I(VI)-total Colorings on the Disjoint Union of 16-Cycles Vertex-distinguished by Multisets

Chen WANG, Xiang'en CHEN

***Abstract*—In the paper, the I-total colorings and VI-total colorings of mC_{16} which are vertex-distinguished by multisets are researched by use of the method of distributing color sets in advance in order to construct colorings. The optimal I-total colorings and VI-total colorings of mC_{16} which are vertex-distinguished by multisets are constructed by means of the matrix whose entries are multiple subsets containing 3 elements (at least two elements are different) or empty subsets of the set composed of all colors available. Thereby we obtain I-total chromatic numbers and VI-total chromatic numbers of mC_{16} which are vertex-distinguished by multisets.**

***Index Terms*—cycle; disjoint union; I-total coloring; VI-total coloring; color set; vertex-distinguished**

I. Introduction and basic concepts

MANY results of point-distinguishing general edge colorings are given in [1-6].

The general edge coloring of a graph which is vertex-distinguished by multiple sets was proposed by M. Aigner et al. in [7], the related results were given in [8-12].

Xiang'en Chen and Zepeng Li proposed vertex-distinguishing I-total coloring and vertex-distinguishing VI-total coloring of a graph in [13].

Let mC_n denote the vertex-disjoint union of m cycle C_n of order n.

In [14-16], the I-total colorings and VI-total colorings of mC_n ($3 \leq n \leq 12$) which are vertex-distinguished by multisets are researched. We will study the optimal I-total colorings and optimal VI-total colorings of mC_{16} which are vertex-distinguished by multisets in detail in this paper.

A general total coloring such that any two adjacent vertices receive different colors and any two adjacent edges receive different colors is called an I-total coloring of G.

A general total coloring such that any two adjacent edges are colored with different colors is called a VI-total coloring of G.

An I-total coloring in which l colors are available is called l-I-total coloring (of G).

A VI-total coloring in which l colors are available is called l-VI-total coloring (of G).

Let f be the I-total coloring (resp. VI-total coloring) of a graph G. A multiple color set of any vertex y of G under f is a multiple set of the colors of the vertex y under f and the colors of all edges incident with y. A multiple color set of y under f is denoted by $\tilde{C}_f(y)$ (or simply $\tilde{C}(y)$ if no confusion arises). In this paper, the multiple color set of any vertex y of G under f is also called the color set of y (under f).

Let f be an I-total coloring (resp. a VI-total coloring) of a graph G, if any two distinct vertices have distinct color sets, then f is called an I-total coloring of G which is vertex-distinguished by multisets (resp. a VI-total coloring of G which is vertex-distinguished by multisets).

The minimum number of colors required in an I-total coloring (resp. a VI-total coloring) of G which is vertex-distinguished by multisets is called the I-total (resp. VI-total) chromatic number of G which is vertex-distinguished by multisets. This number is denoted by $\tilde{\chi}^i_{vt}(G)$ (resp. $\tilde{\chi}^{vi}_{vt}(G)$), i.e.,

$\tilde{\chi}^i_{vt}(G)$ =min$\{l|G$ has an l-I-total coloring which is vertex-distinguished by multisets$\}$;

$\tilde{\chi}^{vi}_{vt}(G)$ =min$\{l|G$ has an l-VI-total coloring which is vertex-distinguished by multisets$\}$.

Let δ and Δ denote the minimum degree and the maximum degree of G, respectively. Whereas $n_i(G)$ denotes the number of vertices of G with degree i, $\delta \leq i \leq \Delta$. Let

$\tilde{\zeta}(G)$ =min$\{l|i\binom{l}{i} + \binom{l}{i+1} \geq n_i(G), \delta \leq i \leq \Delta\}$.

Proposition 1[15] $\tilde{\chi}^i_{vt}(G) \geq \tilde{\chi}^{vi}_{vt}(G) \geq \tilde{\zeta}(G)$.

II. Preliminaries

For any integer $l \geq 3$, the $(l-1) \times l$ matrix A_l is constructed, and the elements of the matrix A_l are 3-subsets (multiple sets, at least two elements are different) containing l or empty subsets of the set $\{1, 2, \cdots, l\}$, where the l elements in the i-th row of A_l are $\{l, l, i\}, \{l, i, i\}, \{l, i, i+1\}, \{l, i, i+2\}, \cdots, \{l, i, l-1\}$, $\emptyset, \emptyset, \cdots, \emptyset$, $i = 1, 2, \cdots, l-1$.

Note that there are $i-1$ empty subsets in the i-th row of A_l, $i = 1, 2, \cdots, l-1$.

The entries of A_l are arranged in the following form:

$$\begin{array}{cccccc}
\{l,l,1\} & \{l,1,1\} & \{l,1,2\} & \cdots & \{l,1,l-2\} & \{l,1,l-1\} \\
\{l,l,2\} & \{l,2,2\} & \{l,2,3\} & \cdots & \{l,2,l-1\} & \emptyset \\
\{l,l,3\} & \{l,3,3\} & \{l,3,4\} & \cdots & \emptyset & \emptyset \\
\vdots & & & & & \vdots \\
\{l,l,l-2\} & \{l,l-2,l-2\} & \{l,l-2,l-1\} & \cdots & \emptyset & \emptyset \\
\{l,l,l-1\} & \{l,l-1,l-1\} & \emptyset & \cdots & \emptyset & \emptyset
\end{array}$$

If 16 elements (non-empty sets) of the matrix A_l are exactly the (multiple) color sets of all vertices of C_{16} under an I-total coloring of C_{16} which is vertex-distinguished by multisets, then we say that these 16 elements form a good group (about C_{16}).

Let f be an k-I-total coloring of mC_{16} which is vertex-distinguished by multisets. If all 3-subsets (of $\{1, 2, \cdots, k\}$) in which at least two colors are different are the color sets of vertices of mC_{16} under f, then 3-subsets are said to be used up under f.

We will introduce 11 types of good groups firstly and then give 4 lemmas.

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Good group of type I.

Let $l \geq 5$, $i \equiv 1 \pmod 4$, $i \leq l-6$. The 16 elements (not empty sets) in $A_l[i, i+1, i+2, i+3|1, 2, 3, 4]$ form a good group about C_{16} (because the 16 elements are color sets of all 16 vertices of C_{16} under the I-total coloring in Fig. 1 which is vertex-distinguished by multisets). The good group composed of 16 elements (not empty sets) in $A_l[i, i+1, i+2, i+3|1, 2, 3, 4]$ is called a good group of type I, and the coloring in Fig. 1 is called a coloring of type I.

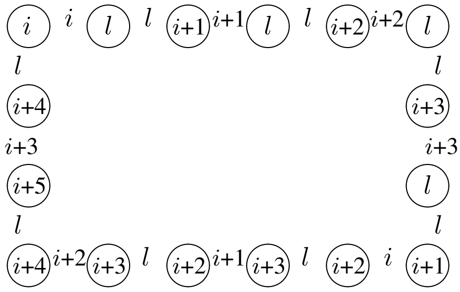


Fig. 1: A coloring of type I

Good group of type II.

Let $l \geq 5$, i and j be positive integers, $i \equiv 1 \pmod 4$, $j \equiv 1 \pmod 4$, $j \geq 5$, $i+j \leq l-5$. Then the 16 elements (not empty sets) in $A_l[i, i+1, i+2, i+3 \; |j, j+1, j+2, j+3]$ form a good group about C_{16} (because the 16 elements are color sets of all 16 vertices of C_{16} under the I-total coloring in Fig. 2 which is vertex-distinguished by multisets). The good group composed of 16 elements in $A_l[i, i+1, i+2, i+3|j, j+1, j+2, j+3]$ is called a good group of type II, and the coloring in Fig. 2 is called a coloring of type II.

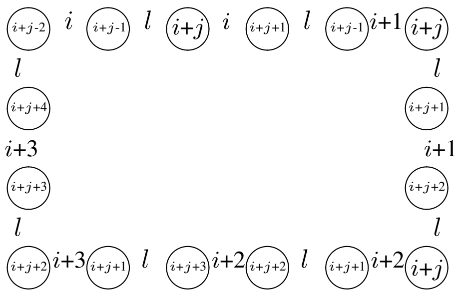


Fig. 2: A coloring of type II

An element (actually a subset) in A_l is said to be active if it is not an empty set and is not an element of any good group of type I or II.

Let the number of active elements in column j of A_l be $t_j, j = 1, 2, \cdots, l$. Let

$$T_l = (t_1, t_2, t_3, \cdots, t_{l-1}, t_l)$$

T_l is called the active sequence of A_l.

