

Study of Dynamics and Synchronization of Uncertain Chaotic Systems with Time-Delay

Vijay Kumar Shukla, Ram Sagar and Gopal Kumar Gupta*

Abstract—In this paper, we are studied the different aspects of time-delay uncertain chaotic systems with uncertainties. Synchronization between time-delay chaotic systems in case of identical and non-identical chaotic systems is discussed. The Lyapunov stability theory is employed to demonstrate the stability of the chaotic systems. In order to synchronize two distinct chaotic systems, the adaptive and active controllers are implemented. By applying adaptive and active control methods the sufficient conditions are derived to synchronize identical and non-identical systems. Numerical simulations and graphical presentation demonstrate the validation and effectiveness of the proposed methods.

Index Terms—Synchronization, Time-delay chaotic system, Uncertainty, Active control, Adaptive control.

I. INTRODUCTION

FROM few decades, several significant and fruitful applications have been seen in the field of nonlinear dynamics. Application of dynamical systems can be found in many engineering and scientific fields. A mathematical model of dynamical system can be stated in the form of either difference equation or differential equation. The nonlinear dynamics has wide uses in various disciplines such as biology, physics, engineering and sociology etc. A nonlinear system with complex nature is known as chaotic system. In 1990, the synchronization between chaotic systems was first time investigated by Pecora and Carroll [1, 2] to discuss the some new ideas of chaotic systems. In addition, Boccaletti et al. [3] also described the synchronization of chaotic systems in broad sense.

Different type of synchronizations have been analyzed by using a lot of methods such as active control [4], backstepping control [5], sliding mode control [6], optimal control [7] and nonlinear feedback control [8] etc. These methods have more applications in the field of control theory. In adaptive control method the parameters are automatically adjustable according to current condition [9] while in active control method the parameters adjust according as desired results. Tarammim and Akter [10] used active control and backstepping control techniques to carry out synchronization. They showed that by using a recursive backstepping approach based on Lyapunov stability theory, the chaotic behavior can be controlled. The active control method also examined by Agrawal et al. [11] in order to synchronize two

distinct fractional-order chaotic systems with consideration of the Lotka–Volterra system as master system while the Newton–Leipnik system as slave system. Similarly, Ahmad et al. [12] looked at the synchronization issue when uncertainties and external disturbances occurred with drive-response systems with varying orders. They provided a robust generalized active control strategy and compute an appropriate controllers that assures the globally exponentially stable synchronization. In the field of secure communications, the information signal becomes more complex due to the anti-synchronization between uncertain chaotic systems. To tackle such type of problems, Ahmad and Shafiq [13] investigated the double combination anti-synchronization of uncertain chaotic systems. In order to achieve the synchronization within a constant-time, Su et al. [14] addressed the fixed-time synchronization for time-delay chaotic neural networks using active control technique by creating a fixed-time control approach.

In synchronization, the adaptive control approach is used when some or all of the system parameters are not measurable and necessary to estimate. Using eigenvalue structure analysis, bifurcation diagrams, Kaplan–Yorke dimension and Lyapunov exponents Saeed et al. [15] examined a chaotic system with six parameters. They obtained even with tiny changes in one or more of its parameter values the suggested dynamical system maintains its chaotic behavior. The study of synchronization phenomena between chaotic systems with parametric uncertainty has been examined by Heidarzadeh et al. [16]. They developed a unique robust controller with free from singularity to ensure boundedness of closed-loop signal. In order to achieve the synchronization between chaotic systems with time-varying delays, Wang and Li [17] used the adaptive intermittent control method. They recommended a novel control strategy to achieve the asymptotic exponential stability. Shukla et al. [18, 19] analyzed the difference synchronization with and without delay terms to study the adaptive difference synchronization for several kinds of chaotic systems using fixed-time control. The Tinkerbell, Henon and Hitzl-Zele chaotic maps have been used to show the synchronization. They also discussed the mechanical analysis of chaotic system. In order to achieve fast convergence and to minimize the amplitude of oscillations in the error signals, Haris et al. [20] proposed an adaptive controller. To deter the synchronization error, this controller combines nonlinear and adaptive control components.

The presence and stability of Hopf bifurcation, fundamental dynamic features of a four-dimensional hyperchaotic system, and the formulas for determining the direction of Hopf bifurcation and the stability of bifurcating periodic solutions is discussed by Du et al. [21]. The uses of chaotic systems with a variety of applications including sensors and cryptography have been shown by Natiq et al. [22].

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In addition, they showed a lot of chaotic systems with low complexity and hyperchaotic behavior, which might be detrimental to the uses of chaos-based sensors. Two meta-heuristic techniques have also used by Dey et al. [23] to analyze the correlation coefficient for gray-level images. This demonstrates the parallelism feature of the discrete time nature of quantum mechanical systems. They also discussed the standard deviation of fitness and the computational time of each technique for every test image. Xu et al. [24] investigated artificial intelligence in the context of image privacy protection. In addition, several researchers studied the different aspects of chaotic system [29-36].

For the past few decades, researchers from a wide range of disciplines including biology, physics, economics, mathematics have been attracted to study delay chaotic systems. Delay differential equations with one or more time-delays are used to simulate natural systems. The existence of time-delay can result in several phenomena including metastability, bifurcation and chaos [37]. Further, several scientists [38-47] used this concept to solve the real world problems.

This paper is organized in the following manner: In Section 2 the systems description is given while in Section 3 synchronization between identical and non-identical time-delay chaotic systems by active control method is analyzed. Section 4 contains the synchronization between identical /non-identical chaotic systems by adaptive control method. Discussion of numerical simulations is explored in Section 5. Finally, in Section 6 conclusion of the paper is revealed.

II. SYSTEM'S DESCRIPTION

In 2019, Sun et al. [25] proposed a new chaotic system and this system with time-delay is given as

$$\begin{aligned}\dot{\xi}_1 &= \alpha \xi_2(t - \tau_1), \\ \dot{\xi}_2 &= \beta(\xi_2 - \xi_1) + \xi_2 \xi_3, \\ \dot{\xi}_3 &= \gamma - \xi_2^2.\end{aligned}\quad (1)$$

The above chaotic system with uncertainty is expressed as

$$\begin{aligned}\dot{\xi}_1 &= \alpha \xi_2(t - \tau_1) + 0.01 \xi_1, \\ \dot{\xi}_2 &= \beta(\xi_2 - \xi_1) + \xi_2 \xi_3 - 0.01 \xi_2, \\ \dot{\xi}_3 &= \gamma - \xi_2^2 - 0.01 \xi_3.\end{aligned}\quad (2)$$

Here, $[0.01 \xi_1, -0.01 \xi_2, -0.01 \xi_3]^T$ represents uncertain terms of system (1) and τ_1 denotes delay term. For parametric values $\alpha = 1, \beta = 1, \gamma = 1$ and at initial condition $(1, 1, 1)$ the above system (2) depicts chaotic behavior. The phase portraits of this system with different parameters and delay term are given in Fig. 1.

