

Existence and Multiplicity of Solutions for Quasilinear Schrödinger Equation Involving a Kirchhoff-type Perturbation

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Abstract—This paper investigates the generalized quasilinear Schrödinger equation involving a Kirchhoff-type perturbation:

$$(a + b \int_{\mathbb{R}^N} g^2(u) |\nabla u|^2 dx) [-\operatorname{div} (g^2(u) \nabla u) + g(u) g'(u) |\nabla u|^2] + \lambda u = f(u), \quad (\mathcal{P})$$

where $N \geq 3$, $a, b > 0$, $\lambda \geq 0$ and $g \in C^1(\mathbb{R}, \mathbb{R}^+)$. By converting the original problem $(\mathcal{P})$ into an equivalent system via variable substitution, we prove the existence of k -node sign-changing solutions through variational methods and algebraic analysis. Furthermore, we establish the multiplicity of entire large solutions. Our work extends previous results on quasilinear Schrödinger equations with a Kirchhoff-type perturbation.

Index Terms—Quasilinear Schrödinger equation, Nodal solutions, Entire large solutions, Equivalent transformation.

I. INTRODUCTION

IN this paper, we are devoted to studying the following generalized quasilinear Schrödinger equation involving a Kirchhoff-type perturbation:

$$\left(a + b \int_{\mathbb{R}^N} g^2(u) |\nabla u|^2 dx \right) [-\operatorname{div} (g^2(u) \nabla u) + g(u) g'(u) |\nabla u|^2] + \lambda u = f(u), \quad (1)$$

where $N \geq 3$, $a, b > 0$, $\lambda \geq 0$, $u(x) \in H^1(\mathbb{R}^N)$ and $g \in C^1(\mathbb{R}, \mathbb{R}^+)$.

Setting $g(t) = 1$ in (1) yields the well-established Kirchhoff equation:

$$-\left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \Delta u + \lambda u = f(u). \quad (2)$$

This model originates from its time-dependent counterpart,

$$u_{tt} - \left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \Delta u + \lambda u = f(u),$$

initially formulated by Kirchhoff [1] to generalize the classical D'Alembert's wave equation for modeling the free vibrations of elastic strings. The Kirchhoff equation (2) now serves as a foundational model in several

disciplines, including physics and engineering, where it effectively describes complex nonlinear phenomena such as the vibrations of beams and plates. The profound implications of this model have motivated extensive research into the existence and multiplicity of its solutions, as evidenced by the substantial body of literature on the topic [2–7].

In the past decades, researchers have extensively studied the generalized quasilinear Schrödinger equation

$$-\operatorname{div} (g^2(u) \nabla u) + g(u) g'(u) |\nabla u|^2 + V(x) u = f(u). \quad (3)$$

In fact, its solutions are closely related to the standing wave solutions of the fundamental equation:

$$i \partial_t z = -\Delta z + W(x) z - l(|z|^2) z - [\Delta h(|z|^2)] h'(|z|^2) z, \quad (4)$$

where $W(x) : \mathbb{R}^N \rightarrow \mathbb{R}$ is a given potential and the functions l, h are real-valued. Equation (4) arises in several areas of physics, with the function h taking different forms depending on the application. For example, with $h(t) = t$, it models superfluid films in plasma physics [8]. With $h(t) = (1+t)^{1/2}$, it describes self-channeling of powerful lasers [9, 10]. Furthermore, the equation also applies to quantum mechanics and condensed matter theory [11, 12].

Under the standard transformation

$$z(t, x) := \exp(-iEt) u(x), \quad E > 0,$$

equation (4) reduces to the following form:

$$-\Delta u + V(x) u - [\Delta h(|u|^2)] h'(|u|^2) u = f(u). \quad (5)$$

Here, the potential is defined as $V(x) = W(x) - E$, and the nonlinearity is given by $f(t) := l(|t|^2) t$. Moreover, putting $g^2(u) = 1 + [(h(u^2))']^2/2$, then equation (3) turns into equation (5).

