Graphs Combining

Ibrahim A. Jawarneh and Bilal N. Al-Hasanat

***Abstract*—A graph operation is a process that creates a new graph from one or more initial graphs. Certainly, there are many operations in which the result on the graph satisfies all or some of the common initial graph properties or produces some new properties depending on the operation description. In this paper, we introduce a new operation on finite simple graphs (both directed and undirected), for which the resulting graph may possess the same properties as the initial graph. This operation is called graphs combining and is denoted by $\boxplus$. In addition, we study the properties of this operation and analyze the characteristics of its graphs.**

***Index Terms*—Simple graph, undirected graph, directed graph, connected graph, graphs operations, adjacent matrix, maximal degree, minimal degree, Hamiltonian graph.**

I. INTRODUCTION

GRAPH theory is a branch of mathematics that studies the relationships between pairs of objects. It is one of the most important sciences since it has been considered by many other sciences. For example, it is used in designing circuit connections in electrical engineering. In addition, it is used to study molecules in physics and chemistry, and also to represent local connections between the dynamics of a physical process in statistical physics. Moreover, graph theory plays a significant role in computer networks, as the study of the relationships among interconnected computers within the network, and to configure the network security.

A graph can be considered as a set of vertices (nodes), connected by a set of edges, corresponding to a specific incidence relation [1]. Therefore, a graph Γ is the triple $\mathcal{V}$, $\mathcal{E}$ and ψ, where V is the set of vertices (denotes the set of elements), E is the set of edges (connect the related elements) and ψ is the incidence relation (assign the appropriate elements to each edge). For more about graph theory and its applications, see [1], [2], [3], [4], [5], [6], [7].

Recently, many studies have been published that focus on using graph theory in real-life problems. For examples in medicine, as we can see in [8], they have shown a graph theory-based approach to model and analyze the human lymphatic network. In [9], They have proposed a graph-based analysis method to quantify the topological properties of the network, both globally and at the nodal level, to detect systemic or single-organ metabolic abnormalities caused by diseases such as lung cancer. Since graphs combine nodes with edges, researchers can examine connections and improve networks in addition to identifying patterns, making thorough data analysis possible. In [10] they have examined essential theories and applications of graphs as well as their analytic methods for analyzing complex systems in this study. Network centrality measures and shortest-path algorithms and graph clustering methods constitute important analytical techniques which the study presents. These discoveries exhibit both the effectiveness and necessity of graph-based models for addressing real-world difficulties through their optimization abilities in structure analysis. Graph theory is used in computer science to model and address problems in many fields, including networking, databases, artificial intelligence, algorithms, and more. Jingjing has studied computer science-based graph theory matching algorithms, solved matching problems on large-scale graphs, reduced computation time, and improved solving efficiency, see [11]. In our work, for finite simple graphs (directed and undirected), we apply a new operation, which permits the resultant graph to retain its features as originally drawn. The symbol for this operation, known as graphs combining, is $\boxplus$. We also examine the features of this process and assess the qualities of its graphs. This can enrich and contribute the research in graph theory to be used in many applications in real-life problems.

Manuscript received January, 2025; revised July, 2025.

Ibrahim A. Jawarneh is an associate professor, Department of Mathematics, Al Hussein Bin Talal University, Ma'an, Jordan (e-mail: ibrahim.a.jawarneh@ahu.edu.jo).

Bilal N. Al-Hasanat is an associate professor, Department of Mathematics, Al Hussein Bin Talal University, Ma'an, Jordan (e-mail: bilal_hasanat@yahoo.com).

II. BASIC DEFINITIONS AND NOTATIONS

This research paper primarily focuses on finite, undirected, simple graphs, which are introduced as follows. A graph Γ is denoted by the pair (V, E), where V is the set of vertices and $E \subseteq \{(u, v) \mid u, v \in V,\, u \neq v\}$ is the set of edges. Noted that $(u, v) = (v, u)$ for undirected graphs.

To keep this paper is self contained, we list some basic concepts of graphs in the next definition.

Definition 1. Let $\Gamma(V, E)$ be a finite undirected simple graphs. Then:

- A **path** of length k in a graph is a sequence of $k + 1$ distinct vertices such that consecutive vertices are adjacent.
- The **distance** $d(u, v)$ between two vertices u and v is the length of the shortest path connecting them.
- The **diameter** of a graph, denoted $\operatorname{diam}(\Gamma)$, is the maximum distance between any pair of vertices in Γ.
- The **radius** of a graph, denoted $\operatorname{rad}(\Gamma)$, is the minimum eccentricity among all vertices, where the eccentricity of a vertex v is $\max\{d(v, u) \mid u \in V\}$.
- The **girth** $g(\Gamma)$ of a graph is the length of its shortest cycle. If Γ is acyclic, then $g(\Gamma) = \infty$.
- The **adjacency matrix** $A(\Gamma)$ of a graph $\Gamma = (V, E)$ is the $|V| \times |V|$ matrix where $A_{ij} = 1$ if $\{v_i, v_j\} \in E$ and 0 otherwise.

