Reduced Second Zagreb Index and Bounds of Graph Products

K.Rengalakshmi, and S.Pethanachi Selvam

Abstract — **In graph theory, binary operations are used to construct new graphs from existing ones, enabling the study of complex structures. Topological indices are numerical descriptors that capture structural features of graphs and have wide applications in chemistry and network analysis. The reduced second Zagreb index is a recently developed degree-based topological index, defined over the edges of a graph using vertex degrees. In this paper, we derive closed-form expressions for the reduced second Zagreb (R_2M) index of the strong, semi-strong, vertex corona, and vertex-edge corona products of two simply connected graphs. We further provide exact upper and lower limits for these indices along with the necessary conditions for equality. The derived results contribute to a deeper mathematical understanding of graph products and may support applications in chemical and communication network modelling.**

***Index Terms*—Strong product, Semi-strong product, vertex corona product, vertex-edge corona product, Reduced second Zagreb index.**

I. INTRODUCTION

In graph theory, a graph G is typically defined as an ordered pair (V, E), where V is the set of vertices and E is the set of edges. In chemical graph theory, graphs serve as models for molecular structures, with vertices representing atoms and edges representing chemical bonds. Binary operations on graphs are used to construct new graphs from existing ones, enabling the modelling of more complex systems and supporting a range of theoretical and applied studies.

Topological indices are quantitative descriptors that contain structural data of chemical compounds and have been used extensively to predict their physicochemical properties. Among them, degree-based indices play a special role. Recent work in mathematics as well as chemistry underscores their profound relation with molecular features, making them important in quantitative structure–property relationship (QSPR) research.

The Zagreb indices, first presented by Gutman and Trinajstić, are leading degree-based topological indices with well-documented uses in boiling point prediction, molecular stability, and toxicity prediction [1], [2]. In order to enhance sensitivity and capture finer structural features, the reduced Zagreb indices were later introduced [3] and has been increasingly utilized in certain chemical modelling contexts. Contemporary research has focused on broadening their applicability and theoretical grounding. For instance, new bounds for variable Zagreb-type and inverse sum-degree indices were established, including extremal graph characterizations and practical applications in modelling molecular structures [4].

Graph products [5], [6] such as strong, semi-strong, and corona products, are powerful tools for modelling, building, and simplifying complicated systems. For example, the strong product of two path graphs gives rise to a mesh-like structure, which is universally found in wireless communication networks. Recent studies have explored graph products through resistance-based indices [7], metric dimensions [8], spectral analysis of wreath products [9], and the stability of Cayley and circulant graphs using algebraic techniques [10]. These developments not only reinforce the enduring relevance of graph products but also showcase their evolving role in addressing complex problems in modern graph theory.

References [11], [12], and [13] are hereby recommended for complete studies on Zagreb and hyper Zagreb indices of graph products, whereas more information on the reduced second Zagreb index can be obtained from [14] and [15].

Inspired by the widespread uses of graph products and the importance of degree-based topological indices, this research seeks to calculate the reduced second Zagreb index of the strong, semi-strong, vertex corona, and vertex-edge corona products of simply connected graphs. In addition, we derive upper and lower bounds for these indices and explore their relevance in real-world contexts. Throughout this work, all graphs considered are assumed to be simple and connected.

II. PRELIMINARIES

In this section, we introduce the fundamental definitions and notations necessary for the subsequent results.

Definition II.1. [16]: Let $Q_1 = (V(Q_1), E(Q_1))$ be a simple connected graph. The degree of a vertex r in G, denoted by $d_{Q_1}(r)$, is the number of edges incident to r .

Definition II.2. [16]: A graph Q_1 is said to be regular if all the vertices in Q_1 are of same degree.

Definition II.3. [2]: The first Zagreb index of a graph Q_1, denoted by $M_1(Q_1)$, is defined as:

$$M_1(Q_1) = \sum_{v_i \in V(Q_1)} d_{v_i}^2$$

Manuscript received May 25, 2025; revised August 29, 2025.

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Definition II.4. [2]: The second Zagreb index of a graph Q_1, denoted by $M_2(Q_1)$, is given by:

$$M_2(Q_1) = \sum_{v_i v_j \in E(Q_1)} d_{v_i} d_{v_j}$$

Definition II.5. [3]: The reduced second Zagreb index of a graph Q_1, denoted by $R_2M(Q_1)$, is defined as:

$$R_2M(Q_1) = \sum_{v_i v_j \in E(Q_1)} \left(d_{v_i} - 1\right)\left(d_{v_j} - 1\right)$$

Definition II.6. [5]: The Strong product of Q_1 and Q_2 denoted by $Q_1 \circledast Q_2$ is defined on the vertex set $V(Q_1) \times V(Q_2)$. (r, f) and (s, g) in $V(Q_1 \circledast Q_2)$ are adjacent iff $r = s$ and $(f, g) \in E(Q_2)$ or $f = g$ and $(r, s) \in E(Q_1)$ or $(f, g) \in E(Q_2)$ and $(r, s) \in E(Q_1)$. Degree of $(r, f) \in V(Q_1 \circledast Q_2)$ can be written as $d_{Q_1}(r) + d_{Q_2}(f) + d_{Q_1}(r)d_{Q_2}(f)$. Strong product of a path on three vertices (P_3) and a path on two vertices(P_2) is graphically depicted in figure 1.

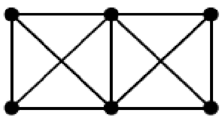

Fig 1. Strong Product of P_3 and P_2.

