The M/M/c/N Vacation Queueing System with Impatient Customers and Repairable Server

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***Abstract*—This study develops a queueing model that incorporates server vacations, customer impatience, and repairable server breakdowns. To enhance customer satisfaction and reduce server operational costs, this study introduces a specific strategy. Servers follow startup and shutdown periods, and some servers continue working during vacation. This design makes the model more practical. Based on the model description, this study derives the transition rate matrix for the system and employs an iterative approach to determine steady-state probabilities and evaluate performance metrics. The impact of various parameters on system performance is investigated using numerical experiments. Finally, the benefits for individuals and the overall system are analyzed, and suggestions are provided to improve personal earnings and overall benefits.**

***Index Terms*—Startup and shutdown periods, repairable server, impatient customers, system optimization.**

I. Introduction

TO minimize the system's operational energy consumption, save the operation costs and maintain the equipment at regular intervals, servers need various modes of vacation. Vacation queueing has become an essential strategy in modern operations management across fields. Its key advantages include better resource use, improved service quality, and lower operating costs. Li et al. [1] introduced a GI/M/1 queueing model that incorporates bernoulli vacations and vacation interruptions. They employed matrix analysis to derive system performance metrics, conducted numerical experiments to evaluate the impact of various parameters on queue indicators, and performed a cost optimization analysis. Saravanan et al. [2] investigated an M/M/m retrial queueing model with simultaneous vacations, impatient customers, and unreliable servers. They linked the model to a practical application in health helpline services and derived the steady-state probabilities for the system's performance metrics. Ma et al. [3] designed and studied a multiple vacation queueing system that accounts for impatient customers and working breakdowns, based on the M/M/1 multiple vacation queueing structure. Laxmi and Jyothsna [4] developed a queueing system incorporating impatient customers and bernoulli vacation interruptions. By utilizing the method of probability generating functions, they obtained explicit expressions for the steady-state probabilities. Zhu and Xu [5] built upon the classic M/M/c queueing model by incorporating impatient customers and a partial working vacation strategy. They also introduced the N-policy as a vacation termination criterion. By applying the quasi-birth-and-death process alongside the matrix-geometric solution approach, they calculated the steady-state queue-length distribution for the system. Furthermore, they established a random decomposition structure under the assumption that all servers are occupied. Yang et al. [6] added startup time, working vacations, and working breakdown policies to a repairable M/M/1/N queueing system and derived significant performance metrics for the system. In their research, Zhang et al. [7] proposed an M/M/c queueing system incorporating a startup period, shutdown period, synchronous multiple working vacations, vacation interruptions, and impatient customer behavior. Through the quasi-birth-and-death process and the matrix-geometric solution method, they calculated the steady-state distribution of the system.

In numerous related studies, authors frequently assume that servers are entirely reliable. In practice, however, servers are prone to failures caused by various factors, such as environmental changes, operational obstructions, upgrades, and natural wear and tear. The restored servers can then resume normal service rates. Many researchers have studied and analyzed repairable queueing models with server failures. Avi-Itzhak and Naor [8] investigated multiclass queueing models that incorporate server breakdowns and repairable strategies, deriving steady state distributions and performance metrics, such as queue length and sojourn time for each model. Seenivasan et al. [9] investigated an M/M/1 single vacation queueing system with variable arrival and service rates, server breakdowns, and a feedback mechanism, deriving the steady-state probabilities for each period of the system. Tian et al. [10] studied a queueing system incorporating working breakdowns and additional services, utilizing the probability generating function method to derive the state probabilities of the service station, and further obtaining performance metrics such as the system's average queue length under steady-state conditions. Ye and Chen [11] investigated an M/M/1 retry queueing system with working breakdowns. Based on the generalized characteristic value method, they derived key performance metrics for the queueing system and applied the L-S transform to the distribution function of arbitrary customer sojourn times to obtain the average sojourn time for any customer. Towsley and Tripathi [12] studied a single-server queueing model with priorities and repairable failures, deriving the expression for the queue length generating function. Falin [13] explored an M/G/1 retrial queueing framework with an unreliable server and repair times governed by a general distribution, deriving the steady-state distribution under the condition that both service and repair times are generally distributed. V. Karthick and V. Suvitha [14] explored repairable Markovian queueing systems

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featuring three service counters where servers are on vacation. Lv et al. [15] examined the impact of varying input and failure rates on queueing systems.

The phenomenon where customers leave before entering the service system is known as balking, and the phenomenon where customers abandon the queue during the waiting process is known as reneging. Both are collectively referred to as impatience phenomena. As researchers in queueing theory have delved deeper into their studies, they have introduced impatience phenomena into their research. In 1963, Ancker and Gafarian [16] studied the M/M/1 queueing model with impatient customers. Subsequently, Stanford [17] discussed the GI/G/1 single-server queueing system with impatient customers, deriving the waiting time and queue length distributions within the system. Bae et al. [18] examined the M/G/1 queueing system with impatient customers, analyzing the waiting time of customers in the system. Altman and Yechiali [19] studied vacation queueing systems such as M/M/1 and M/G/1 with impatient customers. The work of Yue et al. [20] utilized the probability generating function method to explore an M/M/1 queueing framework with impatient customers and K-vacations, calculating relevant indicators such as the steady-state queue length and verifying special cases where K takes specific values. Chen [21] studied a multi-server queueing system with balking, reneging, and negative customers in the context of customers using mobile travel apps. Shan [22] investigated the M/M/1/N queueing inventory system model with impatient customers and multiple working vacations. Based on the (s, S) inventory policy, they used matrix iteration methods to obtain the steady-state probability distribution of the system and provided relevant performance indicators, further establishing the average inventory cost function of the system.

