Signed-strong Domination in Signed Graphs

Diviya K D and Anjaly Kishore

***Abstract*—Domination in signed graphs is a potent extension of traditional domination theory. In a signed graph $\Gamma(G, \sigma)$ with the underlying graph $G(V, E)$, $D \subseteq V$ is said to be a dominating set if for every $v \in V \setminus D$ there exists a vertex $u \in N^+(v) \cap D$. By incorporating the influence of both positive and negative interactions, the notion of domination in signed graphs is extended to signed-strong domination (SS-domination) in signed graphs. The argument behind this concept is that the impact of dominating vertex will be strengthened if it dominates a vertex in all aspects, upholding the consistency and efficiency in its interactions within its closed neighborhood. Some characteristics of SS-dominating sets of various signed graphs are analyzed. The SS-domination number of signed graphs with specific characteristics is also presented.**

***Index Terms*—Signed graph, signed degree, plurality marking, domination, signed-strong domination.**

I. Introduction

SIGNED graphs, which Frank Harary first proposed in 1953 [1], are extensions of traditional graph concepts. Signed graphs add another level of complexity by assigning a sign to each edge, either positive or negative [2]. The structural and theoretical features of the graph are significantly enhanced by this seemingly straightforward addition. This makes signed graphs, an effective model for intricate systems combining cooperation and conflict, including social networks, decision-making processes, etc. [3, 4, 5].

According to traditional graph theory, a dominating set is a collection of vertices in which every vertex in the graph is either part of the set or next to at least one vertex in the set. The idea of influence or control inside a network is encapsulated in this concept. In 2013, B. D. Acharya initiated the study of domination in signed graphs [6]. According to Acharya in [6], a dominating set of a signed graph $\Gamma(G, \sigma)$ is a set $D \subseteq V$ such that all the vertices of Γ are either in D or there exists a function $\mu : V \to \{+1, -1\}$, called a marking of Γ, such that all the vertices $v \in V \setminus D$ are adjacent to at least one vertex $u \in D$ such that $\sigma(uv) = \mu(u)\mu(v)$. In [7], P. Jeyalakshmi defined domination in signed graphs from a perspective that a dominating vertex must have more positive adjacency in the dominating set than its negative adjacency. Comparing it to the notion of domination in usual unsigned graphs, similar definition for domination in signed graphs is given in [8], which is as follows: a vertex is said to dominate another vertex if there exists a positive edge between them. As in unsigned graphs, different types of dominations such as signed domination, Roman domination, restrained domination, open domination, minus domination, double domination, independent domination, connected domination, etc. are defined and studied in literature [9, 8, 10, 11, 12, 13, 14, 15].

E. Sampathkumar in [16] introduced the concept of strong domination in unsigned graphs. According to him, a vertex strongly dominates another vertex if an edge exists between them and the dominating vertex has a degree greater than or equal to that of the other vertex. As per [9], signed dominating function in a signed graph Γ is a function $f : V \to \{-1, +1\}$ such that $f(N[v]) \geq 1$ for every vertex v of Γ, where $f(N[v]) = f(v) + \sum_{u \in N(v)} f(u)\sigma(uv)$ and it has applications in network modeling for decision making. The concept of signed-strong domination in signed graphs is primarily motivated by these two concepts.

The notion of signed-strong domination is introduced in this paper with the aim of emphasizing the impact of dominating vertex in signed graphs. This is done with the perspective that the dominance of a vertex in signed graphs will be enhanced if it dominates a vertex in all aspects, and if it is consistent and efficient in its interactions within its closed neighborhood. Thus, by extending the concept of strong domination [16] in unsigned graphs, and by incorporating the underlying concept of signed dominating function [9], the notion of signed-strong domination is introduced. The introduction of signed-strong domination is mainly evolved from social networks establishing the fact that a person can be dominated by another person with profound impact when the dominating person has an equal or more positivity and an equal or less negativity than the person being dominated, and if the dominating person is consistent and reliable within their closed circle. In signed social networks, this domination concept can be utilized to identify minimum-sized dominating sets that form influential and consistent groups based on interaction patterns. The characteristics of signed-strong dominating sets of various signed graphs are studied here. The signed-strong domination number of signed graphs with particular characteristics is also derived.

II. Preliminaries

This section provides the basic definitions needed for various elements of this research.

Definition 1. [17, 18] “The signed degree or net-degree of a vertex in a signed graph is defined as the number of positive edges incident with it minus the number of negative edges incident with it. Thus, the signed degree of a vertex v is given by $sdeg(v) = d^+(v) - d^-(v)$ where the number of positive edges incident with v is denoted by $d^+(v)$ and the number of negative edges incident with v is denoted by $d^-(v)$.”

Definition 2. [19] “Plurality marking of a vertex v in a signed graph $G = (V, E, \mu)$ is defined as

$$\mu(v) = \begin{cases} + & \text{if } \max\{d^+(v), d^-(v)\} = d^+(v) \\ - & \text{otherwise} \end{cases}$$

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where $d^+(v)$ is the positive degree and $d^-(v)$ is the negative degree of v."

Definition 3. [20] "A signed graph is called net-regular if every vertex of it has the same signed degree or net-degree."

Definition 4. [21] "A signed graph is called signed-regular if the number of positive edges incident with a vertex is independent of the choice of the vertex, and the number of negative edges incident with a vertex is also independent of the choice of the vertex". "A signed graph is said to be (k_1, k_2)-signed regular if k_1 positive edges and k_2 negative edges are incident with every vertex of the signed graph."

Definition 5. [22] "A signed graph $\Gamma(G, \sigma)$ is said to be co-regular if the underlying graph G is r-regular for some positive integer r and Γ is net-regular with net-degree k for some integer k. In this case, the co-regularity pair of Γ is the ordered pair (r, k)."

Definition 6. [8] "Let $\Gamma(G, \sigma)$ be a signed graph. Any set $D \subseteq V$ is said to be a dominating set in Γ if for each vertex $v \in V \setminus D$ there exists a vertex $u \in N^+(v) \cap D$. The minimum cardinality among all the dominating sets of Γ is called the domination number of Γ, denoted by $\gamma(\Gamma)$."

III. The signed-strong domination

The concept of plurality marking of vertices in signed graphs is introduced in [19]. Here, an alternate definition for plurality marking is presented in terms of the signed degree of vertices. This section introduces the notion of signed-strong domination, where the marking of vertices is carried out using this definition of plurality marking. The name "signed-strong domination" is chosen based on the underlying concepts which are used to define this extended version of domination in signed graphs. Throughout this work, the definition given in [8] is followed for domination in signed graphs, and the term proper dominating set is used to denote that the dominating set is a proper subset of the vertex set.