Definition II.7. [17]: The Semi-strong product $Q_1 \ominus Q_2$ is defined on $V(Q_1) \times V(Q_2)$ where (r, f) and (s, g) are adjacent iff $f = g$ and $(r, s) \in E(Q_1)$ or $(f, g) \in E(Q_2)$ and $(r, s) \in E(Q_1)$. Degree of $(r, f) \in V(Q_1 \ominus Q_2)$ is given by $d_{Q_1}(r) + d_{Q_1}(r)d_{Q_2}(f)$. Semi-Strong product of P_3 and P_2 is illustrated in figure 2.

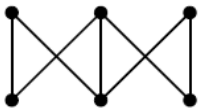

Fig 2. Semi-Strong product of P_3 and P_2.

Definition II.8. [18],[19]: Vertex corona product $Q_1 \odot Q_2$ is constructed by taking one copy of Q_1 and $|V(Q_1)|$ copies of Q_2. Each vertex $r \in V(Q_1)$ is connected to every vertex in a distinct copy of Q_2. Degree of $r \in Q_1 \odot Q_2$ is

$$d_{Q_1 \odot Q_2}(r) = \begin{cases} d_{Q_1}(r) + |V(Q_2)| & if\ r \in V(Q_1) \\ d_{Q_2}(r) + 1 & if\ r \in V(Q_2) \end{cases}.$$

Vertex corona product of P_3 and P_2 is depicted in figure 3.

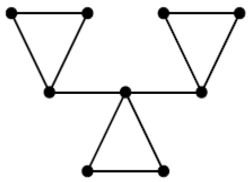

Fig 3. Vertex corona product of P_3 and P_2.

Definition II.9. [6]: Vertex-edge corona product of Q_1 and Q_2 denoted by $Q_1 \bullet Q_2$ is the graph obtained by one copy of Q_1, $|V(Q_1)|$ and $|E(Q_1)|$ copies of Q_2, then joining the i^{th} vertex of Q_1 to every vertex in the i^{th} copy of Q_2 and also joining the end vertices of j^{th} edge of Q_1 to every vertex in the j^{th} edge copy of Q_2, where $1 \le i \le |V(Q_1)|$ and $1 \le j \le |E(Q_1)|$.

Degree of $r \in Q_1 \odot Q_2$ is

$$d_{Q_1 \bullet Q_2}(r) = \begin{cases} (|V(Q_2)| + 1)d_{Q_1}(r) + |V(Q_2)| & \forall\, r \in V(Q_1) \\ d_{Q_2}(r_{ij}) + 2 & \forall\, r_{ij} \in V_{i_e}(Q_2) \\ d_{Q_2}(s_{ij}) + 1 & \forall\, s_{ij} \in V_{i_v}(Q_2) \end{cases}$$

The vertex-edge corona product of P_3 and P_2 is illustrated in figure 4.

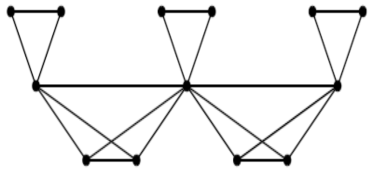

Fig 4. Vertex-edge corona product of P_3 and P_2.

III. Reduced Second Zagreb Index for Graph Products

In this section, we derive general expressions for the R_2M index corresponding to the strong, semi-strong, vertex corona, and vertex-edge corona graph products. Let $Q_1 = \left(V(Q_1), E(Q_1)\right)$ and $Q_2 = \left(V(Q_2), E(Q_2)\right)$ be two given graphs, where $|V(Q_1)| = V_1$, $|E(Q_1)| = E_1$, $|V(Q_2)| = V_2$ and $|E(Q_2)| = E_2$. The degree of a vertex $r \in V(Q_1)$ and $f \in V(Q_2)$ are denoted by $d_{Q_1}(r)$ and $d_{Q_2}(f)$ respectively.

Additionally, we compute the R_2M index for particular instances of these graph products, focusing on standard graphs like paths(P_n) and cycles(C_n) with n vertices.

Theorem III.1.
$R_2M(Q_1 \circledast Q_2) = M_2(Q_1)(2M_2(Q_2) + 3M_1(Q_2) + 6E_2 + V_2) + M_2(Q_2)(3M_1(Q_1) + 6E_1 + V_1) + M_1(Q_1)(M_1(Q_2) - E_2 - V_2) + M_1(Q_2)(M_1(Q_1) - E_1 - V_1) - 6E_1E_2 + E_2V_1 + E_1V_2$

Proof

$$R_2M(Q_1 \circledast Q_2) = \sum_{\substack{rs \in E(Q_1) \\ bc \in E(Q_2)}} \sum_{\substack{r=s \\ fg \in E(Q_2)}} \sum_{\substack{f=g \\ rs \in E(Q_1)}} \left(d_{Q_1 \circledast Q_2}(r, f) - 1\right)\left(d_{Q_1 \circledast Q_2}(s, g) - 1\right)$$

$$= \sum_{\substack{rs \in E(Q_1) \\ fg \in E(Q_2)}} \left(d_{Q_1 \circledast Q_2}(r, f) - 1\right)\left(d_{Q_1 \circledast Q_2}(s, g) - 1\right) + \sum_{\substack{r=s \\ fg \in E(Q_2)}} \left(d_{Q_1 \circledast Q_2}(r, f) - 1\right)\left(d_{Q_1 \circledast Q_2}(r, g) - 1\right) + \sum_{\substack{f=g \\ rs \in E(Q_1)}} \left(d_{Q_1 \circledast Q_2}(r, f) - 1\right)\left(d_{Q_1 \circledast Q_2}(s, g) - 1\right)$$

$$= A_1 + B_1 + C_1$$

$$A_1 = \sum_{\substack{rs \in E(Q_1) \\ fg \in E(Q_2)}} \left(d_{Q_1 \circledast Q_2}(r, f) - 1\right)\left(d_{Q_1 \circledast Q_2}(s, g) - 1\right)$$