Since each item in T_l is in $\{0, 1, 2, 3, 4, 5\}$, the comma between two successive items in T_l is omitted when actually writing, and a space is added between the $4s$-th item and the $(4s+1)$-th item, where $s \geq 1, 4s+1 \leq l$. For example $T_{22} = (5543\ 6543\ 6543\ 6543\ 6543\ 21)$.

When writing T_l, a line segment is drawn under a sequence of 32 consecutive items such as "5432 5432 5432 5432 5432 5432 5432 5432" or "6543 6543 6543 6543 6543 6543 6543 6543" or "3210 3210 3210 3210 3210 3210 3210 3210" or "4321 4321 4321 4321 4321 4321 4321 4321". For example, T_{32t+5}=(4432 $\underline{\text{5432 5432 5432 5432 5432 5432 5432 5432}}$ $\cdots$ $\underline{\text{5432 5432 5432 5432 5432 5432 5432 5432}}$ 1). There are t line segments which are drawn under specific sequence T_{32t+5}.

Good group of type III.

There are 56 active elements in the 16 columns of A_l corresponding to the subsequence (5432 5432 5432 5432) (this subsequence is constructed by the former 16 terms or the back 16 terms in one 32-term subsequence under which a line segment is drawn, and the first term of this subsequence is corresponded to the j-th column of A_l, $j \equiv 5 \pmod{16}$). The 48 elements among these 56 active elements can be divided into three good groups about C_{16} (because the 48 elements are the color sets of all 48 vertices of $3C_{16}$ under its I-total coloring in Fig. 3 which is vertex-distinguished by multisets). These three good groups are all called the good groups of type III, and three colorings in Fig. 3 (a), (b), (c) are all called the colorings of type III.

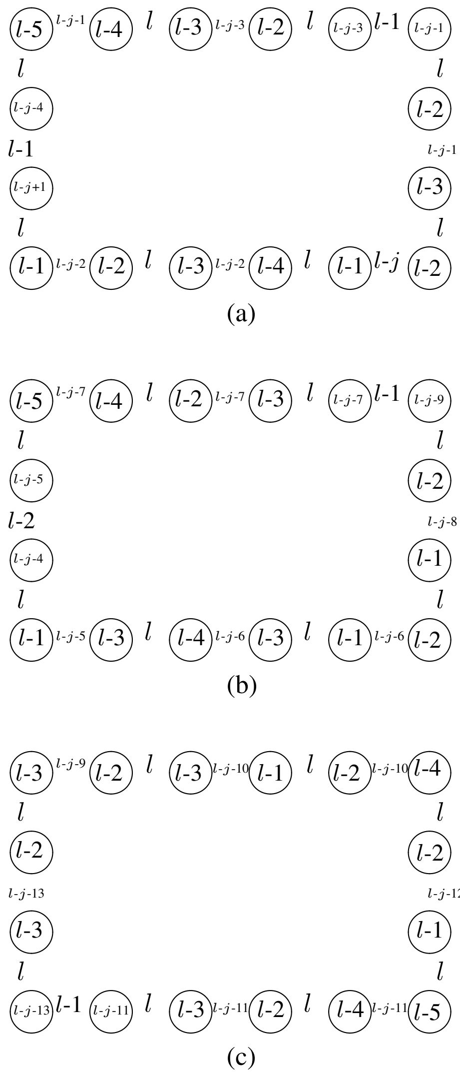


Fig. 3: Colorings of type III

Good group of type IV.

For each 32-term subsequence (5432 5432 5432 5432 5432 5432 5432 5432) under which a line segment is drawn (the first term of this subsequence is corresponded to the j-th column of A_l, $j \equiv 5 \pmod{32}$, $s = j + 16$), there are exactly 16 active elements which are in the columns (from j-th to $(s + 15)$-th column) of A_l corresponding to this subsequence and which are not in any good group of type III. These 16 active elements can form a good group about C_{16} (because the 16 elements are the color sets of all 16 vertices of C_{16} under the I-total coloring in Fig. 4 which is vertex-distinguished by multisets). This good group is called a good group of type IV, and the coloring in Fig. 4 is called a coloring of type IV.

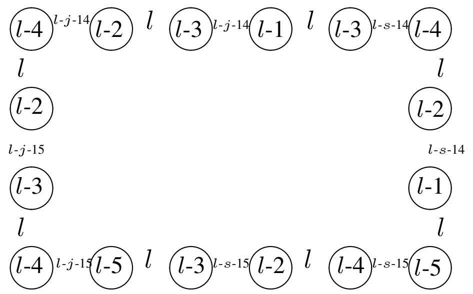


Fig. 4: A coloring of type IV

Good group of type V.

For each subsequence (6543) of four successive terms in T_l (the first term of this subsequence is corresponded to the i-th column of A_l, $i \equiv 5 \pmod{4}$, $i \geq 5$), there are 16 active elements of A_l which are in the columns (from i-th to $(i + 3)$-th column) corresponding to this subsequence and which can form a good group about C_{16} (because the 16 elements are the color sets of all 16 vertices of C_{16} under the I-total coloring in Fig. 5 which is vertex-distinguished by multisets). Such a good group is called a good group of type V, and the coloring in Fig. 5 is called a coloring of type V.

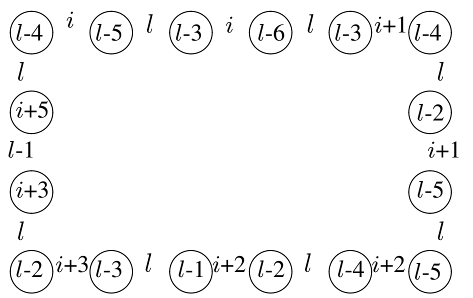


Fig. 5: A coloring of type V

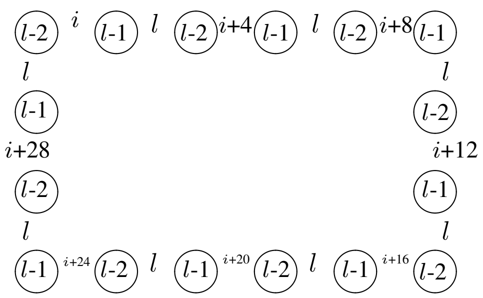


Fig. 6: A coloring of type VI

Good group of type VI.

For each subsequence (6543 6543 6543 6543 6543 6543 6543 6543) of T_l under which a line segment is drawn (the first term of this subsequence is corresponded to the i-th column of A_l, $i \equiv 5 \pmod{32}$, $i \geq 5$), the corresponding 32 columns (from i-th to $(i+31)$-th column) of A_l correspond eight good groups of type V, and another 16 active elements correspond to this subsequence are not in any good groups of type V. These 16 active elements can form a good group about C_{16} (because the 16 elements are color sets of all 16 vertices of C_{16} under the I-total coloring in Fig. 6 which is vertex-distinguished by multisets). This good group is called a good group of type VI, and the coloring in Fig. 6 is called a coloring of type VI.

Good group of type VII.

For each subsequence of the four successive terms (5543) in T_l under which a line segment is not drawn, the 16 active elements (not including $\{l, l-2, l-1\}$) at the intersection of the first four columns and the reciprocal first to fourth rows in A_l can form a good group about C_{16} (because the 16 elements are color sets of all 16 vertices of C_{16} under the I-total coloring in Fig. 7 which is vertex-distinguished by multisets). This good group is called a good group of type VII, and the coloring in Fig. 7 is called a coloring of type VII.

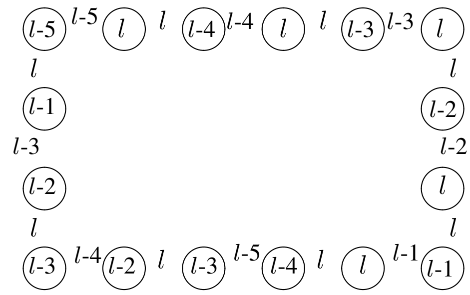


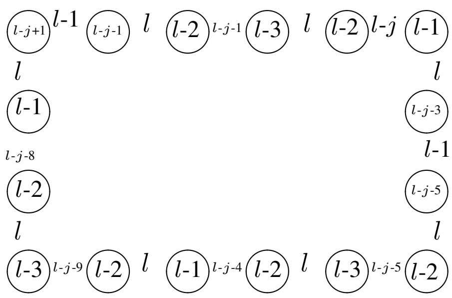
Fig. 7: A coloring of type VII

Good group of type VIII.