Definition 7. For a signed graph $\Gamma(G, \sigma)$ with the underlying graph $G(V, E)$ and signature function σ, the plurality marking of a vertex $v \in V$ is given by

$$\mu(v) = \begin{cases} +1 & \text{if } sdeg(v) \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

Definition 8. Consider a signed graph $\Gamma(G, \sigma)$ with the underlying graph $G(V, E)$ and the signature function σ, where $\sigma : E \to \{+1, -1\}$. Let μ represents the plurality marking of Γ. A set $D \subset V$ is said to be a signed-strong dominating set if for every $v \in V \setminus D$, there exists a vertex $u \in N^+(v) \cap D$ such that the following conditions are satisfied.
(i) $d^+(u) \geq d^+(v)$ and $d^-(u) \leq d^-(v)$
(ii) $\mu(N[u]) \geq 1$
where $\mu(N[u]) = \mu(u) + \sum\limits_{w \in N(u)} \mu(w)\sigma(uw)$
Here, the vertex u is said to SS-dominate the vertex v, and the vertex u is said to be SS-dominated by itself.

If no proper subset of V is a signed-strong dominating set, then V itself is taken as the signed-strong dominating set. A minimal signed-strong dominating set is a signed-strong dominating set such that any proper subset of it is not signed-strong dominating. The signed-strong domination number is defined as the cardinality of a signed-strong dominating set having the minimum number of vertices of Γ. Here, a signed-strong dominating set is denoted by SS-dominating set, and the signed-strong domination number of Γ is denoted by $\gamma_{SS}(\Gamma)$.

Note 1. In the case of an all-positive signed graph, the second condition of SS-domination is always satisfied. In this case, the first condition of SS-domination coincides with the condition of strong domination in unsigned graphs.

Note 2. From the definition of SS-domination, it is clear that all vertices with a positive degree zero will be in every SS-dominating set of a signed graph, as no other vertex will dominate these vertices.

Note 3. Any superset of a proper SS-dominating set in a signed graph is SS-dominating, since the conditions of SS-domination will be satisfied even if any number of vertices outside the SS-dominating set are included in it.

Illustration 1. Consider the following signed graph.

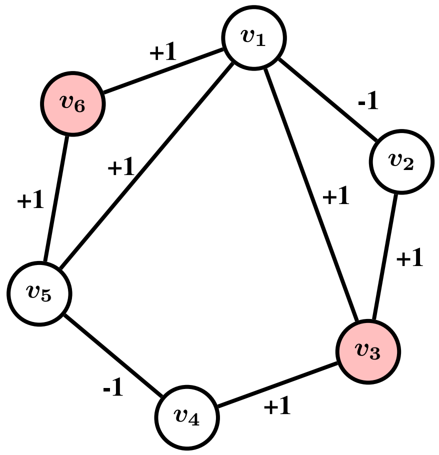

Fig. 1 Example

The vertices v_1 and v_3 have a maximum positive degree of 3. The vertex v_1 dominates the vertices v_3, v_5 and v_6, and the vertex v_3 dominates the vertices v_1, v_2, and v_4 . By verifying the conditions of SS-domination, v_1 SS-dominates v_3 and v_5, and v_3 SS-dominates the vertices v_1, v_2, and v_4. So v_3 is included in the SS-dominating set. Now, among the vertices v_5 and v_6, v_6 SS-dominates v_5. Hence, as shown in Fig. 1, $\{v_3, v_6\}$ is a SS-dominating set. In addition, a single vertex cannot SS-dominate all the other vertices. Hence, $\{v_3, v_6\}$ is a SS-dominating set with minimum cardinality, and therefore the SS-domination number of this signed graph is 2.

IV. Main results

General characterizations of SS-dominating sets of signed graphs are analyzed in this section. The SS-domination number of signed graphs with specific characteristics is also determined. In addition, SS-dominating sets of signed path graphs, signed cycle graphs, and signed complete graphs are analyzed. Also, the SS-domination number of signed star graphs and signed bistar graphs is established under various conditions. The notation $K_{1,n}^{(r)}$ is used to denote a signed star graph with $n+1$ vertices and r negative edges. The notation, $B^{(r_1,r_2)}_{(m,n)}$ is used to denote a signed bistar graph with $m+n+2$ vertices. In the signed bistar graph $B^{(r_1,r_2)}_{(m,n)}$, among the m edges incident with the first central vertex, exactly r_1 are negative, and among the n edges incident with the other central vertex, exactly r_2 are negative. The central edge between the two central vertices is not counted among these r_1+r_2 negative edges. Also, $B^{(r_1,r_2)+}_{(m,n)}$ denotes the signed bistar graph with positive central edge, and $B^{(r_1,r_2)-}_{(m,n)}$ denotes the signed bistar graph with negative central edge. The following proposition is the first primary observation derived from the definition.

Proposition 1. For a signed graph Γ, $1 \leq \gamma_{SS}(\Gamma) \leq n$ where n is the order of the graph

Remark 1. The bounds are sharp in the above proposition. Consider an all-positive signed complete graph in which any singleton vertex set is a SS-dominating set. Hence, the SS-domination number is 1. In the case of an all-negative signed graph of order n, no proper subset of the vertex set is a SS-dominating set, and hence the SS-domination number is n.

Proposition 2. In a signed graph Γ, $\gamma(\Gamma) \leq \gamma_{SS}(\Gamma)$

Proof: The result follows from the fact that every SS-dominating set is a dominating set. ∎

Proposition 3. For a signed cycle graph, every dominating set is SS-dominating if and only if each vertex having positive degree 2 is dominated by a vertex with positive degree 2

Proof: If no proper SS-dominating set exists for a signed cycle graph, then the result follows trivially. If it exists, let D be a proper dominating set with minimum cardinality. Now, for every $v \in V \setminus D$, there exists $u \in N^+(v) \cap D$. Obviously, all vertices with $d^+(v)=0$ must be in D. Let $v \in V \setminus D$. Then $d^+(v)$ is either 1 or 2. Hence, for $u \in N^+(v) \cap D$, $d^+(u)$ is either 1 or 2 so that $d^-(u)$ is either 1 or 0.

Case(i): $d^+(v)=1$

If $d^+(v)=1$ then $d^-(v)=1$. Hence, $d^+(u) \geq d^+(v)$. Also, since $d^-(u)$ is either zero or one, $d^-(u) \leq d^-(v)$.

Case(ii): $d^+(v)=2$

If $d^+(v)=2$ then $d^-(v)=0$. Since $d^-(u)$ is either zero or one, $d^-(v) \leq d^-(u)$. Since $d^+(v)=2$, the condition that $d^+(u) \geq d^+(v)$ is satisfied if and only if $d^+(u)=2$.