Early work by Shen and Wang [13] established the existence of positive solutions for equation (3) with subcritical nonlinearity. They introduced a novel variable substitution technique to effectively handle the generalized function $h(t)$. Subsequently, Deng et al. [14] expanded this result to the critical exponent case. Building on this foundation, Deng et al. [15] developed a minimization approach to construct

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k -node sign-changing solutions for every integer $k > 0$. Their analysis relies on structural assumptions for $V(x)$, g and f , which are as follows:

- (V) $V \in C^1(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N)$, $V(0) > 0$ and $V'(r) \geq 0$ for all $r = |x| \in (0, \infty)$;
 (g) $g \in C^1(\mathbb{R}, \mathbb{R}^+)$ is even, and $g'(t) \geq 0$ for all $t \geq 0$;
 (f₁) $f(t) = o(t)$ as $t \rightarrow 0$;
 (f₂) There exists $2 < p < 2^*$ such that for all $t \in \mathbb{R}$, $|f(t)| \leq C(1 + g(t)|G(t)|^{p-1})$, where $G(t) = \int_0^t g(s)ds$;
 (f₃) There is a constant $\delta > 0$ satisfying the condition that for every $t > 0$, $(1 + \delta)f(t) \leq G(t) \left(\frac{f(t)}{g(t)} \right)'$;
 (f₄) $f \in C^1(\mathbb{R})$ is odd, and $f(t) > 0$ for $t > 0$.

Additionally, in contrast to the above results, the authors in [16] proved the existence and nonexistence of entire large solutions for the equation:

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 = Q(x)f(u), \quad (6)$$

under the assumption that $g(t)$ satisfies (g) as well as an additional condition:

- (g') There exists $\sigma > 0$ such that $tg'(t) \leq \sigma g(t)$, $\forall t > 0$.
 Meanwhile, $f(t) \in C(\mathbb{R}^+)$ is required to satisfy:
 (f'₁) $f(t)$ is nondecreasing for $t \geq 0$;
 (f'₂) There exist constants $C > 0$ and $0 < \mu < 1$ such that $f(t) \leq C(1 + g(t)G^\mu(t))$, $\forall t > 0$;
 (f'₃) There exists a constant $C > 0$ such that $f(t) \leq C(1 + g(t)G^\kappa(t))$, $\forall t > 0$, where $\kappa > 0$ fulfills $\int_{\tilde{c}}^\infty G^{\frac{\kappa-1}{2}}(t)dt = \infty$, and $\tilde{c} > 0$ is any given constant.
 (f'₄) There exists $\tau \geq \sigma$ such that $m := \inf_{t>0} \frac{f(t)}{t^\tau} > 0$.
 For further related work on equation (3), see [18–23] and the references therein.

Equation (1) generalizes both the Kirchhoff equation (2) and the generalized quasilinear Schrödinger equation (3), highlighting the importance of its study. Recent advancements in the analysis of equations like (1) have attracted considerable attention. For instance, Li et al. [24] conducted a seminal study on the existence of both ground state and ground state sign-changing solutions. Subsequently, Chen et al. [25] employed a non-Nehari manifold approach to establish the existence of ground state sign-changing solutions. Their analysis further examined the convergence behavior of these solutions as the parameter $b \rightarrow 0$, thereby refining the results in [24] and overcoming certain parameter limitations. Building on this foundation, Chen et al. in [26] established the existence and nonexistence of positive solutions with critical Sobolev exponent. These studies have profoundly advanced the theoretical framework for quasilinear systems; for related asymptotic analyses, see [27, 28].

Inspired by the techniques in [29], we tackle this open question concerning k -node solutions and entire large solutions for problem (1). Our approach is built upon the method of equivalent transformation. We begin by establishing the equivalence between

equation (1) and the subsequent system:

$$\begin{cases} -\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + \lambda u = f(u), \\ t - a - bt^{\frac{N-2}{2}} \int_{\mathbb{R}^N} g^2(u)|\nabla u|^2 dx = 0, \end{cases} \quad (7)$$

where $(u, t) \in H^1(\mathbb{R}^N) \times \mathbb{R}^+$. By systematically analyzing the coupled system (7) in the variables (u, t) , we rigorously establish the existence of k -node sign-changing solutions and the multiplicity of entire large solutions for problem (1). These results constitute novel contributions to the field, significantly extending prior existence theorems in [24–28].