III. DEFINITION OF GRAPHS COMBINING

A. Undirected simple graphs combining

The considered graphs in this section are finite, undirected, and simple graphs (graphs have neither loops nor parallel edges). Consider the graph Γ associated with a set of vertices

V and set of edges E, then the order of Γ is the number of its vertices and denoted by $O(\Gamma) = |V|$, and the size of Γ is the number of its edges and denoted by $S(\Gamma) = |E|$. The degree of a vertex $v \in V$ is the number of edges incidence with v and denoted by $\deg v$, the maximal degree of Γ is $\max\{\deg v \mid v \in V\}$ and denoted by $\Delta(\Gamma)$, and the minimal degree of Γ is $\delta(\Gamma) = \min\{\deg v \mid v \in V\}$.

The following definition introduces a new operation on given graphs, namely the graph combining which is denoted by $\boxplus$.

Definition 2. Let Γ_1 and Γ_2 be two finite, simple, and undirected graphs associated with sets of vertices V_1 and V_2, and sets of edges E_1 and E_2 (respectively). The combining graph Γ of Γ_1 and Γ_2, denoted by $\Gamma_1 \boxplus \Gamma_2$, is a graph associated with set of vertices V_Γ and set of edges E_Γ, where

$$\begin{aligned} &V_\Gamma = V_1 \cup V_2 \\ \text{and } &E_\Gamma = E_1 \cup E_2 \cup \{uv \mid u \in V_1 \setminus V_2 \text{ and } v \in V_2 \setminus V_1\}. \end{aligned}$$

Example 1. Let $\Gamma_1 = P_3$ and $\Gamma_2 = P_2$. Then $\Gamma = \Gamma_1 \boxplus \Gamma_2$ is the combining graph of P_3 and P_2, which associated with set of vertices V and set of edges E defined as:

$$\begin{aligned} V &= \{v_1, v_2, v_3\} \cup \{u_1, u_2\} = \{v_1, v_2, v_3, u_1, u_2\} \\ E &= \{v_1v_2, v_2v_3\} \cup \{u_1u_2\} \bigcup_{\substack{i=1,2,3 \\ j=1,2}} \{v_iu_j\} \\ &= \{v_1v_2, v_2v_3, u_1u_2, v_1u_1, v_1u_2, v_2u_1, v_2u_2, v_3u_1, v_3u_2\}. \end{aligned}$$

which is shown in the following figure:

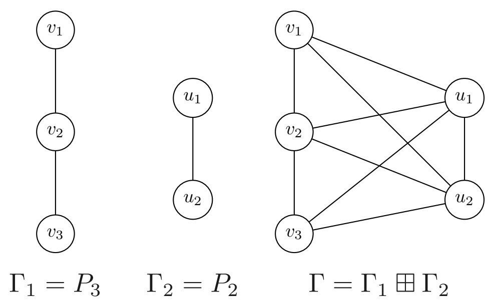


Fig. 1: The combining of disjoint graphs

Example 2. Consider the graphs $\Gamma_1(V_1, E_1) = P_3$ and $\Gamma_2(V_2, E_2) = P_2$ having the vertex v_2 in common as:

$$V_1 = \{v_1, v_2, v_3\}\ ,\ E_1 = \{v_1v_2, v_2v_3\}$$

and

$$V_2 = \{u_1, v_2\}\ ,\ E_2 = \{v_2u_1\}.$$

Then, the set of verities of $\Gamma = \Gamma_1 \boxplus \Gamma_2$ is

$$V = \{v_1, v_2, v_3\} \cup \{u_1, v_2\} = \{v_1, v_2, v_3, u_1\},$$

and the set of edges of Γ is

$$\begin{aligned} E &= E_1 \cup E_2 \cup \{uv \mid u \in V_1 \setminus V_2 \text{ and } v \in V_2 \setminus V_1\} \\ &= \{v_1v_2, v_2v_3\} \cup \{v_2u_1\} \\ &\quad \cup \{uv \mid u \in \{v_1, v_3\} \text{ and } v \in \{u_1\}\} \\ &= \{v_1v_2, v_2v_3\} \cup \{v_2u_1\} \cup \{v_1u_1, v_3u_1\} \\ &= \{v_1u_1, v_1v_2, v_2v_3, v_2u_1, v_3u_1\} \end{aligned}$$

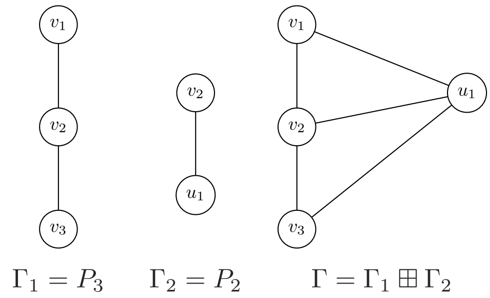


Fig. 2: The combining of none disjoint graphs

Then the combining graph $\Gamma = \Gamma_1 \boxplus \Gamma_2$ is shown on the following figure:

Remark 3. *One can easily show that,* $\Gamma \boxplus \Gamma = \Gamma$ *for any graph* Γ.