To check the second condition, $\mu(u)=+1$ since $d^+(u)$ is either 1 or 2. $\therefore \mu(N[u]) = 1 + \sum\limits_{w\in N(u)} \mu(w)\sigma(uw) \geq 1$ since $\sum\limits_{w\in N(u)} \mu(w)\sigma(uw) \geq 0$. Thus, the second condition of the SS-domination is satisfied by D. Hence, the result follows. ∎

Proposition 4. Every dominating set of a signed path graph is SS-dominating if and only if each vertex having positive degree 2 is dominated by a vertex with positive degree 2

Proof: The proof is similar to that of Proposition 3. ∎

Proposition 5. For a signed star graph $K^{(r)}_{1,n}$ with r negative edges,

$$\gamma_{SS}(K^{(r)}_{1,n}) = \begin{cases} 1 & \text{if } r=0 \\ 2 & \text{if } n=2, r=1 \\ n+1 & \text{otherwise} \end{cases}$$

Proof: Consider a signed star graph $K^{(r)}_{1,n}$ with r negative edges. Let v be the central vertex. Here, two cases arise: $r=0$ or $r \geq 1$. The construction of a SS-dominating set of minimum cardinality is as follows.

Case (i): $r=0$

In this case, the set consisting of the central vertex alone is SS-dominating, and the result follows.

Case(ii): $r \geq 1$

In this case, the pendant vertices of negative pendant edges must be in the SS-dominating set D. Also, the other pendant vertices of positive pendant edges incident with v must also be in D since these vertices are not SS-dominated by v as $d^-(v) \geq 1$, and all these pendant vertices have negative degree zero. Now, consider the central vertex v. There exists a dominating vertex for the central vertex v, say u, if and only if it is the pendant vertex of a positive pendant edge. Then $d^+(u)=1$ and u SS-dominates v if and only if $d^+(v)=1$ and $\mu(N[u]) = \mu(u)+\mu(v)\sigma(uv) \geq 1$. Also, $\mu(N[u]) = \mu(u)+\mu(v)\sigma(uv) \geq 1$ if and only if $\mu(v)=+1$. Now, $\mu(v)=+1$ if and only if $sdeg(v) \geq 0$. If $d^+(v)=1$ then $sdeg(v)=0$ if and only if $r=1$. Thus, u SS-dominates v if and only if $d^+(v)=1$ and $r=1$. In this case, $n=2$ and D consists of the two pendant vertices. If this is not the case, then the central vertex v is not SS-dominated by any other vertex, so that v also must be in D. Hence, the result follows. ∎

Proposition 6. If either $r_1 \neq 0$ or $r_2 \neq 0$, the SS-domination number of the signed bistar graph $B^{(r_1,r_2)}_{(m,n)}$ is as follows.

(i) If $r_1=0$ and $r_2 \neq 0$,

$$\gamma_{SS}(B^{(0,r_2)+}_{(m,n)}) = \begin{cases} n+1 & \text{if } m \geq n-r_2 \\ n+2 & \text{otherwise} \end{cases}$$

and

$$\gamma_{SS}(B^{(0,r_2)-}_{(m,n)}) = m+n+2$$

(ii) If $r_1 \neq 0$ and $r_2=0$,

$$\gamma_{SS}(B^{(r_1,0)+}_{(m,n)}) = \begin{cases} m+1 & \text{if } n \geq m-r_1 \\ m+2 & \text{otherwise} \end{cases}$$

and

$$\gamma_{SS}(B^{(r_1,0)-}_{(m,n)}) = m+n+2$$

Proof: Consider the bistar graph $B^{(r_1,r_2)}_{(m,n)}$. Let u be the central vertex adjacent to m pendant vertices, and let v be the central vertex adjacent to n pendant vertices. The construction of a minimum SS-dominating set D is as given below.

(i) Consider the case when $r_1=0$ and $r_2 \neq 0$. Here, two subcases arise depending on the sign of the central edge.

Subcase(i): The central edge is positive

Here, the central vertex u SS-dominates the m pendant vertices adjacent to it as $d^+(u) \geq 1$ and $\mu(N[u]) \geq 1$. The

r_2 pendant vertices of the r_2 negative pendant edges incident with v must be in the SS-dominating set D as their positive degree is 0. Since $d^-(v) \geq 1$, v will not SS-dominate the other $n - r_2$ pendant vertices adjacent to it and hence, those $n-r_2$ pendant vertices must also be in D. Since $\mu(N[u]) \geq 1$ and $d^-(u) = 0$, u SS-dominates v iff $m \geq n - r_2$. The pendant vertex of a positive pendant edge adjacent to v will not SS-dominate v as $d^+(v) \geq 2$ and the pendant vertex has positive degree 1. Thus

$$\gamma_{SS}(B_{(m,n)}^{(0,r_2)+}) = \begin{cases} n+1 & \text{if } m \geq n - r_2 \\ n+2 & \text{otherwise} \end{cases}$$

Subcase (ii): The central edge is negative

Here, the central vertex u will not SS-dominate the m pendant vertices adjacent to it as $d^-(u) = 1$ and these pendant vertices have negative degree 0. So these m pendant vertices must be in D. The central vertex u will not be SS-dominated by any of the adjacent vertices as the pendant vertices have negative degree 0 and $d^-(u) = 1$. So u must be in D. The central vertex v will not SS-dominate the n pendant vertices adjacent to it as $d^-(v) \geq 2$ and $n - r_2$ pendant vertices have negative degree 0, and r_2 pendant vertices have positive degree 0. Hence, these m pendant vertices must be in D. Now, a pendant vertex of a positive pendant edge adjacent to v SS- dominates v if and only if $d^+(v) = 1$ and $\mu(v) = +1$, which is not possible since $d^-(v) \geq 2$. Hence, v also must be in D. Thus,

$$\gamma_{SS}(B_{(m,n)}^{(0,r_2)-}) = m + n + 2$$

This completes the proof of (i).
(ii) Consider the case when $r_1 \neq 0$ and $r_2 = 0$. By proceeding as in the proof of (i), the following result is obtained.

$$\gamma_{SS}(B_{(m,n)}^{(r_1,0)+}) = \begin{cases} m+1 & \text{if } n \geq m - r_1 \\ m+2 & \text{otherwise} \end{cases}$$

and

$$\gamma_{SS}(B_{(m,n)}^{(r_1,0)-}) = m + n + 2$$

This completes the proof. ■

Proposition 7. For a signed bistar graph $B_{(m,n)}^{(r_1,r_2)}$ with $r_1 \neq 0$ and $r_2 \neq 0$, the SS-domination number is as follows.
(i)

$$\gamma_{SS}(B_{(m,n)}^{(r_1,r_2)+}) = \begin{cases} m+n+1 & \text{if } m - r_1 \geq n - r_2 \text{ and} \\ & r_1 \leq r_2 \\ m+n+1 & \text{if } n - r_2 \geq m - r_1 \text{ and} \\ & r_2 \leq r_1 \\ m+n+2 & \text{otherwise} \end{cases}$$

(ii)

$$\gamma_{SS}(B_{(m,n)}^{(r_1,r_2)-}) = m + n + 2$$

Proof: Consider the signed bistar graph $B_{(m,n)}^{(r_1,r_2)}$ with $r_1 \neq 0$ and $r_2 \neq 0$. Let u be the central vertex incident with m pendant edges, and let v be the central vertex incident with n pendant edges. The construction of a SS dominating set D with minimum cardinality is as follows.

