

Fractional-Order Hyperchaos in Financial Systems: Analysis, Stabilization, and Adaptive Synchronization under Parameter Uncertainties

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Abstract—In the current global economic environment, which is more complex and changeable, and with the factors influencing the financial system being more diverse, this research introduces a novel four-dimensional fractional-order hyperchaotic financial system by adding an external perturbation term and a new parameter. Firstly, the dynamics analysis, such as the Lyapunov exponential spectrum, phase diagram, chaotic attractor, and equilibrium stability is provided for the new system, then the most suitable parameters under the hyperchaotic phenomenon are identified. Secondly, the stability analysis of the new fractional-order system is conducted by using the dynamic feedback gain control method, and the system is dynamically stabilized at its different equilibrium points. Then, an adaptive shot synchronization controller is synthesized to attain system self-synchronization. The controller can adapt to the parameter uncertainties and external disturbances of the new system. Finally, numerical simulations validate all the theoretical results.

Index Terms—financial system, fractional-Order, external Disturbance, dynamic feedback gain, adaptive synchronization.

I. INTRODUCTION

CHAOS theory originated in the early 20th century and focuses on exploring the dynamics of nonlinear systems. In 1963, meteorologist Edward Noren Lorenz first proposed the classical Lorenz system, a three-dimensional model composed of three nonlinear ordinary differential equations to describe atmospheric heat convection. Lorenz further expanded this system by introducing nonlinear terms and additional driving forces, forming a hyperchaotic system that reveals more complex dynamic behaviors. This advancement marks the emergence of four-dimensional chaotic systems, a direct outcome of in-depth research on classical chaotic systems [1-6]. With the continuous development of fractional calculus theory, researchers begin to apply it to the modeling and analysis of chaotic systems [7-11]. Compared with traditional integer-order calculus, fractional calculus introduces non-integer orders (e.g., fractional derivatives and integrals), endowing fractional-order chaotic systems with richer and more complex dynamic characteristics [12-16]. These systems not only enrich the content of chaos theory but

also open up broad prospects for applications in information security, control engineering, image processing, and other fields [17-20].

The international financial landscape is undergoing an unprecedented complex evolution, where factors such as local disturbances, market speculation, and external shocks influence each other through nonlinear transmission. The impact of the pandemic in 2020 led to the Dow Jones index being "fused" four times in a single month. In 2022, as a result of the aggressive interest rate increases by the U.S. Federal Reserve, a wave of capital outflows was unleashed from emerging markets. The Silicon Valley bank incident in 2023 exposed the fragile interconnectedness of the financial system. These crisis events have confirmed that the modern financial system has evolved into a "super network", where even minor local disturbances can potentially trigger a systemic collapse [21-25].

In recent years, nonlinear dynamics has made significant progress in the field of financial modeling [26-29]. The three-dimensional chaotic financial system successfully simulated the nonlinear coupling relationship among price index, investment demand, and interest rate. However, this kind of integer-order model exhibits fundamental flaws when describing the modern market dominated by high-frequency trading: firstly, it cannot simulate the fractional-order memory effect between financial variables, which is particularly prominent in the cryptocurrency market; secondly, the fixed parameter system struggles to adapt to the dynamic shifts in economic cycles, such as abrupt changes in nonlinear relationships caused by global supply chain crises [30-33].

To tackle these challenges, the present study has achieved theoretical advancements in three critical aspects: first, the introduction of a nonlinear quadratic term to simulate the intrinsic endogenous amplification feedback mechanism within financial markets; second, the development of an innovative adaptive parameter to capture dynamic shifts in market behavior under varying economic conditions; third, the identification through parameter analysis that this parameter can real-time regulate chaotic intensity, with appropriate values enabling the system to establish a fractional-order hyperchaotic state. Furthermore, comprehensive dynamic analysis, stability assessment, and synchronization control of the system have been conducted. The findings validate the system's efficacy in capturing complex financial behaviors and its potential for practical economic modeling.

The main structure of this article is organized as follows:

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Section II emphasizes the origin of the novel system and constructs an improved version of system. Section III performs an all-encompassing dynamic analysis studies of the improved system utilizing a range of analytical tools, including Lyapunov exponents, attractors, phase portraits, power spectrum diagrams, and equilibrium point stability. This section determines the parameters and properties of the system across states such as hyperchaos, chaos, periodicity, and stable fixed points. In Section IV, a dynamic feedback gain control strategy is proposed, and it is proven effective in stabilizing the system at multiple equilibrium points. This method provides a theoretical basis for solving financial instability issues. Section V designs an adaptive synchronization controller, which ensures reliable self-synchronization of the system even in the presence of parameter uncertainties and external disturbances. This method holds particular significance for applications related to multi-regional financial coordination and market stability. Additionally, all content in the text has undergone numerical verification and been visualized through graphs. Section VI presents a summary and outlook.

II. DESIGN OF A NEW HYPERCHAOTIC SYSTEM

A. Preliminaries

There are a number of recognized methods for defining fractional order calculus, of which, the more common are the Caputo definition, the Grünwald-Letnikov (G-L) definition, the Fourier / Laplace transform definition and the Riemann-Liouville (R-L) definition [34-37]. For the purpose of analysis, the Caputo fractional derivative will be used, which is defined as follows.

Definition 1 If there is a function $w(t)$, whose fractional integrals is:

$$T_t^q w(t) = \frac{1}{\Gamma(q)} \int_{t_0}^t (t-\tau)^{q-1} w(\tau) d\tau, \quad (1)$$

where $q > 0, t \geq t_0$, $\Gamma(\cdot)$ is a gamma function, and defined as:

$$\Gamma(\nu) = \int_0^\infty t^{\nu-1} e^{-t} dt. \quad (2)$$

Then for the function $w(t)$, the Caputo fractional derivative is:

$$D_t^q w(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^t \frac{w^{(n)}(\tau)}{(t-\tau)^{q-n+1}} d\tau, \quad (3)$$

where $n-1 < q < n, t \geq t_0$, and $n \in \mathbb{N}^*$.

Lemma 1 When $0 < q \leq 1$, $t \geq t_0$, the Caputo fractional derivative $D_t^q w(t)$ is integrable, and $T_t^q D_t^q w(t) = w(t) - w(t_0)$.

Lemma 2 For the following fractional order system:

$$\frac{d^q \varphi(t)}{dt^q} = w(\varphi(t)), \quad (4)$$

where $0 < q < 1$, $\varphi(t) = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T \in \mathbb{R}^n$. Let $w(\varphi(t)) = 0$, then the equilibrium points of a system can be solved by the above equation. Substituting each equilibrium point into the Jacobi matrix, then the eigenvalues of the

matrix can be calculated, and the stability of an equilibrium or system can be determined from the eigenvalues.

B. New System Model

Amidst the digital transformation of the global economy and the overlapping geopolitical risks, the financial system, which is composed of labor (L), capital (M), production (P), and inventory (I), faces triple dynamic coupling challenges. In previous work [1], we improved the three-dimensional classical system into a four-dimensional system by adding variables $D(t)$:

$$\begin{cases} \dot{A}(t) = -oA + AB + C + D, \\ \dot{B}(t) = -A^2 - pB + 1, \\ \dot{C}(t) = -A - rC, \\ \dot{D}(t) = -sAB - tD. \end{cases} \quad (5)$$

The traditional four-dimensional financial model exhibits fundamental limitations: its linear coupling assumption fails to capture the nonlinear transmission of production factors in the digital economy era, and the fixed parameter system cannot account for these nonlinear shifts. To address these issues, this study introduces a "dynamic memory-adaptive" modeling framework. Initially, it overcomes the three constraints inherent in classical models by incorporating a nonlinear quadratic term $B(t)C(t)$ (economically interpreted as the multiplier suppression effect of price overshoot on investment) to the third equation of system (5), thereby reconstructing the variable interaction mechanism. Subsequent dynamic analysis reveals that this modification elevates the system to three positive Lyapunov exponents. Furthermore, a time-varying parameter m (representing the norm of money supply changes) is innovatively designed to facilitate self-sensing adjustments to external disturbance intensity. To depict the path-dependent characteristics of financial variables, Caputo fractional derivatives are employed in lieu of traditional integer derivatives. Ultimately, this establishes a fractional-order hyperchaotic financial system.

$$\begin{cases} D^{q_1} A(t) = -oA + AB + C + D, \\ D^{q_2} B(t) = -A^2 - pB + 1, \\ D^{q_3} C(t) = -A - rC + mBC, \\ D^{q_4} D(t) = -sAB - tD. \end{cases} \quad (6)$$

In the financial system constructed above, the investment demand is replaced by $B(t)$, while the price index is denoted by $C(t)$. When the price index ascends due to inflation or the increase in market demand, it may cause an increase or decrease in investment demand, leading to corresponding positive or negative variations. The new parameter m and the nonlinear quadratic term $B(t)C(t)$ are introduced to capture the impacts of external perturbations on the system and strive to simulate the changing and complex characteristics of the current global economic situation. Additionally, $A(t)$ and $D(t)$ respectively denote the interest rate and the average profitability of the commodity, and the remaining characters o, p, r, s, t, m are control parameters. By adjusting these control parameters, it is feasible to study the dynamic behavior of the system under diverse conditions

and provide policymakers with valuable information regarding the stability of the economic system and regulatory strategies.