Consider the combining graph $\Gamma(V, E)$ of the graphs $\Gamma_1(V_1, E_2)$ and $\Gamma_2(V_2, E_2)$ in which Γ_1 is a subgraph of Γ_2. Then $V = V_1 \cup V_2 = V_2$, because $V_1 \subseteq V_2$. Also, $E = E_1 \cup E_2 \cup \{uv \,|\, u \in V_1 \setminus V_2 \text{ and } v \in V_2 \setminus V_1\} = E_1 \cup E_2 \cup \Phi = E_2$. Therefore, $\Gamma_1 \boxplus \Gamma_2 = \Gamma_2$. In this case, the result of the graphs combining aligns perfectly with the result of the union of these graphs. That is to say, if Γ_1 is a subgraph of Γ_2, then $\Gamma_1 \boxplus \Gamma_2 = \Gamma_1 \cup \Gamma_2 = \Gamma_2$.

Lemma 4. *Let* Γ_1 *and* Γ_2 *be two graphs associated with sets of vertices* V_1 *and* V_2*, and sets of edges* E_1 *and* E_2 *(respectively), and let* $\Gamma = \Gamma_1 \boxplus \Gamma_2$. *Then:*

1) $|V_\Gamma| = |V_1| + |V_2 \setminus V_1| = |V_1 \setminus V_2| + |V_2|$.
2) $|E_\Gamma| = |E_1| + |E_2| + |V_1 \setminus V_2||V_2 \setminus V_1|$.

Proof: Consider the graphs Γ_1 and Γ_2 associated with sets of vertices V_1 and V_2, and sets of edges E_1 and E_2 (respectively). Let $\Gamma = \Gamma_1 \boxplus \Gamma_2$. Then $V_1 \cup V_2 = V_\Gamma$ and

$$|V_\Gamma| = |V_1 \cup V_2| = |V_1| + |V_2| - |V_1 \cap V_2| = |V_1| + |V_2 \setminus V_1|.$$

In addition, $E_1 \cup E_2 \subset E_\Gamma$, also if $u \in V_1 \setminus V_2$ and $v \in V_2 \setminus V_1$, then $uv \in E_\Gamma$ and the number of such edges is $|V_1 \setminus V_2||V_2 \setminus V_1|$. ■

As a direct consequence of the previous lemma, one can find that, if Γ_1 and Γ_2 be two disjoint graphs and $\Gamma = \Gamma_1 \boxplus \Gamma_2$. Then:

1) $|V_\Gamma| = |V_1| + |V_2|$.
2) $|E_\Gamma| = |E_1| + |E_2| + |V_1||V_2|$.

Applying the previous results on example 1 and example 2. In these examples, the combining operation was demonstrated in two different cases. In example 1, the operation was performed on two disjoint graphs, revealing that, the order of the result graph is $5 = 3 + 2 = O(\Gamma_1) + O(\Gamma_2)$ and size 9, given by $9 = 2 + 1 + (3)(2) = S(\Gamma_1) + S(\Gamma_2) + O(\Gamma_1)O(\Gamma_2)$. In contrast, the second example involved the composition of the same graphs as in example 1, but sharing a common vertex, revealing that, the order of the result graph is $4 = O(\Gamma_1) + O(\Gamma_2) - 1$ and size 5, given by $5 = S(\Gamma_1) + S(\Gamma_2) + (O(\Gamma_1) - 1)(O(\Gamma_2) - 1)$. In both examples, the resulting graphs exhibit different orders and sizes.

B. Basic Properties of Undirected Graphs Combining

This section presents some essential properties of undirected graph combining. These properties form the foundation for understanding more advanced graph-theoretic concepts and highlight the key distinctions between graph combining and other graph operations.

Theorem 5. *The graph combining is commutative, that is* $\Gamma_1 \boxplus \Gamma_2 = \Gamma_2 \boxplus \Gamma_1$.

Proof: Let Γ_1 and Γ_2 be two graphs associated with sets of vertices V_1 and V_2, and sets of edges E_1 and E_2 (respectively). Then the set of vertices of $\Gamma = \Gamma_1 \boxplus \Gamma_2$ is $V_\Gamma = V_1 \cup V_2$. In addition, if u and v in V_Γ, then uv in E_Γ if $u, v \in V_1$ and $uv \in E_1$ or $u, v \in V_2$ and $uv \in E_2$, or (without lose of generality) $u \in V_1 \setminus V_2$ and $v \in V_2 \setminus V_1$. Clearly, the set of vertices of $\Gamma_2 \boxplus \Gamma_1$ is $V_2 \cup V_1 = V_1 \cup V_2$ and it also has the same incidence relation as $\Gamma_1 \boxplus \Gamma_2$. ∎

Theorem 6. *The graph combining is associative, that is* $(\Gamma_1 \boxplus \Gamma_2) \boxplus \Gamma_3 = \Gamma_1 \boxplus (\Gamma_2 \boxplus \Gamma_3)$.