Based on the aforementioned literature, this paper establishes a queueing system with impatient customers, startup and shutdown periods, repairable server breakdowns, and partial server working vacations. The aim is to enhance customer satisfaction while reducing server operational costs. This model enriches the theoretical framework and enhances its applicability to real-world scenarios.

II. Model description

1) This is an M/M/c/N queueing model. In this model, customer arrivals obey a Poisson process with a rate of arrival λ.

2) To reduce the system's operating energy consumption, this paper introduces strategies involving startup and shutdown periods (state 1), as well as partial server working vacations (state 0). (i) If no customers are present in the system, it transitions into a shutdown period, where the shutdown time D is exponentially distributed with parameter γ. (ii) If the customer arrives during the system shutdown period, the system immediately enters the busy period (state 2), and the customer's service time obeys an exponential distribution with parameter μ_1. (iii) If no customers arrive during the shutdown period, all servers synchronously initiate a random-length vacation, with the vacation time following an exponential distribution parameterized by θ. During the vacation period, it is assumed that up to d $(0 < d < c)$ servers do not completely cease service and serve customers at a reduced service rate, with service times following an exponential distribution parameterized by μ_2 $(\mu_2 < \mu_1)$. The remaining $c - d$ servers are in a complete vacation state and do not serve customers. If no customers are present in the system at the end of a vacation period, the servers continue into an independent and synchronous vacation process. If customers are present in the system at the end of a vacation period, all servers simultaneously enter the startup phase. (iv) Considering that servers in the working vacation state and those in complete vacation state require a period of startup adaptation to resume normal busy period operations from their vacation states, the startup time U for all servers follows an exponential distribution parameterized by ξ.

3) Server Breakdowns and Repair Process: Failures of the server can occur during regular busy periods, startup/shutdown phases, and vacation periods. These failures result in the server's inability to operate normally, causing service interruptions for customers currently being served. These customers are then placed back at the front of the queue to resume service, with the previously served time becoming invalid. The failure process follows a Poisson distribution with a parameter α. Servers are repairable, upon failure, it is immediately sent for repair. Only one repairman can maintain each server, and the repair time adheres to an exponential distribution characterized by parameter β. The paper assumes that failures occur randomly, mutually independent, and the count of repairmen equals the number of servers.

4) Impatient Waiting Process of Customers: During server vacation periods, the reduced service rate may lead to customer impatience, causing some to leave the system without receiving service. however, customers who are currently being served within the queue will not experience impatience. The impatient waiting time of customers is assumed to follow an exponential distribution characterized by parameter ε.

5) The service follows the First-Come-First-Served (FCFS) queueing rule. Additionally, it is assumed that all processes are independent of each other.

III. Transfer Rate Matrix

Let $L(t)$ represent the number of customers in the system at time t, $Y(t)$ denote the number of normally functioning servers at time t, and $J(t)$ indicate the states of the servers at time t.

Then $\{L(t), Y(t), J(t)\}$ is a three-dimensional Markov process, its state space is

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3.$$

among which

$$\Omega_1 = \{(l, y, j), 0 \le l \le N, 0 \le y \le c, j = 0\},$$

$$\Omega_2 = \{(l, y, j), 0 \le l \le N, 1 \le y \le c, j = 1\},$$

$$\Omega_3 = \{(l, y, j), 1 \le l \le N, 1 \le y \le c, j = 2\}.$$

The state space is arranged in dictionary order to derive the infinitesimal generators of the state transition matrix as follows:

$$
\mathrm{Q} = \begin{pmatrix}
A_0 & C_0 & & & & & & & \\
B_1 & A_1 & C & & & & & & \\
 & B_2 & A_2 & C & & & & & \\
 & & \ddots & \ddots & \ddots & & & & \\
 & & & B_c & A_c & C & & & \\
 & & & & \ddots & \ddots & \ddots & & \\
 & & & & & B_{N-1} & A_{N-1} & C \\
 & & & & & & B_N & A_N
\end{pmatrix}.
$$

In which $A_0, C_0, C, A_k\,(1 \le k \le N), B_k\,(1 \le k \le N)$ denote the state transition rate matrices between the corresponding levels, respectively. To ensure that the matrices are concise and readable, numerical symbols are appended below the matrices. The specific matrices are as follows:

The square matrix A_0 of order $(2c+1)$ is

$$
A_0 = \begin{pmatrix}
D_0 & F_0 & & & & \\
E_1 & D_1 & F_1 & & & \\
 & E_2 & D_2 & F_2 & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & E_{c-1} & D_{c-1} & F_{c-1} \\
 & & & & E_c & D_{c-1}
\end{pmatrix}.
$$

$$
D_0 = a_{0,0},\ F_0 = (c\beta, 0)\,,
$$

$$
E1 = \begin{pmatrix} \alpha \\ 0 \end{pmatrix},\ D_i = \begin{pmatrix} a_{i,0} & 0 \\ \gamma & a_{i,1} \end{pmatrix}, 1 \le i \le c.
$$

$$
E_i = \begin{pmatrix} i\alpha & 0 \\ 0 & i\alpha \end{pmatrix}, 2 \le i \le c.
$$