(i) In this case, the central edge uv is positive. The pendant vertices of negative pendant edges have a positive degree 0, and the pendant vertices of positive pendant edges have a negative degree 0, and $d^-(u) \geq 1$. Hence, the central vertex u will not SS-dominate these m pendant vertices, so that the m pendant vertices adjacent to u must be in D. Similarly, the n pendant vertices adjacent to v also must be in D. None of the pendant vertices of a positive pendant edge incident with u will SS-dominate u since all these vertices have positive degree 1 and $d^+(u) > 1$. Similarly, none of the pendant vertices of positive pendant edges incident with v will SS-dominate v. Now, it is to be checked whether u SS-dominates v or vice versa. As each pendant vertex adjacent to u adds 1 to the sum $\mu(N[u])$, $\mu(N[u]) \geq 1$. From the definition, it follows that u SS-dominates v if and only if $m-r_1 \geq n-r_2$ and $r_1 \leq r_2$. Similarly, v SS-dominates u if and only if $n - r_2 \geq m - r_1$ and $r_2 \leq r_1$. Thus

$$\gamma_{SS}(B_{(m,n)}^{(r_1,r_2)+}) = \begin{cases} m+n+1 & \text{if } m - r_1 \geq n - r_2 \text{ and} \\ & r_1 \leq r_2 \\ m+n+1 & \text{if } n - r_2 \geq m - r_1 \text{ and} \\ & r_2 \leq r_1 \\ m+n+2 & \text{otherwise} \end{cases}$$

(ii) Here, the central edge uv is negative. If no positive edge is in the graph, then the result follows trivially. Otherwise, as proceeding in the proof of (i), m pendant vertices adjacent to u and n pendant vertices adjacent to v must be in the SS-dominating set D. Now, it is to be checked whether a pendant vertex of a positive pendant edge SS-dominates the central vertex u. Let w be such a pendant vertex. This pendant vertex SS-dominates u if and only if $d^+(u) = 1$ and $\mu(u) = +1$, which is not possible since $d^-(u) \geq 2$. Hence, none of the pendant vertices of positive pendant edges adjacent to u will SS-dominate u. Similarly, none of the pendant vertices of positive pendant edges adjacent to v will SS-dominate v. Obviously neither u dominates v nor v dominates u. Hence

$$\gamma_{SS}(B_{(m,n)}^{(r_1,r_2)-}) = m + n + 2$$

Thus the result follows. ■

Remark 2. Even though Proposition 6 is a special case of Proposition 7, it cannot be directly deduced from Proposition 7. In Proposition 7, every SS-dominating set necessarily includes all pendant vertices, whereas in Proposition 6, every SS-dominating set must include the central vertex incident only with positive edges, rather than the pendant vertices adjacent to it.

Proposition 8. For a signed complete graph of order n with $\delta^+(\Gamma) \geq \lfloor \frac{n}{2} \rfloor$, every proper dominating set is SS-dominating if and only if each vertex has its positive degree less than or equal to that of its dominating vertex

Proof: Consider a signed complete graph of order n with $\delta^+(\Gamma) \geq \lfloor \frac{n}{2} \rfloor$. Let D be a proper dominating set. Consider $v \in V \setminus D$. Then there exists a $u \in N^+(v) \cap D$. Since the graph is complete, if $d^+(u) \geq d^+(v)$ then $d^-(u) \leq d^-(v)$. So u satisfies the first condition of SS-domination over v if and only if $d^+(u) \geq d^+(v)$. Now, the second condition has to be verified. Since $\delta^+(\Gamma) \geq \lfloor \frac{n}{2} \rfloor$, all the vertices of Γ

have a signed degree greater than or equal to zero. Thus, all vertices are marked with $+1$ under plurality marking.

$\therefore \mu(N[u]) = \mu(u) + \sum_{w\in N(u)} \mu(w)\sigma(uw)$
$= 1 + sdeg(u) \geq 1.$

Thus, u SS-dominates v if and only if $d^+(u) \geq d^+(v)$. This completes the proof. ■

Proposition 9. If the vertex of maximum positive degree is unique in a signed graph, then every SS-dominating set contains that vertex.

Proof: Consider the signed graph $\Gamma(G,\sigma)$ with a unique vertex of maximum positive degree. Let v be that vertex. Also, let $D \subseteq V$ be a SS-dominating set of Γ. If the vertex $v \notin D$, then by definition, there exists a $u \in N^+(v) \cap D$ such that $d^+(u) > d^+(v)$, which leads to a contradiction. Hence, $v \in D$. This completes the proof. ■

Theorem 1. For a signed graph $\Gamma(G,\sigma)$ of order n, $\gamma_{SS}(\Gamma)$ is 1 if and only if $\Delta^+(\Gamma) = n-1$ and the vertex with positive degree $n-1$ is adjacent to at least $\lceil \frac{n}{2} \rceil$ vertices of signed degree greater than or equal to zero

Proof: Consider a signed graph $\Gamma(G,\sigma)$ of order n. Let $G(V,E)$ be its underlying graph. First, assume that $\gamma_{SS}(\Gamma) = 1$. Then, a SS-dominating set with a single vertex, say u, exists. Then for every $v \in V \setminus D$, there exists an edge uv with $\sigma(uv) = +1$. Hence, $d^+(u) = n-1$. Therefore, $\Delta^+(\Gamma) = n-1$. Now, by the second condition of the SS-dominating set, $\mu(N[u]) \geq 1$. Since $\mu(u) = +1$ and for every $w \in N(u)$, $\sigma(uw) = +1$, the condition $\mu(N[u]) \geq 1$ implies $\sum_{w\in N(u)} \mu(w) \geq 0$. Now, $\mu(w) \geq 0$ if and only if $sdeg(w) \geq 0$. Thus, u is adjacent to at least $\lceil \frac{n}{2} \rceil$ vertices of signed degree greater than or equal to zero.