$= \sum_{\substack{rs\in E(Q_1)\\ fg\in E(Q_2)}} \big(d_{Q_1}(r) + d_{Q_2}(f) + d_{Q_1}(r)d_{Q_2}(f) - 1\big)\big(d_{Q_1}(s) + d_{Q_2}(g) + d_{Q_1}(s)d_{Q_2}(g) - 1\big)$

$= 2\sum_{rs\in E(Q_1)}\sum_{fg\in E(Q_2)} \big(d_{Q_1}(r) + d_{Q_2}(f) + d_{Q_1}(r)d_{Q_2}(f) - 1\big)\big(d_{Q_1}(s) + d_{Q_2}(g) + d_{Q_1}(s)d_{Q_2}(g) - 1\big)$

$= 2\big(M_2(Q_1)\big(E_2 + M_2(Q_2) + M_1(Q_2)\big) + M_2(Q_2)\big(E_1 + M_1(Q_1)\big) - E_2M_1(Q_1) - E_1M_1(Q_2) + E_1E_2\big)$

Now,

$$B_1 = \sum_{\substack{r=s\\ fg\in E(Q_2)}} \big(d_{Q_1\circledast Q_2}(r,f) - 1\big)\big(d_{Q_1\circledast Q_2}(r,g) - 1\big)$$

$= \sum_{r=s}\sum_{fg\in E(Q_2)} \big(d_{Q_1}(r) + d_{Q_2}(f) + d_{Q_1}(r)d_{Q_2}(f) - 1\big)\big(d_{Q_1}(r) + d_{Q_2}(g) + d_{Q_1}(r)d_{Q_2}(g) - 1\big)$

$= M_1(Q_1)\big(E_2 + M_1(Q_2)\big) + M_2(Q_2)\big(V_1 + 4E_1 + M_1(Q_1)\big) - V_1M_1(Q_2) - 4E_1E_2 + E_2V_1$

Similarly, for

$$C_1 = \sum_{\substack{f=g\\ rs\in E(Q_1)}} \big(d_{Q_1\circledast Q_2}(r,f) - 1\big)\big(d_{Q_1\circledast Q_2}(s,g) - 1\big)$$

we get,

$C_1 = M_1(Q_2)\big(E_1 + M_1(Q_1)\big) + M_2(Q_1)\big(V_2 + 4E_2 + M_1(Q_2)\big) - V_2M_1(Q_1) - 4E_1E_2 + E_1V_2$

Adding A_1, B_1, and C_1, the result follows.

Corollary III.1.1.

For $n, m \geq 3$,

(i) $R_2M(P_m \circledast P_n) = 196mn - 339(m+n) + 566$

(ii) $R_2M(P_m \circledast C_n) = 196mn - 290n + 5$

(iii) $R_2M(C_m \circledast C_n) = 196mn$

Proof

It is known that for any path graph P_n and any cycle graph C_n, the number of vertices and edges are $|V(P_n)| = n, |V(C_n)| = n$ and $|E(P_n)| = (n-1), |E(C_n)| = n$ respectively.

Moreover, the first and second Zagreb indices of P_n are $M_1(P_n) = (4n-6)$ and $M_2(P_n) = (4n-8)$

For C_n, the first and second Zagreb indices are $M_1(C_n) = 4n$ and $M_2(C_n) = 4n$

Substituting these values in Theorem III.1 yields the desired result.

Theorem III.2.

$R_2M(Q_1 \ominus Q_2) = M_2(Q_1)\big(3M_1(Q_2) + 6E_2 + V_2 + 2M_2(Q_2)\big) - M_1(Q_1)(M_1(Q_2) + V_2 + 4E_2) + 2E_1E_2 + E_1V_2$

Proof

$R_2M(Q_1 \ominus Q_2) = \sum_{\substack{rs\in E(Q_1)\\ f\in V(Q_2)}} \big(d_{Q_1\ominus Q_2}(r,f) - 1\big)\big(d_{Q_1\ominus Q_2}(s,f) - 1\big) + \sum_{\substack{rs\in E(Q_1)\\ fg\in E(Q_2)}} \big(d_{Q_1\ominus Q_2}(r,f) - 1\big)\big(d_{Q_1\ominus Q_2}(s,g) - 1\big)$

$= A_2 + B_2$

$A_2 = \sum_{\substack{rs\in E(Q_1)\\ f\in V(Q_2)}} \big(d_{Q_1\ominus Q_2}(r,f) - 1\big)\big(d_{Q_1\ominus Q_2}(s,f) - 1\big)$

$= \sum_{\substack{rs\in E(Q_1)\\ f\in V(Q_2)}} \big(d_{Q_1}(r)d_{Q_2}(f) + d_{Q_1}(r) - 1\big)\big(d_{Q_1}(s)d_{Q_2}(f) + d_{Q_1}(s) - 1\big)$