Proof: Let Γ_1, Γ_2 and Γ_3 be three graphs associated with sets of vertices V_1, V_2 and V_3, and sets of edges E_1, E_2 and E_3 (respectively), and let:

$$\Gamma = (\Gamma_1 \boxplus \Gamma_2) \boxplus \Gamma_3$$

and

$$\Gamma' = \Gamma_1 \boxplus (\Gamma_2 \boxplus \Gamma_3) .$$

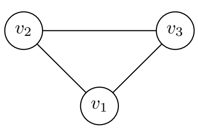
Then, $V_\Gamma = V_{\Gamma'} = V_1 \cup V_2 \cup V_3 = V$.
Let $V_i^U = \{v \in V_i \mid v \notin Vj \ , \ i \neq j = 1, 2, 3\}$. Then, for any $u, v \in V$, we have the following cases:

- Case 1: If $u, v \in V_i$ and $uv \in E_i$ for some $i = 1, 2, 3$, then $uv \in E_\Gamma$ and $E_{\Gamma'}$ using Lemma 4.
- Case 2: If $u, v \in V_i$ and $uv \notin E_i$, $i = 1, 2, 3$, then uv not in E_Γ and not in $E_{\Gamma'}$ using Lemma 4.
- Case 3: If $u \in V_i^U$ and $v \in V_j^U$ for $i \neq j = 1, 2, 3$, then $uv \in \Gamma_i \boxplus \Gamma_j$ which is a subgraph of Γ and also a subgraph of Γ'.

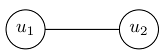
Thus Γ and Γ' have the same set of vertices and the same incidence relation. Hence, $\Gamma = \Gamma'$. ∎

The previous theorems highlight several key properties of the graphs combining operation, namely the commutative and associative properties. While these properties may hold for some other operations, such as the union and intersection of graphs, they certainly do not apply to all graphs operations. For instance, the Corona product is not commutative. See the following example.

Example 3. Consider the next graphs Γ_1 and Γ_2:

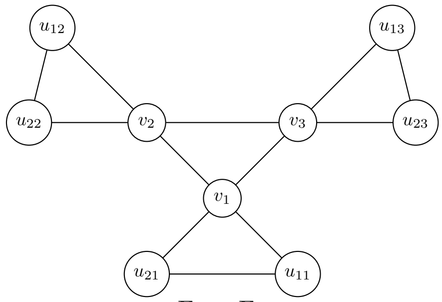


Γ_1

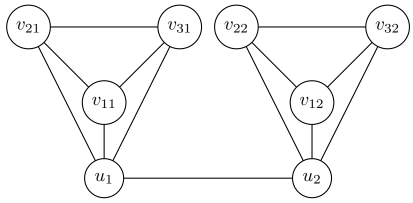


Γ_2

Then:



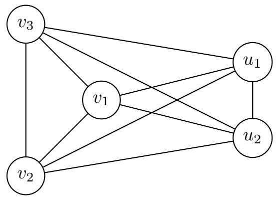
$\Gamma_1 \odot \Gamma_2$



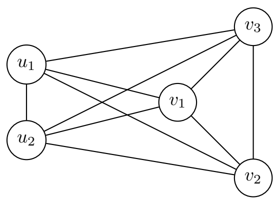
$\Gamma_2 \odot \Gamma_1$

Fig. 3: None commutativity of the Corona product

On the other hand:



$\Gamma_1 \boxplus \Gamma_2$



$\Gamma_2 \boxplus \Gamma_1$

Fig. 4: The commutativity of the graphs combining

This shows that $\Gamma_1 \odot \Gamma_2 \neq \Gamma_2 \odot \Gamma_1$, but $\Gamma_1 \boxplus \Gamma_2 = \Gamma_2 \boxplus \Gamma_1$.

C. Structural Analysis of Undirected Graphs Combining

This section is devoted to the structural analysis of graphs formed by combining two or more undirected simple graphs.

Graph combining methods—such as the union, intersection, Corona product, Cartesian product, and other binary operations—offer powerful tools for constructing complex networks from simpler components. Understanding the structural properties of these combined graphs is essential for both theoretical investigations and practical applications, including network design, modeling biological systems, and studying algebraic or topological properties of graphs.

Theorem 7. *Let* Γ_1 *and* Γ_2 *be two disjoint graphs associated with sets of vertices* V_1 *and* V_2*, and sets of edges* E_1 *and*

E_2 (respectively), and let $\Gamma = \Gamma_1 \boxplus \Gamma_2$. Then

$$\Delta(\Gamma) = \max\{\Delta(\Gamma_1) + |V_2|, \Delta(\Gamma_2) + |V_1|\}.$$

Proof: Consider the disjoint graphs Γ_1 and Γ_2 which associated with sets of vertices V_1 and V_2, and sets of edges E_1 and E_2 (respectively). Let $\Gamma = \Gamma_1 \boxplus \Gamma_2$. If $v \in V_\Gamma$, then, we have two cases:

Case1: $v \in V_1$, then v is adjacent with all $u \in V_2$. Which implies that $\deg_\Gamma v = \deg_{\Gamma_1} v + |V_2|$.

Case2: $v \in V_2$, then v is adjacent with all $u \in V_1$. This implies that $\deg_\Gamma v = \deg_{\Gamma_2} v + |V_1|$.

Thus,

$$\begin{aligned}\Delta(\Gamma) &= \max \quad \{\deg v \mid v \in V_\Gamma\} \\ &= \max \quad \{\max\{\deg v \mid v \in V_1\} + |V_2| \\ &\qquad\quad , \max\{\deg v \mid v \in V_2\} + |V_1|\} \\ &= \max \quad \{\Delta(\Gamma_1) + |V_1|, \Delta(\Gamma_2) + |V_1|\}.\end{aligned}$$

■

Theorem 8. *Let Γ_1 and Γ_2 be two disjoint graphs associated with sets of vertices V_1 and V_2, and sets of edges E_1 and E_2 (respectively), and let $\Gamma = \Gamma_1 \boxplus \Gamma_2$. Then*

$$\delta(\Gamma) = \min\{\delta(\Gamma_1) + |V_2|, \delta(\Gamma_2) + |V_1|\}.$$

Proof: The proof has the same outlines as the previous proof and so it is emitted. ■

Recall that, an empty graph Γ is $\Gamma(V, E)$ for which $|V| \geq 1$ and $|E| = 0$.