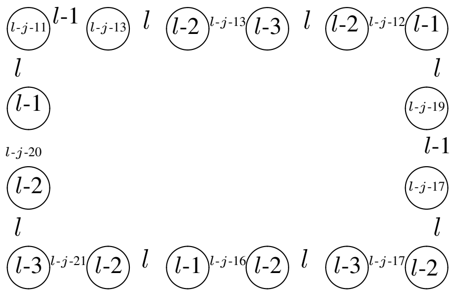
For each 32-term subsequence (3210 3210 3210 3210 3210 3210 3210 3210) under which a line segment is drawn (the first term of this subsequence is corresponded to the j-th column of A_l, $j \equiv 5 \pmod{32}$, $j \geq 5$), there are 48 active elements in 32 columns of A_l (from j-th to $(j + 31)$-th column), which can be divided into three good groups about C_{16} (because the 48 elements are the color sets of all 48 vertices of $3C_{16}$ under its I-total coloring in Fig. 8 which is vertex-distinguished by multisets), and these three good groups are called the good groups of type VIII. The colorings in Fig. 8(a), (b), (c) are called the colorings of type VIII.

Good group of type IX.

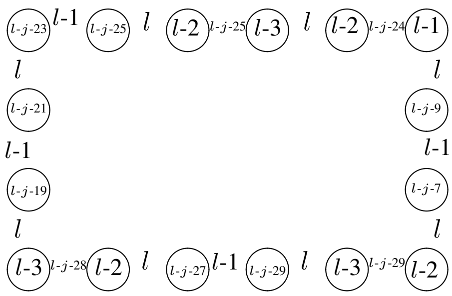
For each subsequence (4321 4321) of the eight successive terms in T_l under which a line segment is drawn (the first term of this subsequence is corresponded to the j-th column of A_l, $j \equiv 5 \pmod{8}$, $j \geq 5$), there are 16 active elements in 8 columns of A_l (from j-th to $(j + 7)$-th column), which can form a good group about C_{16} (because the 16 elements are the color sets of all 16 vertices of C_{16} under the I-total coloring in Fig. 9 which is vertex-distinguished by multisets).



(a)



(b)



(c)

Fig. 8: Colorings of type VIII

This good group is called a good group of type IX, and the coloring in Fig. 9 is called a coloring of type IX.

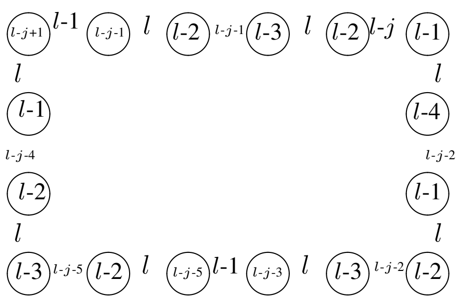


Fig. 9: A coloring of type IX

Good group of type X.

For each 32-term subsequence (4321 4321 4321 4321 4321 4321 4321 4321) under which a line segment is drawn (the first term of this subsequence is corresponded to the j-th column of A_l, $j \equiv 5 \,(\mathrm{mod}\ 32)$), there are 16 active elements in 32 columns of A_l (from j-th to $(j+31)$-th column) are not in any good group of type IX. These 16 active elements can form a good group about C_{16} (because the 16 elements are the color sets of all 16 vertices of C_{16} under the I-total coloring in Fig. 10 which is vertex-distinguished by multisets), which is called a good group of type X, and the coloring in Fig. 10 is called a coloring of type X.

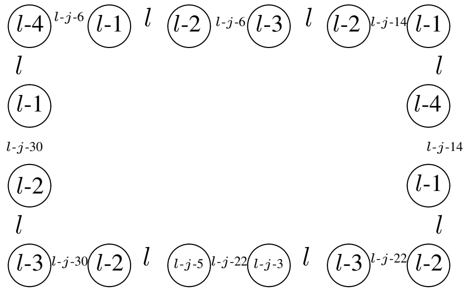


Fig. 10: A coloring of type X

Good group of type XI.

Consider subsequence (33 4321) of T_l under each term of which a line segment is not drawn. The first and second term of T_l are 3 and the penultimate first to fourth term of T_l are 1, 2, 3, 4 respectively. There are 6 active elements in first two columns of A_l and there are 10 active elements in the penultimate first to fourth column of T_l. These 16 active elements can form a good group about C_{16} (because the 16 elements are the color sets of all 16 vertices of C_{16} under the I-total coloring in Fig. 11 which is vertex-distinguished by multisets), which is called a good group of type XI, and the coloring in Fig. 10 is called a coloring of type XI.

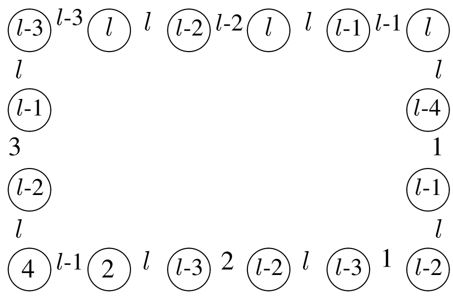


Fig. 11: A coloring of type XI

By the above 11 types of good groups and 11 types of colorings, we can obtain the following 4 lemmas easily and omit their proofs.

Lemma 1. Let $l \geq 5$ and $l \equiv 1 \,(\mathrm{mod}\ 4)$.

(1) When $l \equiv 5 \,(\mathrm{mod}\ 32)$, $\binom{l}{2} - 14$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2} - 14)$ good groups about C_{16}, and another 14 elements (not the empty subset) that are not in these good groups are $\{l, l, l-4\}$, $\{l, l-4, l-4\}$, $\{l, l-4, l-3\}$, $\{l, l-4, l-2\}$, $\{l, l, l-3\}$, $\{l, l-3, l-3\}$, $\{l, l-3, l-2\}$, $\{l, l-3, l-1\}$, $\{l, l, l-2\}$, $\{l, l-2, l-2\}$, $\{l, l-2, l-1\}$, $\{l, l, l-1\}$, $\{l, l-1, l-1\}$, $\{l, 1, l-1\}$.

(2) When $l \equiv 9 \,(\mathrm{mod}\ 32)$, we have that $\binom{l}{2} - 12$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2} - 12)$ good groups, and another 12 elements (not the empty subset) that are not in these good groups are $\{l, 1, l-5\}$, $\{l, 1, l-4\}$, $\{l, 1, l-3\}$, $\{l, 1, l-2\}$, $\{l, 1, l-1\}$, $\{l, 2, l-4\}$, $\{l, 2, l-3\}$,$\{l, 2, l-2\}$, $\{l, 2, l-1\}$, $\{l, 3, l-3\}$, $\{l, 3, l-2\}$, $\{l, 3, l-1\}$.

(3) When $l \equiv 13 \,(\mathrm{mod}\ 32)$, $\binom{l}{2} - 10$ elements (each of which is not the empty subset) in A_l can be divided into

$\frac{1}{16}(\binom{l}{2}-10)$ good groups, and another 10 elements (not the empty subset) that are not in these good groups are $\{l,l,l-4\}$, $\{l,l-4,l-4\}$, $\{l,l-4,l-3\}$, $\{l,l,l-3\}$, $\{l,l-3,l-3\}$, $\{l,l-3,l-2\}$, $\{l,l,l-2\}$, $\{l,l-2,l-2\}$, $\{l,l,l-1\}$, $\{l,l-1,l-1\}$.

(4) When $l\equiv 17\,(\mathrm{mod}\ 32)$, $\binom{l}{2}-8$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-8)$ good groups, and another 8 elements (not the empty subset) that are not in these good groups are $\{l,l,l-4\}$, $\{l,l-4,l-4\}$, $\{l,l-4,l-3\}$, $\{l,l-4,l-2\}$, $\{l,l,l-3\}$, $\{l,l-3,l-3\}$, $\{l,l-3,l-2\}$, $\{l,l-3,l-1\}$.

(5) When $l\equiv 21\,(\mathrm{mod}\ 32)$, $\binom{l}{2}-6$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-6)$ good groups, and another 6 elements (not the empty subset) that are not in these good groups are $\{l,l,l-4\}$, $\{l,l-4,l-4\}$, $\{l,l,l-3\}$, $\{l,l-3,l-3\}$, $\{l,l,l-2\}$, $\{l,l-2,l-2\}$.