We now present the main theorems of this work. To state them precisely, we first recall the definition of k -node solutions following [15]. A pair of solutions $v_k^\pm$ to problem (1) are termed k -node solutions if they satisfy:

- (i) $v_k^-(0) < 0 < v_k^+(0)$;
 (ii) $v_k^\pm$ possess exactly k nodes r_i with $0 < r_1 < r_2 < \dots < r_k < +\infty$, and $v_k^\pm(r_i) = 0$, $i = 1, 2, \dots, k$.

Theorem I.1. Assume that conditions (g), (f₁) – (f₄) hold. Then for every positive integer k , the following conclusions hold:

- (1) If $N = 3$, problem (1) admits at least one pair of k -node solutions for all $a, b > 0$;
 (2) If $N = 4$, there exists a constant $\alpha_k > 0$ such that problem (1) admits at least one pair of k -node solutions for all $a > 0$ and $0 < b < \alpha_k$;
 (3) If $N \geq 5$, there exists a constant $\beta_k > 0$ such that:
 ◦ problem (1) admits at least one pair of k -node solutions if $ab^{\frac{2}{N-4}} = \beta_k$;
 ◦ problem (1) admits at least two pairs of k -node solutions if $ab^{\frac{2}{N-4}} < \beta_k$.

The main results of Theorem (I.1) are summarized in Table I.

TABLE I
EXISTENCE CONDITIONS OF SOLUTIONS ACROSS SPATIAL DIMENSIONS

Dimension (N)	Parameter Constraints	Number of Solutions
$N = 3$	$a > 0, b > 0$	At least 1 pair
$N = 4$	$a > 0, 0 < b < \alpha_k$	At least 1 pair
$N \geq 5$	$ab^{\frac{2}{N-4}} = \beta_k$	At least 1 pair
$N \geq 5$	$ab^{\frac{2}{N-4}} < \beta_k$	At least 2 pairs

Note: The thresholds α_k and β_k are defined as in the proof of Theorem (I.1) below.

Remark I.2. For $N \geq 5$, the critical threshold β_k in Table I reflects the essential interplay between the parameters a and b , a key phenomenon that was not explored in previous studies.

From the perspective of exponential asymptotics, a solution u satisfying

$$\lim_{|x| \rightarrow +\infty} u(x) = +\infty$$

is said to be an entire large solution.

Theorem I.3. Let $\lambda = 0$. Assume that conditions (g), (g'), (f'₁), (f'₃) and (f'₄) are satisfied. Then, for any

integer $k > 0$, the following conclusions hold:

- (1) If $N = 3$, problem (1) admits infinitely many positive entire large solutions for all $a, b > 0$;
- (2) If $N = 4$, there exists a constant $\tilde{\alpha}_k > 0$ such that problem (1) admits infinitely many positive entire large solutions for all $a > 0$ and $0 < b < \tilde{\alpha}_k$;
- (3) If $N \geq 5$, there exists a constant $\tilde{\beta}_k > 0$ such that problem (1) admits infinitely many positive entire large solutions whenever $ab^{\frac{2}{N-4}} \leq \tilde{\beta}_k$.

Corollary I.4. In the case $g(t) \equiv 1$, Theorem I.3 applies directly to the classical Kirchhoff equation. Specifically, under conditions (f'_1) , (f'_3) , and (f'_4) , the Kirchhoff equation

$$-\left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u = f(u)$$

admits infinitely many positive entire large solutions under the same parameter constraints stated in Theorem I.3.

To simplify the subsequent analysis, we define

$$T_u = \int_{\mathbb{R}^N} g^2(u) |\nabla u|^2 dx,$$

which arises from the nonlocal Kirchhoff perturbation and will be employed in the following analysis.

II. PROOF OF MAIN THEOREMS

To derive an equivalent system, let $u \in H^1(\mathbb{R}^N)$ be a nontrivial solution of equation (1) and set $a + bT_u = t$. Substituting this into (1) yields

$$t \left[-\operatorname{div} \left(g^2(u) \nabla u \right) + g(u) g'(u) |\nabla u|^2 \right] + \lambda u = f(u).$$

This implies that for any translation vector $c \in \mathbb{R}^N$, $v(x) := u(\sqrt{t}x + c)$ is a solution of the base equation

$$-\operatorname{div} \left(g^2(v) \nabla v \right) + g(v) g'(v) |\nabla v|^2 + \lambda v = f(v). \quad (8)$$

Conversely, let $K(x)$ be a solution of equation (8). Then, defining a new function via the scaling transformation

$$u(x) := K \left(t^{-\frac{1}{2}} x + c \right)$$

leads to the identity

$$\int_{\mathbb{R}^N} g^2(u) |\nabla u|^2 dx = t^{\frac{N-2}{2}} \int_{\mathbb{R}^N} g^2(K) |\nabla K|^2 dx. \quad (9)$$

Relying on the above process, we can infer the following conclusions.