Lemma 9. *If Γ_1 and Γ_2 are two disjoint empty graphs of orders m and n, respectively, then $\Gamma_1 \boxplus \Gamma_2 \cong K_{m,n}$ (the complete bipartite graph).*

Proof: Let Γ_1 and Γ_2 be two disjoint empty graphs for which $|V_1| = m$ and $|V_2| = n$. Set $\Gamma = \Gamma_1 \boxplus \Gamma_2$ and let $u, v \in V_\Gamma$. Then, we have the following cases:

Case 1: If $u, v \in V_1$, then u and v are not adjacent, because Γ_1 is empty graph.

Case 2: If $u, v \in V_2$, then u and v are not adjacent, because Γ_2 is empty graph.

Case 3: If $u \in V_1$ and $v \in V_2$, or $u \in V_2$ and $v \in V_1$, then v and u are adjacent.

Therefore, V_Γ the set of vertices of Γ is partitioned into two disjoint subsets V_1 and V_2, for which every vertex in one part is adjacent to all vertices in the other part and not adjacent to any vertex on the same part. Hence, $\Gamma \cong K_{m,n}$. ■

Corollary 10. *Consider the disjoint graphs Γ_1 and Γ_2 of orders m and n, respectively. Then, $K_{m,n}$ is a spanning subgraph of $\Gamma_1 \boxplus \Gamma_2$.*

Proof: Let Γ_1 and Γ_2 be two disjoint graphs of orders m and n respectively and $\Gamma = \Gamma_1 \boxplus \Gamma_2$. Since the empty graph H_k of order k is a subgraph of any graph of order $n \geq k$. Then $H_m \boxplus H_n \subseteq \Gamma_1 \boxplus \Gamma_2$. Using Lemma 9, we have $K_{m,n} \cong H_m \boxplus H_n$ and thus $K_{m,n} \subseteq \Gamma_1 \boxplus \Gamma_2$ and since the order of $K_{m,n} = m + n = |V_1| + |V_2|$. Then $K_{m,n}$ is a spanning subgraph of $\Gamma_1 \boxplus \Gamma_2$. ■

Remark 11. *Consider any two non-disjoint graphs $\Gamma_1(V_1, E_1)$ and $\Gamma_2(V_2, E_2)$, and let $V = V_1 \cap V_2$ with $|V_1| = n_1$ $|V_2| = n_2$, and $|V| = k$. Then, both complete bipartite graphs K_{n_1, n_2-k} and K_{n_2, n_1-k} are subgraphs of $\Gamma_1 \boxplus \Gamma_2$.*

Clearly, the total number of distinct vertices in $\Gamma_1 \boxplus \Gamma_2$ is given by

$$|V_1 \cup V_2| = |V_1| + |V_2| - |V_1 \cap V_2| = n_1 + n_2 - k,$$

which implies that the complete bipartite graphs in both cases span the vertex set of $\Gamma_1 \boxplus \Gamma_2$. Therefore, they are spanning subgraphs of $\Gamma_1 \boxplus \Gamma_2$.

Corollary 12. *For any disjoint graphs Γ_1 and Γ_2, then $\Gamma_1 \boxplus \Gamma_2$ is a connected graph.*

Proof: Since, $K_{m,n}$ is a spanning subgraph of $\Gamma_1 \boxplus \Gamma_2$ for any disjoint graphs Γ_1 and Γ_2 of orders m and n respectively, and $K_{m,n}$ is a connected spanning subgraph of Γ. Then, the graph $K_{m,n} \cup E$, where E is a non-empty set of edges is also connected graph. Hence, $\Gamma_1 \boxplus \Gamma_2$ is also connected. ■

Corollary 13. *Let Γ_1 and Γ_2 be two graphs. Then for any $u, v \in \Gamma_1 \boxplus \Gamma_2$, we have $d(u, v) \leq 2$.*

Proof: Let Γ_1 and Γ_2 be two graphs. Note that the complete bipartite graph $K_{m,n}$ is a spanning subgraph of $\Gamma_1 \boxplus \Gamma_2$, where $n + m = |V_1 \cup V_2|$ (See Remark 11). Consider any two vertices u and v in $\Gamma_1 \boxplus \Gamma_2$. Then $u, v \in K_{m,n}$, whose vertex set is partitioned into two disjoint subsets V_1 and V_2.

- If $u \in V_1$ and $v \in V_2$ (or vice versa), then u and v are adjacent, so $d(u, v) = 1$.
- If $u, v \in V_1$ or $u, v \in V_2$, then there exists a vertex w in the opposite subset such that u, uw, w, wv, v is the shortest path joining u and v, which implies $d(u, v) = 2$.