Conversely, suppose that $\Delta^+(\Gamma) = n-1$ and the vertex u with positive degree $n-1$ is adjacent to at least $\lceil \frac{n}{2} \rceil$ vertices of signed degree greater than or equal to zero. Consider the set $D = \{u\}$. Clearly, as $d^+(u) = n-1$, for every $v \in V \setminus D$ there exists a positive edge uv. Hence, $d^+(u) \geq d^+(v)$ and $d^-(u) = 0 \leq d^-(v)$. Now, the second condition of SS-domination is to be verified. Since $\mu(u) = +1$ and $\sigma(uw) = +1$ for every $w \in N(u)$,

$\mu(N[u]) = \mu(u) + \sum_{w\in N(u)} \mu(w)\sigma(uw) = 1 + \sum_{w\in N(u)} \mu(w).$

As u is adjacent to at least $\lceil \frac{n}{2} \rceil$ vertices of signed degree greater than or equal to zero, $\sum_{w\in N(u)} \mu(w) \geq 0$ so that $\mu(N[u]) \geq 1$. Hence, $D = \{u\}$ is a SS-dominating set, and hence $\gamma_{SS}(\Gamma) = 1$. Thus, the theorem follows. ■

Proposition 10. For a signed path graph Γ of order n, $\gamma_{SS}(\Gamma) = n-1$ if and only if it has exactly one positive edge

Proof: Consider the signed path graph Γ of order n. Assume that $\gamma_{SS}(\Gamma) = n-1$. Let D be a SS-dominating set of $n-1$ vertices. Also, let $V \setminus D = \{v\}$. Then there exists $u \in D$ such that $\sigma(uv) = +1$ and

(i) $d^+(u) \geq d^+(v)$, $d^-(u) \leq d^-(v)$, and

(ii) $\mu(N[u]) = \mu(u) + \sum_{w\in N(u)} \mu(w)\sigma(uw) \geq 1$

We prove that uv is the only positive edge of Γ.

Suppose v is incident with another positive edge, say vu'. Then $d^+(v) = 2$ so that $d^-(v) = 0$. Hence, $d^+(u) = 2$ and $d^-(u) = 0$. Here, $\mu(u) = \mu(v) = \mu(v') = 1$ since all these vertices have signed degree non-negative. By computation, $\mu(N[v]) = \mu(v) + \mu(u)\sigma(uv) + \mu(u')\sigma(vu') = 3$. Hence, the vertex v SS-dominates the vertices u and u' and hence, $D\setminus\{u,u'\}\cup\{v\}$ is SS-dominating with cardinality less than that of D, leading to a contradiction. Hence, there exists exactly one positive edge between the vertices of D and $V \setminus D$.

Now, suppose there exists a positive edge, say yz, between the vertices y and z of D. If $y = u$, then $d^+(y) = 2$ so that $d^-(y) = 0$. Also, $d^+(z) = 1$ so that $d^-(z) = 1$. Here, $\mu(y) = \mu(v) = \mu(z) = 1$ since all these vertices have signed degree non-negative. $\mu(N[y]) = \mu(y) + \mu(v)\sigma(vy) + \mu(z)\sigma(yz) = 3$. Hence, y SS-dominates both v and z. Hence, $D\setminus\{z\}$ is SS-dominating, which is a contradiction. Suppose $y \neq u$. Then $d^+(y) = 1$ so that $d^-(y)$ is either 0 or 1. Similarly, $d^+(z) = 1$ so that $d^-(z)$ is either 0 or 1. Also, $\mu(N[y]) \geq 1$ since $\mu(y) = 1$ and $\mu(z)\sigma(yz) = 1$. Similarly, $\mu(N[z]) \geq 1$ since $\mu(z) = 1$ and $\mu(y)\sigma(yz) = 1$. Thus, either y SS-dominates z or z SS-dominates y. Hence, either $D \setminus \{y\}$ or $D \setminus \{z\}$ is SS-dominating, leading to a contradiction. Thus, there does not exist a positive edge between the vertices in D. Hence, Γ has exactly one positive edge.

Conversely, suppose that the signed path graph Γ has exactly one positive edge, say uv. Then $d^+(u) = 1$ and $d^-(u)$ is either 0 or 1. Similarly, $d^+(v) = 1$ and $d^-(v)$ is either 0 or 1. Also, the values of $\mu(N[u])$ and $\mu(N[v])$ are greater than or equal to one since $\mu(u) = \mu(v) = +1$ and $\sigma(uv) = +1$. Thus, u SS-dominates v if $d^-(u) \leq d^-(v)$ or v SS-dominates u if $d^-(v) \leq d^-(u)$. All the other $n-2$ vertices of positive degree 0 must be in any SS-dominating set of Γ. Thus, these $n-2$ vertices together with the SS-dominating vertex among u and v form an SS-dominating set of minimum cardinality. Hence, $\gamma_{SS}(\Gamma) = n-1$.

Thus, the result follows. ■

Theorem 2. Let Γ be a signed graph of order n which is not a signed path graph and has exactly one positive edge. Then $\gamma_{SS}(\Gamma) = n-1$ if and only if $d^-(u) \neq 0, 2$ where u is the dominating vertex such that uv is the positive edge.

Proof: Consider the signed graph Γ, which is not a signed path graph and has exactly one positive edge uv. The $n-2$ vertices of Γ other than the end vertices u and v of the edge uv must be in any SS-dominating set since all these vertices have positive degree 0.

Assume that $\gamma_{SS}(\Gamma) = n-1$. Without loss of generality, assume that among the end vertices of the positive edge uv, u SS-dominates v. Suppose $d^-(u) = 0$. As $d^+(u) = 1$, it follows that $sdeg(u) = 1$ so that $\mu(u) = +1$. Since $d^-(v) \geq d^-(u)$ and Γ is not a signed path graph, $d^-(v) \geq 2$ so that $\mu(v) = -1$. Now, $\mu(N[u]) = \mu(u) + \mu(v)\sigma(uv) = 0$, a contradiction to the second condition of SS-domination that $\mu(N[u]) \geq 1$. Now, suppose $d^-(u) = 2$. Hence, $sdeg(u) = -1$ so that $\mu(u) = -1$. Since $d^-(v) \geq d^-(u)$, $d^-(v) \geq 2$ so that $sdeg(v) \leq -1$. Hence, $\mu(v) = -1$. Thus, $\mu(N[u]) = \mu(u) + \mu(v)\sigma(uv) + \sum_{\substack{w\in N(u)\\ w\neq v}} \mu(w)\sigma(uw) = 0$ since $\sum_{\substack{w\in N(u)\\ w\neq v}} \mu(w)\sigma(uw) = 2$, which is also a contradiction. Thus, if $\gamma_{SS}(\Gamma) = n-1$ then $d^-(u) \neq 0, 2$, where u is the dominating vertex such that uv is the positive edge.

Conversely, assume that the dominating vertex among u and v has negative degree neither 0 nor 2. Since uv is the only positive edge, $d^+(u) = d^+(v) = 1$. Now, either $d^-(v) \geq d^-(u)$ or $d^-(u) < d^-(v)$.

Case (i): $d^-(v) \geq d^-(u)$

According to the definition of domination in signed graphs, either u or v is dominating. Since $d^-(v) \geq d^-(u)$, the vertex u is taken as the dominating vertex. Then the first condition of SS-domination is satisfied by u. By the assumption, $d^-(u) \neq 0, 2$. To verify the second condition of SS-domination, two subcases arise: either $\mu(u) = +1$ or $\mu(u) = -1$.