Veeman et al. [26] explored another chaotic system and this system with time-delay can be written in following form

$$\begin{aligned}\dot{\eta}_1 &= \eta_1 \eta_3 - 7 \eta_3, \\ \dot{\eta}_2 &= -b \eta_2^2 + \eta_3^2, \\ \dot{\eta}_3 &= -\eta_1^2 + a \eta_2^2 + 14 \eta_1(t - \omega_1) + \eta_3 - 49,\end{aligned}\quad (3)$$

This chaotic system with time-delay and uncertainty is given as

$$\begin{aligned}\dot{\eta}_1 &= \eta_1 \eta_3 - 7 \eta_3 - 0.02 \eta_1, \\ \dot{\eta}_2 &= -b \eta_2^2 + \eta_3^2 + 0.01 \eta_2, \\ \dot{\eta}_3 &= -\eta_1^2 + a \eta_2^2 + 14 \eta_1(t - \omega_1) + \eta_3 - 49 - 0.01 \eta_3,\end{aligned}\quad (4)$$

where η_1, η_2 and η_3 are the state variables and a and b are the real constants parameters and ω_1 denotes delay term.

Here, $[-0.02 \eta_1, 0.01 \eta_2, -0.01 \eta_3]^T$ represents uncertain terms of system (4). The phase portraits of this system (4) are given in Fig. 2.

III. SYNCHRONIZATION BETWEEN CHAOTIC SYSTEMS BY ACTIVE CONTROL METHOD

Now, we discuss synchronization between time-delay chaotic systems with and without uncertainty by active control method. Here, we analyze the synchronization between time-delay chaotic systems with uncertainties. We take the system (2) as master system and following system as slave system.

$$\begin{aligned}\dot{\bar{\xi}}_1 &= \alpha \bar{\xi}_2(t - \tau_2) + 0.01 \bar{\xi}_1 + \Omega_1(t), \\ \dot{\bar{\xi}}_2 &= \beta(\bar{\xi}_2 - \bar{\xi}_1) + \bar{\xi}_2 \bar{\xi}_3 - 0.01 \bar{\xi}_2 + \Omega_2(t), \\ \dot{\bar{\xi}}_3 &= \gamma - \bar{\xi}_2^2 - 0.01 \bar{\xi}_3 + \Omega_3(t),\end{aligned}\quad (5)$$

where, $\Omega_1(t), \Omega_2(t), \Omega_3(t)$ are controller function. Define the synchronization error as

$$\begin{aligned}e_1(t) &= \bar{\xi}_1(t) - \xi_1(t), \\ e_2(t) &= \bar{\xi}_2(t) - \xi_2(t), \\ e_3(t) &= \bar{\xi}_3(t) - \xi_3(t).\end{aligned}\quad (6)$$

Using (2) and (5), we get the error system as

$$\begin{aligned}\dot{e}_1 &= \alpha\{\bar{\xi}_2(t - \tau_2) - \xi_2(t - \tau_1)\} + 0.01 \bar{\xi}_1 - 0.01 \xi_1 + \Omega_1(t) \\ \dot{e}_2 &= \beta(e_2 - e_1) + \bar{\xi}_2 \bar{\xi}_3 - \xi_2 \xi_3 + 0.01 \xi_2 - 0.01 \bar{\xi}_2 + \Omega_2(t) \\ \dot{e}_3 &= -\xi_2^2 + \bar{\xi}_2^2 + 0.01 \xi_3 - 0.01 \bar{\xi}_3 + \Omega_3(t)\end{aligned}\quad (7)$$

Suppose, the control functions are given as

$$\begin{aligned}\Omega_1(t) &= -\alpha\{\bar{\xi}_2(t - \tau_2) - \xi_2(t - \tau_1)\} - \bar{\xi}_1 + \xi_1 + \\ &\quad 0.01 \xi_1 - 0.01 \bar{\xi}_1 + \Psi_1(t) \\ \Omega_2(t) &= -\bar{\xi}_2 \bar{\xi}_3 + \xi_2 \xi_3 - 0.01 \xi_2 + 0.01 \bar{\xi}_2 + \Psi_2(t) \\ \Omega_3(t) &= \xi_2^2 + \bar{\xi}_2^2 - \bar{\xi}_3 + \xi_3 - 0.01 \xi_3 + 0.01 \bar{\xi}_3 + \Psi_3(t)\end{aligned}\quad (8)$$

After putting the control function (8) in (7), the error system is expressed as

$$\begin{aligned}\dot{e}_1 &= -e_1 + \Psi_1(t) \\ \dot{e}_2 &= \beta(e_2 - e_1) + \Psi_2(t) \\ \dot{e}_3 &= -e_3 + \Psi_3(t)\end{aligned}\quad (9)$$

Here $\Psi_1(t), \Psi_2(t), \Psi_3(t)$ are the controller. To stabilize the above error system we design

$$\begin{bmatrix} \Psi_1(t) \\ \Psi_2(t) \\ \Psi_3(t) \end{bmatrix} = P \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

where P is a 3×3 matrix and given as

$$P = \begin{bmatrix} -1 & 0 & 0 \\ \beta & -2\beta & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

So, the system (9) is reduces in following form

$$\dot{e}_1 = -2e_1, \dot{e}_2 = -\beta e_2, \dot{e}_3 = -3e_3 \quad (10)$$

Now all eigenvalue of error system (10) are negative hence required condition of synchronization are achieved [27]. To

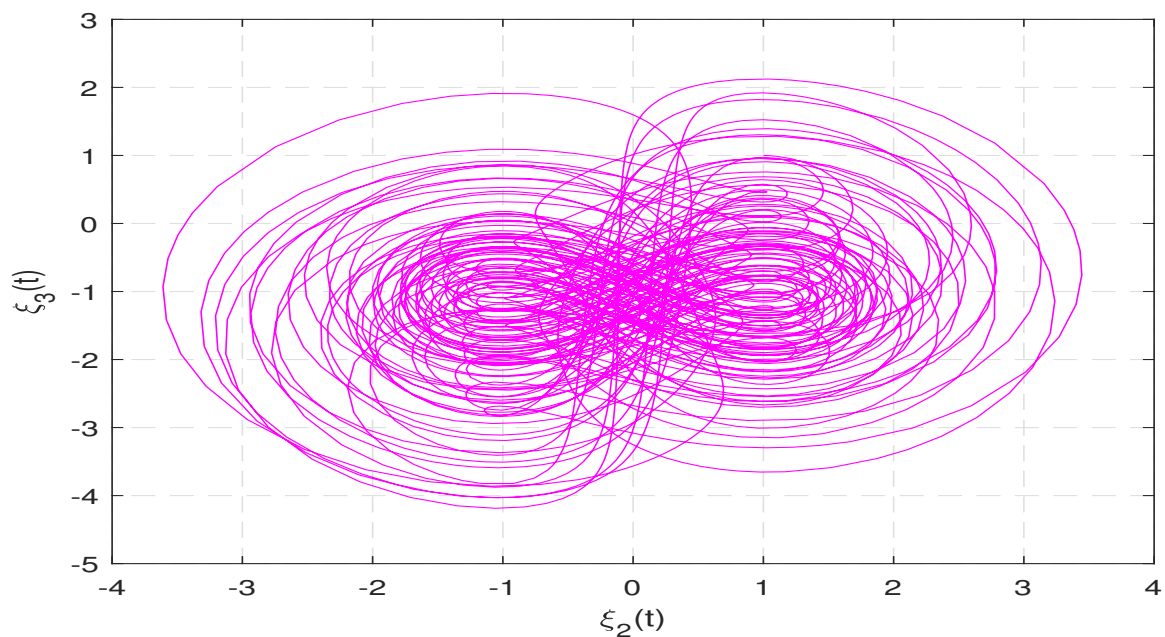


Fig. 1. Phase portrait of system (1) with delay and uncertain terms in $\xi_2 - \xi_3$ plane.