$$
F_i = \begin{pmatrix} (c-1)\,\beta & 0 \\ 0 & (c-1)\,\beta \end{pmatrix}, 1 \le i \le c-1.
$$

where

$$
a_{i,0} = -\left[(c-i)\,\beta + i\alpha + \lambda\right], 0 \le i \le c,
$$

$$
a_{i,1} = -\left[(c-i)\,\beta + i\alpha + r + \lambda\right], 2 \le i \le c,
$$

$$
a_{1,1} = -\left[(c-1)\,\beta + r + \lambda\right].
$$

The $(2c+1) \times (3c+1)$-order matrix C_0 is

$$
C_0 = \begin{pmatrix}
\lambda & & & & & & \\
 & \lambda & & & & & \\
 & & 0 & \lambda & & & \\
 & & & & \ddots & & \\
 & & & & & \lambda & \\
 & & & & & 0 & \lambda
\end{pmatrix}.
$$

The $(3c+1) \times (2c+1)$-order matrix B_1 is

$$
B_1 = \begin{pmatrix}
0 & & & & & & & \\
 & \mu_2 & & & & & & \\
 & & 0 & & & & & \\
 & & \mu_1 & & & & & \\
 & & & \mu_2 & & & & \\
 & & & & 0 & & & \\
 & & & & \mu_1 & & & \\
 & & & & & \ddots & & \\
 & & & & & & \mu_2 & \\
 & & & & & & & 0 \\
 & & & & & & & \mu_1
\end{pmatrix}.
$$

The square matrix $B_i\ (2 \le i < c)$ of order $(3c+1)$ is

$$
B_i = \begin{pmatrix}
0 & & & & & \\
 & b_1 & & & & \\
 & & b_2 & & & \\
 & & & \ddots & & \\
 & & & & b_i & \\
 & & & & & b_i
\end{pmatrix}.
$$

where

$$
b_k = \begin{pmatrix} k\mu_2 + (i-k)\,\varepsilon & & \\ & 0 & \\ & & k\mu_1 \end{pmatrix}, k \le d,
$$

$$
b_k = \begin{pmatrix} d\mu_2 + (i-d)\,\varepsilon & & \\ & 0 & \\ & & k\mu_1 \end{pmatrix}, d < k < c.
$$

The square matrix $B_i\ (c \le i \le N)$ of order $(3c+1)$ is

$$
B_i = \begin{pmatrix}
0 & & & & & & & \\
 & b_1 & & & & & & \\
 & & \ddots & & & & & \\
 & & & b_d & & & & \\
 & & & & b_{d+1} & & & \\
 & & & & & \ddots & & \\
 & & & & & & b_{c-1} & \\
 & & & & & & & b_c
\end{pmatrix}.
$$

where

$$
b_k = \begin{pmatrix} k\mu_2 + (i-k)\,\varepsilon & & \\ & 0 & \\ & & k\mu_1 \end{pmatrix}, k \le d,
$$

$$
b_k = \begin{pmatrix} d\mu_2 + (i-d)\,\varepsilon & & \\ & 0 & \\ & & k\mu_1 \end{pmatrix}, d < k \le c.
$$

The square matrix $A_i\,(1 \le i \le N)$ of order $(3c+1)$ is

$$A_i = \begin{pmatrix} A^1{}_{i,0} & A^2{}_{i,0} & & & & \\ A^0{}_{i,1} & A^1{}_{i,1} & A^2{}_{i,1} & & & \\ & A^0{}_{i,2} & A^1{}_{i,2} & A^2{}_{i,2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & A^0{}_{i,c-1} & A^1{}_{i,c-1} & A^2{}_{i,c-1} \\ & & & & A^0{}_{i,c} & A^1{}_{i,c} \end{pmatrix}.$$

where j is $1 \le j \le c$, defines the symbol:

$$n^1{}_{i,j} = \begin{cases} -\lambda - \theta - \min\{i,j,d\}\,\mu_2 - (c-j)\,\beta - \\ (i - \min\{i,j,d\})\,\varepsilon - j\alpha, 1 \le i \le N-1, \\ -\theta - \min\{i,j,d\}\,\mu_2 - (i - \min\{i,j,d\})\,\varepsilon \\ -(c-j)\,\beta - j\alpha, i = N. \end{cases}$$

$$n^2{}_{i,j} = \begin{cases} -[\lambda + \xi + (c-j)\,\beta + j\alpha]\,, 1 \le i \le N-1, \\ -[\xi + (c-j)\,\beta + j\alpha]\,, i = N. \end{cases}$$

$$n^3{}_{i,j} = \begin{cases} -[\lambda + \min\{i,j\}\,\mu_1 + (c-j)\,\beta + j\alpha]\,, \\ 1 \le i \le N-1, \\ -[\min\{i,j\}\,\mu_1 + (c-j)\,\beta + j\alpha]\,, i = N. \end{cases}$$

where

$$A^1{}_{i,0} = \begin{cases} -(\lambda + c\beta)\,, 1 \le i \le N-1, \\ -c\beta, i = N. \end{cases}$$

$$A^2{}_{i,0} = (c\beta, 0, 0)\,, A^0{}_{i,1} = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix},$$

$$A^0{}_{i,j} = \begin{pmatrix} j\alpha & & \\ & j\alpha & \\ & & j\alpha \end{pmatrix} (2 \le j \le c)\,,$$

$$A^1{}_{i,j} = \begin{pmatrix} n^1{}_{i,j} & \theta & \\ & n^2{}_{i,j} & \xi \\ & & n^3{}_{i,j} \end{pmatrix} (1 \le j \le c)\,,$$

$$A^2{}_{i,j} = \begin{pmatrix} (c-j)\,\beta & & \\ & (c-j)\,\beta & \\ & & (c-j)\,\beta \end{pmatrix},$$