Subcase (i): $\mu(u) = +1$

Since $d^-(u) \neq 0$, this subcase occurs when $d^-(u) = 1$ so that $sdeg(u) = 0$. Since Γ is not a signed path graph, $d^-(v) \neq 1$. Hence, $d^-(v) \geq 2$ so that $\mu(v) = -1$. Now, $\mu(N[u]) = \mu(u) + \mu(v)\sigma(uv) + \mu(w)\sigma(uw) = 1$.

Subcase (ii): $\mu(u) = -1$

Since $\mu(u) = -1$ and $d^-(u) \neq 2$, it follows that $d^-(u) \geq 3$. Then $d^-(v) \geq 3$ since $d^-(v) \geq d^-(u)$, and hence, $\mu(v)$ is -1. Now, if uw is the negative edge incident with u, then $\mu(N[u]) = \mu(u) + \mu(v)\sigma(uv) + \sum_{\substack{w \in N(u) \\ w \neq v}} \mu(w)\sigma(uw) \geq 1$ since $\sum_{\substack{w \in N(u) \\ w \neq v}} \mu(w)\sigma(uw) \geq 3$. Hence, in case(i), u SS-dominates the vertex v.

Case (ii): $d^-(u) > d^-(v)$

Since $d^-(u) > d^-(v)$, the vertex v is taken as the dominating vertex. Also, the vertex v satisfies the first condition of SS-domination. Here also, two subcases arise: $\mu(v) = +1$ or $\mu(v) = -1$. Proceed as in case (i). Since $d^-(v) \neq 0$, $\mu(v)$ is $+1$ only if $d^-(v) = 1$. If $d^-(v) = 1$ then $d^-(u) > 1$ so that $\mu(u) = -1$. If vw is the negative edge incident to v, then $\mu(N[v]) = \mu(v) + \mu(u)\sigma(uv) + \mu(w)\sigma(vw) = 1$. Now, since $d^-(v) \neq 2$, it follows that $\mu(v) = -1$ when $d^-(v) \geq 3$. Thus $d^-(u) \geq 4$ and hence, $\mu(v) = -1$. As in subcase(ii) above, $\mu(N[v]) \geq 1$. Thus, in case (ii), v SS-dominates u.

Hence, the set consisting of the $n-2$ vertices of positive degree 0 and the vertex u or v depending on whether $d^-(v) \geq d^-(u)$ or $d^-(u) > d^-(v)$ is the SS-dominating set of minimum cardinality so that $\gamma_{SS}(\Gamma) = n - 1$.

Hence the proof. ■

Remark 3. The above theorem is a general characterization of signed graphs of order n other than signed path graphs, to have the SS-domination number $n-1$. For the signed path graph with exactly one positive edge uv, $\gamma_{SS}(\Gamma) = n - 1$ even if the dominating vertex u has $d^-(u) = 0$. For example, consider the signed path graph Γ given below.

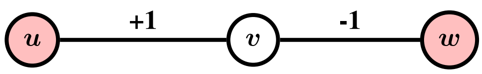

Fig. 2 Example: $d^-(u) = 0$, $\gamma_{SS}(\Gamma) = n - 2$

Here, even though the dominating vertex u of the positive edge uv has $d^-(u) = 0$, $\gamma_{SS}(\Gamma) = 2$ as shown in Fig. 2.

Theorem 3. A signed graph $\Gamma(G, \sigma)$ of order n has $\gamma_{SS}(\Gamma) = n$ if any of the following conditions hold

(i) $\Gamma(G, \sigma)$ has no proper dominating set

(ii) The signed degree of each vertex is -1

Proof: Consider the signed graph $\Gamma(G, \sigma)$ of order n, and let $G(V, E)$ be its underlying graph.

Suppose the condition (i) of the statement of the theorem holds. By the definition of SS-domination, $\gamma_{SS}(\Gamma) = n$ if and only if it has no proper subset for V as the SS-dominating set. Hence, if $\Gamma(G, \sigma)$ has no proper subset of the vertex set as a dominating set, then it has no proper SS-dominating set, and the result follows.

Suppose that the signed degree is -1 for each vertex of the signed graph Γ. Let D be a proper dominating set of Γ, if it exists. Let $v \in V \setminus D$. Then there exists a dominating vertex, say u. By the assumption, $\mu(u) = -1$. Since $sdeg(u)$ is -1, $\sum_{w \in N(u)} \sigma(uw) = -1$. Hence, $\sum_{w \in N(u)} \mu(u)\sigma(uw) = 1$ so that $\mu(u) + \sum_{w \in N(u)} \mu(u)\sigma(uw) = 0$. Thus, the second condition of the SS-domination will not be satisfied by D and the result follows. ■

Theorem 4. If a positive pendant edge is added to a SS-dominating vertex of a signed graph, then the SS-domination number remains unchanged

Proof: Consider a signed graph $\Gamma(G, \sigma)$ with $G(V, E)$ as its underlying graph. Let $D \subseteq V$ be SS-dominating. Consider $u \in D$. Also, let Γ' be the graph derived from Γ by adding a positive pendant edge uv at u. Then the positive degree of u increases by one, and the negative degree remains unchanged in Γ'. Hence, the first condition of the SS-domination is also satisfied by D in Γ'. The value of $\mu(N[u])$ in Γ will be increased by one in Γ' so that the second condition of SS-domination is also satisfied by D in Γ'. Hence, every SS-dominating set of Γ is SS-dominating in Γ'. In addition, all minimal SS-dominating sets in Γ' are SS-dominating in Γ as minimal SS-dominating sets of Γ' will not contain the pendant vertex of the newly added pendant edge. Hence, the theorem. ■

Theorem 5. Every SS-dominating set of a signed graph contains a vertex with the maximum positive degree and a vertex with the minimum negative degree

Proof: Consider a signed graph $\Gamma(G, \sigma)$ with the underlying graph $G(V, E)$ of order n. Let $\gamma_{SS}(\Gamma) = k$, where k is a positive integer such that $1 \leq k \leq n$. Let $D = \{u_1, u_2, ..., u_k\}$ be a SS-dominating set of Γ.

$\therefore V \setminus D \subseteq N^+(u_1) \cup N^+(u_2) \cup ... \cup N^+(u_k)$.

Also, for every $v \in V \setminus D$, there exists at least one $u_i \in D$ such that $\sigma(u_i v) = +1$, $d^+(u_i) \geq d^+(v)$, $d^-(u_i) \leq d^-(v)$ and $\mu(N[u_i]) \geq 1$ where $i \in \{1, 2, ..., k\}$. Hence, for every $v \in V \setminus D$,

$\max\{d^+(u_1), d^+(u_2), ..., d^+(u_k)\} \geq d^+(v)$, and

$\min\{d^-(u_1), d^-(u_2), ..., d^-(u_k)\} \leq d^-(v)$.