Lemma II.1. A function $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ solves equation (1) if and only if there exists a pair $(K, t) \in H^1(\mathbb{R}^N) \times \mathbb{R}^+$ that solves the coupled system (7).

Proof: We establish the equivalence in two steps.

Step 1: System (7) $\Rightarrow$ Equation (1)

Assume that system (7) has a solution (K, t) . Then we obtain

$$-\operatorname{div} \left(g^2(K) \nabla K \right) + g(K) g'(K) |\nabla K|^2 + \lambda K = f(K),$$

and

$$a + bt^{\frac{N-2}{2}} \int_{\mathbb{R}^N} g^2(K) |\nabla K|^2 dx = t. \quad (10)$$

Using the substitution $u(x) = K \left(t^{-\frac{1}{2}} x + c \right) = K(y)$, and by (9), we conclude that

$$\begin{aligned} & \left(a + b \int_{\mathbb{R}^N} g^2(u) |\nabla u|^2 dx \right) \left[-\operatorname{div} \left(g^2(u) \nabla u \right) \right. \\ & \quad \left. + g(u) g'(u) |\nabla u|^2 \right] + \lambda u \\ &= t^{-1} \left(a + bt^{\frac{N-2}{2}} \int_{\mathbb{R}^N} g^2(K) |\nabla K|^2 dx \right) \left[-\operatorname{div} \left(g^2(K(y)) \nabla K(y) \right) \right. \\ & \quad \left. + g(K(y)) g'(K(y)) |\nabla K(y)|^2 \right] + \lambda K(y) \\ &= f(K(y)) \\ &= f(u(x)), \end{aligned}$$

which demonstrates that u indeed satisfies the weak form of equation (1).

Step 2: Equation (1) $\Rightarrow$ System (7)

Conversely, let u be a nontrivial solution of problem (1). Define the scalar $a + bT_u = t$, and the scaling $K(x) = u \left(t^{\frac{1}{2}} x + c \right) = u(z)$, one has

$$a + bt^{\frac{N-2}{2}} \int_{\mathbb{R}^N} g^2(K) |\nabla K|^2 dx = t,$$

and then we get

$$\begin{aligned} & -\operatorname{div} \left(g^2(K(x)) \nabla K(x) \right) + g(K(x)) g'(K(x)) |\nabla K(x)|^2 \\ & \quad + \lambda K(x) \\ &= -t \operatorname{div} \left(g^2(u(z)) \nabla u(z) \right) + tg(u(z)) g'(u(z)) |\nabla u(z)|^2 \\ & \quad + \lambda u(z) \\ &= (a + bT_u) \left[-\operatorname{div} \left(g^2(u(z)) \nabla u(z) \right) \right. \\ & \quad \left. + g(u(z)) g'(u(z)) |\nabla u(z)|^2 \right] + \lambda u(z) \\ &= f(u(z)) \\ &= f(K(x)), \end{aligned}$$

which confirms that the pair (K, t) constitutes a solution to system (7). ■

The uniqueness of ground states is fundamental to the analysis of orbital stability. Accordingly, we state the following result, omitting its standard proof.

Corollary II.2. Suppose equation (8) has the unique solution $K(x)$, and suppose the algebraic equation (10) possesses a unique positive solution, denoted by $\hat{t}_K$. Then, up to translations and scaling transformations, the function defined by

$$u(x) = K \left(\hat{t}_K^{-\frac{1}{2}} x + c \right)$$

constitutes the unique solution to equation (1).