Hence, $d(u, v) \leq 2$. ■

Theorem 14. *Let Γ_1 and Γ_2 be two disjoint graphs of order n and m, respectively. The adjacency matrix A of $\Gamma_1 \boxplus \Gamma_2$ can be written as:*

$$A = \begin{bmatrix} A_1 & \mathbf{1} \\ \mathbf{1} & A_2 \end{bmatrix},$$

where A_1 and A_2 are the $n \times n$ and $m \times m$ adjacency matrices of Γ_1 and Γ_2, respectively.

Proof: Let Γ_1 and Γ_2 be two disjoint graphs of order n and m associated with sets of vertices $V_1 = \{u_1, u_2, \ldots, u_n\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$, and sets of edges E_1 and E_2 (respectively). Then u_i is adjacent with v_j in $\Gamma_1 \boxplus \Gamma_2$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, this impose 1 on all entries corresponding the raw u_i with the column v_j in A. In addition, u_i adjacent with u_j in $\Gamma_1 \boxplus \Gamma_2$ if $u_i u_j \in E_1$ for all $i, j = 1, 2, \ldots, n$. Also, v_i adjacent with v_j in $\Gamma_1 \boxplus \Gamma_2$ if $v_i v_j \in E_2$ for all $i, j = 1, 2, \ldots, m$. Then, A consists of 4 blocks, the first indicates the entries corresponding to the set V_1 which is A_1, the second indicates the entries corresponding to the raw u_i with the column v_j for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, the third indicates the entries corresponding to the raw v_j with the column u_i for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and the last block indicates the entries corresponding to the set V_2 which is A_2. That is:

$$A = \begin{array}{c|cccc|cccc|} & u_1 & u_2 & \cdots & u_n & v_1 & v_2 & \cdots & v_m \\ \hline u_1 & & & & & 1 & 1 & \cdots & 1 \\ u_2 & & & & & 1 & 1 & \cdots & 1 \\ \vdots & & A_1 & & & \vdots & \vdots & \cdots & \vdots \\ u_n & & & & & 1 & 1 & \cdots & 1 \\ \hline v_1 & 1 & 1 & \cdots & 1 & & & & \\ v_2 & 1 & 1 & \cdots & 1 & & & & \\ \vdots & \vdots & \vdots & \cdots & \vdots & & A_2 & & \\ v_m & 1 & 1 & \cdots & 1 & & & & \end{array} = \begin{bmatrix} A_1 & \mathbf{1} \\ \mathbf{1} & A_2 \end{bmatrix}.$$

■

Theorem 15. *For any disjoint graphs Γ_1 and Γ_2, the graph $\Gamma_1 \boxplus \Gamma_2$ is Hamiltonian graph.*

Proof: Set $V_1 = \{u_1, u_2, \ldots, u_n\}$ to be the set of vertices of Γ_1 and $V_2 = \{v_1, v_2, \ldots, v_m\}$ be the set of vertices of Γ_2. Then,

$$u_1, u_1v_1, v_1, v_1u_2, u_2, u_2v_3, v_3, \ldots, v_{m-1}, v_{m-1}u_1, u_1$$

is a cycle in $\Gamma = \Gamma_1 \boxplus \Gamma_2$ that visits each vertex in Γ exactly once. Hence, Γ is Hamiltonian graph. ■

Remark 16. *Since, every edge in the combining of any disjoint graphs is in a cycle. Then, the combining operation on distinct graphs produce a graph that has no bridges. That is, if $\Gamma_1 \neq \Gamma_2$ are two graphs, then every edge in $\Gamma = \Gamma_1 \boxplus \Gamma_2$ is not a bridge.*

Recall that, in a connected graph $\Gamma(V, E)$, the eccentricity $e(v)$ of a vertex $v \in V_\Gamma$ is the maximum distance between v and any other vertex $u \in V_\Gamma$, where the distance between u and v is the length of the shortest path between u and v. The radius of the connected graph Γ is

$$rad(\Gamma) = \min\{e(v) \mid v \in V_\Gamma\}$$

and the diameter of Γ is

$$diam(\Gamma) = \max\{e(v) \mid v \in V_\Gamma\}.$$

Eccentricity, distance, radius, and diameter are fundamental concepts in graph theory, and they have significant importance in understanding the structure and properties of graphs. These concepts are crucial for understanding and optimizing the structure and performance of networks and systems modeled by graphs. They provide valuable insights into vertex centrality, network efficiency, and overall connectivity, making them indispensable in fields like communication theory, network analysis, and complex system modeling.

Given the importance of these measures, we study them in the context of the graph combining operation.

Theorem 17. *Let $\Gamma_1 \neq \Gamma_2$ be two graphs and $\Gamma = \Gamma_1 \boxplus \Gamma_2$. Then, the radius of Γ is $rad(\Gamma) = 1$ and the diameter of Γ is $diam(\Gamma) \leq 2$.*

Proof: Let $\Gamma_1 \neq \Gamma_2$ be two graphs and $\Gamma = \Gamma_1 \boxplus \Gamma_2$. Then, for any vertex u in V_Γ, we have two cases:

Case 1: If $u \in V_1$ and for any $u \neq v \in V_1$ which is adjacent to u, then the distance between u and v is 1.

Case 2: If $u \in V_1$ and for any $u \neq v \in V_1$ which is not adjacent to u, then, the shortest path between u and v is: u, uw, w, wv, v for $w \in V_2$. So, the distance between u and v is 2.