III. QUANTITATIVE ANALYSIS OF SYSTEM (6)

A. Lyapunov Exponent

The Lyapunov exponent is a measure of the speed of two neighboring trajectories moving away from or towards each other at an exponential rate over time in phase space. If the Lyapunov exponent of a system is positive, then the system is judged to be chaotic. For the purpose of examining the impact of parameter m on the system's stability, when m changes, and parameters $o = 1.1$, $p = 0.2$, $r = 1.85$, $s = 0.17$, $t = 0.2$, the Lyapunov exponent diagram is drawn as shown in Fig. 1.

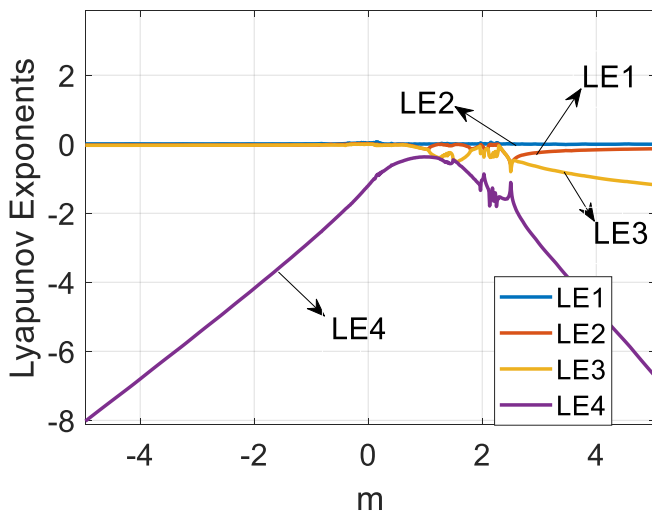


Fig. 1. Lyapunov exponent diagram as m varies.

As is apparent from Fig. 1, the system exhibits distinct dynamical states across different ranges of m : it is in a hyperchaotic state when m ranges from -5 to 0.1 ; in a chaotic state when m is between 0.1 and 1.5 ; in a weak chaos state for m values from 1.5 to 2.5 ; in a periodic state as m varies from 2.5 to 3.5 ; and converges to a stable fixed point when m is in the range of 3.5 to 5 . Consequently, the state of the financial system can be regulated by appropriately selecting the value of m .

Theory 1 (Existence Condition of Hyperchaos): when the system parameter satisfies $m \leq 0.1$, and fractional order $q_i \in (0,1)$, the system has at least two positive Lyapunov exponents and enters hyper chaos.

Theory 2 (Scale-Invariant Fractal Properties): The calculation formula for Lyapunov dimension is:

$$D_L = n + \frac{1}{|LE_{n+1}|} \sum_{i=1}^n LE_i. \quad (7)$$

TABLE I

LYAPUNOV INDEX AND SYSTEM CHARACTERISTICS

m	Lyapunov exponents	Lyapunov dimensions and Dynamic behavior
0.1	(0.047229, 0.008012, 0.000901, -1.026344), (+, +, +, -)	$D_L = 3.0547$, Hyperchaos
1.5	(0.010507, -0.024053, -0.445416, -0.464988), (+, -, -, -)	$D_L = 1.437$, Chaos
2.5	(0.013720, -0.575315, -0.787052, -1.110497), (+, -, -, -)	$D_L = 1.024$, Periodic states
3.5	(-0.007120, -0.186831, -0.834882, -3.853732), (-, -, -, -)	$D_L = 0$, Stable and immobile point

When $D_L > 3$, it can be confirmed that the system has formed a strange attractor with Hausdorff-Besicovitch characteristics, indicating sustained hyperchaotic behavior across time scales.

A characterization of the system via Lyapunov exponents is presented in what follows. Numerical computations were performed using MATLAB, with parameters fixed as $o = 1.1$, $p = 0.2$, $r = 1.85$, $s = 0.17$, $t = 0.2$, while m was assigned values of 0.1 , 1.5 , 2.5 , and 3.5 . To investigate the system's chaotic properties, multiple analytical approaches were employed, including evaluations of Lyapunov exponents and Lyapunov dimensions. Detailed results of these computations are summarized in Table 1, whereas Figs. 2–5 illustrate the time-evolution curves of Lyapunov exponents corresponding to each m value.

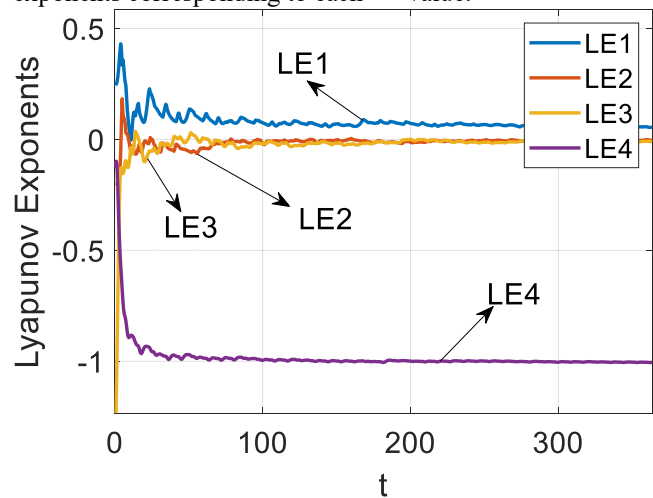


Fig. 2. Lyapunov exponential diagram with $m=0.1$.

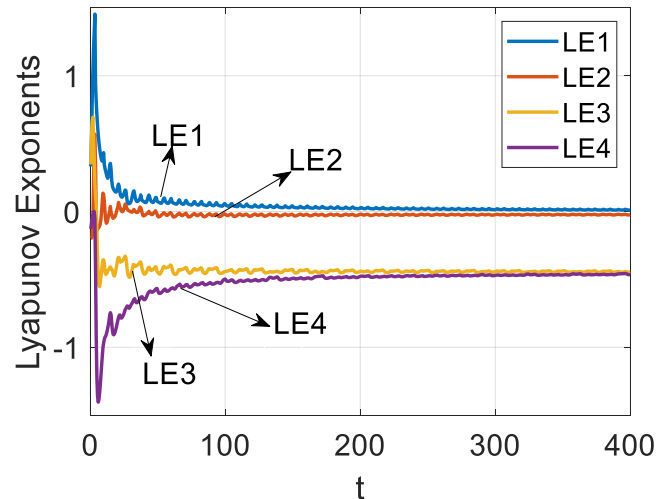
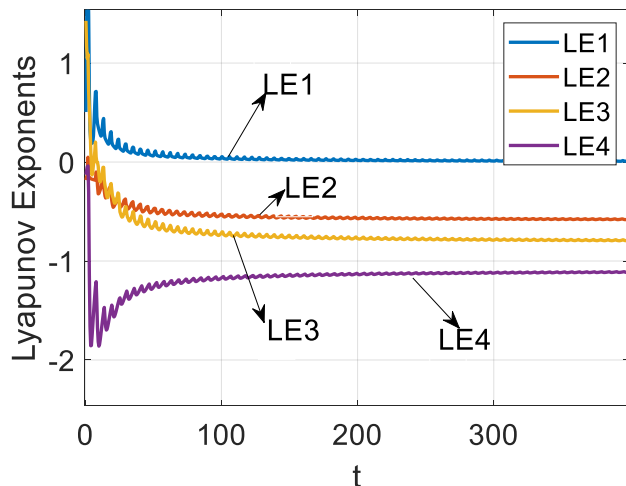
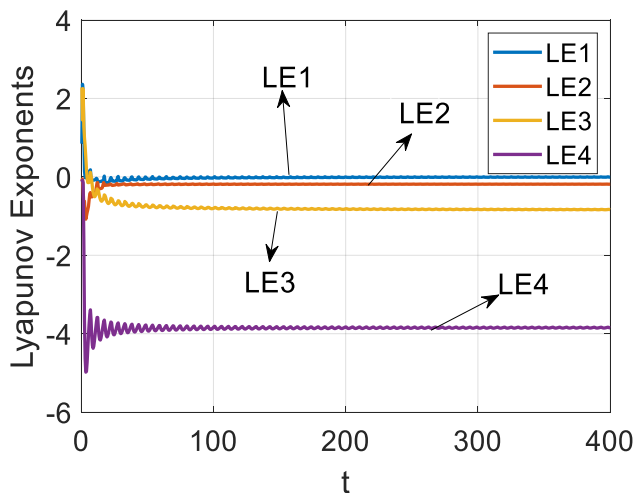


Fig. 3. Lyapunov exponential diagram with $m=1.5$.