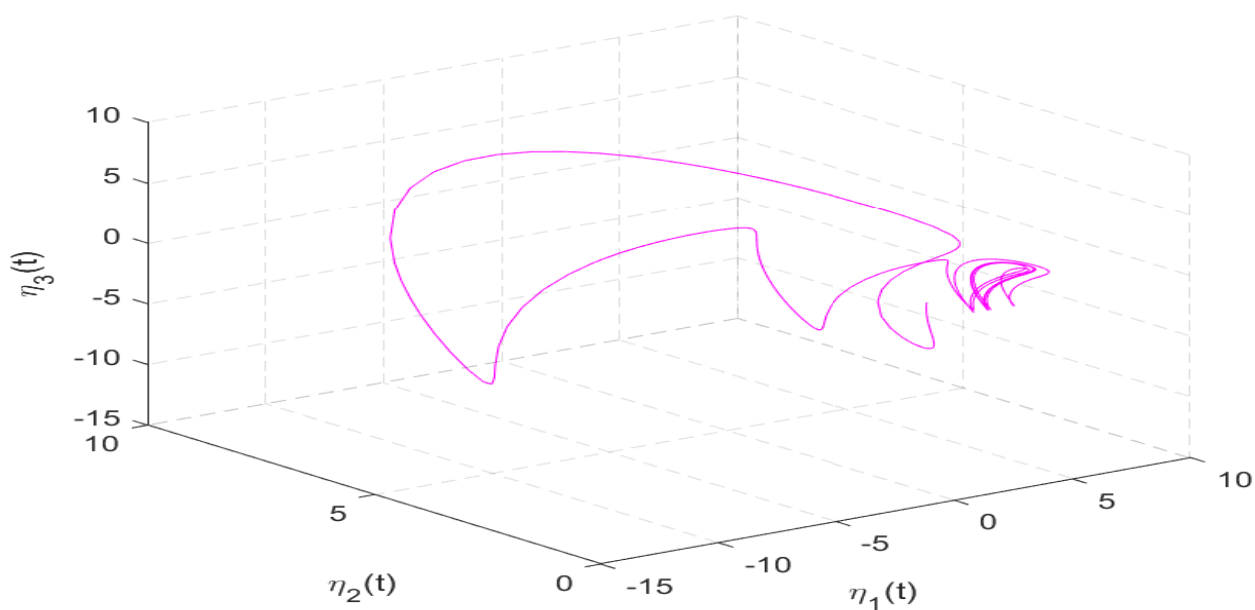


Fig. 2. Phase portraits of system (4) with delay and uncertain terms in $\eta_1 - \eta_2 - \eta_3$ space.

study the synchronization between another identical time-delay chaotic system (4), we consider the controlled slave system as

is expressed as

$$\begin{aligned} e_1(t) &= \chi_1(t) - \eta_1(t), \\ e_2(t) &= \chi_2(t) - \eta_2(t), \\ e_3(t) &= \chi_3(t) - \eta_3(t) \end{aligned} \quad (12)$$

$$\begin{aligned} \dot{\chi}_1 &= \chi_1\chi_3 - 7\chi_3 - 0.02\chi_1 + \lambda_1(t), \\ \dot{\chi}_2 &= -b\chi_2^2 + \chi_3^2 + 0.01\chi_2 + \lambda_2(t), \\ \dot{\chi}_3 &= -\chi_1^2 - a\chi_2^2 + 14\chi_1(t - \omega_2) + \chi_3 - 49 - 0.01\chi_3 \\ &\quad + \lambda_3(t), \end{aligned} \quad (11)$$

where, $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$ are the control function and χ_1, χ_2, χ_3 are the state variables. Now, the error dynamics

Using equation (4) and (11) in (12), we get

$$\begin{aligned} \dot{e}_1(t) &= \chi_1\chi_3 - \eta_1\eta_3 - 7e_3 - 0.02\chi_1 + 0.02\eta_1 + \lambda_1(t), \\ \dot{e}_2(t) &= -b(\chi_2^2 - \eta_2^2) + \chi_3^2 - \eta_3^2 + 0.01\chi_2 - 0.01\eta_2 + \lambda_2(t), \\ \dot{e}_3(t) &= -\chi_1^2 + \eta_3^2 - a(\chi_2^2 - \eta_2^2) + 14\chi_1(t - \omega_2) - \\ &\quad 14\eta_1(t - \omega_1) + e_3 - 0.01\chi_3 + 0.01\eta_3 + \lambda_3(t), \end{aligned} \quad (13)$$

Further, the control functions are assumed as

$$\begin{aligned}\lambda_1(t) &= -\chi_1\chi_3 + \eta_1\eta_3 + 0.02\chi_1 - 0.02\eta_1 + U_1(t), \\ \lambda_2(t) &= b(\chi_2^2 - \eta_2^2) - \chi_3^2 + \eta_3^2 - \chi_2 + \eta_2 - 0.01\chi_2 + \\ &\quad 0.01\eta_2 + U_2(t), \\ \lambda_3(t) &= \chi_1^2 - \eta_3^2 + a(\chi_2^2 - \eta_2^2) - 14\chi_1(t - \omega_2) + \\ &\quad 14\eta_1(t - \omega_1) + 0.01\chi_3 - 0.01\eta_3 + U_3(t),\end{aligned}\quad (14)$$

So, the error system is obtained as

$$\begin{aligned}\dot{e}_1(t) &= -7e_3 + U_1(t), \\ \dot{e}_2(t) &= -e_2 + U_2(t), \\ \dot{e}_3(t) &= e_3 + U_3(t)\end{aligned}\quad (15)$$

Now, we choose $\begin{bmatrix} U_1(t) \\ U_2(t) \\ U_3(t) \end{bmatrix} = S \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$, where, S is a 3×3 matrix and expressed as

$$S = \begin{bmatrix} -1 & 0 & 7 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Now, we get

$$\dot{e}_1 = -e_1, \dot{e}_2 = -e_2, \dot{e}_3 = -e_3 \quad (16)$$

All eigenvalues of error system are negative which ensure that stability of error system are achieved.

