and A. Sezgin, “Characterizations of right weakly regular semigroups in terms of generalized cubic soft sets,” Mathematics, vol. 6, pp. 293, 2018.
[63] A. O. Atagün and A. Sezgin, “Int-soft substructures of groups and semirings with applications,” Applied athematics & information sciences 11, no. 1 (2017): 105-113.
[64] A. Sezgin, “A new view on AG-groupoid theory via soft sets for uncertainty modeling,” Filomat, vol. 32, no. 8, pp. 2995–3030, 2018.
[65] A. Sezgin, N. Çağman and A. O. Atagün, “A completely new view to soft intersection rings via soft uni-int product,” Applied Soft Computing, vol. 54, pp. 366-392, 2017.
[66] A. Sezgin, A. O. Atagün, N. Çağman and H. Demir, “On near-rings with soft union ideals and applications,” New Mathematics and Natural Computation, vol. 18, no. 2, pp. 495-511, 2022.
[67] A. O. Atagün and A. Sezgin, “Int-soft substructures of groups and semirings with applications,” Applied Mathematics & Information Sciences, vol. 11, no. 1, pp. 105-113, 2017.
[68] A. S. Sezer, A. O. Atagün and , N. Çağman, “N-group SI-action and its applications to n-group theory,” Fasciculi Mathematici, vol. 52, pp. 139-153, 2014.
[69] M. M. K. Rao, “A Study of bi-quasi-interior ideal as a new generalization of ideal of generalization of semiring,” Journal of International Mathematical Virtual Institute, vol. 9, pp. 19-35, 2019.
[70] M. M. K. Rao., “Bi-quasi-interior Ideals,” Annals of Communications in Mathematics, vol.7, no. 3, pp. 296-309, 2024.

[71] M. M. K. Rao., “Bi-quasi-interior Ideals of Semiring,” Journal of International Mathematical Virtual Institute, vol. 14, no. 1, pp. 17-31, 2024.
[72] A. H. Clifford and G. B. Preston, “The Algebraic Theory of Semigroups Vol. I (Second Edition)” American Mathematical Society, 1964.
[73] A. İlgin, “Soft Intersection (Almost) Bi-interior Ideals and Weak-interior ideals, Soft Intersection Almost Subsemigroups and Ideals, Bi-quasi ideals and Tri-bi-ideals of Semigroups” Master’s Thesis. Amasya University Graduate School of Natural and Applied Sciences. Türkiye. 2025.
[74] F. Z. Kocakaya, “Soft Intersection (Almost) Quasi-interior Ideals and Soft Intersection Almost Quasi-ideals of Semigroups” Master’s Thesis. Amasya University Graduate School of Natural and Applied Sciences. Türkiye. 2025.
[75] B. Onur, “Soft Intersection Bi-quasi Ideals, Soft Intersection Almost Bi-ideals and Tri-ideals of Semigroups.” Master’s Thesis. Amasya University Graduate School of Natural and Applied Sciences. Türkiye. 2025.