$= M_2(Q_1)(M_1(Q_2) + 4E_2 + V_2) - M_1(Q_1)(V_2 + 2E_2) + V_2E_1$

$B_2 = \sum_{\substack{rs\in E(Q_1)\\ fg\in E(Q_2)}} \big(d_{Q_1\ominus Q_2}(r,f) - 1\big)\big(d_{Q_1\ominus Q_2}(s,g) - 1\big)$

$= \sum_{\substack{rs\in E(Q_1)\\ fg\in E(Q_2)}} \big(d_{Q_1}(r)d_{Q_2}(f) + d_{Q_1}(r) - 1\big)\big(d_{Q_1}(s)d_{Q_2}(g) + d_{Q_1}(s) - 1\big) + \big(d_{Q_1}(r)d_{Q_2}(g) + d_{Q_1}(r) - 1\big)\big(d_{Q_1}(s)d_{Q_2}(f) + d_{Q_1}(s) - 1\big)$

$= M_2(Q_1)\big(2(M_1(Q_2) + M_2(Q_2) + E_2)\big) - M_1(Q_1)(M_1(Q_2) + 2E_2) + 2E_1E_2$

On adding A_2 and B_2, we arrive at the result.

Corollary III.2.1.

For $n, m \geq 3$,

(i) $R_2M(P_m \ominus P_n) = 75mn - 122m - 165n + 262$

(ii) $R_2M(P_m \ominus C_n) = 31mn - 16m^2 - 32m - 29n + 60n^2 + 46$

(iii) $R_2M(C_m \ominus C_n) = 75mn$

Proof

The proof proceeds in the same manner as that of Corollary 3.1.1, with the application of Theorem III.2.

Theorem III.3.

$R_2M(Q_1 \odot Q_2) = M_1(Q_1)(V_2 - 1) + M_2(Q_1) + V_1M_2(Q_2) + E_1{V_2}^2 - 2E_1V_2 + E_1 + 4E_1E_2 + 2V_1V_2E_2 - 2V_1E_2$

Proof

$$R_2M(Q_1 \odot Q_2) = \sum_{rs\in E(Q_1)}\sum_{\substack{r\in V(Q_1)\\ fg\in E(Q_2)}}\sum_{\substack{r\in V(Q_1)\\ f\in V(Q_2)}} \big(d_{Q_1\odot Q_2}(r,f) - 1\big)\big(d_{Q_1\odot Q_2}(s,g) - 1\big)$$

$= \sum_{rs\in E(Q_1)} \big(d_{Q_1}(r) - 1\big)\big(d_{Q_1}(s) - 1\big) + \sum_{\substack{r\in V(Q_1)\\ fg\in E(Q_2)}} \big(d_{Q_2}(f) - 1\big)\big(d_{Q_2}(g) - 1\big) + \sum_{\substack{r\in V(Q_1)\\ f\in V(Q_2)}} \big(d_{Q_1}(r) - 1\big)\big(d_{Q_2}(f) - 1\big)$

$= A_3 + B_3 + C_3$

Consider $A_3 = \sum_{rs\in E(Q_1)} \big(d_{Q_1}(r) - 1\big)\big(d_{Q_1}(s) - 1\big)$

$= \sum_{rs\in E(Q_1)} \big(d_{Q_1}(r) + V_2 - 1\big)\big(d_{Q_1}(s) + V_2 - 1\big)$

$= M_2(Q_1) + V_2M_1(Q_1) - M_1(Q_1) + E_1{V_2}^2 - 2E_1V_2 + E_1$

Likewise, $B_3 = V_1\sum_{fg\in E(Q_2)} \big(d_{Q_2}(f) - 1\big)\big(d_{Q_2}(g) - 1\big)$

$= V_1\sum_{fg\in E(Q_2)} \big(d_{Q_2}(f) + 1 - 1\big)\big(d_{Q_2}(g) + 1 - 1\big)$

$= V_1M_2(Q_2)$

In a similar fashion,

$C_3 = \sum_{\substack{r\in V(Q_1)\\ f\in V(Q_2)}} \big(d_{Q_1}(r) - 1\big)\big(d_{Q_2}(f) - 1\big)$

$= \sum_{\substack{r\in V(Q_1)\\ f\in V(Q_2)}} \big(d_{Q_1}(r) + V_2 - 1\big)\big(d_{Q_2}(f) + 1 - 1\big)$

$= 4E_1E_2 + 2V_1V_2E_2 - 2V_1E_2$

The final expression is derived by summing A_3, B_3 and C_3.

Corollary III.3.1.

For $n, m \geq 3$,

(i) $R_2M(P_m \odot P_n) = 8mn - 11m - 12n + 3mn^2 - n^2 + 1$

(ii) $R_2M(P_m \odot C_n) = 8mn + 3mn^2 - n^2 - 7m - 4n + 5$

(iii) $R_2M(C_m \odot C_n) = 8mn + m + 3mn^2$

Proof
Following the strategy employed in Corollary III.1.1, we obtain the result by utilizing Theorem III.3.