(6) When $l\equiv 25\,(\mathrm{mod}\ 32)$, $\binom{l}{2}-4$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-4)$ good groups, and another 4 elements (not the empty subset) that are not in these good groups are $\{l,l,l-4\}$, $\{l,l-4,l-4\}$, $\{l,l,l-3\}$, $\{l,l-3,l-3\}$.

(7) When $l\equiv 29\,(\mathrm{mod}\ 32)$, $\binom{l}{2}-2$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-2)$ good groups, and another 2 elements (not the empty subset) that are not in these good groups are $\{l,1,l-2\}$, $\{l,1,l-1\}$.

(8) When $l\equiv 1\,(\mathrm{mod}\ 32)$, all elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}\binom{l}{2}$ good groups, all elements (each of which is not the empty subset) in A_l are in these good groups.

Lemma 2. Let $l\geq 6$ and $l\equiv 2\,(\mathrm{mod}\ 4)$.

(1) When $l\equiv 6\,(\mathrm{mod}\ 32)$, $\binom{l}{2}-4$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-4)$ good groups, and another 4 elements (not the empty subset) that are not in these good groups are $\{l,l-2,l-1\}$, $\{l,1,l-2\}$, $\{l,1,l-1\}$, $\{l,2,l-1\}$.

(2) When $l\equiv 10\,(\mathrm{mod}\ 32)$, $\binom{l}{2}-6$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-6)$ good groups, and another 6 elements (not the empty subset) that are not in these good groups are $\{l,l-2,l-1\}$, $\{l,1,l-2\}$, $\{l,1,l-1\}$, $\{l,2,l-1\}$, $\{l,5,l-2\}$, $\{l,5,l-1\}$.

(3) When $l\equiv 14\,(\mathrm{mod}\ 32)$, $\binom{l}{2}-8$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-8)$ good groups, and another 8 elements (not the empty subset) that are not in these good groups are $\{l,l-2,l-1\}$, $\{l,1,l-2\}$, $\{l,1,l-1\}$, $\{l,2,l-1\}$, $\{l,5,l-2\}$, $\{l,5,l-1\}$, $\{l,9,l-2\}$, $\{l,9,l-1\}$.

(4) When $l\equiv 18\,(\mathrm{mod}\ 32)$, $\binom{l}{2}-10$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-10)$ good groups, and another 10 elements (not the empty subset) that are not in these good groups are $\{l,l-2,l-1\}$, $\{l,1,l-2\}$, $\{l,1,l-1\}$, $\{l,2,l-1\}$, $\{l,5,l-2\}$, $\{l,5,l-1\}$, $\{l,9,l-2\}$, $\{l,9,l-1\}$, $\{l,13,l-2\}$, $\{l,13,l-1\}$.

(5) When $l\equiv 22\,(\mathrm{mod}\ 32)$, $\binom{l}{2}-12$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-12)$ good groups, and another 12 elements (not the empty subset) that are not in these good groups are $\{l,l-2,l-1\}$, $\{l,1,l-2\}$, $\{l,1,l-1\}$, $\{l,2,l-1\}$, $\{l,5,l-2\}$, $\{l,5,l-1\}$, $\{l,9,l-2\}$, $\{l,9,l-1\}$, $\{l,13,l-2\}$, $\{l,13,l-1\}$, $\{l,17,l-2\}$, $\{l,17,l-1\}$.

(6) When $l\equiv 26\,(\mathrm{mod}\ 32)$, $\binom{l}{2}-14$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-14)$ good groups, and another 14 elements (not the empty subset) that are not in these good groups are $\{l,l-2,l-1\}$, $\{l,1,l-2\}$, $\{l,1,l-1\}$, $\{l,2,l-1\}$, $\{l,5,l-2\}$, $\{l,5,l-1\}$, $\{l,9,l-2\}$, $\{l,9,l-1\}$, $\{l,13,l-2\}$, $\{l,13,l-1\}$, $\{l,17,l-2\}$, $\{l,17,l-1\}$, $\{l,21,l-2\}$, $\{l,21,l-1\}$.

(7) When $l\equiv 30\,(\mathrm{mod}\ 32)$, all elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}\binom{l}{2}$ good groups, all elements (each of which is not the empty subset) in A_l are in these good groups.

(8) When $l\equiv 2\,(\mathrm{mod}\ 32)$, $\binom{l}{2}-2$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-2)$ good groups, and another 2 elements (not the empty subset) that are not in these good groups are $\{l,2,l-1\}$, $\{l,l-2,l-1\}$.

Lemma 3. Let $l\geq 7$ and $l\equiv 3\,(\mathrm{mod}\ 4)$.

(1) When $l\equiv 7\,(\mathrm{mod}\ 32)$, $\binom{l}{2}-11$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-11)$ good groups, and another 11 elements (not the empty subset) that are not in these good groups are $\{l,l,l-1\}$, $\{l,l-1,l-1\}$, $\{l,l,l-2\}$, $\{l,l-2,l-2\}$, $\{l,l-2,l-1\}$, $\{l,1,l-3\}$, $\{l,1,l-2\}$, $\{l,1,l-1\}$, $\{l,2,l-2\}$, $\{l,2,l-1\}$, $\{l,3,l-1\}$.

(2) When $l\equiv 11\,(\mathrm{mod}\ 32)$, $\binom{l}{2}-1$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-1)$ good groups, and another one element (not the empty subset) that are not in these good groups are $\{l,1,l-1\}$.

(3) When $l\equiv 15\,(\mathrm{mod}\ 32)$, $\binom{l}{2}-7$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-7)$ good groups, and another 7 elements (not the empty subset) that are not in these good groups are $\{l,l,l-1\}$, $\{l,l-1,l-1\}$, $\{l,l,l-2\}$, $\{l,l-2,l-2\}$, $\{l,l-2,l-1\}$, $\{l,1,l-3\}$, $\{l,1,l-2\}$.

(4) When $l\equiv 19\,(\mathrm{mod}\ 32)$, $\binom{l}{2}-13$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-13)$ good groups, and another 13 elements (not the empty subset) that are not in these good groups are $\{l,l,l-1\}$, $\{l,l-1,l-1\}$, $\{l,l,l-2\}$, $\{l,l-2,l-2\}$, $\{l,l-2,l-1\}$, $\{l,1,l-3\}$, $\{l,1,l-2\}$, $\{l,1,l-1\}$, $\{l,2,l-2\}$, $\{l,2,l-1\}$, $\{l,3,l-1\}$, $\{l,5,l-3\}$, $\{l,5,l-2\}$.

(5) When $l\equiv 23\,(\mathrm{mod}\ 32)$, $\binom{l}{2}-3$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-3)$ good groups, and another 3 elements (not the empty subset) that are not in these good groups are $\{l,l,l-2\}$, $\{l,l-2,l-2\}$, $\{l,l-2,l-1\}$.

(6) When $l\equiv 27\,(\mathrm{mod}\ 32)$, $\binom{l}{2}-9$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-9)$ good groups, and another 9 elements (not the empty subset) that are not in these good groups are $\{l,l,l-1\}$, $\{l,l-1,l-1\}$, $\{l,l,l-2\}$, $\{l,l-2,l-2\}$, $\{l,l-2,l-1\}$, $\{l,1,l-3\}$, $\{l,1,l-2\}$, $\{l,1,l-1\}$, $\{l,3,l-1\}$.

(7) When $l\equiv 31\,(\mathrm{mod}\ 32)$, $\binom{l}{2}-15$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-15)$ good groups, andanother 15 elements (not the empty subset) that are not in these good groups are $\{l,l,l-1\}$, $\{l,l-1,l-1\}$, $\{l,l,l-2\}$, $\{l,l-2,l-2\}$, $\{l,l-2,l-1\}$, $\{l,1,l-3\}$, $\{l,1,l-2\}$, $\{l,1,l-1\}$, $\{l,2,l-2\}$, $\{l,2,l-1\}$,

$\{l,3,l-1\}$, $\{l,5,l-3\}$, $\{l,5,l-2\}$, $\{l,5,l-1\}$, $\{l,7,l-1\}$.

(8) When $l \equiv 3 \,(\mathrm{mod}\ 32)$, $\binom{l}{2}-5$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-5)$ good groups, and another 5 elements (not the empty subset) that are not in these good groups are $\{l,l,l-1\}$, $\{l,l-1,l-1\}$, $\{l,l,l-2\}$, $\{l,l-2,l-2\}$, $\{l,l-2,l-1\}$.

Lemma 4. Let $l \geq 8$ and $l \equiv 0 \,(\mathrm{mod}\ 4)$.