Finally, we synchronize time-delay non-identical chaotic systems with uncertainty by active control method. Further, we suppose system (2) as master system and (4) as slave system. The controlled slave system is given as

$$\begin{aligned}\dot{\eta}_1 &= \eta_1\eta_3 - 7\eta_3 - 0.02\eta_1 + Y_1(t), \\ \dot{\eta}_2 &= -b\eta_2^2 + \eta_3^2 + 0.01\eta_2 + Y_2(t), \\ \dot{\eta}_3 &= -\eta_1^2 - a\eta_2^2 + 14\eta_1(t - \omega_1) + \eta_3 - 49 - 0.01\eta_3 \\ &\quad + Y_3(t),\end{aligned}\quad (17)$$

where $Y_1(t)$, $Y_2(t)$ and $Y_3(t)$ are control functions. The error system can be written as

$$\begin{aligned}e_1(t) &= \eta_1(t) - \xi_1(t), \\ e_2(t) &= \eta_2(t) - \xi_2(t), \\ e_3(t) &= \eta_3(t) - \xi_3(t)\end{aligned}\quad (18)$$

Hence the error system is given as

$$\begin{aligned}\dot{e}_1 &= \eta_1\eta_3 - 7e_3 - 7\xi_3 - \alpha\xi_2(t - \tau_1) - 0.02\eta_1 - 0.01\xi_1 + \\ &\quad Y_1(t), \\ \dot{e}_2 &= -b\eta_2^2 + \eta_3^2 - \beta(e_2 + e_1) - \xi_2\xi_3 + 0.01\xi_2 + 0.01\eta_2 - \\ &\quad \beta(\eta_2 - \eta_1) + Y_2(t), \\ \dot{e}_3 &= -\eta_1^2 + a\eta_2^2 + 14\eta_1(t - \omega_1) + e_3 - 49 + \xi_3 - \gamma + \xi_2^2 \\ &\quad - 0.01\xi_3 + Y_3(t)\end{aligned}\quad (19)$$

Now the control function is supposed as

$$\begin{aligned}Y_1(t) &= -\eta_1\eta_3 + 0.01\xi_1 + 7\xi_3 + \alpha\xi_2(t - \tau_1) + 0.02\eta_1 + \\ &\quad Q_1(t), \\ Y_2(t) &= b\eta_2^2 - \eta_3^2 - 0.01\xi_2 + \xi_2\xi_3 - 0.01\eta_2 + \beta(\eta_2 - \eta_1) \\ &\quad + Q_2(t), \\ Y_3(t) &= \eta_1^2 - a\eta_2^2 + 49 - 14\eta_1(t - \omega_1) - \xi_3 + \gamma - \xi_2^2 \\ &\quad + 0.01\xi_3 + Q_3(t)\end{aligned}\quad (20)$$

where, control inputs $Q_1(t)$, $Q_2(t)$, $Q_3(t)$ are the function of e_1 , e_2 and e_3 . After putting (20) in (19), we have

$$\begin{aligned}\dot{e}_1 &= -7e_3 + Q_1(t), \\ \dot{e}_2 &= -\beta(e_2 + e_1) + Q_2(t), \\ \dot{e}_3 &= e_3 + Q_3(t).\end{aligned}\quad (21)$$

Now, suppose $\begin{bmatrix} Q_1(t) \\ Q_2(t) \\ Q_3(t) \end{bmatrix} = H \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$, where H is a 3×3

matrix and given as $H = \begin{bmatrix} -1 & 0 & 7 \\ \beta & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

So, we get

$$\dot{e}_1 = -e_1, \dot{e}_2 = -\beta e_2, \dot{e}_3 = -e_3$$

According similar fashion we can say that required synchronization is achieved.

IV. SYNCHRONIZATION OF CHAOTIC SYSTEM BY ADAPTIVE CONTROL METHOD

In this section, we will discuss the synchronization between time-delay chaotic systems by adaptive control method. We assume time-delay chaotic system with uncertainties (4) as master system and following system as slave system.

$$\begin{aligned}\dot{\bar{\xi}}_1 &= \alpha\bar{\xi}_2(t - \tau_2) + 0.01\bar{\xi}_1 + \mu_1(t), \\ \dot{\bar{\xi}}_2 &= \beta(\bar{\xi}_2 - \bar{\xi}_1) + \bar{\xi}_2\bar{\xi}_3 - 0.01\bar{\xi}_2 + \mu_2(t), \\ \dot{\bar{\xi}}_3 &= \gamma - \bar{\xi}_2^2 - 0.01\bar{\xi}_3 + \mu_3(t),\end{aligned}\quad (22)$$

where, $\mu_1(t)$, $\mu_2(t)$ and $\mu_3(t)$ are the control function. Now, the error system is defined as

$$\begin{aligned}e_1(t) &= \bar{\xi}_1(t) - \xi_1(t), \\ e_2(t) &= \bar{\xi}_2(t) - \xi_2(t), \\ e_3(t) &= \bar{\xi}_3(t) - \xi_3(t)\end{aligned}\quad (23)$$

In the term of state variables the error system is expressed as

$$\begin{aligned}\dot{e}_1 &= \alpha\{\bar{\xi}_2(t - \tau_2) - \xi_2(t - \tau_1)\} + 0.01\bar{\xi}_1 - 0.01\xi_1 + \mu_1(t) \\ \dot{e}_2 &= \beta(e_2 - e_1) + \bar{\xi}_2\bar{\xi}_3 - \xi_2\xi_3 - 0.01\bar{\xi}_2 + 0.01\xi_2 + \mu_2(t) \\ \dot{e}_3 &= -\bar{\xi}_2^2 + \xi_2^2 - 0.01\bar{\xi}_3 + 0.01\xi_3 + \mu_3(t)\end{aligned}\quad (24)$$

From above equation we can design the adaptive parameters as

$$\begin{aligned}\mu_1(t) &= -\hat{\alpha}\{\bar{\xi}_2(t - \tau_2) - \xi_2(t - \tau_1)\} - 0.01\bar{\xi}_1 + 0.01\xi_1 \\ &\quad - p_1e_1, \\ \mu_2(t) &= -\hat{\beta}(e_2 - e_1) - \bar{\xi}_2\bar{\xi}_3 + \xi_2\xi_3 + 0.01\bar{\xi}_2 - 0.01\xi_2 \\ &\quad - p_2e_2, \\ \mu_3(t) &= \bar{\xi}_2^2 - \xi_2^2 + 0.01\bar{\xi}_3 - 0.01\xi_3 - p_3e_3,\end{aligned}\quad (25)$$

where, $\hat{\alpha}$ and $\hat{\beta}$ are the estimation parameters of α and β respectively. Further, we suppose

$$e_\alpha = \alpha - \hat{\alpha}, e_\beta = \beta - \hat{\beta}, \dot{e}_\alpha = -\dot{\hat{\alpha}}, \dot{e}_\beta = -\dot{\hat{\beta}} \quad (26)$$