Theorem III.4.
$$R_2M(Q_1 \bullet Q_2) = 2M_1(Q_1)(V_2^2 + V_2E_2 + E_2 + V_2) + V_1M_2(Q_2) + M_2(Q_1)(V_2+1)^2 + E_1\big(M_2(Q_2) + M_1(Q_2)\big) + E_1E_2(1+4V_2) + 2V_2^2 + 4E_2V_2 + 2V_1V_2E_2 - 2V_1E_2 + E_1(1+V_2^2)$$

Proof
$$R_2M(Q_1 \bullet Q_2) = \sum_{rs\in E(Q_1)}\big(d_{Q_1}(r)-1\big)\big(d_{Q_1}(s)-1\big) + E_1\sum_{fg\in E(Q_2)}\big(d_{Q_2}(f)+1\big)\big(d_{Q_2}(g)+1\big) + \sum_{\substack{rs\in E(Q_1)\\ f\in V(Q_2)}}\big(d_{Q_1}(r)+d_{Q_1}(s)-1\big)\big(d_{Q_2}(f)-1\big) + \sum_{\substack{r\in V(Q_1)\\ fg\in E(Q_2)}} d_{Q_2}(f)d_{Q_2}(g) + \sum_{\substack{r\in V(Q_1)\\ f\in V(Q_2)}}\big(d_{Q_1}(r)-1\big)\big(d_{Q_2}(f)-1\big)$$

$$= A_4 + B_4 + C_4 + D_4 + E_4$$

$$A_4 = \sum_{rs\in E(Q_1)}\big(d_{Q_1}(r)-1\big)\big(d_{Q_1}(s)-1\big)$$
$$= \sum_{rs\in E(Q_1)}\Big((V_2+1)d_{Q_1}(r) + V_2 - 1\Big)\Big((V_2+1)d_{Q_1}(s) + V_2 - 1\Big)$$
$$= (V_2+1)^2M_2(Q_1) + M_1(Q_1)(V_2^2-1) + E_1V_2^2 - 2E_1V_2 + E_1$$

$$B_4 = E_1\sum_{fg\in E(Q_2)}\big(d_{Q_2}(f)+1\big)\big(d_{Q_2}(g)+1\big)$$
$$= E_1\big(M_2(Q_2) + M_1(Q_2) + E_2\big)$$

$$C_4 = \sum_{\substack{rs\in E(Q_1)\\ f\in V(Q_2)}}\big(d_{Q_1}(r)+d_{Q_1}(s)-1\big)\big(d_{Q_2}(f)-1\big)$$
$$= \sum_{\substack{rs\in E(Q_1)\\ f\in V(Q_2)}}\Big((V_2+1)d_{Q_1}(r) + V_2 - 1 + (V_2+1)d_{Q_1}(s) + V_2 - 1\Big)\big(d_{Q_2}(f)-1\big)$$
$$= (V_2+1)M_1(Q_1)(2E_2+V_2) - 2E_1(2E_2+V_2)$$

$$D_4 = \sum_{\substack{r\in V(Q_1)\\ fg\in E(Q_2)}} d_{Q_2}(f)d_{Q_2}(g)$$
$$= V_1M_2(Q_2)$$

$$E_4 = \sum_{\substack{r\in V(Q_1)\\ f\in V(Q_2)}}\big(d_{Q_1}(r)-1\big)\big(d_{Q_2}(f)-1\big)$$
$$= \sum_{\substack{r\in V(Q_1)\\ f\in V(Q_2)}}\big((V_2+1)d_{Q_1}(r) + V_2 - 1\big)\big(d_{Q_2}(f)\big)$$
$$= 4E_1E_2V_2 + 4E_1E_2 + 2V_1V_2E_2 - 2E_2V_1$$

Summing up A_4, B_4, C_4, D_4, E_4, the result follows.

Corollary III.4.1.
For $n, m \geq 3$,
(i) $R_2M(P_m \bullet P_n) = -31n^2 - 35n - 24m + 21mn + 27mn^2 + 18$
(ii) $R_2M(P_m \bullet C_n) = -27n^2 + 36mn + 5m + 27mn^2 - 49n - 9$
(iii) $R_2M(C_m \bullet C_n) = 6n^2 + 21mn^2 + 31mn + 5m$

Proof
The result is obtained by applying Theorem III.4 through the same technique as in Corollary III.1.1.

We next focus on the determination of some upper and lower bounds for the R_2M index of these graph products.

IV. BOUNDS FOR THE REDUCED SECOND ZAGREB INDEX

In this section, we establish upper and lower bounds for the R_2M index of the strong, semi-strong, vertex corona and vertex-edge corona products of Q_1 and Q_2. The maximum and minimum degrees of Q_1 and Q_2 are denoted by Δ_{Q_1}, Δ_{Q_2} and δ_{Q_1}, δ_{Q_2} respectively.

Theorem IV.1.
$$\big(\delta_{Q_1} + \delta_{Q_2} + \delta_{Q_1}\delta_{Q_2} - 1\big)^2(E_1E_2 + V_1E_2 + E_1V_2) \leq R_2M(Q_1 \circledast Q_2) \leq \big(\Delta_{Q_1} + \Delta_{Q_2} + \Delta_{Q_1}\Delta_{Q_2} - 1\big)^2(E_1E_2 + V_1E_2 + E_1V_2)$$
The bounds are tight if and only if Q_1 and Q_2 are regular.