 Fig. 4. Lyapunov exponential diagram with $m=2.5$.

 Fig. 5. Lyapunov exponential diagram with $m=3.5$.

Conclusion 1 When $m = 0.1$, three of the Lyapunov exponents are positive, and $D_L = 3.0547$ is the fractional dimension, indicating that the system (6) exhibits highly abundant hyperchaotic behaviors.

B. Attractors and Phase Diagrams

Chaotic attractors refer to a distinct class of complex geometric entities within phase diagrams. During system evolution, trajectories are "attracted" into this finite region, where they exhibit non-repetitive behavior and extreme sensitivity to initial conditions. Phase diagram is a geometric representation that employs the state variables of a chaotic system as coordinate axes, visually depicting all feasible states and evolutionary trajectories of the system via points or curves. It enables the transformation of abstract system dynamics into an observable spatial structure.

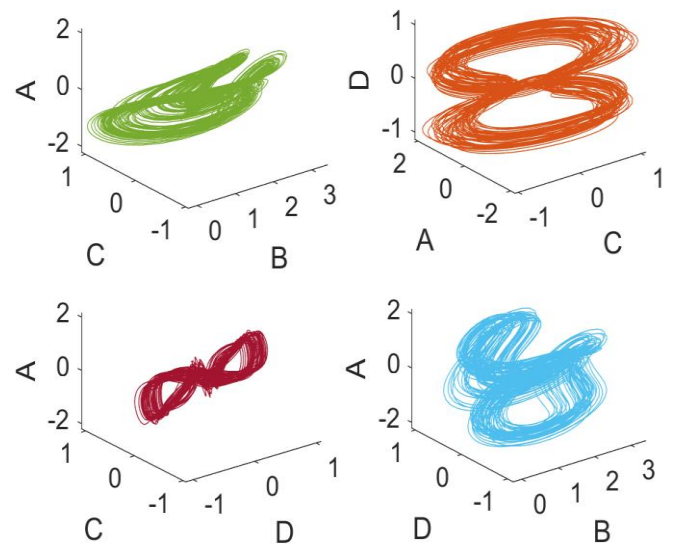
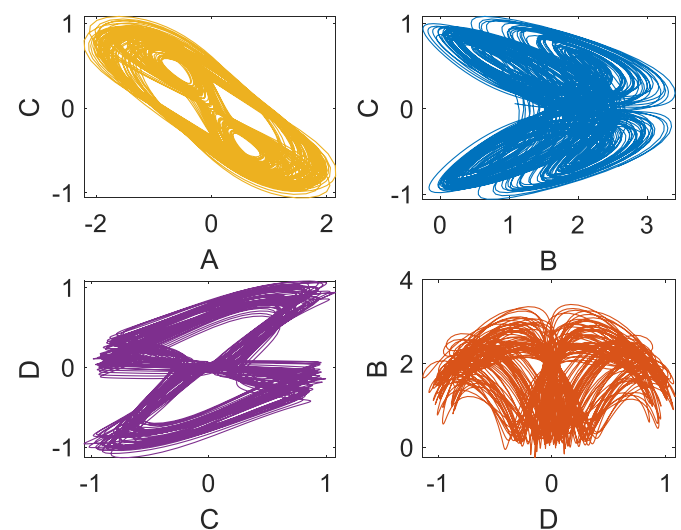
Through chaotic attractors and phase diagram analysis, the dynamical behaviors of nonlinear systems, including limit loops, periodic motions and chaotic motions, can be visually and initially determined. To verify the influence of varying parameter values on the system's chaotic properties, we will present the attributes of the system under various parameter settings.

In the previous section, we have provided the Lyapunov exponent diagrams corresponding to parameter values of $m = 0.1, 1.5, 2.5, 3.5$, these diagrams offer quantitative insights into the system's chaos characteristics. To

complement this quantitative analysis and more intuitively validate the system's chaotic states, we will further present dynamic demonstration diagrams of the system in 3-dimensional and 2-dimensional planes. Such visualizations will not only corroborate the conclusions derived from the Lyapunov exponent analysis but also facilitate a more comprehensive understanding of how parameter variations modulate the system's transition between periodic and chaotic behaviors.

Fixed fractional order $q_i = 0.995$ ($i = 1, 2, 3, 4$), parameters $o = 1.1$, $p = 0.2$, $r = 1.85$, $s = 0.17$, $t = 0.2$, and initial values $(A(t), B(t), C(t), D(t)) = (0.01, 1.08, 0.06, 0.01)$.

When $m = 0.1$, system (6) is in a hyperchaotic state, and its chaotic attractor exhibits a fractal structure. As can be observed from Fig. 6 (3D attractor) and Fig. 7 (2D phase diagrams), the attractor fills a very wide range in the phase space, with more complex folding and stretching of the orbits, indicating that the system displays relatively abundant chaotic behaviors.


 Fig. 6. 3D attractors with $m=0.1$.

 Fig. 7. Phase diagrams with $m=0.1$.

When $m = 1.5$, the system is in a chaotic state. However, it has only one positive Lyapunov exponent. Compared with the system at $m = 0.1$, its degree of chaos is relatively lower, and the complexity and unpredictability of the system are

relatively weaker. The corresponding 3D attractor plots and 2D planar graphs are presented in Figs. 8 and 9, respectively.

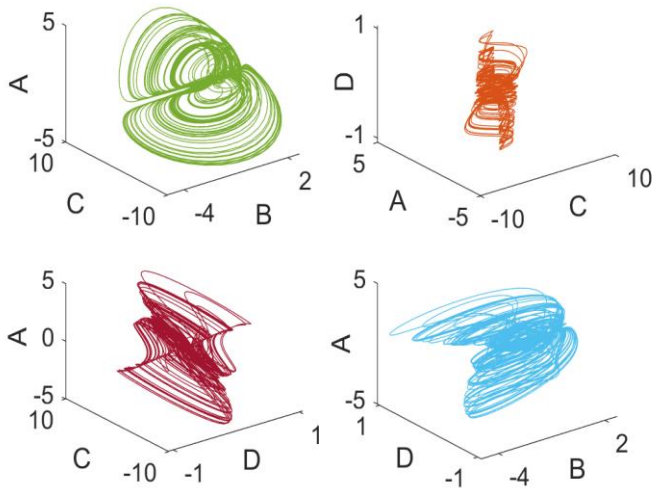


Fig. 8. 3D attractors with $m=1.5$.

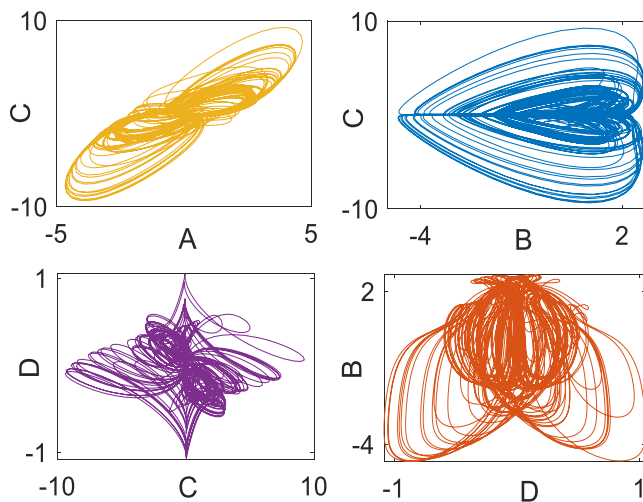


Fig. 9. Phase diagrams with $m=1.5$.