$(1 \le j \le c-1)\,.$

The square matrix C of order $(3c+1)$ is

$$C = \begin{pmatrix} \lambda & & & & \\ & \lambda & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & \lambda \end{pmatrix}.$$

IV. Matrix Solutions for Steady State Probability

From the structure of matrix Q, it can be seen that the state process of the system is a horizontally dependent proposed birth and death process, and when the Markov process returns normally, the steady state distribution is defined as follows:

$$P_{k,y,j} = \lim_{t \to \infty} P\{L(t) = k, Y(t) = y, J(t) = j\},$$

$(k, y, j) \in \Omega.$

To accommodate the structure of matrix Q, the steady-state probability vector P is segmented as follows:

$$P = (P_0, P_1, P_2, ..., P_N)\,,$$

$$P_0 = (P_{0,0,0}, P_{0,1,0}, P_{0,1,1}, \cdots, P_{0,c,0}, P_{0,c,1})\,,$$

where $k \ge 1$,

$$P_k = (P_{k,0,0}, P_{k,1,0}, P_{k,1,1}, P_{k12}, \cdots, P_{k,c,0}, P_{k,c,1}, P_{k,c,2})\,.$$

The steady-state probability vector P satisfies the system of equations

$$\begin{cases} PQ = 0, \\ Pe = 1. \end{cases}$$

where e represents a column vector of suitable dimension, with every element being 1.

Expanding the system of equations satisfied by the steady-state probability vector P yields,

$$P_0 A_0 + P_1 B_1 = 0, \tag{1}$$

$$P_0 C_0 + P_1 A_1 + P_2 B_2 = 0, \tag{2}$$

$$P_k C + P_{k+1} A_{k+1} + P_{k+2} B_{k+2} = 0, 1 \le k \le N-2, \tag{3}$$

$$P_{N-1} C + P_N A_N = 0, \tag{4}$$

$$P_0 e_{2c+1} + \sum_{k=1}^{N} p_k e_{3c+1} = 1, \tag{5}$$

where e_{2c+1} denotes a $2c+1$ -dimensional column vector with all entries being 1, and e_{3c+1} is a $3c+1$ -dimensional column vector with all elements set to 1.

The iterative formula for the steady state probability vector p can be obtained by solving Eq. (1) to Eq. (5) jointly.

Theorem 1. The steady-state probability vector

$$P_k = P_N R_k\,(0 \le k \le N)\,.$$

where P_N satisfies

$$\begin{cases} P_N\,(R_0 C_0 + R_1 A_1 + R_2 C_2) = 0, \\ P_N \left(R_0 e_{2c+1} + \sum\limits_{k=1}^{N} R_k e_{3c+1} \right) = 1. \end{cases}$$

$R_k\,(0 \le k \le N)$ is

$$\begin{cases} R_0 = -R_1 B_1 A_0{}^{-1}, \\ R_k = -(R_{k+1} A_{k+1} + R_{k+2} B_{k+2})\,C^{-1}, 1 \le k \le N-2 \\ R_{N-1} = -A_N C^{-1}, \\ R_N = I. \end{cases}$$

Lemma 1. Let $A = (a_{ij})$ be a square matrix of order n over the field of real numbers, if $|a_{ii}| > \sum\limits_{j \ne i} |a_{ij}|, i = 1, 2, ..., n,$ Then $|A| \ne 0$. For proof, see [23].

Proof 1) Prove that the matrix A_0 is invertible.

Since $a_{0,0}, a_{1,0}, a_{1,1}, ...a_{c,0}, a_{c,1}$ is on the diagonal of the matrix A_0 and there are

$$|a_{i,0}| = (c-i)\,\beta + i\alpha + \lambda > (c-i)\,\beta + i\alpha, 0 \le i \le c,$$

$$|a_{1,1}| = (c-1)\,\beta + \gamma + \lambda > (c-1)\,\beta + \gamma,$$

$$|a_{i,1}| = (c-i)\,\beta + i\alpha + \gamma + \lambda > (c-i)\,\beta + i\alpha + \gamma, 2 \le i \le c.$$

So A_0 satisfies Lemma 1. Then $|A_0| \ne 0$, A_0 is invertible.

2) Prove that the matrix C is invertible.

Since the matrix $C = \lambda I$, I is a unitary matrix of order 3c+1, the matrix C is also invertible.

3) When $k = N$, $P_N = P_N R_N = P_N I$ is clearly fulfilled.

4) When $k = N-1$, from Eq. (4), $P_{N-1} = -P_N A_N C^{-1}$, such that $R_{N-1} = -A_N C^{-1}$, then $P_{N-1} = P_N R_{N-1}$.

5) When $1 \le k \le N-2$, we can use mathematical induction to prove that $P_k = P_N R_k$ is valid.

Assuming that $P_k = P_N R_k$ holds for $k = i+1$ and $k = i+2$, then

$$\begin{aligned}
&P_i \\
&= -\left(P_{i+1}A_{i+1} + P_{i+2}B_{i+2}\right)C^{-1} \\
&= -\left(P_N R_{i+1}A_{i+1} + P_N R_{i+2}B_{i+2}\right)C^{-1} \\
&= -P_N\left(R_{i+1}A_{i+1} + R_{i+2}B_{i+2}\right)C^{-1} \\
&= P_N R_i
\end{aligned}$$

So the formula $P_k = P_N R_k$ holds for $k = i$.