(1) When $l \equiv 8 \,(\mathrm{mod}\ 32)$, $\binom{l}{2}-3$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-3)$ good groups, and another 3 elements (not the empty subset) that are not in these good groups are $\{l,l-3,l-2\}$, $\{l,l-3,l-1\}$, $\{l,l-2,l-1\}$.

(2) When $l \equiv 12 \,(\mathrm{mod}\ 32)$, $\binom{l}{2}-13$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-13)$ good groups, and another 13 elements (not the empty subset) that are not in these good groups are $\{l,l-3,l-2\}$, $\{l,l-3,l-1\}$, $\{l,l-2,l-1\}$, $\{l,5,l-4\}$, $\{l,5,l-3\}$, $\{l,5,l-2\}$, $\{l,5,l-1\}$, $\{l,6,l-3\}$, $\{l,6,l-2\}$, $\{l,6,l-1\}$, $\{l,7,l-2\}$, $\{l,7,l-1\}$, $\{l,8,l-1\}$.

(3) When $l \equiv 16 \,(\mathrm{mod}\ 32)$, $\binom{l}{2}-7$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-7)$ good groups, and another 7 elements (not the empty subset) that are not in these good groups are $\{l,l-3,l-2\}$, $\{l,l-3,l-1\}$, $\{l,l-2,l-1\}$, $\{l,5,l-4\}$, $\{l,5,l-3\}$, $\{l,5,l-2\}$, $\{l,5,l-1\}$.

(4) When $l \equiv 20 \,(\mathrm{mod}\ 32)$, $\binom{l}{2}-1$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-1)$ good groups, and another one element (not the empty subset) that are not in these good groups are $\{l,l-2,l-1\}$.

(5) When $l \equiv 24 \,(\mathrm{mod}\ 32)$, $\binom{l}{2}-11$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-11)$ good groups, and another 11 elements (not the empty subset) that are not in these good groups are $\{l,l-3,l-2\}$, $\{l,l-3,l-1\}$, $\{l,l-2,l-1\}$, $\{l,5,l-4\}$, $\{l,5,l-3\}$, $\{l,5,l-2\}$, $\{l,5,l-1\}$, $\{l,13,l-4\}$, $\{l,13,l-3\}$, $\{l,13,l-2\}$, $\{l,13,l-1\}$.

(6) When $l \equiv 28 \,(\mathrm{mod}\ 32)$, $\binom{l}{2}-5$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-5)$ good groups, and another 5 elements (not the empty subset) that are not in these good groups are $\{l,l,l-3\}$, $\{l,l-3,l-3\}$, $\{l,l-3,l-2\}$, $\{l,l-3,l-1\}$, $\{l,l-2,l-1\}$.

(7) When $l \equiv 0 \,(\mathrm{mod}\ 32)$, $\binom{l}{2}-15$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-15)$ good groups, and another 15 elements (not the empty subset) that are not in these good groups are $\{l,l-3,l-2\}$, $\{l,l-3,l-1\}$, $\{l,l-2,l-1\}$, $\{l,5,l-4\}$, $\{l,5,l-3\}$, $\{l,5,l-2\}$, $\{l,5,l-1\}$, $\{l,13,l-4\}$, $\{l,13,l-3\}$, $\{l,13,l-2\}$, $\{l,13,l-1\}$, $\{l,21,l-4\}$, $\{l,21,l-3\}$, $\{l,21,l-2\}$, $\{l,21,l-1\}$.

(8) When $l \equiv 4 \,(\mathrm{mod}\ 32)$, $\binom{l}{2}-9$ elements (each of which is not the empty subset) in A_l can be divided into $\frac{1}{16}(\binom{l}{2}-9)$ good groups, and another 9 elements (not the empty subset) that are not in these good groups are $\{l,l,l-3\}$, $\{l,l-3,l-3\}$, $\{l,l-3,l-2\}$, $\{l,l-3,l-1\}$, $\{l,l,l-2\}$, $\{l,l-2,l-2\}$, $\{l,l-2,l-1\}$, $\{l,l,l-1\}$, $\{l,l-1,l-1\}$.

III. Main Results

Theorem 1. If $2\binom{l-1}{2}+\binom{l-1}{3} < 16m \leq 2\binom{l}{2}+\binom{l}{3}$, $m \geq 1$, $l \geq 4$, then $\tilde{\chi}^i_{vt}(mC_{16}) = l$.

Proof. Obviously, $\tilde{\chi}^i_{vt}(mC_{16}) \geq \tilde{\zeta}(mC_{16})$=$l$. Therefore, it suffices to give the l-I-total coloring of mC_{16} which is vertex-distinguished by multisets when $2\binom{l-1}{2}+\binom{l-1}{3} < 16m \leq 2\binom{l}{2}+\binom{l}{3}$.

We will abbreviate " l-I-total coloring of mC_{16} which is vertex-distinguished by multisets" to " l-coloring of mC_{16}" in the following.

For each $l \geq 4$, only the l-coloring of $\lfloor \frac{1}{16}[2\binom{l}{2}+\binom{l}{3}] \rfloor$ C_{16} is given below. Then the l-coloring of mC_{16} can be obtained immediately by restriction when $2\binom{l-1}{2}+\binom{l-1}{3} < 16m \leq 2\binom{l}{2}+\binom{l}{3}$ and $m \neq \lfloor \frac{1}{16}[2\binom{l}{2}+\binom{l}{3}] \rfloor$.

When $m=1, l=4$, sixteen 3-subsets $\{1,1,2\}$, $\{2,2,1\}$, $\{1,4,4\}$, $\{4,3,1\}$, $\{1,1,4\}$, $\{4,3,3\}$, $\{3,1,1\}$, $\{3,3,1\}$, $\{2,3,4\}$, $\{1,2,3\}$, $\{3,3,2\}$, $\{2,2,3\}$, $\{3,4,4\}$, $\{4,4,2\}$, $\{4,2,2\}$, $\{4,2,1\}$ are the color sets of vertices of C_{16} under 4-coloring depicted in Fig. 12, and there is no remaining 3-subset.

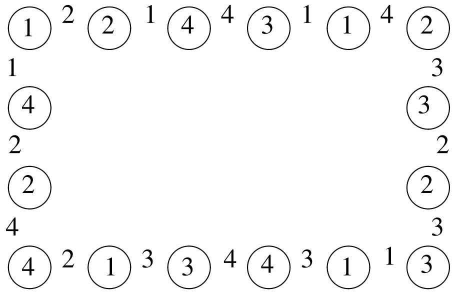


Fig. 12: A coloring of C_{16}

We execute the following 32 procedures recursively when l increases from 5.

In every **procedure**, we will construct colorings on the basis of the constructed $(l-1)$-coloring of the graph $\lfloor \frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}] \rfloor$ C_{16}.

Procedure 1 $l \equiv 5$(mod 32).

Based on the conclusion of the Lemma 1 (1), we can obtain the colorings of cycles from the $\{\lfloor \frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}] \rfloor + 1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-14]$-th C_{16}. There are fourteen 3-subsets that are not the color sets of any vertices. These fourteen 3-subsets are $\{l,l,l-4\}$, $\{l,l-4,l-4\}$, $\{l,l-4,l-3\}$, $\{l,l-4,l-2\}$, $\{l,l,l-3\}$, $\{l,l-3,l-3\}$, $\{l,l-3,l-2\}$, $\{l,l-3,l-1\}$, $\{l,l,l-2\}$, $\{l,l-2,l-2\}$, $\{l,l-2,l-1\}$, $\{l,l,l-1\}$, $\{l,l-1,l-1\}$, $\{l,1,l-1\}$, and also say that these fourteen 3-subsets remain.

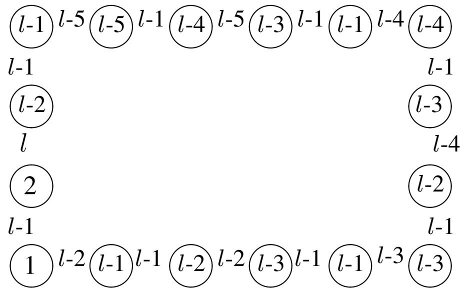


Fig. 13: A new coloring of C_{16} appears in **Procedure 2**

Procedure 2 $l \equiv 6$(mod 32).

Based on the conclusion of the Lemma 2 (1), we can obtain the colorings of cycles from the $\{\lfloor \frac{1}{16}[2\binom{l-1}{2}+$

$\binom{l-1}{3})]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-18]$-th C_{16} and four 3-subsets remain. A new good group for C_{16} is formed by combining two 3-subsets of the four 3-subsets above and fourteen 3-subsets which remain in Procedure 1 (this good group is constituted from the color sets of all vertices in C_{16} under the coloring depicted in Fig. 13), and the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-2]$-th C_{16} is colored properly. Consequently, $\{l,1,l-1\}$, $\{l,1,l-2\}$ remain.