When $m = 2.5$, the system exhibits periodic behaviors. As illustrated in Figs. 10 and 11, the system demonstrates a relatively low level of chaos and tends toward periodic motion—a dynamic feature that carries significant implications for real economic systems.

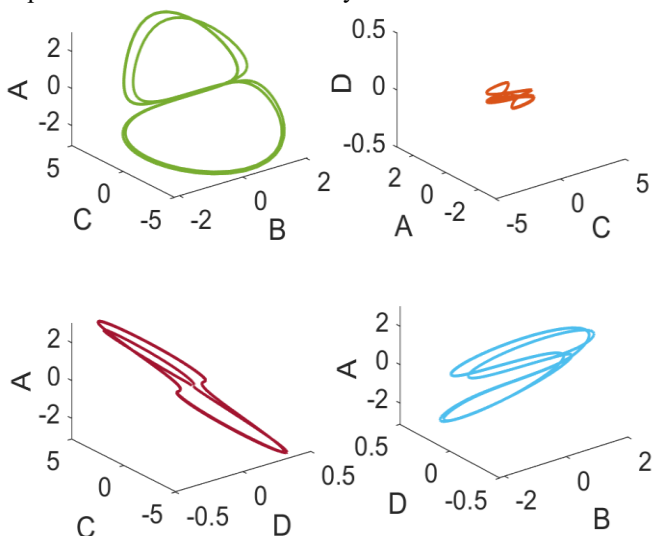


Fig. 10. 3D attractors with $m=2.5$.

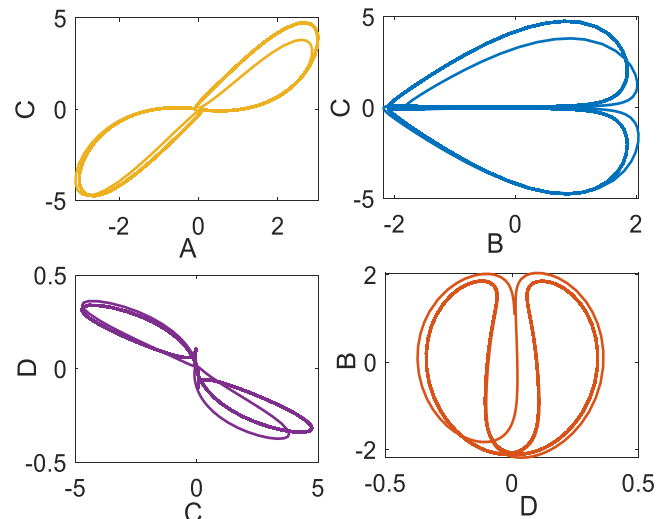


Fig. 11. Phase diagrams with $m=2.5$.

When $m = 3.5$, as indicated by Table 1, all Lyapunov exponents of the system are negative. This implies that any infinitesimal initial perturbation of the system will decay over time, and the system will eventually converge to a stable fixed point or periodic orbit. Figs. 12 and 13 confirm that the system does not exhibit chaos and is in a completely stable and ordered state.

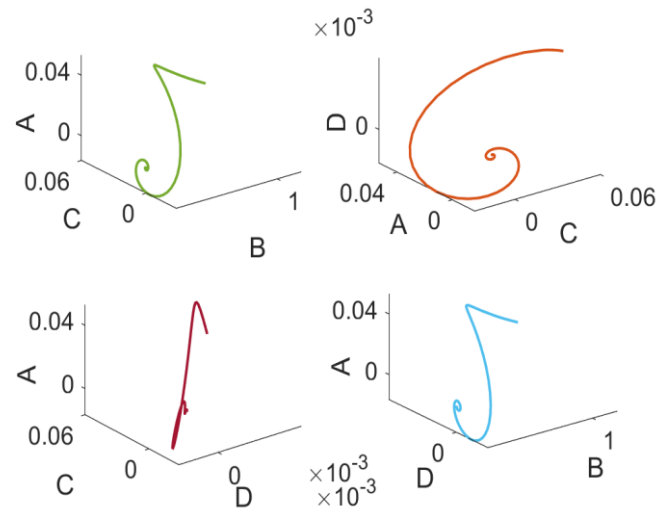


Fig. 12. 3D attractors with $m=3.5$.

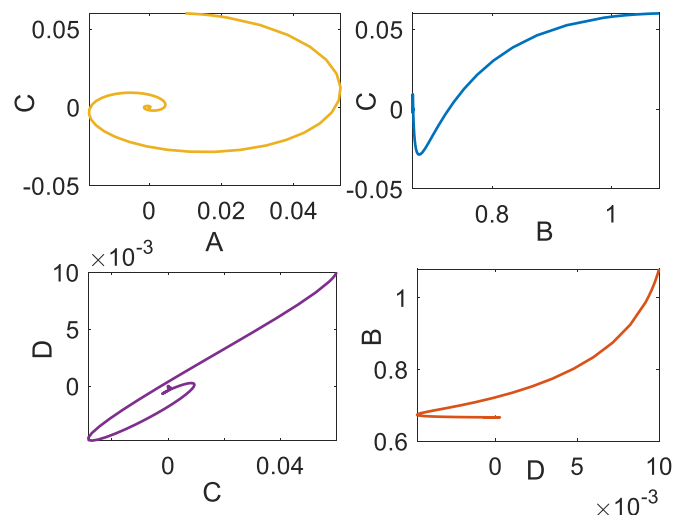


Fig. 13. Phase diagrams with $m=3.5$.

As evident from Figs. 6–13, the system undergoes substantial state transitions with variations in the parameter m . The influence of such changes in the chaotic behavior of dynamic systems on economic systems essentially manifests as a game between uncertainty and adaptability: moderate chaos serves as a driver of economic vitality, excessive chaos induces disorder, and insufficient chaos results in rigidity.

C. Initial Conditions and Fractional-Order

When $m = 0.1$, the system exhibits hyperchaotic behavior. A well-documented characteristic of hyperchaotic systems is their extreme sensitivity to initial conditions: even minor disturbances, such as variations in initial values or fractional orders, can be rapidly amplified. This amplification ultimately causes the system's evolution trajectory to deviate completely from its originally expected path, leading to dynamic behaviors that are distinctly different from those observed in the initial state.

To explore the system's dependence on its initial state, the fractional order and all other parameters were kept constant while the system's initial value was modified to $(1.61, 1.58, 1.06, 0.80)$, the corresponding attractors and phase portraits are depicted in Figs. 14–15, respectively.

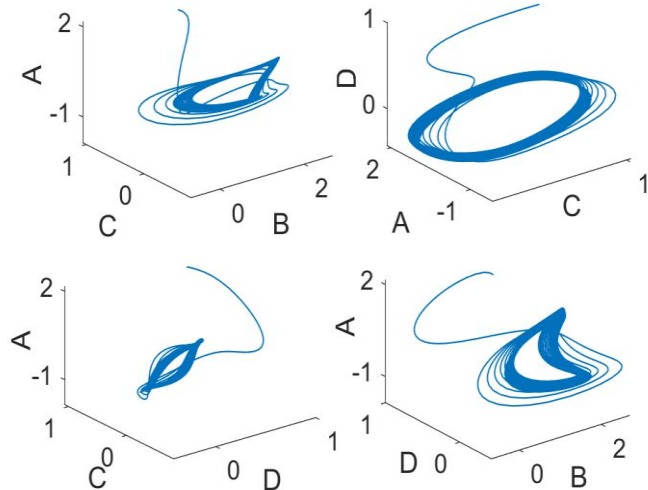


Fig. 14. 3D attractors with initial values $(1.61, 1.58, 1.06, 0.80)$.

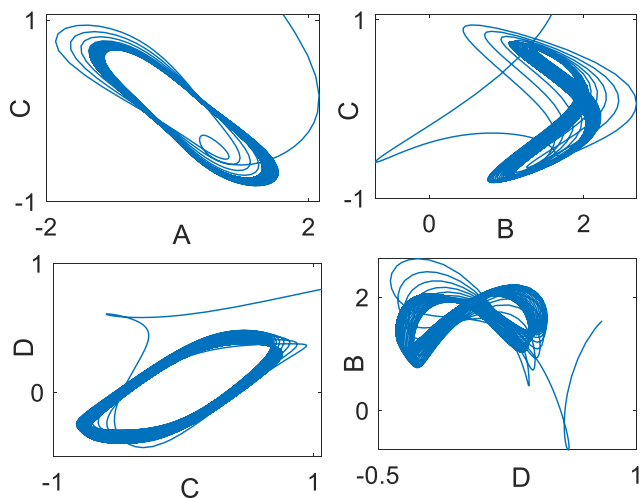


Fig. 15. Phase diagrams with initial values $(1.61, 1.58, 1.06, 0.80)$.