6) From Eq. (1), $P_0 = -P_1 B_1 {A_0}^{-1} = -P_N R_1 B_1 {A_0}^{-1}$, let $R_0 = -R_1 B_1 {A_0}^{-1}$, then $P_0 = P_N R_0$.

7) Bringing $P_1 = P_N R_1$ and $P_2 = P_N R_2$ into Eq. (2) yields $P_N\left(R_0 C_0 + R_1 A_1 + R_2 B_2\right) = 0$.

8) The normalization condition is satisfied by $P_0 e_{2c+1} + \sum\limits_{k=1}^{N} p_k e_{3c+1} = 1$, So we conclude that $P_N\left(R_0 e_{2c+1} + \sum\limits_{k=1}^{N} R_k e_{3c+1}\right) = 1$.

V. System Steady-state Performance Measures

According to Theorem 1, we can find the steady-state probability of the system, and then we get the performance indexes of the system, which are expressed as follows:

1) The average length of the queue in the system at steady-state is

$$\begin{aligned}
E\left(L\right) &= \sum_{k=0}^{N} kP\left(L=k\right) \\
&= \sum_{k=1}^{N}\sum_{y=1}^{c}\sum_{j=0}^{2} kP_{k,y,j} + \sum_{k=1}^{N} kP_{k,0,0}.
\end{aligned}$$

2) The average sojourn time is

$$\begin{aligned}
E\left(W\right) &= \frac{1}{\lambda}E\left(L\right) \\
&= \frac{1}{\lambda}\left[\sum_{k=0}^{N} kP\left(L=k\right)\right] \\
&= \frac{1}{\lambda}\left[\sum_{k=1}^{N}\sum_{y=1}^{c}\sum_{j=0}^{2} kP_{k,y,j} + \sum_{k=1}^{N} kP_{k,0,0}\right].
\end{aligned}$$

3) The probability that the servers are in the vacation state is

$$P_v = \sum_{k=0}^{N}\sum_{y=0}^{c} P_{k,y,0}.$$

4) The probability that the servers are in the startup or shutdown phase is

$$P_r = \sum_{k=0}^{N}\sum_{y=1}^{c} P_{k,y,1}.$$

5) The probability that the servers are in a regular busy period is

$$P_m = \sum_{k=1}^{N}\sum_{y=1}^{c} P_{k,y,2}.$$

6) The average number of servers breakdown is

$$\begin{aligned}
E\left(Y\right) = &\sum_{k=0}^{N}\sum_{y=0}^{c}\left(c-y\right)P_{k,y,0} + \sum_{k=0}^{N}\sum_{y=1}^{c}\left(c-y\right)P_{k,y,1} \\
&+ \sum_{k=1}^{N}\sum_{y=1}^{c}\left(c-y\right)P_{k,y,2}.
\end{aligned}$$

7) The average probability of customer abandonment during the waiting process in queue is

$$P_l = \sum_{y=1}^{d}\sum_{k>y}^{N}(k-y)\varepsilon P_{k,y,0} + \sum_{y=d+1}^{c}\sum_{k>d}^{N}(k-d)\varepsilon P_{k,y,0}.$$

VI. Numerical Experiments

A. Performance Indicator Analysis

In this section, we utilize MATLAB to conduct numerical experiments, resulting in visual images that depict the variations of key system performance indicators with different parameters. These images aid our in-depth analysis of the impact of parameter changes on system performance. However, due to the inability to derive an explicit formula for the steady-state probability vector $P = (P_0, P_1, P_2, P_3, \cdots, P_N)$, we focus our discussion on the case where $N = 10$. By solving for the values of $P = (P_0, P_1, P_2, P_3, \cdots, P_N)$ through the aforementioned reasoning process, we then proceed to create graphs that illustrate the trends in performance changes. In different queueing models, altering parameters has a direct impact on system performance metrics. Specific numerical examples are provided in this section to validate the model's validity and operational feasibility.

Assuming $c = 5, \mu_2 = 2, \alpha = 2, \beta = 1, \xi = 1, \gamma = 1, \varepsilon = 5, \theta = 2, d = 3$. Figure 1 shows the effect of arrival rate λ and the service rate μ_1 on the average queue length $E\left(L\right)$ during peak periods. during normal busy periods in average queue length. When μ_1 remains constant, $E\left(L\right)$ increases as λ increases. This is attributed to the fact that a rise in λ leads to an increased customer count in the system, thereby causing an increase in $E\left(L\right)$. Conversely, when λ remains constant, $E\left(L\right)$ decreases as μ_1 increases. The reason for this is that an increase in μ_1 reflects the ability to serve more customers per unit of time, resulting in a reduction in $E\left(L\right)$. In summary, reducing the arrival rate λ or increasing the service rate μ_1 during normal busy periods can, to a certain degree, decrease the average queue length $E\left(L\right)$ of the system.