Procedure 3 $l\equiv 7(\text{mod } 32)$.

Based on the conclusion of the Lemma 3 (1), we can obtain the colorings of cycles from the $\{\lfloor\frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-13]$-th C_{16}. Consequently, $\{l,l,l-1\}$, $\{l,l-1,l-1\}$, $\{l,l,l-2\}$, $\{l,l-2,l-2\}$, $\{l,l-2,l-1\}$, $\{l,1,l-3\}$, $\{l,1,l-2\}$, $\{l,1,l-1\}$, $\{l,2,l-2\}$, $\{l,2,l-1\}$, $\{l,3,l-1\}$, $\{l-1,1,l-2\}$, $\{l-1,1,l-3\}$ remain.

Procedure 4 $l\equiv 8(\text{mod } 32)$.

Based on the conclusion of the Lemma 4 (1), we can obtain the colorings of cycles from the $\{\lfloor\frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3})]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-16]$-th C_{16} and three 3-subsets remain. A new good group for C_{16} is formed by combining these three 3-subsets and thirteen 3-subsets which remain in Procedure 3 (this good group is constituted from the color sets of all vertices in C_{16} under the coloring depicted in Fig. 14), and the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}]$-th C_{16} is colored properly. Consequently, all 3-subsets in $\{1,2,\cdots,l\}$ are used up.

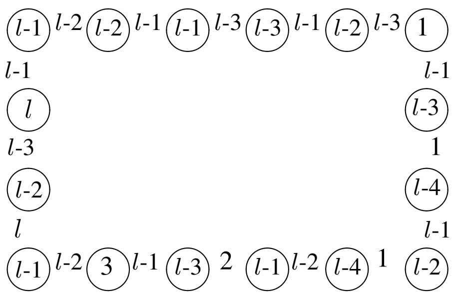


Fig. 14: A new coloring of C_{16} appears in **Procedure 4**

Procedure 5 $l\equiv 9(\text{mod } 32)$.

Based on the conclusion of the Lemma 1 (2), we can obtain the colorings of cycles from the $\{\lfloor\frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3})]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-12]$-th C_{16}. Finally, $\{l,1,l-5\}$, $\{l,1,l-4\}$, $\{l,1,l-3\}$, $\{l,1,l-2\}$, $\{l,1,l-1\}$, $\{l,2,l-4\}$, $\{l,2,l-3\}$, $\{l,2,l-2\}$, $\{l,2,l-1\}$, $\{l,3,l-3\}$, $\{l,3,l-2\}$, $\{l,3,l-1\}$ remain.

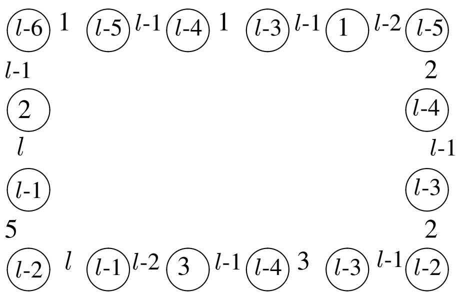


Fig. 15: A new coloring of C_{16} appears in **Procedure 6**

Procedure 6 $l\equiv 10(\text{mod } 32)$.

Based on the conclusion of the Lemma 2 (2), we can obtain the colorings of cycles from the $\{\lfloor\frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3})]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-18]$-th C_{16} and six 3-subsets remain. A new good group for C_{16} is formed by combining four 3-subsets among these six 3-subsets above and twelve 3-subsets which remain in Procedure 5 (this good group is constituted from the color sets of all vertices in C_{16} under the coloring depicted in Fig. 15), and the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-2]$-th C_{16} is colored properly. Consequently, $\{l,1,l-1\}$, $\{l,1,l-2\}$ remain.

Procedure 7 $l\equiv 11(\text{mod } 32)$.

Based on the conclusion of the Lemma 3 (2), we can obtain the colorings of cycles from the $\{\lfloor\frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-3]$-th C_{16}. Finally, $\{l,1,l-1\}$, $\{l-1,1,l-2\}$, $\{l-1,1,l-3\}$ remain.

Procedure 8 $l\equiv 12(\text{mod } 32)$.

Based on the conclusion of the Lemma 4 (2), we can obtain the colorings of cycles from the $\{\lfloor\frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-16]$-th C_{16} and thirteen 3-subsets remain. A new good group for C_{16} is formed by combining these thirteen 3-subsets and three 3-subsets which remain in Procedure 7 (this good group is constituted from the color sets of all vertices in C_{16} under the coloring depicted in Fig. 16), and the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}]$-th C_{16} is colored properly. Consequently, all 3-subsets in $\{1,2,\cdots,l\}$ are used up.

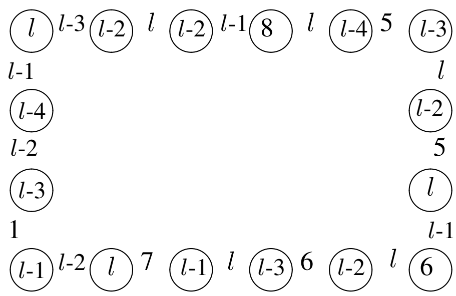


Fig. 16: A new coloring of C_{16} appears in **Procedure 8**

Procedure 9 $l\equiv 13(\text{mod } 32)$.

Based on the conclusion of the Lemma 1 (3), we can obtain the colorings of cycles from the $\{\lfloor\frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3})]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-10]$-th C_{16}. Finally, $\{l,l,l-4\}$, $\{l,l-4,l-4\}$, $\{l,l-4,l-3\}$, $\{l,l,l-3\}$, $\{l,l-3,l-3\}$, $\{l,l-3,l-2\}$, $\{l,l,l-2\}$, $\{l,l-2,l-2\}$, $\{l,l,l-1\}$, $\{l,l-1,l-1\}$ remain.

Procedure 10 $l\equiv 14(\text{mod } 32)$.

Based on the conclusion of the Lemma 2 (3), we can obtain the colorings of cycles from the $\{\lfloor\frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3})]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-18]$-th C_{16} and eight 3-subsets remain. A new good group for C_{16} is formed by combining six 3-subsets of the eight 3-subsets above and ten 3-subsets which remain in Procedure 9 (this good group is constituted from the color sets of all vertices in C_{16} under the coloring depicted in Fig. 17), and the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-2]$-th C_{16} is colored properly. Consequently, $\{l,1,l-1\}$, $\{l,1,l-2\}$ remain.

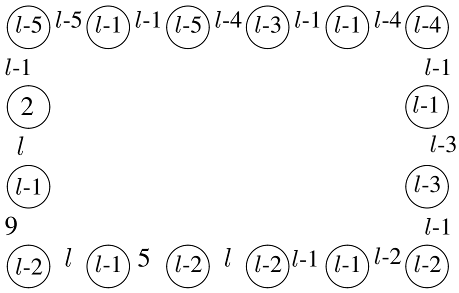


Fig. 17: A new coloring of C_{16} appears in **Procedure 10**

Procedure 11 $l \equiv 15(\text{mod } 32)$.

Based on the conclusion of the Lemma 3 (3), we can obtain the colorings of cycles from the $\{\lfloor \frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-9]$-th C_{16}. Finally, $\{l,l,l-1\}$, $\{l,l-1,l-1\}$, $\{l,l,l-2\}$, $\{l,l-2,l-2\}$, $\{l,l-2,l-1\}$, $\{l,1,l-3\}$, $\{l,1,l-2\}$, $\{l-1,1,l-2\}$, $\{l-1,1,l-3\}$ remain.

Procedure 12 $l \equiv 16(\text{mod } 32)$.

Based on the conclusion of the Lemma 4 (3), we can obtain the colorings of cycles from the $\{\lfloor \frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-16]$-th C_{16} and seven 3-subsets remain. A new good group for C_{16} is formed by combining these seven 3-subsets and nine 3-subsets which remain in Procedure 11 (this good group is constituted from the color sets of all vertices in C_{16} under the coloring depicted in Fig. 18), and the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}]$-th C_{16} is colored properly. Consequently, all 3-subsets in $\{1,2,\cdots,l\}$ are used up.