To analyze how changes in the fractional order affect the system's trajectory, the system's parameters and initial values were held unchanged, with only the fractional order

adjusted to $q_1 = 0.75$, $q_2 = 0.80$, $q_3 = 0.85$, $q_4 = 0.90$, the corresponding attractors and phase portraits are illustrated in Figs. 16–17.

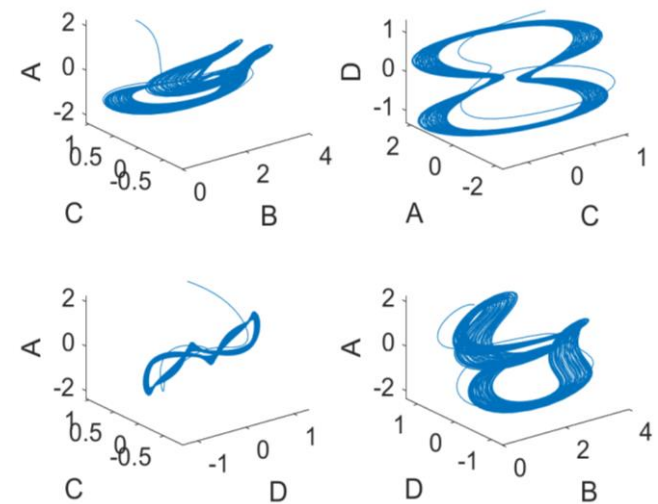


Fig. 16. 3D attractors with fractional orders $q_1 = 0.75, q_2 = 0.80, q_3 = 0.85, q_4 = 0.90$.

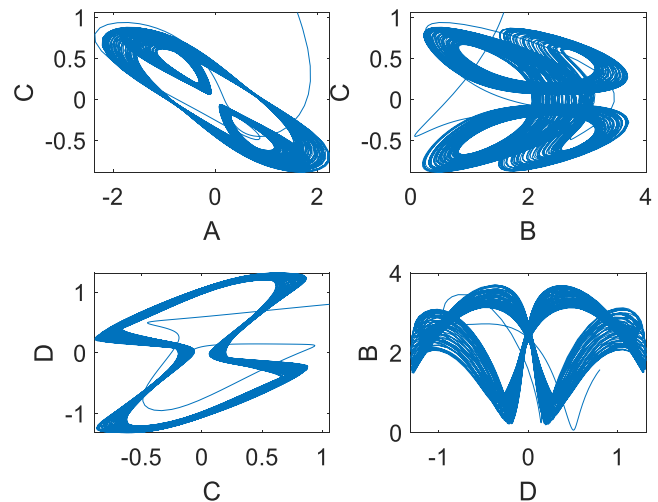


Fig. 17. Phase diagrams with fractional orders $q_1 = 0.75, q_2 = 0.80, q_3 = 0.85, q_4 = 0.90$.

As evident from the aforementioned figures, the system's trajectory has undergone substantial changes—this observation indicates that modifications to initial conditions exert a significant impact on the economic system. Within an economic framework, external shocks such as technological breakthroughs or policy pilots can induce notable deviations in the economy's future growth trajectory. In other words, economic growth exhibits a "path-dependent" pattern, wherein the evolution of the economic system is strongly constrained by its initial development conditions and historical evolutionary processes.

D. Power Spectrum Diagrams

The morphology of the power spectrum can reflect the state of a chaotic system (e.g., the transition from order to chaos, changes in chaos intensity). By analyzing the frequency distribution characteristics of the power spectrum, one can distinguish between chaotic, stochastic, and periodic systems.

In Fig. 18, the continuous spectrum of the power spectrum has the widest range, with a high proportion of energy in high-frequency components, indicating that the system

exhibits high chaos intensity and is in a hyperchaotic state. The continuous spectrum in Fig. 19 is relatively narrow, corresponding to a chaotic state of the system. In Fig. 20, the fluctuations of state variables form a fixed period, suggesting that the system eventually reaches a stable periodic state. In Fig. 21, the state variables gradually converge to a stable fixed point.

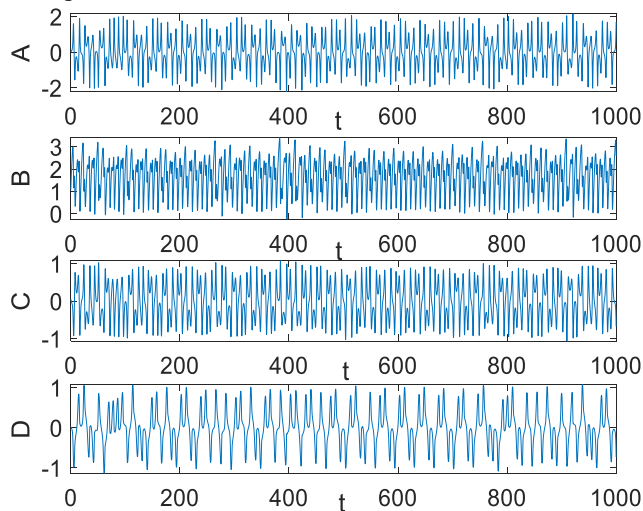


Fig. 18. Power spectrum diagrams with $m=0.1$.

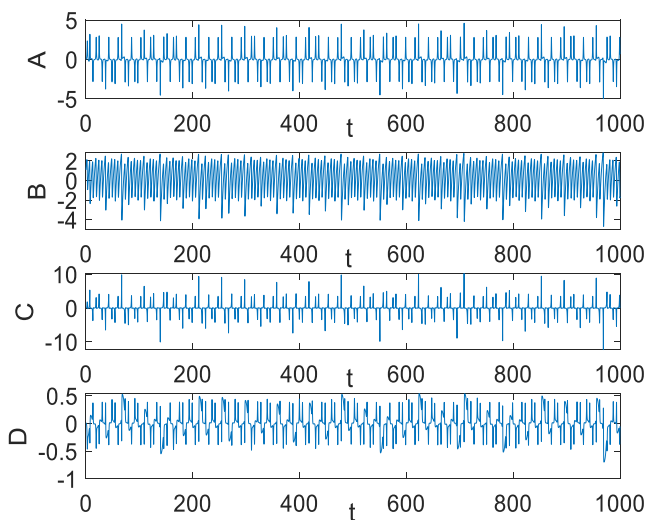


Fig. 19. Power spectrum diagrams with $m=1.5$.

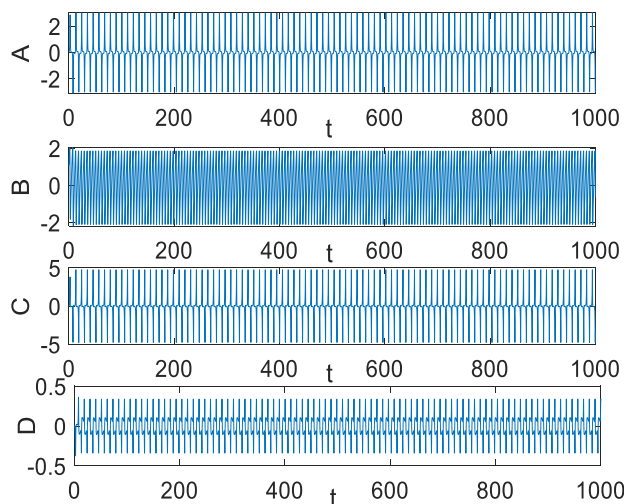


Fig. 20. Power spectrum diagrams with $m=2.5$.