Now, equation (24) can be expressed as

$$\begin{aligned}\dot{e}_1 &= e_\alpha\{\bar{\xi}_2(t - \tau_2) - \xi_2(t - \tau_1)\} - p_1e_1, \\ \dot{e}_2 &= e_\beta(e_2 - e_1) - p_2e_2, \\ \dot{e}_3 &= -p_3e_3,\end{aligned}\quad (27)$$

where p_1 , p_2 , p_3 are positive constants. Choose a Lyapunov's function as

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_\alpha^2 + e_\beta^2) \quad (28)$$

Further, the time derivative of the Lyapunov function will be expressed as

$$\begin{aligned}\dot{V} = & e_1(e_\alpha\{\bar{\xi}_2(t-\tau_2) - \xi_2(t-\tau_1)\} - p_1e_1) + \\ & e_2(e_\beta(e_2 - e_1) - p_2e_2) + e_3(-p_3e_3) + e_\alpha(-\dot{\hat{\alpha}}) \\ & + e_\beta(-\dot{\hat{\beta}}), \\ = & -p_1e_1^2 - p_2e_2^2 - p_3e_3^2 - e_\alpha(-e_1\{\xi_2(t-\tau_2) - \\ & \xi_1(t-\tau_1)\} + \dot{\hat{\alpha}}) - e_\beta(e_1e_2 - e_2^2 + \dot{\hat{\beta}})\end{aligned}\quad (29)$$

From equation (29) the derivative of estimated parameters can be achieved as

$$\begin{aligned}\dot{\hat{\alpha}} &= e_1\{\bar{\xi}_2(t-\tau_2) - \xi_2(t-\tau_1)\} + p_4e_\alpha, \\ \dot{\hat{\beta}} &= -e_1e_2 + e_2^2 + p_5e_\beta,\end{aligned}\quad (30)$$

where p_4 and p_5 are positive constants. After putting the values of $\dot{\hat{\alpha}}$ and $\dot{\hat{\beta}}$ from (30) in (29) we obtain

$$\dot{V} = -p_1e_1^2 - p_2e_2^2 - p_3e_3^2 - p_4e_\alpha^2 - p_5e_\beta^2 \quad (31)$$

From above equation we can say that the error system (24) achieved the synchronization according to stability theory [28]. Now, we analyze the synchronization between another time-delay chaotic systems by same method. For this purpose, we assume the system (4) as master system and following system as slave system.

$$\begin{aligned}\dot{\chi}_1 &= \chi_1\chi_3 - 7\chi_3 - 0.02\chi_1 + \rho_1(t), \\ \dot{\chi}_2 &= -b\chi_2^2 + \chi_3^2 + 0.01\chi_2 + \rho_2(t), \\ \dot{\chi}_3 &= -\chi_1^2 + a\chi_2^2 + 14\chi_1(t - \omega_2) + \chi_3 - 49 - 0.01\chi_3 + \\ & \rho_3(t),\end{aligned}\quad (32)$$

where, χ_1 , χ_2 and χ_3 are the state variables and a , b are the parameters and $\rho_1(t)$, $\rho_2(t)$, $\rho_3(t)$ are control function. For above systems, the error system is define as

$$\begin{aligned}e_1(t) &= \chi_1(t) - \eta_1(t), \\ e_2(t) &= \chi_2(t) - \eta_2(t), \\ e_3(t) &= \chi_3(t) - \eta_3(t),\end{aligned}\quad (33)$$

Using equation (4) and (32) in (33), we have

$$\begin{aligned}\dot{e}_1(t) &= \chi_1\chi_3 - \eta_1\eta_3 - 7e_3 - 0.02\chi_1 + 0.02\eta_1 + \rho_1(t), \\ \dot{e}_2(t) &= -b(\chi_2^2 - \eta_2^2) + \chi_3^2 - \eta_3^2 + 0.01\chi_2 - 0.01\eta_2 \\ & + \rho_2(t), \\ \dot{e}_3(t) &= -\chi_1^2 + \eta_1^2 + a(\chi_2^2 - \eta_2^2) + 14\chi_1(t - \omega_2) - 0.01\chi_3 \\ & - 14\eta_1(t - \omega_1) + e_3 + \rho_3(t),\end{aligned}\quad (34)$$

Now, the control functions are expressed as

$$\begin{aligned}\rho_1(t) &= -\chi_1\chi_3 + \eta_1\eta_3 + 7e_3 + 0.02\chi_1 - 0.02\eta_1 - q_1e_1, \\ \rho_2(t) &= \hat{b}(\chi_2^2 - \eta_2^2) - \chi_3^2 + \eta_3^2 - 0.01\chi_2 + 0.01\eta_2 - q_2e_2, \\ \rho_3(t) &= \chi_1^2 - \eta_1^2 - \hat{a}(\chi_2^2 - \eta_2^2) - 14\chi_1(t - \omega_2) + 0.01\chi_3 \\ & - 0.01\eta_3 + 14\eta_1(t - \omega_1) - e_3 - q_3e_3,\end{aligned}\quad (35)$$

where, $\hat{a}$, $\hat{b}$ are estimation of parameters of a and b respectively. Substituting equation (35) in (34), we get

$$\begin{aligned}\dot{e}_1(t) &= -q_1e_1, \\ \dot{e}_2(t) &= -(b - \hat{b})(\chi_2^2 - \eta_2^2) - q_2e_2, \\ \dot{e}_3(t) &= (a - \hat{a})(\chi_2^2 - \eta_2^2) - q_3e_3\end{aligned}\quad (36)$$

Now, error dynamics and time derivative of estimated parameters are given as

$$e_a = a - \hat{a}, e_b = b - \hat{b}, \dot{e}_a = -\dot{\hat{a}}, \dot{e}_b = -\dot{\hat{b}} \quad (37)$$

So equation (36) can be written as

$$\begin{aligned}\dot{e}_1(t) &= -q_1e_1, \\ \dot{e}_2(t) &= -e_b(\chi_2^2 - \eta_2^2) - q_2e_2, \\ \dot{e}_3(t) &= e_a(\chi_2^2 - \eta_2^2) - q_3e_3\end{aligned}\quad (38)$$

The Lyapunov's function and its derivative will be expressed as

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_a^2 + e_b^2) \quad (39)$$

$$\begin{aligned}\dot{V} = & e_1(-q_1e_1) + e_2\{-e_b(\chi_2^2 - \eta_2^2) - q_2e_2\} + \\ & e_3\{e_a(\chi_2^2 - \eta_2^2) - q_3e_3\} + e_a(-\dot{\hat{a}}) + e_b(-\dot{\hat{b}}), \\ = & -q_1e_1^2 - q_2e_2^2 - q_3e_3^2 - e_a(-e_3(\chi_2^2 - \eta_2^2) + \dot{\hat{a}}) - \\ & e_b(e_2(\chi_2^2 - \eta_2^2) + \dot{\hat{b}})\end{aligned}\quad (40)$$

The estimated parameters can be obtained from equation (40) as

$$\begin{aligned}\dot{\hat{a}} &= e_3(\chi_2^2 - \eta_2^2) + q_4e_a, \\ \dot{\hat{b}} &= -e_2(\chi_2^2 - \eta_2^2) + q_5e_b,\end{aligned}\quad (41)$$

where, q_4 and q_5 are positive constants. Finally, the Lyapunov's function will be written as

$$\dot{V} = -q_1e_1^2 - q_2e_2^2 - q_3e_3^2 - q_4e_a^2 - q_5e_b^2 \quad (42)$$

Hence, from above equation we can say that the error system has achieved synchronization.