Proof
$$R_2M(Q_1 \circledast Q_2) = \sum_{\substack{rs\in E(Q_1)\\ fg\in E(Q_2)}}\sum_{\substack{r=s\\ fg\in E(Q_2)}}\sum_{\substack{f=g\\ rs\in E(Q_1)}}\big(d_{Q_1\circledast Q_2}(r,f) - 1\big)\big(d_{Q_1\circledast Q_2}(s,g) - 1\big)$$
$$= \sum_{\substack{rs\in E(Q_1)\\ fg\in E(Q_2)}}\sum_{\substack{r=s\\ fg\in E(Q_2)}}\sum_{\substack{f=g\\ rs\in E(Q_1)}}\big(d_{Q_1}(r) + d_{Q_2}(f) + d_{Q_1}(r)d_{Q_2}(f) - 1\big)\big(d_{Q_1}(s) + d_{Q_2}(g) + d_{Q_1}(s)d_{Q_2}(g) - 1\big)$$
$$\leq \big(\Delta_{Q_1} + \Delta_{Q_2} + \Delta_{Q_1}\Delta_{Q_2} - 1\big)^2(E_1E_2 + V_1E_2 + E_1V_2)$$
Also,
$$R_2M(Q_1 \circledast Q_2) \geq \big(\delta_{Q_1} + \delta_{Q_2} + \delta_{Q_1}\delta_{Q_2} - 1\big)^2(E_1E_2 + V_1E_2 + E_1V_2)$$
Clearly the bounds are tight if and only if Q_1 and Q_2 are regular graphs.

Theorem IV.2.
$\big(\delta_{Q_1}(\delta_{Q_2}+1) - 1\big)^2(E_1V_2 + E_1E_2) \leq R_2M(Q_1 \ominus Q_2) \leq \big(\Delta_{Q_1}(\Delta_{Q_2}+1) - 1\big)^2(E_1V_2 + E_1E_2)$ where the bounds are tight if and only if the graphs Q_1 and Q_2 are regular.

Proof
$$R_2M(Q_1 \ominus Q_2) = \sum_{\substack{rs\in E(Q_1)\\ f\in V(Q_2)}}\big(d_{Q_1\ominus Q_2}(r,f) - 1\big)\big(d_{Q_1\ominus Q_2}(s,f) - 1\big) + \sum_{\substack{rs\in E(Q_1)\\ fg\in E(Q_2)}}\big(d_{Q_1\ominus Q_2}(r,f) - 1\big)\big(d_{Q_1\ominus Q_2}(s,g) - 1\big)$$
$$= \sum_{\substack{rs\in E(Q_1)\\ f\in V(Q_2)}}\sum_{\substack{rs\in E(Q_1)\\ fg\in E(Q_2)}}\big(d_{Q_1}(r)(d_{Q_2}(f)+1) - 1\big)\big(d_{Q_1}(s)(d_{Q_2}(g)+1) - 1\big)$$
$$\leq \big(\Delta_{Q_1}(\Delta_{Q_2}+1) - 1\big)^2(E_1V_2 + E_1E_2)$$
Similarly,
$$\big(\delta_{Q_1}(\delta_{Q_2}+1) - 1\big)^2(E_1V_2 + E_1E_2) \leq R_2MQ_1 \ominus Q_2$$

Theorem IV.3.
$$E_1\big(\delta_{Q_1} + V_2 - 1\big)^2 + V_1E_2\delta_{Q_2}^2 + V_1V_2\delta_{Q_2}\big(\delta_{Q_1} + V_2 - 1\big) \leq R_2M(Q_1 \odot Q_2) \leq E_1\big(\Delta_{Q_1} + V_2 - 1\big)^2 + V_1E_2\Delta_{Q_2}^2 + V_1V_2\Delta_{Q_2}\big(\Delta_{Q_1} + V_2 - 1\big)$$
The bounds are tight if and only if Q_1 and Q_2 are regular graphs.

Proof
$$R_2M(Q_1 \odot Q_2) = \sum_{rs\in E(Q_1)}\big(d_{Q_1}(r)-1\big)\big(d_{Q_1}(s)-1\big) + \sum_{\substack{r\in V(Q_1)\\ fg\in E(Q_2)}}\big(d_{Q_2}(f)-1\big)\big(d_{Q_2}(g)-1\big) + \sum_{\substack{r\in V(Q_1)\\ f\in V(Q_2)}}\big(d_{Q_1}(r)-1\big)\big(d_{Q_2}(f)-1\big)$$

$$= \sum_{rs\in E(Q_1)} \left(d_{Q_1}(r) + V_2 - 1\right)\left(d_{Q_1}(s) + V_2 - 1\right) + V_1 \sum_{fg\in E(Q_2)} \left(d_{Q_2}(f)\right)\left(d_{Q_2}(g)\right) + \sum_{\substack{r\in V(Q_1)\\ f\in V(Q_2)}} \left(d_{Q_1}(r) + V_2 - 1\right)\left(d_{Q_2}(f) + 1 - 1\right)$$

$$\le E_1\left(\Delta_{Q_1} + V_2 - 1\right)^2 + V_1 E_2 \Delta_{Q_2}^2 + V_1 V_2 \Delta_{Q_2}\left(\Delta_{Q_1} + V_2 - 1\right)$$

Likely,

$$R_2M(Q_1 \odot Q_2) \ge E_1\left(\delta_{Q_1} + V_2 - 1\right)^2 + V_1 E_2 \delta_{Q_2}^2 + V_1 V_2 \delta_{Q_2}\left(\delta_{Q_1} + V_2 - 1\right)$$

Theorem IV.4.