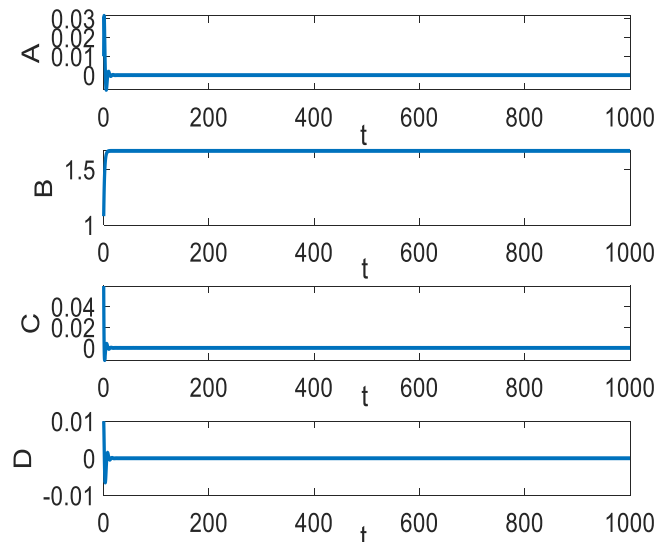


Fig. 21. Power spectrum diagrams with $m=3.5$.

When the initial values are changed, the power spectrum in Fig. 22 also undergoes significant changes: chaotic characteristics are present in the early stage and eventually reaches a stable periodic state. Fig. 23 presents the power spectrum after adjusting the fractional order. The figure shows that the chaotic intensity of the system is reduced.

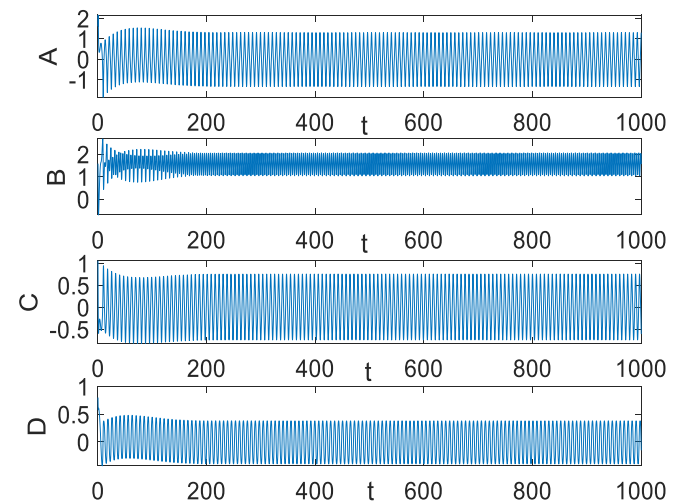


Fig. 22. Power spectrum diagrams with initial values (1.61,1.58,1.06,0.80).

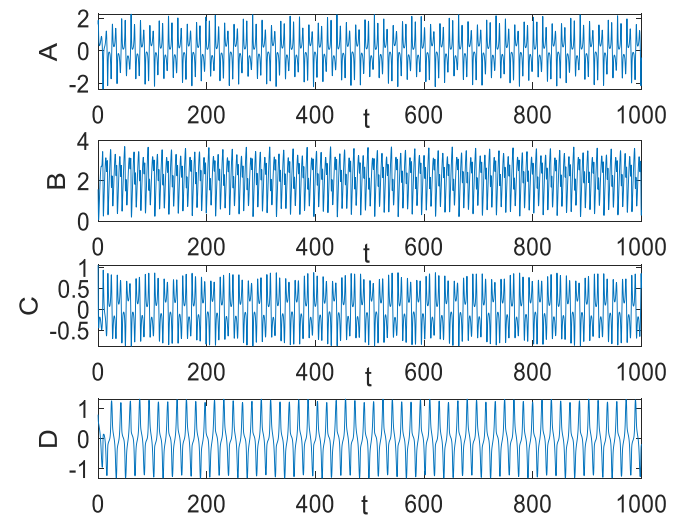


Fig. 23. Power spectrum diagrams with fractional orders $q_1 = 0.75, q_2 = 0.80, q_3 = 0.85, q_4 = 0.90$.

E. Equilibrium and Stability

By solving the following equation:

$$\begin{cases} 0 = -oA + AB + C + D, \\ 0 = -A^2 - pB + 1, \\ 0 = -A - rC + mBC, \\ 0 = -sAB - tD. \end{cases} \quad (8)$$

There are three points of equilibrium in the system:

$$Y_1 = (0, 1/p, 0, 0),$$

$$Y_{2,3} = (\pm\zeta, \frac{1-\zeta^2}{p}, \pm\frac{\zeta(o pt + (t-s)(\zeta^2-1))}{pt}, \pm\frac{s\zeta(\zeta^2-1)}{pt}),$$

where ζ satisfies:

$$((t-s)(\zeta^2-1) + opt)/pt = -p/(m(\zeta^2-1) + pr).$$

To find the specific equilibrium point value, we present the Jacobian matrix of the system at $Y_j = (\phi_1, \phi_2, \phi_3, \phi_4)$.

$$J = \begin{pmatrix} \phi_2 - o & \phi_1 & 1 & 1 \\ -2\phi_1 & -p & 0 & 0 \\ -1 & m\phi_3 & m\phi_2 - r & 0 \\ -s\phi_2 & -s\phi_1 & 0 & -t \end{pmatrix}. \quad (9)$$

We will study the equilibrium point stability of the system in a hyperchaotic state. When the parameters of the system satisfy $m = 0.1$, $o = 1.1$, $p = 0.2$, $r = 1.85$, $s = 0.17$, $t = 0.2$, the eigenvalues of Jacobi matrix of the equilibrium point Y_1 are -0.8179 , -0.2000 , 3.4975 , 0.2004 , which is unstable by Lyapunov's theory. The eigenvalues of $Y_{2,3}$ are -8.8951 , -1.7880 , $0.1892 - 1.0252i$, $0.1892 + 1.0252i$, and $Y_{2,3}$ are also instabilities.

With the advancement of the global economy, new challenges have emerged. Firstly, during the transition between old and new growth drivers, the stickiness of labor costs and the time lag in technological innovation have triggered dynamic imbalances between production and inventory. Secondly, the interaction between the Federal Reserve's quantitative tightening measures and the internationalization of the RMB has caused irregular fluctuations in capital liquidity, thereby exacerbating the chaotic nature of monetary liquidity. Third, the restructuring of the global supply chain has historically reversed the correlation coefficient between the price index and investment demand—shifting it from a positive to a negative correlation.

Thus, through the dynamic analysis above, the system we have constructed can effectively fit the current economic landscape and supplement chaotic economic theory.

IV. CONTROL ANALYSIS OF HYPERCHAOTIC SYSTEM (6)

In the present section, we propose a dynamic feedback gain design, so that the system can automatically converge to the appropriate balance according to its initial state. The advantage of this method is that it is independent of the initial value of the system and can quickly converge to the equilibrium point.

Theorem 1: Consider the following controlled fractional-order system:

$$D_t^\alpha X = f(x) + Bu. \quad (10)$$

Where $f(x): \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Lipschitz-continuous vector

field, $B \in \mathbb{R}^{n \times m}$ is constant gain matrix, $u = Kx$ is a dynamic gain feedback controller with $K = k(t)B^T$, and the feedback gain $k(t)$ evolves according to the adaptive law:

$$D_t^\alpha k(t) = -\hbar \|x\|^2, \hbar > 0. \quad (11)$$

If a matrix B can be found, and $f(x)$ satisfies the dissipative condition:

$$x^T f(x) \leq -\lambda \|x\|^2, \lambda > 0. \quad (12)$$

Then, the closed-loop system achieves asymptotic stability.

Proof. A Lyapunov function candidate is considered herein, which is:

$$V = \frac{1}{2} x^T x + \frac{1}{2\hbar} (k(t) - k^*)^2, \quad (13)$$

where k^* is the ideal steady-state gain.

Using the properties of the Caputo fractional derivative:

$$D_t^\alpha V \leq x^T D_t^\alpha x + \frac{1}{\hbar} (k(t) - k^*) D_t^\alpha k(t). \quad (14)$$

Substituting dynamics:

$$D_t^\alpha X = f(x) + Bk(t)B^T x, \quad (15)$$

and adaptive law:

$$D_t^\alpha k(t) = -\hbar \|x\|^2. \quad (16)$$

Get:

$$D_t^\alpha V \leq x^T f(x) + k(t)x^T B B^T x - (k(t) - k^*) \|x\|^2. \quad (17)$$

Under the dissipative condition:

$$x^T f(x) \leq -\lambda \|x\|^2, \quad (18)$$

$$D_t^\alpha V \leq -\lambda \|x\|^2 + k(t) \|B^T x\|^2 - (k(t) - k^*) \|x\|^2. \quad (19)$$

Selecting B such that $\|B^T x\|^2 \leq \ell \|x\|^2$ ($\ell > 0$) and $k^* > \lambda$, we obtain:

$$D_t^\alpha V \leq -(\lambda - k^* + \ell k(t)) \|x\|^2 \leq -\alpha V, \alpha > 0. \quad (20)$$

By the fractional Lyapunov stability criterion, the system is asymptotically stabilized.