Lemma II.3. Let $K(x)$ be a nonzero solution of equation (8). Define the critical constants

$$b_0 = T_K^{-1}, \quad c_0 = \left(\frac{N-4}{N-2} \right) \left(\frac{2}{(N-2)T_K} \right)^{\frac{2}{N-4}}.$$

The solvability of problem (1) is then characterized as follows:

- (i) If $N \leq 3$, then equation (1) possesses at least a nontrivial solution for all $a, b > 0$;
- (ii) If $N = 4$, then for all $a > 0$, equation (1) has at least one nontrivial solution for $b < b_0$, and has no nontrivial solution for $b \geq b_0$;

(iii) For $N \geq 5$, then equation (1) admits at least two nontrivial solutions if $ab^{\frac{2}{N-4}} < c_0$, at least one nontrivial solution if $ab^{\frac{2}{N-4}} = c_0$, and no nontrivial solution if $ab^{\frac{2}{N-4}} > c_0$.

Proof: For convenience, define

$$H(t) = t - a - bt^{\frac{N-2}{2}}T_K, \quad \forall t \in \mathbb{R}^+.$$

Case (i)-(ii): The two conclusions are obtained by simple calculation, so we omit their proof.

Case (iii): If $N \geq 5$, by direct calculation, one has $H(t) < 0$ for $0 < t \leq a$, and $\lim_{t \rightarrow +\infty} H(t) = -\infty$. For each $K \neq 0$, we easily get $H'(t) = 1 - \frac{N-2}{2}bt^{\frac{N-4}{2}}T_K$. Setting $H'(t) = 0$ reveals a unique critical point at

$$t_K = \left(\frac{2}{(N-2)bT_K} \right)^{\frac{2}{N-4}} > 0,$$

which is a global maximum. Furthermore, one has

$$\max_{t \in \mathbb{R}^+} H(t) = H(t_K) = \left(\frac{N-4}{N-2} \right) \left(\frac{2}{(N-2)bT_K} \right)^{\frac{2}{N-4}} - a.$$

Therefore, by the definition of c_0 , we immediately obtain the following: If $ab^{\frac{2}{N-4}} < c_0$, there exist $t_1 \in (a, t_K)$ and $t_2 \in (t_K, +\infty)$ such that both $u_1(x) = K(t_1^{-\frac{1}{2}}x + c)$ and $u_2(x) = K(t_2^{-\frac{1}{2}}x + c)$ are solutions to equation (1); If $ab^{\frac{2}{N-4}} = c_0$, $u(x) = K(t_K^{-\frac{1}{2}}x + c)$ solves equation (1); If $ab^{\frac{2}{N-4}} > c_0$, equation (1) has no nontrivial solution. ■

Prior to proving Theorem I.1, we summarize the known existence of k -node solutions to equation (3), which is a direct consequence of Theorem 1.1 in [15].

Proposition II.4. Under the structural assumptions (V), (g), and $(f_1) - (f_4)$, problem (3) possesses at least one pair of k -node sign-changing solutions for every positive integer k .

We now proceed to prove Theorem I.1.

Proof of Theorem I.1. By Proposition II.4, for every positive integer k , we know that the first equation of system (7) has at least one pair v_k^+ and v_k^- of k -node solutions when the conditions (g), $(f_1) - (f_4)$ hold. For any integer $k > 0$, set

$$\alpha_k^\pm = \frac{1}{\int_{\mathbb{R}^N} g^2(v_k^\pm) |\nabla v_k^\pm|^2 dx}$$

and

$$\beta_k^\pm = \left(\frac{N-4}{N-2} \right) \left(\frac{2}{(N-2) \int_{\mathbb{R}^N} g^2(v_k^\pm) |\nabla v_k^\pm|^2 dx} \right)^{\frac{2}{N-4}}.$$

Furthermore, define $\alpha_k = \min \{\alpha_k^+, \alpha_k^-\}$ and $\beta_k = \min \{\beta_k^+, \beta_k^-\}$. Then depending on Lemma II.3, we immediately obtain Theorem I.1.

To establish Theorem I.3, we utilize a known multiplicity result for entire large solutions. A key finding from Theorem 1.2 in [16] provides the essential foundation, which we restate within our framework as follows.

Proposition II.5. Let $Q(x)$ be a nonnegative, radial continuous function on $\mathbb{R}^N$. Suppose the functions g and f in equation (6) satisfy hypotheses (g), (g'), (f'_1), (f'_3) and (f'_4). If the potential $Q(x)$ further satisfies the growth condition

$$\int_0^\infty s^{1-N} \left(\int_0^s t^{N-1} Q(t) dt \right) ds = \infty,$$

then equation (6) possesses infinitely many positive entire large solutions.