Now, we will discuss synchronization between two non-identical time-delay chaotic systems by adaptive control method. We consider the time-delay uncertain chaotic systems (4) as master system and chaotic system (2) will assume as slave system. So, the controlled slave system is expressed as

$$\begin{aligned}\dot{\eta}_1 &= \eta_1\eta_3 - 7\eta_3 - 0.02\eta_1 + \delta_1(t), \\ \dot{\eta}_2 &= -b\eta_2^2 + \eta_3^2 + 0.01\eta_2 + \delta_2(t), \\ \dot{\eta}_3 &= -\eta_1^2 + a\eta_2^2 + 14\eta_1(t - \omega_1) + \eta_3 - 49 - 0.01\eta_3 + \delta_3(t),\end{aligned}\quad (43)$$

where, $\delta_1(t)$, $\delta_2(t)$, $\delta_3(t)$ are adaptive control function. Define the error system as

$$\begin{aligned}e_1(t) &= \eta_1(t) - \xi_1(t), \\ e_2(t) &= \eta_2(t) - \xi_2(t), \\ e_3(t) &= \eta_3(t) - \xi_3(t)\end{aligned}\quad (44)$$

Hence, the error system for master system (4) and slave system (43) according to error system (44) can be expressed as

$$\begin{aligned}\dot{e}_1 &= \eta_1\eta_3 - 7\eta_3 - \alpha\xi_2(t - \tau_1) - 0.02\eta_1 - 0.01\xi_1 + \delta_1(t), \\ \dot{e}_2 &= -b\eta_2^2 + \eta_3^2 - \beta(\xi_2 - \xi_1) - \xi_2\xi_3 + 0.01\eta_2 + 0.01\xi_2 \\ & + \delta_2(t), \\ \dot{e}_3 &= -\eta_1^2 + a\eta_2^2 + 14\eta_1(t - \omega_1) + \eta_3 - 49 - \gamma + \xi_2^2 - \\ & 0.01\eta_3 + 0.01\xi_3 + \delta_3(t),\end{aligned}\quad (45)$$

Here, the control function are supposed that

$$\begin{aligned}\delta_1(t) &= -\eta_1\eta_3 + 7\eta_3 + \hat{\alpha}\xi_2(t - \tau_1) + 0.02\eta_1 + 0.01\xi_1 - \\ & r_1e_1, \\ \delta_2(t) &= \hat{b}\eta_2^2 - \eta_3^2 + \hat{\beta}(\xi_2 - \xi_1) + \xi_2\xi_3 - 0.01\eta_2 - 0.01\xi_2 \\ & - r_2e_2, \\ \delta_3(t) &= \eta_1^2 - \hat{a}\eta_2^2 - 14\eta_1(t - \omega_1) - \eta_3 + 49 + \hat{\gamma} - \xi_2^2 + \\ & 0.01\eta_3 - 0.01\xi_3 - r_3e_3,\end{aligned}\quad (46)$$

where $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ and $\hat{a}$, $\hat{b}$ are the estimation of parameters of α , β , γ and a , b respectively and r_1 , r_2 , r_3 are positive

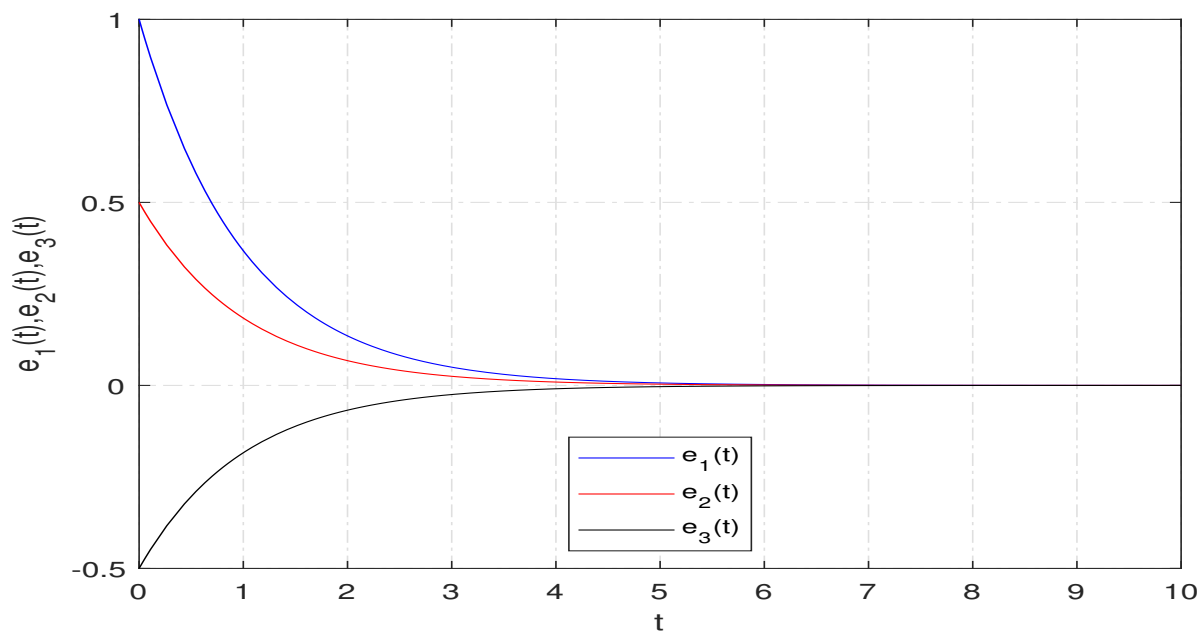


Fig. 3. The time evolution of state of error system $e_1(t), e_2(t), e_3(t)$ by active control method.