Here, $\Delta^+(\Gamma) = \max\{d^+(u_1), d^+(u_2), ..., d^+(u_k)\}$ and

$\delta^-(\Gamma) = \min\ \{d^-(u_1), d^-(u_2), ..., d^-(u_k)\}$.

Hence the theorem. ■

Theorem 6. For a (k_1, k_2)-signed regular signed graph

$\Gamma(G,\sigma)$ of order n,

$$\gamma_{SS}(\Gamma) = \begin{cases} \gamma(\Gamma) & \text{if } k_1 \geq k_2 \\ \gamma(\Gamma) & \text{if } k_1 < k_2, k_2 - k_1 \neq 1 \\ n & \text{if } k_2 - k_1 = 1 \end{cases}$$

Proof: Let $\Gamma(G,\sigma)$ be a (k_1,k_2)-signed regular signed graph of order n. Let $G(V,E)$ be its underlying graph. From the definition itself, it follows that every SS-dominating set of a signed graph is also dominating. Now, let $D \subseteq V$ be a dominating set of Γ. Hence, for every $v \in V \setminus D$, there exists $u \in N^+(v) \cap D$. Since $d^+(u) = d^+(v) = k_1$ and $d^-(u) = d^-(v) = k_2$, the first condition of SS-domination is satisfied by D. Then either $k_1 \geq k_2$ or $k_1 < k_2$. Also, let $k_1 - k_2 = k$.

Case(i): $k_1 \geq k_2$

Here, $0 \leq k \leq (n-1)$. Since the signed degree of each vertex is greater than or equal to zero, all vertices are marked with $+1$ under plurality marking. Hence,

$\mu(N[u]) = \mu(u) + \sum_{w \in N(u)} \mu(w)\sigma(uw) = 1 + sdeg(u) \geq 1.$

Thus, the second condition of SS-domination is also satisfied.

Case(ii): $k_1 < k_2$

Here, $-(n-1) \leq k < 0$. As the signed degree of each vertex is less than zero, all vertices are marked with -1 under plurality marking. Hence, $\sum_{w \in N(u)} \mu(w)\sigma(uw) = -sdeg(u)$. Thus,

$$\sum_{w \in N(u)} \mu(w)\sigma(uw) = \begin{cases} 1 & \text{when } |k| = 1 \\ \geq 1 & \text{when } |k| \neq 1 \end{cases}$$

Hence,

$$\mu(N[u]) = \begin{cases} 0 & \text{when } |k| = 1 \\ \geq 1 & \text{when } |k| \neq 1 \end{cases}$$

Thus, D is a SS-dominating set if and only if $|k| \neq 1$. When $|k| = 1$, V is the SS-dominating set.

This completes the proof. ■

Corollary 1. For a co-regular signed graph $\Gamma(G,\sigma)$ of order n with co-regularity pair (k,r),

$$\gamma_{SS}(\Gamma) = \begin{cases} \gamma(\Gamma) & \text{if } r + k \geq r - k \\ \gamma(\Gamma) & \text{if } r + k < r - k, k \neq -1 \\ n & \text{if } r + k < r - k, k = -1 \end{cases}$$

Proof: Let $\Gamma(G,\sigma)$ be a co-regular signed graph with co-regularity pair (k,r). Now, by definition, $\Gamma(G,\sigma)$ is also (k_1,k_2)-signed regular where $k_1 = \frac{r+k}{2}$ and $k_2 = \frac{r-k}{2}$. Then the result follows from Theorem 6. ■

Remark 4. For a (k_1,k_2)-signed regular signed graph $\Gamma(G,\sigma)$, the SS-domination number and the domination number may not be equal. For example, consider the $(1,2)$-signed regular signed graph given in Fig. 3. It can be easily verified that $\{v_3, v_5, v_6\}$, as shown in Fig. 3, is a dominating set of minimum cardinality. Hence, $\gamma(\Gamma) = 3$. But, by Theorem 6, $\gamma_{SS}(\Gamma) = 6$.

Remark 5. For a co-regular signed graph, the SS-domination number need not be equal to the domination number. For example, consider the co-regular signed graph $\Gamma(G,\sigma)$ of co-regularity pair $(3,-1)$ given in Fig. 3 . Here, $\gamma(\Gamma) = 3$, and by Corollary 1 of Theorem 6, $\gamma_{SS}(\Gamma) = 6$.

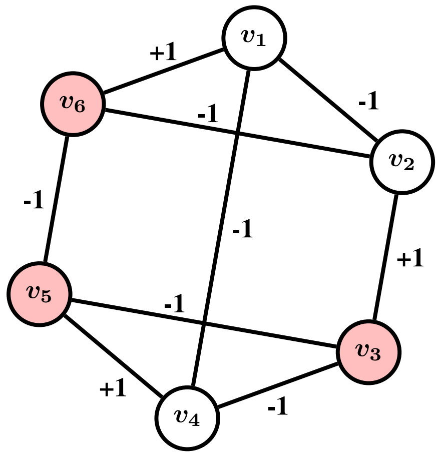

Fig. 3 Example

Theorem 7. For a net-regular signed graph $\Gamma(G,\sigma)$ of signed degree r, a proper dominating set of Γ is SS-dominating if and only if $r \neq -1$ and each vertex of Γ and its dominating vertex have an equal degree.

Proof: Consider a net-regular signed graph $\Gamma(G,\sigma)$ with the underlying graph $G(V,E)$ and signed degree r. It is obvious that every SS-dominating set of a signed graph is dominating. Now, let D be a proper dominating set of Γ. Let $v \in V \setminus D$. Then there exists $u \in N^+(v) \cap D$. Now, two cases arise: either $r \geq 0$ or $r < 0$.

Case (i): $r \geq 0$

Here, $\mu(w) = +1$ for every $w \in V$. Now, two sub cases have to be considered: (i) $r = 0$ and (ii) $r > 0$.

Subcase (i): $r = 0$

Here, $\mu(N[u]) = \mu(u) + \sum_{w \in N(u)} \mu(w)\sigma(uw)$

$$= 1 + \sum_{w \in N(u)} \sigma(uw) = 1 + sdeg(u) = 1$$

Hence, the second condition of SS-domination is satisfied by D. Since $d^+(u) = d^-(u)$ and $d^+(v) = d^-(v)$, if $d^+(u) > d^+(v)$ then $d^-(u) > d^-(v)$. Hence, the first condition of SS-domination will be satisfied if and only if $d^+(u) = d^+(v)$ so that $d^-(u) = d^-(v)$ also.