Assuming $c = 5, \mu_1 = 4, \mu_2 = 2, \alpha = 2, \beta = 1, \xi = 1, \gamma = 1, \varepsilon = 5, \theta = 2, d = 3$. Figure 2 illustrates the trend of probability variations with respect to the arrival rate λ for

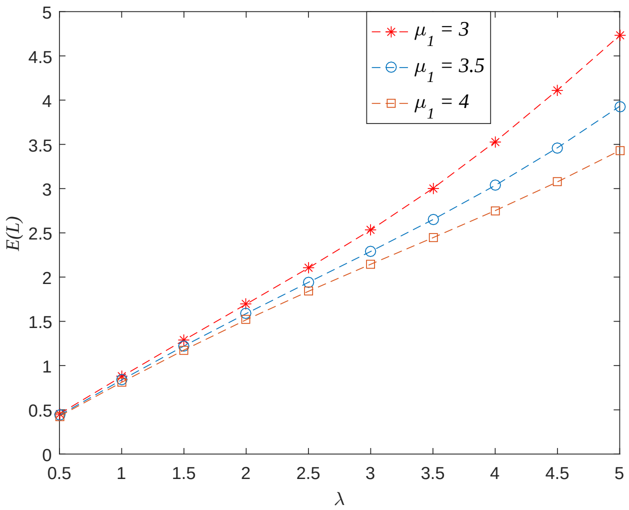

Fig. 1. The trend of $E(L)$ versus λ and μ_1.

different system states. Observations reveal that, with other parameters remaining constant, An increase in the arrival rate λ leads to a higher probability P_m of the system being in the busy period and a lower probability P_v of being in the vacation period, and the probability of being in the startup/shutdown period P_r exhibits an initial increase followed by a decrease. A low arrival rate causes a slight rise in the probability of the system entering the startup/shutdown period. However, as the arrival rate continues to rise, the proportion of time the system spends serving customers increases accordingly, reducing the probability of the system being empty. Consequently, the probability of being in the startup/shutdown period gradually decreases, the probability of entering the vacation period diminishes, and the probability of being in the busy period enhances.

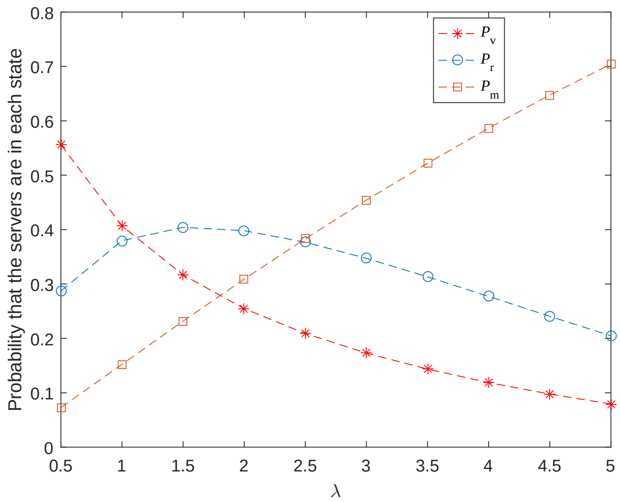

Fig. 2. The probability of each state versus λ .

Assuming $c = 5,6,7; d = 3, \lambda = 4, \mu_2 = 2, \alpha = 2, \beta = 1, \xi = 1, \gamma = 1, \varepsilon = 5, \theta = 2$. Figure 3 demonstrates the relationship between the average number of customers in the system, denoted as $E(L)$, and the variations in the service rate during busy periods, μ_1, as well as the number of servers, c. Observations indicate that $E(L)$ gradually decreases as both μ_1 and c increase. When μ_1 and c increase, it enhances the opportunity for customers to receive service, resulting in a progressive reduction in the average sojourn time of customers within the system. Consequently, $E(L)$ decreases as system and c increase.

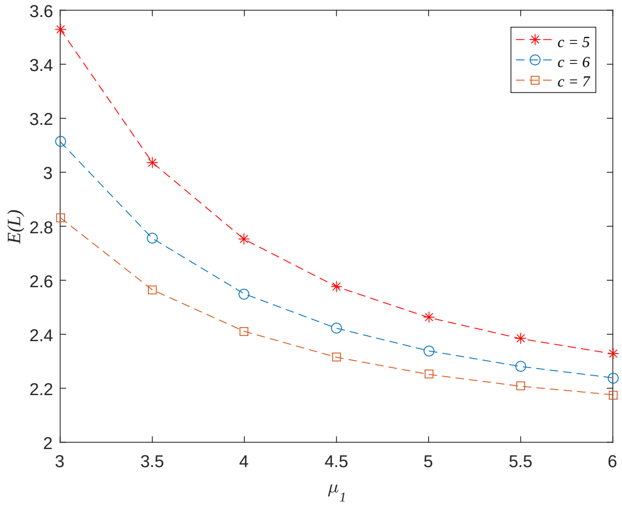

Fig. 3. The trend of $E(L)$ versus c and μ_1.

Assuming $c = 6, d = 3, \lambda = 2, \mu_2 = 2, \alpha = 2, \xi = 1, \gamma = 1, \theta = 2$. Figure 4 demonstrates the relationship between the average queue length of customers $E(L)$ and the variables μ_1 and β. When β is constant, as μ_1 increases, the opportunity for customers to receive service enhances. Consequently, $E(L)$ decreases gradually with the increase of μ_1. When μ_1 is constant, an increase in the repair rate β leads to an increase in the number of servers available for normal operation. This, in turn, improves customers' access to service, causing the average sojourn time in the system to decrease. Therefore, $E(L)$ decreases as β increases.