Thus, $e(u) = 1, 2$ for all $u \in V_1$. Similarly, $e(u) = 1, 2$ for all $u \in V_2$. Note that, if Γ is a complete graph, so every two vertices are adjacent, then $e(u) = 1$ for all $u \in V_\Gamma$. Therefore:

$$rad(\Gamma) = \min\{e(u) \mid u \in V_\Gamma\} = \min\{1, 2\} = 1$$

and

$$\begin{aligned} diam(\Gamma) &= \max\{e(u) \mid u \in V_\Gamma\} \\ &= \max\{1, 2\} = 2 \\ \text{or } diam(\Gamma) &= 1 \text{ if } \Gamma \text{ is complete graph} \end{aligned}$$

■

Theorem 18. *Let $g(\Gamma)$ denote the girth of a graph Γ. Then,*

$$g(\Gamma_1 \boxplus \Gamma_2) \leq \min\{g(\Gamma_1),\ g(\Gamma_2),\ 4\}.$$

Proof: If $V_1 \cap V_2 = \emptyset$, the new edges form a complete bipartite subgraph $K_{n,m}$ between V_1 and V_2. Such a graph contains 4-cycles but no 3-cycles. Thus, the girth is at most 4. If either of Γ_1 or Γ_2 has a shorter cycle, it will appear in the union, hence the bound. ■

IV. Combining Directed Graphs

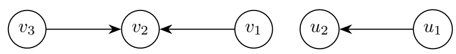
In this section, we consider finite, simple, and directed graphs.

Definition 19. Let Γ_1 and Γ_2 be two finite, simple, directed graphs associated with sets of vertices V_1 and V_2, and sets of arrows E_1 and E_2 (respectively). The combining graph of Γ_1 and Γ_2, denoted by $\Gamma_1 \boxplus \Gamma_2$, is the graph Γ associated with set of vertices V_Γ and set of arrows E_Γ, defined as follows:

$$\begin{aligned} &V_\Gamma = V_1 \cup V_2, \\ &E_\Gamma = E_1 \cup E_2 \cup \{(u, v) \mid u \in V_1 \setminus V_2 \text{ and } v \in V_2 \setminus V_1\}. \end{aligned}$$

The combining operation defined above allows for the construction of a new graph that preserves the structure of the original components while introducing directed connections between disjoint parts. This operation may be relevant in modeling interactions between two systems or layers in a network.

Example 4. Consider the next graphs Γ_1 and Γ_2:



Γ_1 Γ_2

Then, the combining graph $\Gamma_1 \boxplus \Gamma_2$ and $\Gamma_2 \boxplus \Gamma_1$ are shown on the next figure:

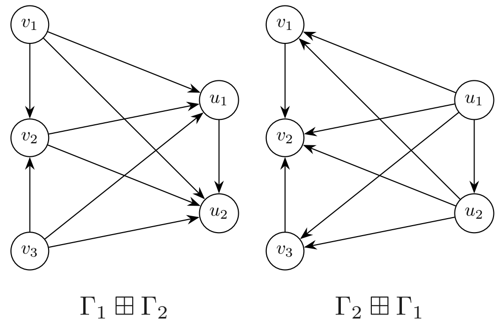


Fig. 5: The none commutativity of directed graphs combining

A. Properties of Directed Graphs Combining

We present basic structural properties of the combining operation $\Gamma_1 \boxplus \Gamma_2$ on finite, simple, directed graphs.

Let $\Gamma_1(V_1, E_1)$ and $\Gamma_2(V_2, E_2)$ be two finite, simple, directed graphs. The next result show the basic properties of $\Gamma_1 \boxplus \Gamma_2$.

Proposition 20. *The combining operation $\boxplus$ of two graphs is not commutative. That is, in general,*

$$\Gamma_1 \boxplus \Gamma_2 \neq \Gamma_2 \boxplus \Gamma_1.$$

Proof: By definition, the additional arrows go from $V_1 \setminus V_2$ to $V_2 \setminus V_1$. Swapping Γ_1 and Γ_2 reverses the direction of these additional arrows. Therefore, the resulting arrow sets differ, see Example 4. ■

Proposition 21. *In general, the operation $\boxplus$ is not associative; that is,*

$$(\Gamma_1 \boxplus \Gamma_2) \boxplus \Gamma_3 \neq \Gamma_1 \boxplus (\Gamma_2 \boxplus \Gamma_3).$$

Proof: The intermediate graph in each case alters the direction and presence of new arrows added during the second application of $\boxplus$. This causes the final edge set to differ depending on the grouping. ■

Remark 22. *The vertex set of $\Gamma_1 \boxplus \Gamma_2$ is*

$$V_{\Gamma_1 \boxplus \Gamma_2} = V_1 \cup V_2.$$

$$O(\Gamma_1 \boxplus \Gamma_2) = \begin{cases} O(\Gamma_1) + O(\Gamma_2), & \textit{if } V_1 \cap V_2 = \emptyset, \\ O(\Gamma_1) + O(\Gamma_2) - O(\Gamma_1 \cap \Gamma_2), & \textit{otherwise}. \end{cases}$$