$E_1\left((V_2+1)\delta_{Q_1} + V_2\right)^2 - 2V_2E_1(V_2+1)\delta_{Q_1} + E_1\left(1 + (\delta_{Q_2}+1)^2\right) + 2V_2E_1(\delta_{Q_2}+1)\left(V_2(1+\delta_{Q_1})\right) + V_1E_2\delta_{Q_2}^2 + \delta_{Q_2}\left((V_2+1)\delta_{Q_1} + V_2 - 1\right) \le R_2M(Q_1 \bullet Q_2) \le E_1\left((V_2+1)\Delta_{Q_1} + V_2\right)^2 - 2V_2E_1(V_2+1)\Delta_{Q_1} + E_1\left(1 + (\Delta_{Q_2}+1)^2\right) + 2V_2E_1(\Delta_{Q_2}+1)\left(V_2(1+\Delta_{Q_1})\right) + V_1E_2\Delta_{Q_2}^2 + \Delta_{Q_2}\left((V_2+1)\Delta_{Q_1} + V_2 - 1\right)$ where the bounds are sharp if and only if Q_1 and Q_2 are regular graphs.

Proof

$$R_2M(Q_1 \bullet Q_2) = \sum_{rs\in E(Q_1)}\left(d_{Q_1}(r) - 1\right)\left(d_{Q_1}(s) - 1\right) + E_1\sum_{fg\in E(Q_2)}\left(d_{Q_2}(f) + 1\right)\left(d_{Q_2}(g) + 1\right) + \sum_{\substack{rs\in E(Q_1)\\ f\in V(Q_2)}}\left(d_{Q_1}(r) + d_{Q_1}(s) - 1\right)\left(d_{Q_2}(f) - 1\right) + \sum_{\substack{r\in V(Q_1)\\ fg\in E(Q_2)}} d_{Q_2}(f)d_{Q_2}(g) + \sum_{\substack{r\in V(Q_1)\\ f\in V(Q_2)}}\left(d_{Q_1}(r) - 1\right)\left(d_{Q_2}(f) - 1\right)$$

$$= \sum_{rs\in E(Q_1)}\left((V_2+1)d_{Q_1}(r) + V_2 - 1\right)\left((V_2+1)d_{Q_1}(s) + V_2 - 1\right) + E_1\sum_{fg\in E(Q_2)}\left(d_{Q_2}(f) + 1\right)\left(d_{Q_2}(g) + 1\right) + \sum_{\substack{rs\in E(Q_1)\\ f\in V(Q_2)}}\left((V_2+1)d_{Q_1}(r) + V_2 - 1 + (V_2+1)d_{Q_1}(s) + V_2 - 1\right)\left(d_{Q_2}(f) - 1\right) + \sum_{\substack{r\in V(Q_1)\\ fg\in E(Q_2)}} d_{Q_2}(f)d_{Q_2}(g) + \sum_{\substack{r\in V(Q_1)\\ f\in V(Q_2)}}\left((V_2+1)d_{Q_1}(r) + V_2 - 1\right)\left(d_{Q_2}(f)\right)$$

$$\le E_1\left((V_2+1)\Delta_{Q_1} + V_2\right)^2 - 2V_2E_1(V_2+1)\Delta_{Q_1} + E_1\left(1 + (\Delta_{Q_2}+1)^2\right) + 2V_2E_1(\Delta_{Q_2}+1)\left(V_2(1+\Delta_{Q_1})\right) + V_1E_2\Delta_{Q_2}^2 + \Delta_{Q_2}\left((V_2+1)\Delta_{Q_1} + V_2 - 1\right)$$

Similarly,

$$R_2M(Q_1 \bullet Q_2) \ge E_1\left((V_2+1)\delta_{Q_1} + V_2\right)^2 - 2V_2E_1(V_2+1)\delta_{Q_1} + E_1\left(1 + (\delta_{Q_2}+1)^2\right) + 2V_2E_1(\delta_{Q_2}+1)\left(V_2(1+\delta_{Q_1})\right) + V_1E_2\delta_{Q_2}^2 + \delta_{Q_2}\left((V_2+1)\delta_{Q_1} + V_2 - 1\right)$$

The equality holds when both Q_1 and Q_2 are regular graphs.

V. Numerical Illustrations and Observations

In this section, we present numerical evaluations of the R_2M index for various graph products involving paths and cycles. The computed values and corresponding plots illustrate the behaviour of R_2M index across different graph combinations and product types. These numerical findings not only validate the theoretical bounds but also reveal structural influences on the behaviour of the reduced second Zagreb index.

TABLE I

R_2M Index values for path–path graphs under strong, semi-strong, vertex corona and vertex-edge corona products.

(m,n)	Strong	Semi-strong	Vertex corona	VE corona
(3,3)	296	76	-86	978
(3,4)	545	136	-144	858
(3,5)	794	196	-222	1832
(4,4)	990	314	-171	1350
(5,4)	1435	492	-198	1563
(5,5)	2076	702	-314	2092
(4,5)	1435	449	-268	4398
(5,6)	2717	912	-462	4062

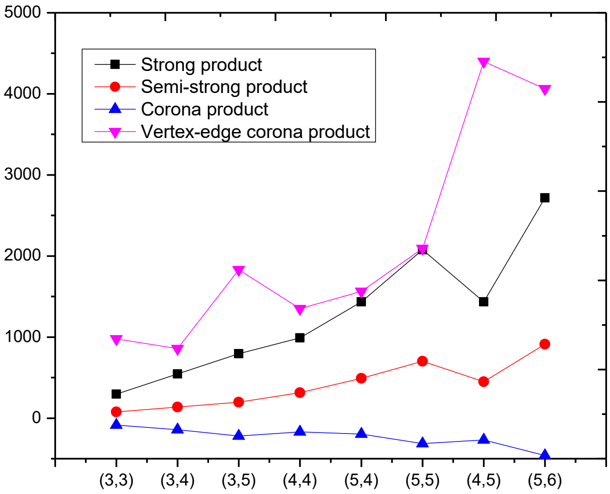

Fig 5. Graphical representation for Table 1.