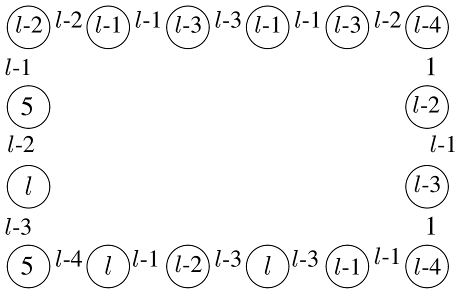


Fig. 18: A new coloring of C_{16} appears in **Procedure 12**

Procedure 13 $l \equiv 17(\text{mod } 32)$.

Based on the conclusion of the Lemma 1 (4), we can obtain the colorings of cycles from the $\{\lfloor \frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-8]$-th C_{16}. Finally, $\{l,l,l-4\}$, $\{l,l-4,l-4\}$, $\{l,l-4,l-3\}$, $\{l,l-4,l-2\}$, $\{l,l,l-3\}$, $\{l,l-3,l-3\}$, $\{l,l-3,l-2\}$, $\{l,l-3,l-1\}$ remain.

Procedure 14 $l \equiv 18(\text{mod } 32)$.

Based on the conclusion of the Lemma 2 (4), we can obtain the colorings of cycles from the $\{\lfloor \frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-18]$-th C_{16} and ten 3-subsets remain. A new good group for C_{16} is formed by combining eight 3-subsets among the ten 3-subsets above and eight 3-subsets which remain in Procedure 13 (this good group is constituted from the color sets of all vertices in C_{16} under the coloring depicted in Fig. 19), and the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-2]$-th C_{16} is colored properly. Consequently, $\{l,1,l-1\}$, $\{l,1,l-2\}$ remain.

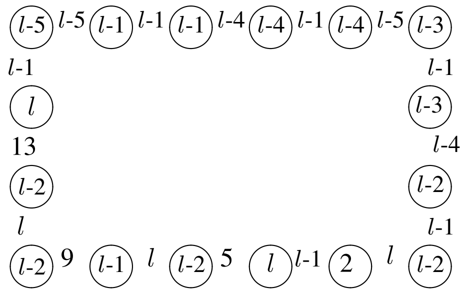


Fig. 19: A new coloring of C_{16} appears in **Procedure 14**

Procedure 15 $l \equiv 19(\text{mod } 32)$.

Based on the conclusion of the Lemma 3 (4), we can obtain the colorings of cycles from the $\{\lfloor \frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-15]$-th C_{16}. Finally, $\{l,l,l-1\}$, $\{l,l-1,l-1\}$, $\{l,l,l-2\}$, $\{l,l-2,l-2\}$, $\{l,l-2,l-1\}$, $\{l,1,l-3\}$, $\{l,1,l-2\}$, $\{l,1,l-1\}$, $\{l,2,l-2\}$, $\{l,2,l-1\}$, $\{l,3,l-1\}$, $\{l,5,l-3\}$, $\{l,5,l-2\}$, $\{l-1,1,l-2\}$, $\{l-1,1,l-3\}$ remain.

Procedure 16 $l \equiv 20(\text{mod } 32)$.

Based on the conclusion of the Lemma 4 (4), we can obtain the colorings of cycles from the $\{\lfloor \frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-16]$-th C_{16} and one 3-subset remain. A new good group for C_{16} is formed by combining this one 3-subset and fifteen 3-subsets which remain in Procedure 15 (this good group is constituted from the color sets of all vertices in C_{16} under the coloring depicted in Fig. 20), and the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}]$-th C_{16} is colored properly. Consequently, all 3-subsets in $\{1,2,\cdots,l\}$ are used up.

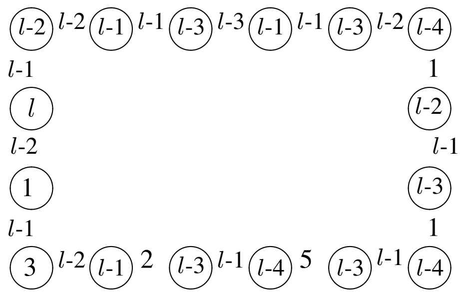


Fig. 20: A new coloring of C_{16} appears in **Procedure 16**

Procedure 17 $l \equiv 21(\text{mod } 32)$.

Based on the conclusion of the Lemma 1 (5), we can obtain the colorings of cycles from the $\{\lfloor \frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-6]$-th C_{16}. Finally, $\{l,l,l-4\}$, $\{l,l-4,l-4\}$, $\{l,l,l-3\}$, $\{l,l-3,l-3\}$, $\{l,l,l-2\}$, $\{l,l-2,l-2\}$ remain.

Procedure 18 $l \equiv 22(\text{mod } 32)$.

Based on the conclusion of the Lemma 2 (5), we can obtain the colorings of cycles from the $\{\lfloor \frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-18]$-th C_{16} and twelve 3-subsets remain. A new good group for C_{16} is formed by combining ten 3-subsets among the twelve 3-subsets above and six 3-subsets which remain in Procedure 17 (this good group is constituted from the color sets of all vertices in C_{16} under the coloring depicted in Fig. 21), and the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-2]$-th C_{16} is colored properly. Consequently, $\{l,1,l-1\}$, $\{l,1,l-2\}$ remain.

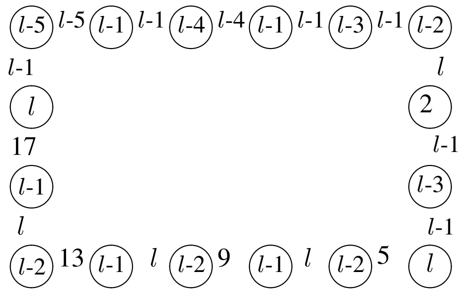


Fig. 21: A new coloring of C_{16} appears in **Procedure 18**

Procedure 19 $l \equiv 23 (\mathrm{mod}\ 32)$.

Based on the conclusion of the Lemma 3 (5), we can obtain the colorings of cycles from the $\{\lfloor \frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-5]$-th C_{16}. Finally, $\{l, l, l-2\}$, $\{l, l-2, l-2\}$, $\{\{l, l-2, l-1\}$, $\{l-1, 1, l-2\}$, $\{l-1, 1, l-3\}$ remain.

Procedure 20 $l \equiv 24 (\mathrm{mod}\ 32)$.

Based on the conclusion of the Lemma 4 (5), we can obtain the colorings of cycles from the $\{\lfloor \frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-16]$-th C_{16} and eleven 3-subsets remain. A new good group for C_{16} is formed by combining these eleven 3-subsets and five 3-subsets which remain in Procedure 19 (this good group is constituted from the color sets of all vertices in C_{16} under the coloring depicted in Fig. 22), and the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}]$-th C_{16} is colored properly. Consequently, all 3-subsets in $\{1, 2, \cdots, l\}$ are used up.

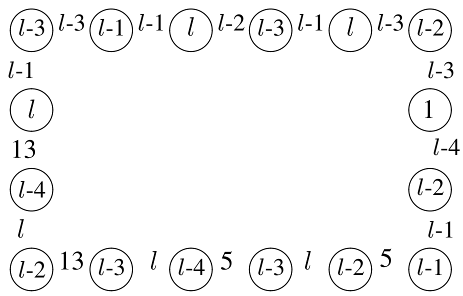


Fig. 22: A new coloring of C_{16} appears in **Procedure 20**

Procedure 21 $l \equiv 25 (\mathrm{mod}\ 32)$.

Based on the conclusion of the Lemma 1 (6), we can obtain the colorings of cycles from the $\{\lfloor \frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-4]$-th C_{16}. Finally, $\{l, l, l-4\}$, $\{l, l-4, l-4\}$, $\{l, l, l-3\}$, $\{l, l-3, l-3\}$ remain.

Procedure 22 $l \equiv 26 (\mathrm{mod}\ 32)$.

Based on the conclusion of the Lemma 2 (6), we can obtain the colorings of cycles from the $\{\lfloor \frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-18]$-th C_{16}. A new good group for C_{16} is formed by combining twelve 3-subsets among the fourteen 3-subsets above and four 3-subsets which remain in Procedure 21 (this good group is constituted from the color sets of all vertices in C_{16} under the coloring depicted in Fig. 23), and the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-2]$-th C_{16} is colored properly. Consequently, $\{l, 1, l-1\}$, $\{l, 1, l-2\}$ remain.