Subcase (ii): $r > 0$

Here, $\mu(N[u]) = \mu(u) + \sum_{w \in N(u)} \mu(w)\sigma(uw)$

$$= 1 + \sum_{w \in N(u)} \sigma(uw) \geq 2$$

$\therefore$ the second condition of SS-domination is satisfied by D. Let $d^+(v) = k$, a positive integer, Then $d^-(v) = k - r$. Now, the first condition of SS-domination will be satisfied only when $d^+(u) \geq k$. First, let $d^+(u) = k$, then $d^-(u) = k - r$ so that the first condition of SS-domination will be satisfied. But when $d^+(u) = k_1 > k$ where k_1 is a positive integer, then $d^-(u) = k_1 - r > d^-(v)$. So the first condition will not be satisfied. Hence, in this subcase, the first condition of SS-domination will be satisfied by D if and only if $d^+(u) = d^+(v)$ so that $d^-(u) = d^-(v)$ also.

Case (ii): $r < 0$

In this case $\mu(w) = -1$ for every $w \in V$.

Hence, $\mu(N[u]) = \mu(u) + \sum_{w \in N(u)} \mu(w)\sigma(uw)$

$$= -1 - \sum_{w \in N(u)} \sigma(uw) = -1 - r$$

In this case, $\mu(N[u]) \geq 1$ only when $r \neq -1$.

Now, the first condition of SS-domination is to be verified. Here, $d^+(u) < d^-(u)$ and $d^+(v) < d^-(v)$ since $r < 0$. Let $d^+(v) = l$, a positive integer. Then $d^-(v) = l - r$. Now, the first condition of SS-domination is satisfied only if $d^+(u) \geq l$. When $d^+(u) = l$, then $d^-(u) = l - r$ and the first condition of SS-domination is satisfied by D. To check the possibility when $d^+(u) > l$, let $d^+(u) = l_1 > l$, where l_1 is a positive integer. Then $d^-(u) = l_1 - r > d^-(v)$. Thus, when $d^-(u) > l$, D will not satisfy the first condition of SS-domination. Hence, when $r < 0$, D will be a SS-dominating set if and only if $r \neq 1$ and $d^+(v) = d^+(u)$ so that $d^-(u) = d^-(v)$ also.

This completes the proof of the theorem. ∎

V. Conclusion

By incorporating the influence of edge signs to domination theory in signed graphs, the concept of signed-strong domination is explored in this work. The influence of the dominating sets in signed graphs are significantly enhanced by the notion of signed-strong domination. According to our studies, the domination parameters are considerably changed by the existence of negative edges. This work has scope for further investigations, such as algorithmic advancement and applications in fields where signed interactions are a natural occurrence.

References

[1] F. Harary, "On the notion of balance of a signed graph." *Michigan Mathematical Journal*, vol. 2, no. 2, pp. 143–146, 1953.

[2] T. Zaslavsky, "Signed graphs," *Discrete Applied Mathematics*, vol. 4, no. 1, pp. 47–74, 1982.

[3] N. Girdhar and K. Bharadwaj, "Signed social networks: a survey," in *Advances in Computing and Data Sciences: First International Conference, ICACDS 2016, Ghaziabad, India, November 11-12, 2016, Revised Selected Papers 1*. Springer, 2017, pp. 326–335.

[4] A.-W. Li, J. Xiao, and X.-K. Xu, "The family of assortativity coefficients in signed social networks," *IEEE Transactions on Computational Social Systems*, vol. 7, no. 6, pp. 1460–1468, 2020.

[5] R. Sun, Y. Wu, X. Wang, C. Chen, W. Zhang, and X. Lin, "Clique identification in signed graphs: a balance theory based model," *IEEE Transactions on Knowledge and Data Engineering*, 2023.

[6] B. D. Acharya, "Domination and absorbance in signed graphs and digraphs, i: Foundations," *J. Combin. Math. Combin. Comput*, vol. 84, pp. 5–20, 2013.

[7] P. Jeyalakshmi, "Domination in signed graphs," *Discrete Mathematics, Algorithms and Applications*, vol. 13, no. 01, p. 2050094, 2021.

[8] J. Joseph and M. Joseph, "Roman domination in signed graphs," *Communications in Combinatorics and Optimization*, vol. 8, no. 2, pp. 349–358, 2023.

[9] H. Walikar, S. V. Motammanavar, and B. Acharya, "Signed domination in signed graphs." *Journal of Combinatorics, Information & System Sciences*, vol. 40, 2015.

[10] A. J. Mathias, V. Sangeetha, and M. Acharya, "Restrained domination in signed graphs," *Acta Univ. Sapientiae Math*, vol. 12, no. 1, pp. 155–163, 2020.

[11] K. Germina and P. Ashraf, "On open domination and domination in signed graphs," in *International Mathematical Forum*, vol. 8, no. 38, 2013, pp. 1863–1872.

[12] B. Acharya, "Minus domination in a signed graph." *Journal of Combinatorics, Information & System Sciences*, vol. 37, 2012.

[13] P. Ashraf and K. Germina, "Double domination in signed graphs," *Cogent Mathematics*, vol. 3, no. 1, p. 1186135, 2016.

[14] P. Jeyalakshmi, K. Karuppasamy, and S. Arockiaraj, "Independent domination in signed graphs," *Discrete Mathematics, Algorithms and Applications*, vol. 13, no. 05, p. 2150065, 2021.

[15] P. Jeyalakshmi and K. Karuppasamy, "Connected domination in a signed graph and its complement," *Journal of Intelligent & Fuzzy Systems*, vol. 45, no. 1, pp. 345–356, 2023.

[16] E. Sampathkumar and L. P. Latha, "Strong weak domination and domination balance in a graph," *Discrete Mathematics*, vol. 161, no. 1-3, pp. 235–242, 1996.

[17] G. Chartrand, H. Gavlas, F. Harary, and M. Schultz, "On signed degrees in signed graphs," *Czechoslovak Mathematical Journal*, vol. 44, no. 4, pp. 677–690, 1994.

[18] S. Pirzada, T. Naikoo, and F. Dar, "Signed degree sets in signed graphs," *Czechoslovak Mathematical Journal*, vol. 57, pp. 843–848, 2007.

[19] B. Adhikari, A. Singh, and S. K. Yadav, "Corona product of signed graphs and its application to modeling signed networks," *Discrete Mathematics, Algorithms and Applications*, vol. 15, no. 01, p. 2250062, 2023.

[20] N. G. Nayak, *On net-regular signed graphs*. Infinite Study, 2016.

[21] D. Sinha, P. Garg, and S. Arumugam, "On the regularity of some signed graph structures," *AKCE International Journal of Graphs and Combinatorics*, vol. 8, no. 1, pp. 63–74, 2011.

[22] K. Shahul Hameed, V. Paul, and K. Germina, "On co-regular signed graphs," *Australasian Journal of Combinatorics*, vol. 62, no. 1, pp. 8–17, 2015.