Numerical Verification: The controlled system is defined as follows:

$$\begin{cases} D_t^{\alpha_1} A(t) = -oA + AB + C + D - \kappa_1(A - \phi_1), \\ D_t^{\alpha_2} B(t) = -A^2 - pB + 1 - \kappa_2(B - \phi_2), \\ D_t^{\alpha_3} C(t) = -A - rC + mBC - \kappa_3(C - \phi_3), \\ D_t^{\alpha_4} D(t) = -sAB - tD - \kappa_4(D - \phi_4). \end{cases} \quad (21)$$

Where $\kappa_1, \kappa_2, \kappa_3, \kappa_4$, represents the feedback control coefficient of the controlled system (21) and indicates the control strength, and $(\phi_1, \phi_2, \phi_3, \phi_4)$ represents an arbitrary set of equilibrium points of the system. The system can be stabilized at an arbitrary equilibrium point or period by appropriately selecting the feedback factor.

The eigenvalues are obtained by solving the Jacobian matrix of the controlled system (21):

$$J = \begin{pmatrix} B - o - \kappa_1 & A & 1 & 1 \\ -2A & -p - \kappa_2 & 0 & 0 \\ -1 & mC & mB - r - \kappa_3 & 0 \\ -sB & -sA & 0 & -t - \kappa_4 \end{pmatrix}. \quad (22)$$

In order to find its characteristic equation, let $|J - \lambda I| = 0$, and get a quadratic equation as:

$$\lambda^4 + \xi_3 \lambda^3 + \xi_2 \lambda^2 + \xi_1 \lambda + \xi_0 = 0. \quad (23)$$

When parameters $o = 1.1, p = 0.2, r = 1.85, m = 0.1, s = 0.17, t = 0.2$, assuming the positive feedback gain coefficient $(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (3.5, 1.2, 1.2, 1.2)$.

Regarding the equilibrium Y_1 , the characteristic equation takes the form:

$$\lambda^4 + 4.4000\lambda^3 + 7.5775\lambda^2 + 7.1810\lambda + 3.4335 = 0. \quad (24)$$

The eigenvalues are $-1.9022, -1.4000, -0.5489 + 0.9940i, -0.5489 - 0.9940i$, the eigenvalues all have negative real parts.

For equilibrium $Y_{2,3}$, the characteristic equation takes the form:

$$\lambda^4 + 16.2683\lambda^3 + 76.5644\lambda^2 + 134.2870\lambda + 76.6307 = 0. \quad (25)$$

The eigenvalues are $-9.7402, -1.1793, -1.9807, -3.3681$, and the eigenvalues are all negative.

Due to all eigenvalues of the Jacobi matrix being either negative or having negative real parts, the system exhibits asymptotic stability in the vicinity of this equilibrium point. To provide a more intuitive demonstration of the designed controller's effectiveness, Figs. 24 and 25 illustrate the response time curves of the controlled system described by system (21).

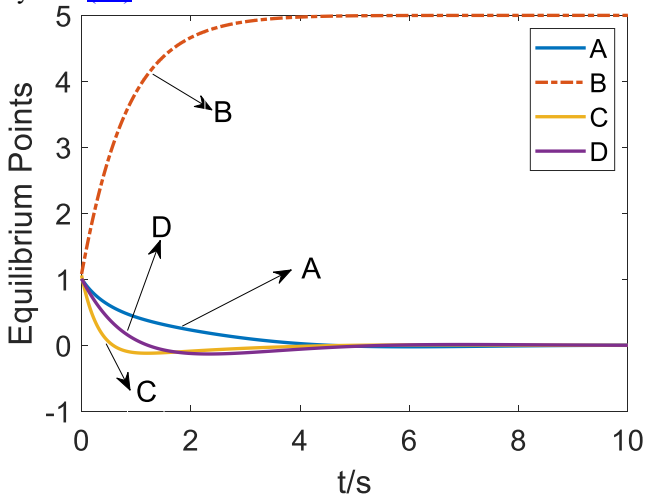


Fig. 24. The curve of response time at equilibrium Y_1 .

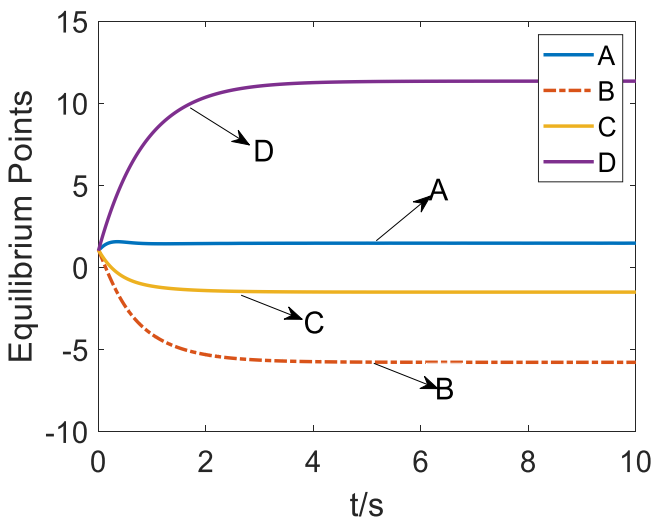


Fig. 25. The curve of response time at equilibrium $Y_{2,3}$.

It is well known that chaotic systems operate in an intricate and variable manner. In financial markets, stabilizing or

controlling the system by identifying and exploiting equilibrium points is critical to maintaining market stability and effectively managing risk, which helps to mitigate market volatility and uncertainty. An in-depth analysis of these equilibrium points enables investors and policy makers to more accurately predict market trends and demand movements, and thus adjust or optimize their investment strategies to achieve their stated objectives. In addition, since the equilibrium points in a financial chaotic system are not unique, and with the change of initial conditions or related parameters, the equilibrium points will change accordingly. Therefore, in practice, we must closely integrate with the current market environment and data characteristics to conduct more detailed and in-depth research and analysis.

V. SYNCHRONIZATION ANALYSIS OF THE HYPERCHAOTIC SYSTEM (6)

Taking system (6) as the driving system, the response system is specified as follows:

$$\begin{cases} D^{q_1} A_1(t) = -o_1 A_1 + A_1 B_1 + C_1 + D_1 + v_1, \\ D^{q_2} B_1(t) = -A_1^2 - p_1 B_1 + 1 + v_2, \\ D^{q_3} C_1(t) = -A_1 - r_1 C_1 + m_1 B_1 C_1 + v_3, \\ D^{q_4} D_1(t) = -s_1 A_1 B_1 - t_1 D_1 + v_4. \end{cases} \quad (26)$$

Where, $o_1, p_1, r_1, m_1, s_1, t_1$ is the estimated value of unknown parameters o, p, r, m, s, t in the drive system (6), v_1, v_2, v_3, v_4 is the adaptive controller to be designed.

The parameter estimation error is:

$$\tilde{o} = o_1 - o, \tilde{p} = p_1 - p, \tilde{r} = r_1 - r, \tilde{s} = s_1 - s, \tilde{m} = m_1 - m, \tilde{t} = t_1 - t. \quad (27)$$

And the synchronization error is:

$$\begin{cases} e_1(t) = A_1(t) - A(t), \\ e_2(t) = B_1(t) - B(t), \\ e_3(t) = C_1(t) - C(t), \\ e_4(t) = D_1(t) - D(t). \end{cases} \quad (28)$$

So, the error system is:

$$\begin{cases} D^{q_1} e_1(t) = -\tilde{o} A_1 - o e_1 + e_3 + e_4 + A_1 B_1 - A B + v_1, \\ D^{q_2} e_2(t) = -\tilde{p} B_1 - p e_2 - (A_1^2 - A^2) + v_2, \\ D^{q_3} e_3(t) = -e_1 - \tilde{r} C_1 - r e_3 + \tilde{m} B_1 C_1 + m(B_1 C_1 - B C) + v_3, \\ D^{q_4} e_4(t) = -\tilde{t} D_1 - t e_4 - \tilde{s} A_1 B_1 - s(A_1 B_1 - A B) + v_4. \end{cases} \quad (29)$$

Theorem 2. When the adaptive control rate is:

$$\begin{cases} v_1 = A B - A_1 B_1 - e_3 - e_4 - \kappa_1 e_1, \\ v_2 = (A_1^2 - A^2) - \kappa_2 e_2, \\ v_3 = e_1 - m(B_1 C_1 - B C) - \kappa_3 e_3, \\ v_4 = s(A_1 B_1 - A B) - \kappa_4 e_4. \end{cases} \quad (30)$$

The derivation of parameter estimation error is:

$$\begin{cases} D^q \tilde{o} = A_1 e_1, \\ D^q \tilde{p} = B_1 e_2, \\ D^q \tilde{r} = C_1 e_3, \\ D^q \tilde{m} = -B_1 C_1 e_3, \\ D^q \tilde{s} = A_1 B_1 e_4, \\ D^q \tilde{t} = D_1 e_4. \end{cases} \quad (31)$$

Then, drive system (6) and the response system (26) reach a state of synchronization.