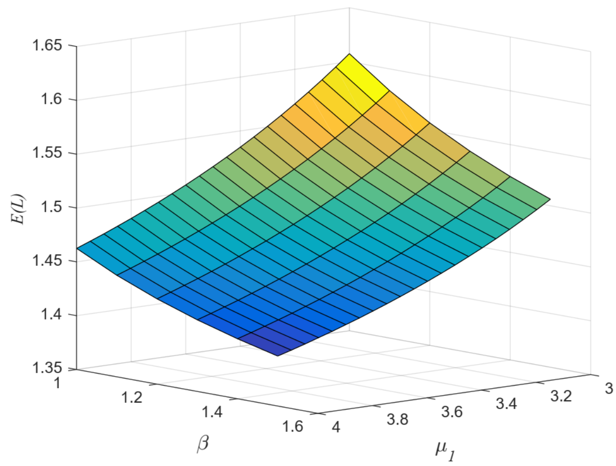

Fig. 4. The trend of $E(L)$ versus β and μ_1.

Assuming $c = 6, d = 3, \lambda = 3, \mu_1 = 4, \mu_2 = 2, \xi = 1, \gamma = 1, \varepsilon = 5, \theta = 2$. In Figure 5, the relationship between the average number of server breakdowns $E(Y)$ in the system and the breakdown rate α as well as the repair rate β is illustrated. When α remains constant and β increases, the servers are more likely to be repaired promptly after a breakdown, leading to a reduction in the average number of server breakdowns $E(Y)$ in the system. Conversely, when β

remains constant and α increases, it indicates that the servers are more susceptible to breakdowns, leading to an increase in the average number of server breakdowns $E(Y)$. In summary, reducing the breakdown rate α or increasing the repair rate β can, to a certain extent, diminish the average number of server breakdowns in the system.

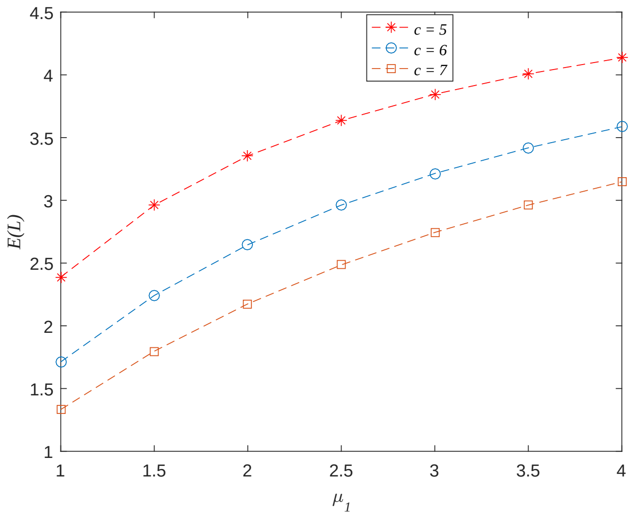

Fig. 5. The trend of $E(Y)$ versus α and β.

Assuming $c = 5, d = 3, \lambda = 4, \mu_1 = 6, \alpha = 2, \beta = 1, \xi = 1, \gamma = 1, \theta = 2$. Figure 6 illustrates the influence of the impatience rate of the customers ε and the service rate μ_2 of the servers on vacation on the average abandonment rate of customers $P(l)$ in the system. As observed in Figure 6, when μ_2 is constant, $P(l)$ increases with increase in ε; conversely, when ε remains constant, $P(l)$ decreases as μ_2 increases. This is mainly because a higher ε implies that customers are less willing to wait in the queue, causing them to be more prone to exit the system, thus leading to an increase in $P(l)$. When μ_2 increases, the server service rate for customers accelerates, enabling customers to receive service within a shorter period during their waiting period. Consequently, customers are less inclined to abandon the system, leading to a decrease in $P(l)$.

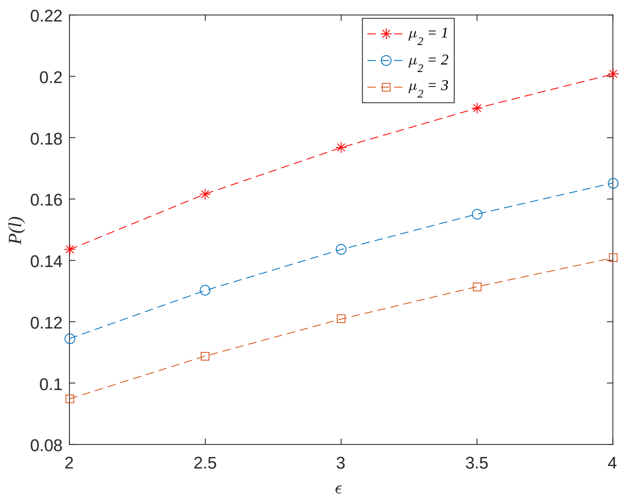

Fig. 6. The trend of $P(l)$ versus μ_2 and ε.

Assuming $c = 7, \lambda = 3, \mu_1 = 4, \mu_2 = 2, \xi = 1, \gamma = 1, \varepsilon = 5, \beta = 1, \alpha = 2$. Based on Table 1, it can be concluded that the vacation rate θ and the number of servers d on working vacation have minor effects on the system's mean queue length $E(L)$, mean sojourn time $E(W)$, and customer abandonment rate $P(l)$. When d remains constant, both $E(L)$ and $E(W)$ increase as θ increases, whereas $P(l)$ decreases with the increase of θ. Conversely, when θ remains constant, both $E(L)$ and $E(W)$ increase as d increases, and $P(l)$ also increases with the increase of d.