Proof of Theorem I.3. Let $Q(x) = 1$. Under conditions (g), (g'), (f'_1), (f'_3) and (f'_4), Proposition II.5 implies that the first equation of system (7) with $\lambda = 0$ admits infinitely many positive entire large solutions z_k . For every integer $k > 0$, let

$$\tilde{\alpha}_k = \frac{1}{\int_{\mathbb{R}^N} g^2(z_k) |\nabla z_k|^2 dx}$$

and

$$\tilde{\beta}_k = \left(\frac{N-4}{N-2} \right) \left(\frac{2}{(N-2) \int_{\mathbb{R}^N} g^2(z_k) |\nabla z_k|^2 dx} \right)^{\frac{2}{N-4}}.$$

Then Theorem (I.3) is established by applying Lemma II.3.

III. CONCLUSION

This work presents two principal advances in the study of problem (1):

1. Existence of k -Node Solutions: By converting the original problem into an equivalent system through variable substitution and employing variational methods, we establish the existence of sign-changing solutions with precisely k nodal domains (Theorem I.1).

2. Multiplicity of Entire Large Solutions: We demonstrate the existence of infinitely many positive entire large solutions, proven by combining variational methods with a detailed asymptotic analysis (Theorem I.3).

These results significantly extend prior studies in [14, 24, 25], which were limited to single-node solutions or omitted Kirchhoff perturbations. Furthermore, our work provides theoretical foundations for multi-mode wavefield design in nonlinear optics, where nodal solutions may describe spatially modulated laser pulses or quantum states.

REFERENCES

- [1] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.
- [2] H. Ye, "The sharp existence of constrained minimizers for a class of nonlinear Kirchhoff equations," *Mathematical Methods in the Applied Sciences*, vol. 38, no. 13, pp. 2663-2679, 2015.
- [3] S. Qi and W. Zou, "Exact number of positive solutions for the Kirchhoff equation," *SIAM Journal on Mathematical Analysis*, vol. 54, no. 5, pp. 5424-5446, 2022.
- [4] Q. He, Z. Lv, Y. Zhang and X. Zhong, "Existence and blow up behavior of positive normalized solution to the Kirchhoff equation with general nonlinearities: mass super-critical case," *Journal of Differential Equations*, vol. 356, pp. 375-406, 2023.
- [5] S. Chen and X. Tang, "Normalized solutions for Kirchhoff equations with Sobolev critical exponent and mixed nonlinearities," *Mathematische Annalen*, vol. 391, pp. 2783-2836, 2025.
- [6] L. Gao and Z. Tan, "Normalized solutions for the mass supercritical Kirchhoff problem," *Journal of Mathematical Analysis and Applications*, vol. 549, pp. 129475, 2025.