Proof. Substitute (31) into (30), then, the error system is:

$$\begin{cases} D^q e_1(t) = -\tilde{o} A_1 - o e_1 - \kappa_1 e_1, \\ D^q e_2(t) = -\tilde{p} B_1 - p e_2 - \kappa_2 e_2, \\ D^q e_3(t) = -\tilde{r} C_1 - r e_3 + \tilde{m} B_1 C_1 - \kappa_3 e_3, \\ D^q e_4(t) = -\tilde{t} D_1 - t e_4 - \tilde{s} A_1 B_1 - \kappa_4 e_4. \end{cases} \quad (32)$$

Take the Lyapunov function as follow:

$$V = \frac{e_1^2 + e_2^2 + e_3^2 + e_4^2 + \tilde{o}^2 + \tilde{p}^2 + \tilde{r}^2 + \tilde{m}^2 + \tilde{s}^2 + \tilde{t}^2}{2}. \quad (33)$$

Take the derivative of function (33):

$$\begin{aligned} D_t^q V &= e_1 D^q e_1 + e_2 D^q e_2 + e_3 D^q e_3 + e_4 D^q e_4 \\ &\quad + \tilde{o} D^q \tilde{o} + \tilde{p} D^q \tilde{p} + \tilde{r} D^q \tilde{r} + \tilde{m} D^q \tilde{m} \\ &\quad + \tilde{s} D^q \tilde{s} + \tilde{t} D^q \tilde{t}. \end{aligned} \quad (34)$$

Substitute (32), (33) into (34), then get the follow:

$$\begin{aligned} D_t^q V &= e_1 (-\tilde{o} A_1 - o e_1 - \kappa_1 e_1) + e_2 (-\tilde{p} B_1 - p e_2 - \kappa_2 e_2) \\ &\quad + e_3 (-\tilde{r} C_1 + \tilde{m} B_1 C_1 - r e_3 - \kappa_3 e_3) + e_4 (-\tilde{t} D_1 \\ &\quad - \tilde{s} A_1 B_1 - t e_4 - \kappa_4 e_4) + \tilde{o} A_1 e_1 + \tilde{p} B_1 e_2 \\ &\quad + \tilde{r} C_1 e_3 - \tilde{m} B_1 C_1 e_3 + \tilde{s} A_1 B_1 e_4 + \tilde{t} D_1 e_4 \\ &= -o e_1^2 - p e_2^2 - r e_3^2 - t e_4^2 - \kappa_1 e_1^2 \\ &\quad - \kappa_2 e_2^2 - \kappa_3 e_3^2 - \kappa_4 e_4^2 \\ &\leq 0. \end{aligned} \quad (35)$$

Drawing on Lyapunov's stability criterion, the error system (29) demonstrates asymptotic stability characteristics. Consequently, the drive system (6) and the response system (26) achieve synchronization.

Numerical Simulation: When system's parameters $o = 1.1$, $p = 0.2$, $r = 1.85$, $m = 0.1$, $s = 0.17$, $t = 0.2$, and the fractional order derivative is assigned a specific value of $q = 0.995$, the initial values of the error system is $(2, 2, 2, 2)$, and the adaptive control factor is $\kappa_i = 1.5$, $i = 1, 2, 3, 4$. The time response curve of error system (29) is given as Fig. 26, and the time response curve of predicted parameters are shown in Fig. 27

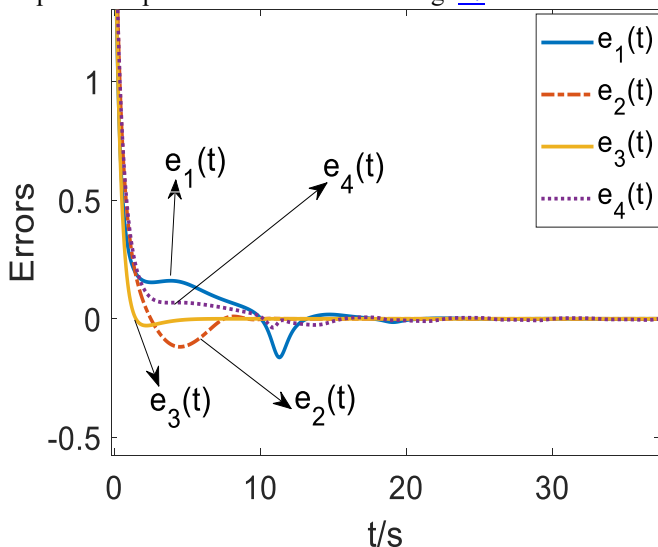


Fig. 26. The time response curve of the error system (6).

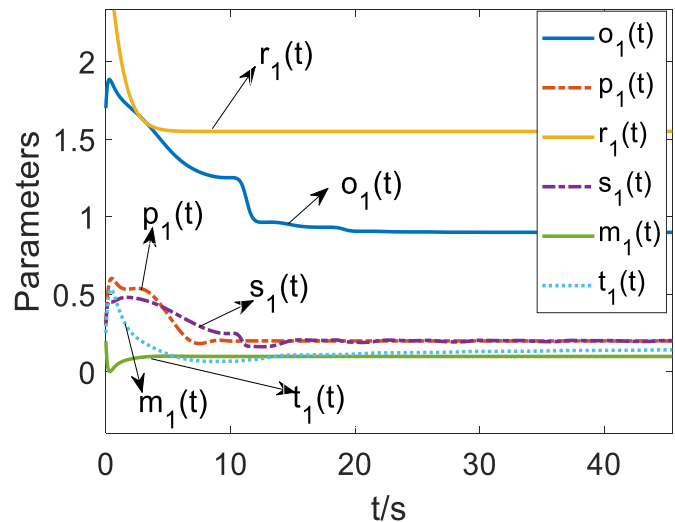


Fig. 27. The time response curve of the predicted parameters.

Against the backdrop of surging complexity and escalating uncertainties in global financial markets, the research on adaptive synchronization of hyperchaotic financial systems with four-dimensional fractional-order under unknown parameter conditions offers innovative solutions for financial risk prediction, information security, and macroeconomic policy regulation. By bridging fractional-order dynamics with adaptive control, the research paves the way for data-driven, intelligent solutions to address emerging challenges in complex financial ecosystems.

VI. CONCLUSIONS

In light of the current intricate and evolving international and domestic economic environment, as well as the frequent emergence of market risk potential, the present paper develops a fractional-order hyperchaotic financial system with four dimensions, which includes new parameters and a quadratic term standing for external perturbations. In this paper, the selection of the new parameters is achieved through Lyapunov exponential spectrum analysis. Moreover, the dynamics analysis of chaotic attractor, phase diagram, influence of initial value, and equilibrium point is presented, providing a theoretical foundation to support studies on controlling and synchronizing the hyper chaotic system. Subsequently, the dynamic feedback gain controller designed in this paper can rapidly facilitate the transition from a chaotic state to a stable equilibrium point. Regarding chaotic synchronization, the adaptive parameter synchronization method proposed herein can not only assess the synchronization performance of the newly constructed chaotic financial system but also estimate the parameter estimates. In the context of the practical application of financial systems, it can offer a theoretical and scientific basis for the formulation of financial market control and synchronization strategies in the current global setting. Additionally, all the conclusions of this paper not only provide theoretical data and proofs, but also give furnish intuitive results by numerical simulation software. This study focuses primarily on theoretical analysis and numerical simulations. Considering the increasing complexity of current financial markets, future research could further promote the translation of these findings into practical applications in the financial industry.

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