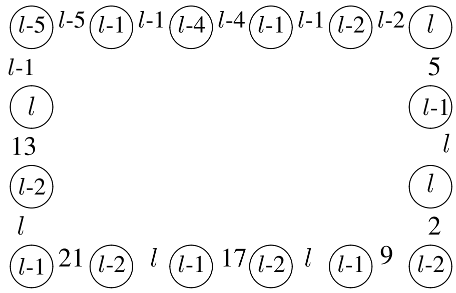


Fig. 23: A new coloring of C_{16} appears in **Procedure 22**

Procedure 23 $l \equiv 27 (\mathrm{mod}\ 32)$.

Based on the conclusion of the Lemma 3 (6), we can obtain the colorings of cycles from the $\{\lfloor \frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-11]$-th C_{16}. Finally, $\{l, l, l-1\}$, $\{l, l-1, l-1\}$, $\{l, l, l-2\}$, $\{l, l-2, l-2\}$, $\{l, l-2, l-1\}$, $\{l, 1, l-3\}$, $\{l, 1, l-2\}$, $\{l, 1, l-1\}$, $\{l, 3, l-1\}$, $\{l-1, 1, l-2\}$, $\{l-1, 1, l-3\}$ remain.

Procedure 24 $l \equiv 28 (\mathrm{mod}\ 32)$.

Based on the conclusion of the Lemma 4 (6), we can obtain the colorings of cycles from the $\{\lfloor \frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-16]$-th C_{16} and five 3-subsets remain. A new good group for C_{16} is formed by combining these five 3-subsets and eleven 3-subsets which remain in Procedure 23 (this good group is constituted from the color sets of all vertices in C_{16} under the coloring depicted in Fig. 24), and the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}]$-th C_{16} is colored properly. Consequently, all 3-subsets in $\{1, 2, \cdots, l\}$ are used up.

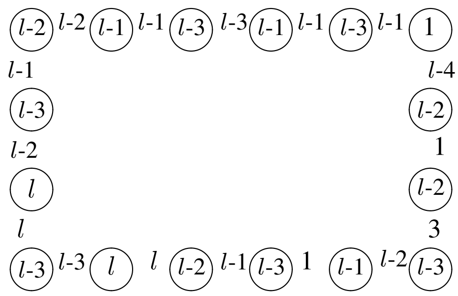


Fig. 24: A new coloring of C_{16} appears in **Procedure 24**

Procedure 25 $l \equiv 29 (\mathrm{mod}\ 32)$.

Based on the conclusion of the Lemma 1 (7), we can obtain the colorings of cycles from the $\{\lfloor \frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-2]$-th C_{16}. Finally, $\{l, 1, l-1\}$, $\{l, 1, l-2\}$ remain.

Procedure 26 $l \equiv 30 (\mathrm{mod}\ 32)$.

Based on the conclusion of the Lemma 2 (7), we can obtain the colorings of cycles from the $\{\lfloor \frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-2]$-th C_{16}. Finally, $\{l-1, 1, l-3\}$, $\{l-1, 1, l-2\}$ remain.

Procedure 27 $l \equiv 31 (\mathrm{mod}\ 32)$.

Based on the conclusion of the Lemma 3 (7), we can obtain the colorings of cycles from the $\{\lfloor \frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-17]$-th C_{16} and fifteen 3-subsets remain. A new good group for C_{16} is formed by combining fourteen 3-subsets of the fifteen 3-subsets above and two 3-subsets which remain in Procedure 26 (this good group is constituted from the color sets of all vertices in C_{16} under the coloring depicted in Fig. in Fig. 25), and the

$\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-1]$-th C_{16} is colored properly. Consequently, $\{l,1,l-1\}$ remains.

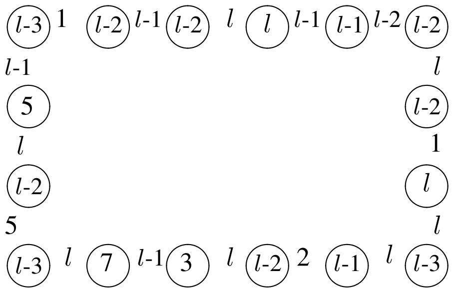


Fig. 25: A new coloring of C_{16} appears in **Procedure 27**

Procedure 28 $l \equiv 0(\text{mod } 32)$.

Based on the conclusion of the Lemma 4 (7), we can obtain the colorings of cycles from the $\{\lfloor\frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-16]$-th C_{16} and fifteen 3-subsets remain. A new good group for C_{16} is formed by combining the fifteen 3-subsets and one 3-subset which remain in Procedure 27 (this good group is constituted from the color sets of all vertices in C_{16} under the coloring depicted in Fig. 26), and the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}]$-th C_{16} is colored properly. Consequently, all 3-subsets in $\{1,2,\cdots,l\}$ are used up.

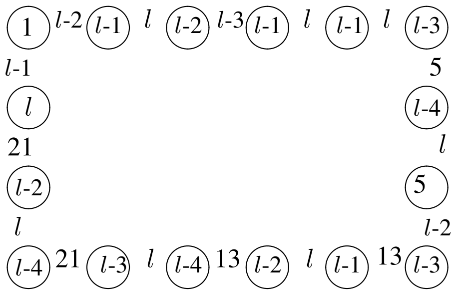


Fig. 26: A new coloring of C_{16} appears in **Procedure 28**

Procedure 29 $l \equiv 1(\text{mod } 32)$.

Based on the conclusion of the Lemma 1 (8), we can obtain the colorings of cycles from the $\{\lfloor\frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}]$-th C_{16}. Finally, all 3-subsets in $\{1,2,\cdots,l\}$ are used up.

Procedure 30 $l \equiv 2(\text{mod } 32)$.

Based on the conclusion of the Lemma 2 (8), we can obtain the colorings of cycles from the $\{\lfloor\frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-2]$-th C_{16}. Finally, $\{l,2,l-1\}$, $\{l,l-2,l-1\}$ remain.

Procedure 31 $l \equiv 3(\text{mod } 32)$.

Based on the conclusion of the Lemma 3 (8), we can obtain the colorings of cycles from the $\{\lfloor\frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-7]$-th C_{16}. Finally, $\{l,l,l-1\}$, $\{l,l-1,l-1\}$, $\{l,l,l-2\}$, $\{l,l-2,l-2\}$, $\{l,l-2,l-1\}$, $\{l-1,2,l-2\}$, $\{l-1,l-3,l-2\}$ remain.

Procedure 32 $l \equiv 4(\text{mod } 32)$.

Based on the conclusion of the Lemma 4 (8), we can obtain the colorings of cycles from the $\{\lfloor\frac{1}{16}[2\binom{l-1}{2}+\binom{l-1}{3}]\rfloor+1\}$-th C_{16} to the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}-16]$-th C_{16} and nine 3-subsets remain. A new good group for C_{16} is formed by combining these nine 3-subsets and seven 3-subsets which remain in Procedure 31 (this good group is constituted from the color sets of all vertices in C_{16} under the coloring depicted in Fig. 27), and the $\frac{1}{16}[2\binom{l}{2}+\binom{l}{3}]$-th C_{16} is colored properly. Consequently, all 3-subsets in $\{1,2,\cdots,l\}$ are used up.

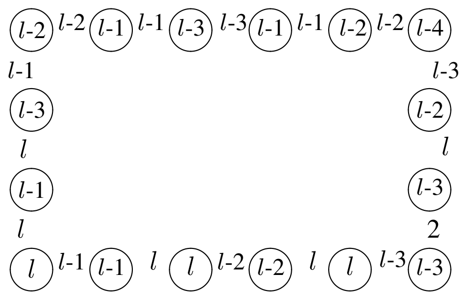


Fig. 27: A new coloring of C_{16} appears in **Procedure 32**

The proof of Theorem 1 is completed.

Note that $\tilde{\zeta}(mC_{16}) = l$ when $2\binom{l-1}{2}+\binom{l-1}{3} < 16m \le 2\binom{l}{2}+\binom{l}{3}$, where $m \ge 1$, $l \ge 4$. Thus by means of Proposition 1 in Section I, from **Theorem 1** in Section III, we may obtain the following theorem.

Theorem 2. If $2\binom{l-1}{2}+\binom{l-1}{3} < 16m \le 2\binom{l}{2}+\binom{l}{3}$, $m \ge 1$, $l \ge 4$, then $\tilde{\chi}_{vt}^{vi}(mC_{16}) = l$.

Remark: The discussions on the I(VI)-total chromatic numbers of mC_n $(13 \le n \le 15)$ which are vertex-distinguished by multisets are very long, so we will state these results and their proofs in another paper.

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