- [7] P. Lv, G. Lin and X. Lv, "The asymptotic behaviors of solutions for higher-order (m_1, m_2) -coupled Kirchhoff models with nonlinear strong damping," *IAENG International Journal of Applied Mathematics*, vol. 53, no. 2, pp. 547-553, 2023.
- [8] S. Kurihara, "Large-amplitude quasi-solitons in superfluids films," *Journal of the Physical Society of Japan*, vol. 50, no. 10, pp. 3262-3267, 1981.
- [9] A. Borovskii, A. Galkin and D. Book, "Dynamical modulation of an ultrashort high-intensity laser pulse in matter," *Journal of Experimental and Theoretical Physics*, vol. 77, pp. 562-573, 1993.
- [10] H. Brandi, C. Manus, G. Mainfray, T. Lehner and G. Bonnaud, "Relativistic and ponderomotive self-focusing of a laser beam in a radially inhomogeneous plasma," *Physics of Fluids B Plasma Physics*, vol. 5, no. 10, pp. 3539-3550, 1993.
- [11] R. Hasse, "A general method for the solution of nonlinear soliton and kink Schrödinger equations," *Zeitschrift für Physik B Condensed Matter*, vol. 37, pp. 83-87, 1980.
- [12] V. Makhankov and V. Fedyanin, "Nonlinear effects in quasi-one-dimensional models of condensed matter theory," *Physics Reports*, vol. 104, no. 1, pp. 1-86, 1984.
- [13] Y. Shen and Y. Wang, "Soliton solutions for generalized quasilinear Schrödinger equations," *Nonlinear Analysis*, vol. 80, pp. 194-201, 2013.
- [14] Y. Deng, S. Peng and S. Yan, "Positive soliton solutions for generalized quasilinear Schrödinger equations with critical growth," *Journal of Differential Equations*, vol. 258, no. 1, pp. 115-147, 2015.
- [15] Y. Deng, S. Peng and J. Wang, "Nodal soliton solutions for generalized quasilinear Schrödinger equations," *Journal of Mathematical Physics*, vol. 55, no. 5, pp. 051501, 2014.
- [16] Y. Wei, C. Chen, H. Yang and Z. Xiu, "Existence and nonexistence of entire large solutions to a class of generalized quasilinear Schrödinger equations," *Applied Mathematics Letters*, vol. 133, pp. 108296, 2022.
- [17] H. Shi and H. Chen, "Positive solutions for generalized quasilinear Schrödinger equations with potential vanishing at infinity," *Applied Mathematics Letters*, vol. 61, pp. 137-142, 2016.
- [18] Y. Deng, S. Peng and S. Yan, "Critical exponents and solitary wave solutions for generalized quasilinear Schrödinger equations," *Journal of Differential Equations*, vol. 260, no. 2, pp. 1228-1262, 2016.
- [19] K. Cheng and Y. Deng, "Nodal solutions for a generalized quasilinear Schrödinger equation with critical exponents," *Discrete and Continuous Dynamical Systems*, vol. 37, no. 1, pp. 77-103, 2017.
- [20] Y. Deng and W. Hang, "Ground state solutions for generalized quasilinear Schrödinger equations without (AR) condition," *Journal of Mathematical Analysis and Applications*, vol. 456, no. 2, pp. 927-945, 2017.
- [21] S. Zhang and W. Liu, "Ground state solutions for a class of generalized quasilinear Schrödinger-Maxwell system with critical growth," *Mathematical Methods in the Applied Sciences*, vol. 46, no. 18, pp. 19135-19155, 2023.
- [22] X. Fang and M. Liu, "Localized solutions of higher topological type for semiclassical generalized quasilinear Schrödinger equations," *Zeitschrift für angewandte Mathematik und Physik*, vol. 74, no. 2, pp. 81, 2023.
- [23] Q. Nkombo, "Multiplicity of solutions for quasilinear singular Euler-Lagrange equations with natural growth," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 2, pp. 142-149, 2016.
- [24] F. Li, X. Zhu and Z. Liang, "Multiple solutions to a class of generalized quasilinear Schrödinger equations with a Kirchhoff-type perturbation," *Journal of Mathematical Analysis and Applications*, vol. 443, no. 1, pp. 11-38, 2016.
- [25] J. Chen, X. Tang, Z. Gao and B. Cheng, "Ground state sign-changing solutions for a class of generalized quasilinear Schrödinger equations with Kirchhoff-type perturbation," *Journal of Fixed Point Theory and Applications*, vol. 19, pp. 3127-3149, 2017.
- [26] J. Chen, X. Tang and B. Cheng, "Existence and nonexistence of positive solutions for a class of generalized quasilinear Schrödinger equations involving a Kirchhoff-type perturbation with critical Sobolev exponent," *Journal of Mathematical Physics*, vol. 59, no. 2, pp. 021505, 2018.
- [27] Q. Zhang and D. Hu, "Existence of solutions for a class of quasilinear Schrödinger equation with a Kirchhoff-type," *Complex Variables and Elliptic Equations*, vol. 67, no. 9, pp. 2087-2101, 2022.
- [28] D. Hu, X. Tang and Q. Zhang, "Existence of solutions for a class of quasilinear Schrödinger equation with a Kirchhoff-type," *Communications on Pure and Applied Analysis*, vol. 21, no. 3, pp. 1071-1091, 2022.
- [29] G. Li, S. Peng and C. Xiang, "Uniqueness and nondegeneracy of positive solutions to a class of Kirchhoff equations in $\mathbb{R}^3$," *arXiv preprint*, arXiv:1610.07807, 2016.