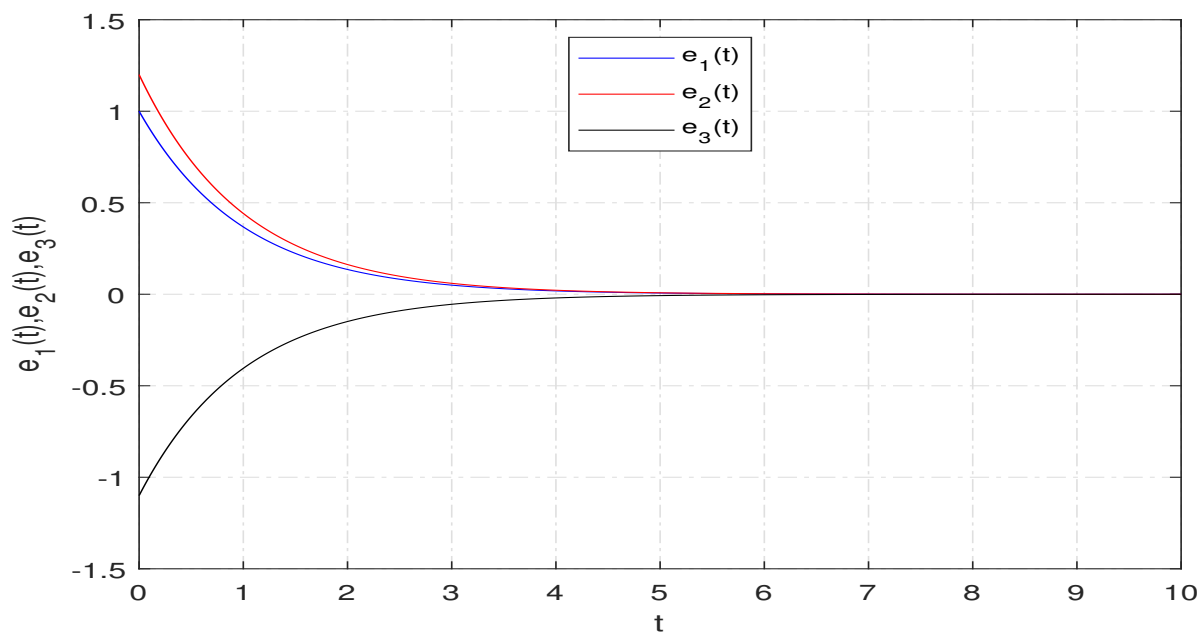


Fig. 4. The time evolution of state of error system $e_1(t), e_2(t), e_3(t)$.

constants.

After substituting the value of control function from (46) in (45), we get the error system as

$$\begin{aligned} \dot{e}_1 &= -(\alpha - \hat{\alpha})\xi_2(t - \tau_1) - r_1 e_1, \\ \dot{e}_2(t) &= -(b - \hat{b})\eta_2^2 - (\beta - \hat{\beta})(\xi_2 - \xi_1) - r_2 e_2, \\ \dot{e}_3(t) &= (a - \hat{a})\eta_2^2 - (\gamma - \hat{\gamma}) - r_3 e_3, \end{aligned} \quad (47)$$

Now, the parameters estimation can be written in succinct form as

$$e_\alpha = \alpha - \hat{\alpha}, e_\beta = \beta - \hat{\beta}, e_\gamma = \gamma - \hat{\gamma}, e_a = a - \hat{a}, e_b = b - \hat{b} \quad (48)$$

The time derivative of above equation will be written in the

following form

$$\dot{e}_\alpha = -\dot{\hat{\alpha}}, \dot{e}_\beta = -\dot{\hat{\beta}}, \dot{e}_\gamma = -\dot{\hat{\gamma}}, \dot{e}_a = -\dot{\hat{a}}, \dot{e}_b = -\dot{\hat{b}} \quad (49)$$

Putting the derivative of estimated parameters from (49) in (47), we have

$$\begin{aligned} \dot{e}_1 &= -e_\alpha \xi_2(t - \tau_1) - r_1 e_1, \\ \dot{e}_2(t) &= -e_b \eta_2^2 - e_\beta (\xi_2 - \xi_1) - r_2 e_2, \\ \dot{e}_3(t) &= e_a \eta_2^2 - e_\gamma - r_3 e_3, \end{aligned} \quad (50)$$

Now we assume the Lyapunov's function and its derivative are explored as

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_\alpha^2 + e_\beta^2 + e_\gamma^2 + e_a^2 + e_b^2) \quad (51)$$

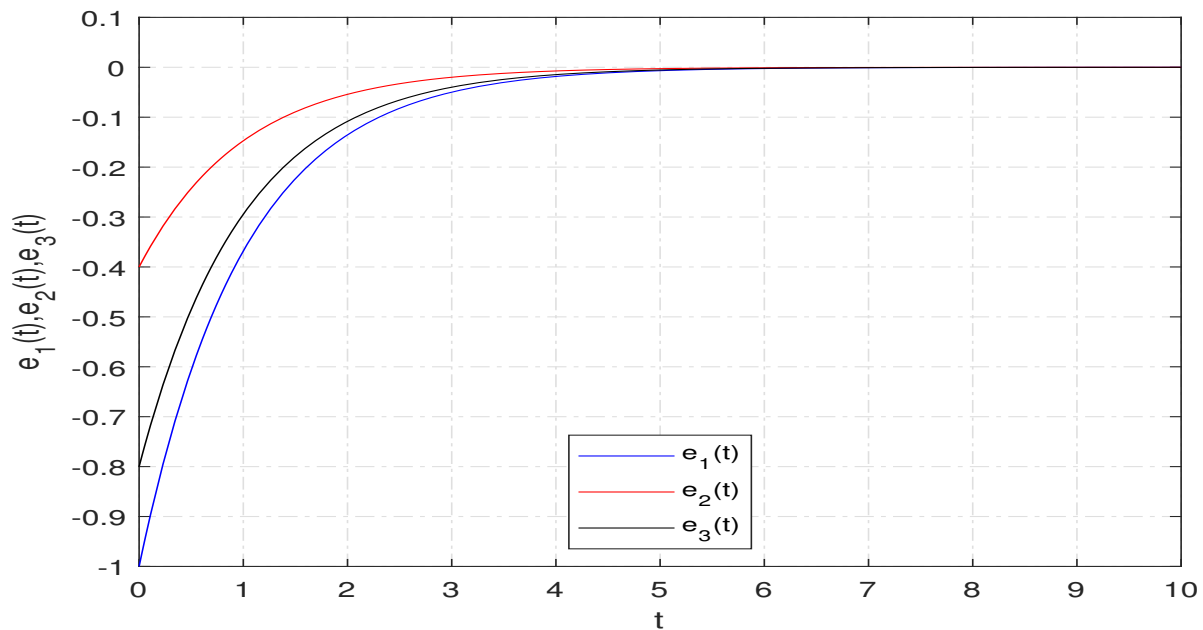


Fig. 5. The time evolution of state of error system $e_1(t), e_2(t), e_3(t)$ between non-identical chaotic systems.

$$\begin{aligned} \dot{V} = & e_1 \{-e_\alpha \xi_2(t - \tau_1) - r_1 e_1\} + e_2 \{-e_b \eta_2^2 - e_\beta (\xi_2 - \xi_1) \\ & - r_2 e_2\} + e_3 \{e_a \eta_2^2 - e_\gamma - r_3 e_3\} + e_\alpha (-\dot{\hat{\alpha}}) + e_\beta (-\dot{\hat{\beta}}) \\ & + e_\gamma (-\dot{\hat{\gamma}}) + e_a (-\dot{\hat{a}}) + e_b (-\dot{\hat{b}}), \\ = & -r_1 e_1^2 - r_2 e_2^2 - r_3 e_3^2 - e_\alpha (e_1 \xi_2(t - \tau_2) + \dot{\hat{\alpha}}) - \\ & e_\beta (e_2 (\xi_2 - \xi_1) + \dot{\hat{\beta}}) - e_\gamma (\dot{\hat{\gamma}}) - e_a (e_3 \eta_2^2 + \dot{\hat{a}}) - \\ & e_b (e_2 \eta_2^2 + \dot{\hat{b}}) \end{aligned} \quad (52)$$