Remark 23. *The arrow set of the combining graph satisfies*

$$E_{\Gamma_1 \boxplus \Gamma_2} \supseteq E_1 \cup E_2,$$

with additional arrows from $V_1 \setminus V_2$ to $V_2 \setminus V_1$.

$$S(\Gamma_1 \boxplus \Gamma_2) = \begin{cases} S(\Gamma_1) + S(\Gamma_2) + |V_2|, & \textit{if } V_1 \cap V_2 = \emptyset, \\ S(\Gamma_1) + S(\Gamma_2) - S(\Gamma_1 \cap \Gamma_2), & \textit{otherwise}. \\ +|V_2 \setminus V_1| \end{cases}$$

Proposition 24. *If Γ_1 and Γ_2 are simple (i.e., have no loops or multiple arrows), then $\Gamma_1 \boxplus \Gamma_2$ is also simple.*

Proof: The added arrows connect distinct vertices and no multiple arrows are introduced. ■

Proposition 25. *If Γ_1 and Γ_2 are disjoint, i.e., $V_1 \cap V_2 = \emptyset$, then $\Gamma_1 \boxplus \Gamma_2$ is weakly connected.*

Proof: Every vertex $u \in V_1$ is connected by a directed edge to every vertex $v \in V_2$, forming paths between the two vertex sets. Thus, ignoring directions, the resulting undirected graph is connected. ■

Proposition 26. *If both Γ_1 and Γ_2 are acyclic, then $\Gamma_1 \boxplus \Gamma_2$ is also acyclic.*

Proof: The additional arrows only go from $V_1 \setminus V_2$ to $V_2 \setminus V_1$, and not in both directions. Since there are no cycles in Γ_1 or Γ_2, and no arrows go back from V_2 to V_1, no directed cycles can be formed. ■

Proposition 27. *Let $deg_\Gamma^+(v)$ and $deg_\Gamma^-(v)$ denote the out-degree and indegree of a vertex v in $\Gamma = \Gamma_1 \boxplus \Gamma_2$. Then:*

- *If $v \in V_1 \setminus V_2$, then*
$$deg_\Gamma^+(v) = deg_{\Gamma_1}^+(v) + |V_2 \setminus V_1|,$$
$$deg_\Gamma^-(v) = deg_{\Gamma_1}^-(v).$$
- *If $v \in V_2 \setminus V_1$, then*
$$deg_\Gamma^+(v) = deg_{\Gamma_2}^+(v),$$
$$deg_\Gamma^-(v) = deg_{\Gamma_2}^-(v) + |V_1 \setminus V_2|.$$
- *If $v \in V_1 \cap V_2$, then*
$$deg_\Gamma^+(v) = deg_{\Gamma_1}^+(v) + deg_{\Gamma_2}^+(v),$$
$$deg_\Gamma^-(v) = deg_{\Gamma_1}^-(v) + deg_{\Gamma_2}^-(v).$$

Proof: The additional arrows connect vertices from $V_1 \setminus V_2$ to $V_2 \setminus V_1$, thereby increasing outdegrees in $V_1 \setminus V_2$ and indegrees in $V_2 \setminus V_1$ accordingly. Vertices in the intersection remain unaffected by the added connections. ■

References

[1] J. A. Bondy, U. S. R. Murty, "Graph Theory with Applications," The Macmillan Press Ltd., Great Britain, 1976.
[2] J. N. Cortes, "History of graph theory," 2014, https://prezi.com/pfyjgp86zlmg/history-of-graph-theory.
[3] Mathegon, "Graphs and networks," 2017, https://mathigon.org/course/graphs-and-networks#full.
[4] J. A. Bondy, U. S. Murty, "Graph Theory," Springer, ISBN 978-1-84628-969-9, 2008.
[5] R. Wilson, "Introduction to Graph Theory," Addison Wesley Longman Limited, Edinburgh Gate, Harlow, Essex CM20 2JE, England, 1996.
[6] T. Harju, "Lecture Notes on Graph Theory," University of Turku, Finalnd, 1994.
[7] D. Shijie and L. Feng, "On the Wiener Index of the Strong Product of Paths," IAENG International Journal of Computer Science, vol. 49, no.2, pp401-409, 2022.
[8] Savinkov Rostislav, Grebennikov Dmitry, Puchkova Darya, Chereshnev Valery, Sazonov Igor and Bocharov Gennady, "Graph Theory for Modeling and Analysis of the Human Lymphatic System," Mathematics, vol. 8, no.12, 2020.
[9] Jingxuan Ruan, Yaping Wu, Haiyan Wang, Zhenxing Huang, Ziwei Liu, Xinlan Yang, Yongfeng Yang, Hairong Zheng, Dong Liang, Meiyun Wang and Zhanli Hu, "Graph theory analysis of a human body metabolic network: A systematic and organ-specific study," Medical Physics, vol. 52, no.4, 2025.
[10] J. Satish Kumar, B. Archana, K. Muralidharan and V. Senthil Kumar, "Graph Theory: Modelling and Analyzing Complex System," Metallurgical and Materials Engineering, vol. 31, no.3, pp70-77, 2025.
[11] Jin Jingjing, "Algorithms for graph theory matching problems based on computer science," International Journal of Modern Physics C, vol. 36, no.8, pp1-19, 2025.