TABLE I
θ AND d VALUE-RELATED PERFORMANCE INDICATORS

(θ, d)	$E(L)$	$E(W)$	$P(l)$
(1,4)	1.9019838	0.6339946	0.0307275
(2,4)	1.9569146	0.6523049	0.0146257
(3,4)	1.9718950	0.6572983	0.0088396
(2,3)	1.9559894	0.6519965	0.0145479
(2,5)	1.9569671	0.6523224	0.0146338

B. *Benefit Analysis*

From numerical analysis, it is evident that any change in a parameter within the model impacts the queuing system. Therefore, we can optimize the model by altering parameter values. In this section, benefit functions are formulated from both individual and overall perspectives. Through numerical analysis, the optimal parameter values for maximizing system benefit are obtained.

Assuming that upon completion of one service, the individual benefit received by a customer is Z, and the expenditure per unit time for a customer within the system is G. Let U_I represent the individual benefit of the customer, then

$$U_I = Z - GE(W)$$

Assuming that $Z = 50, G = 20, N = 10, c = 5, d = 3, \lambda = 3, \varepsilon = 3, \beta = 1, \gamma = 1, \xi = 1, \mu_2 = 2$. As illustrated in Figure 7, the individual benefit of customers increases with the normal busy period service rate μ_1, but decreases with the vacation rate θ and the failure rate α. This is because an increase in the normal busy period service rate μ_1 leads to a reduction in the average queue length, which in turn decreases the average sojourn time, ultimately increasing the individual expected benefit for customers. A higher failure rate α means more frequent service failures. This reduces the number of servers available for normal service, raising the average queue length. Longer queues lengthen customers' average sojourn time and ultimately cut their expected individual benefit.. An increase in the failure rate α indicates a higher likelihood of services failures, reducing the number of services available for normal service, which leads to an increase in the average queue length, subsequently increasing the average sojourn time and ultimately decreasing the individual expected benefit for customers.

Based on the individual benefit functions of customers, the aggregate benefit function of customers can be defined as

$$U_S = \lambda(Z - GE(W))$$

Assuming that $Z = 50, G = 20, N = 10, c = 5, d = 3, \lambda = 3, \varepsilon = 3, \beta = 1, \gamma = 1, \xi = 1, \mu_2 = 2$. As depicted in Figure 8, the overall benefit increases with the rise of

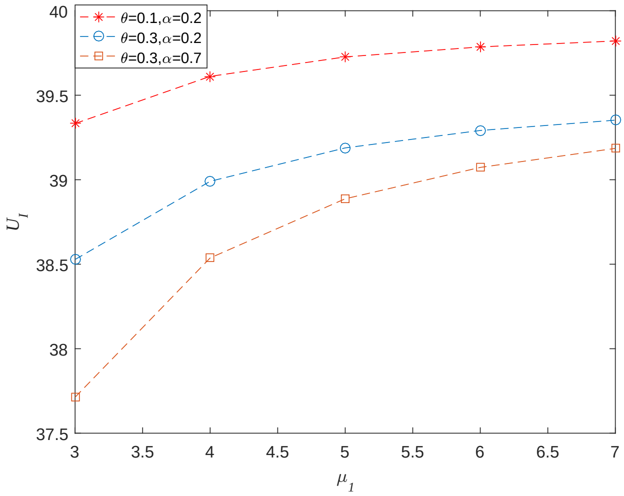

Fig. 7. The individual benefit of U_I versus θ and α.

normal busy period μ_1, while it decreases with the increase of vacation rate θ and failure rate α. Therefore, we can enhance the overall benefit by increasing the service rate μ_1 during busy periods, and reducing the vacation rate θ and failure rate α.

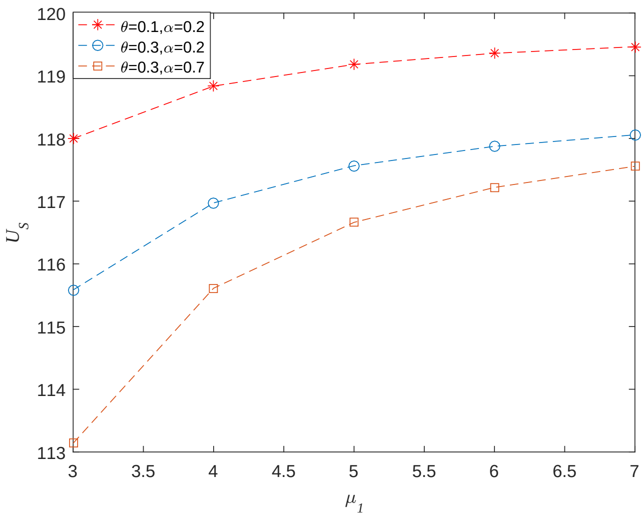

Fig. 8. The overall benefit of U_S versus θ and α.

VII. Conclusion

This paper introduces an M/M/c/N queueing system incorporating impatient customers, setup and shutdown times, repairable failures, and partial work vacations for service stations. The state transition rate matrix for the system is established, and the iterative expression for the steady-state vector is obtained by solving the steady-state equations. Subsequently, the expressions for the system's performance metrics are derived. Additionally, numerical experiments are conducted on this queuing system to intuitively observe the impact of various parameters on system performance. In the final section of this paper, expressions for individual and overall benefit functions are formulated. By assigning specific values to the parameters, methods and suggestions for improving benefits are obtained.

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