TABLE II

R_2M Index values for path–cycle graphs under strong, semi-strong, vertex corona and vertex-edge corona products.

(m,n)	Strong	Semi-strong	Vertex corona	VE corona
(3,3)	899	538	116	669
(3,4)	1197	1034	192	1106
(3,5)	1495	1626	284	1651
(4,4)	1981	1002	265	1687
(5,4)	2765	950	338	2268
(5,5)	3455	1616	500	2511
(4,5)	2475	1637	396	3371
(5,6)	4145	2402	690	4690

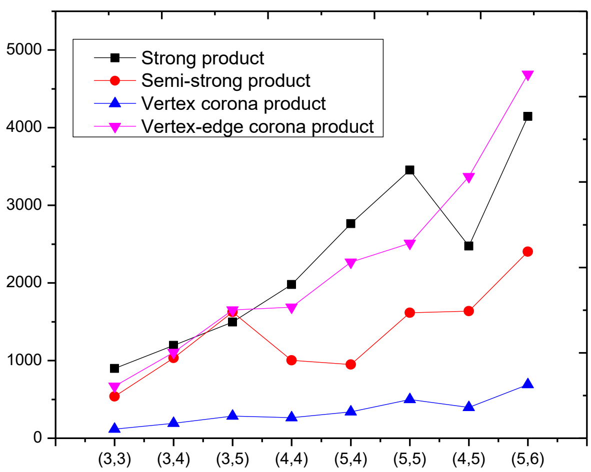

Fig 6. Graphical representation for Table 2

TABLE III
R_2M Index values for cycle–cycle graphs under strong, semi-strong, vertex corona and vertex-edge corona products.

(m,n)	Strong	Semi-strong	Vertex corona	VE corona
(3,3)	1764	675	156	915
(3,4)	2352	900	243	1491
(3,5)	2940	1125	348	2205
(4,4)	3136	1200	324	1956
(5,4)	3920	1500	405	2421
(5,5)	4900	1875	580	2890
(4,5)	3920	1500	464	3575
(5,6)	5880	2250	785	4951

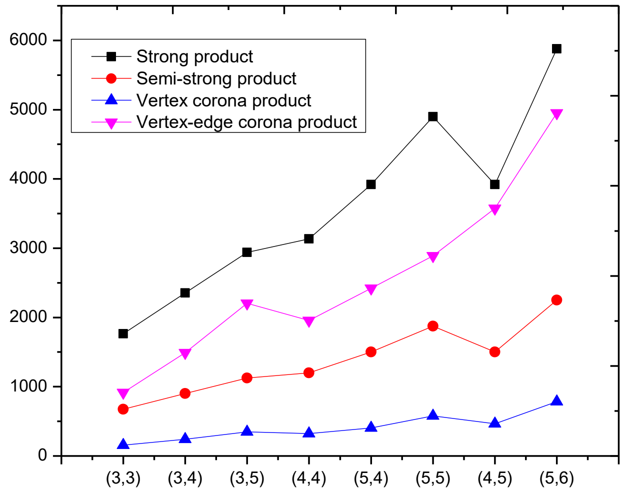

Fig 7. Graphical representation for Table 3

VI. Application

Graph products have significant real-world applications, particularly in the modelling of chemical compounds and the design of communication networks. Computing the reduced second Zagreb index for these graph products provides valuable insights into the structural resilience and connectivity of complex networks.

The strong product of two path graphs, for example P_3 and P_3, is a 3×3 mesh graph with 9 vertices. This configuration represents wireless communication networks where data exchange is possible through several paths. The reduced second Zagreb index assists in determining the strength and reliability of such networks based on internal connectivity and load balancing.

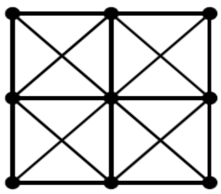
Fig 8. Mesh network on nine vertices.

The corona product of any cycle graph with a null graph on two isolated vertices can represent cycloalkane structures, utilized as high-energy fuels, gasoline blending agents, and starting materials for the synthesis of alcohols. For example, the corona product produces a ring structure with pendant vertices, like a decorated cyclopentane. Cyclopentane and its derivatives also find applications as solvents and eco-friendly blowing agents in the production of foam. In this context, the reduced second Zagreb index captures the influence of pendant groups on molecular connectivity and branching, which are closely linked to the structural and stability characteristics of the molecule.

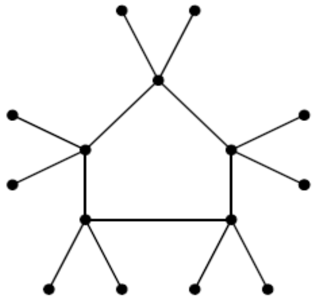
Fig 9. Structure of Cyclopentane.

VII. Conclusion

In this work, we investigated the reduced second Zagreb index for three fundamental graph operations: the strong, semi-strong, vertex corona and vertex-edge corona products of simple connected graphs. For each product, we derived explicit expressions for the index and established strong upper and lower bounds, along with conditions for equality. The mathematical understanding of degree-based topological indices under graph operations has been strengthened by these findings. Our results have significance as they can be used in network modelling and chemical graph theory. As a sensitive structural descriptor, the reduced second Zagreb index provides insight into characteristics including molecule stability, durability, and connectivity. In the future, one may extend this study to other types of graph products or operations, such as splice, link, or composition graphs, and explore their relevance in modelling more complex real-world systems.

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