From equation (52), we get

$$\begin{aligned} \dot{\hat{\alpha}} &= -e_1 \xi_2(t - \tau_2) + r_4 e_\alpha, \\ \dot{\hat{\beta}} &= -e_2 (\xi_2 - \xi_1) + r_5 e_\beta, \\ \dot{\hat{\gamma}} &= r_6 e_\gamma, \\ \dot{\hat{a}} &= -e_3 \eta_2^2 + r_7 e_a, \\ \dot{\hat{b}} &= -e_2 \eta_2^2 + r_8 e_b, \end{aligned} \quad (53)$$

where r_4, r_5, r_6, r_7 and r_8 are positive constants. Substituting equation (53) in (52), we get

$$\dot{V} = -r_1 e_1^2 - r_2 e_2^2 - r_3 e_3^2 - r_4 e_\alpha^2 - r_5 e_\beta^2 - r_6 e_\gamma^2 - r_7 e_a^2 - r_8 e_b^2 \quad (54)$$

Thus we can say that error system has achieved the synchronization.

V. NUMERICAL RESULTS AND DISCUSSION

For numerical simulation MATLAB software is used to depict the basic features of chaotic systems. The initial condition of the drive system (1) and (2) are taken as (1, 1, 1) while the initial condition for the response system is considered as (0, 0.6, 0.6). The parametric values of the system (2) and (4) are taken as $\alpha = 1, \beta = 1, \gamma = 1$ and $a = 3, b = 1$ respectively. Fig. 1 and Fig. 2 depict the phase portraits of systems (2) and (4) respectively while Fig. 3, Fig. 4 and Fig. 5 show the state of error w.r.t. time by active control method. Similarly, Fig. 6, Fig. 7 and Fig. 8 represent the parameter estimation and error system w.r.t. time by adaptive control method. Further, to study the

time-delay effects we consider the values of parameters as $\tau_1 = 0.1, \omega_1 = 0.01, \tau_2 = 0.2$ and $\omega_2 = 0.02$.

VI. CONCLUSION

In this paper, we analyze time-delay chaotic systems with uncertainties. We proposed a control law to synchronize the chaotic systems incase of identical and non-identical systems by using active and adaptive control methods. These analysis are more useful in engineering, cryptosystems and secure communication. Lyapunov stability theory, is employed to calculate the controllers. For effectiveness and validation of theory the numerical and graphical simulations are also demonstrated.

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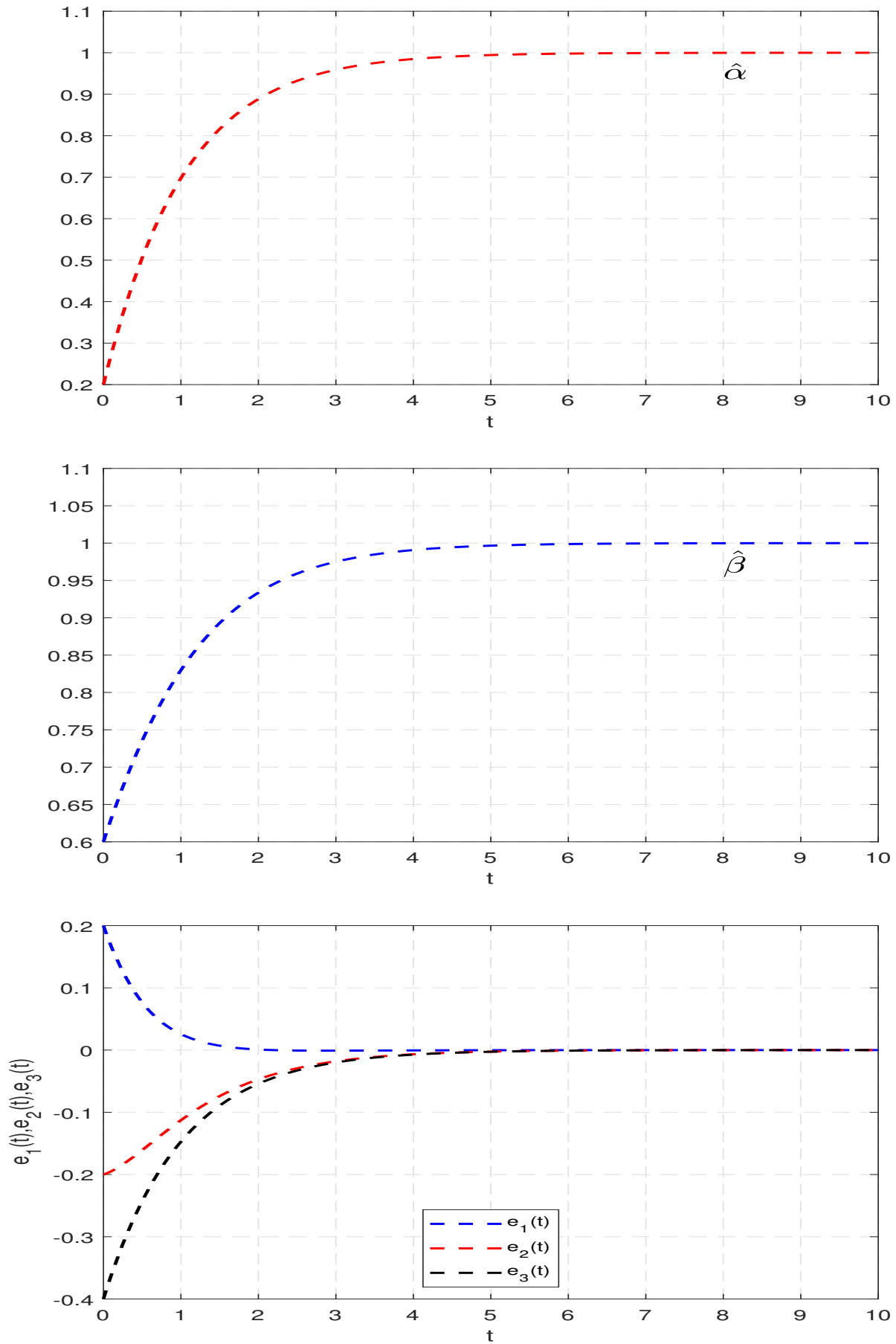


Fig. 6. Parameter estimations of $\hat{\alpha}, \hat{\beta}$ and the time evolution of error state $e_1(t), e_2(t), e_3(t)$ between identical chaotic systems.

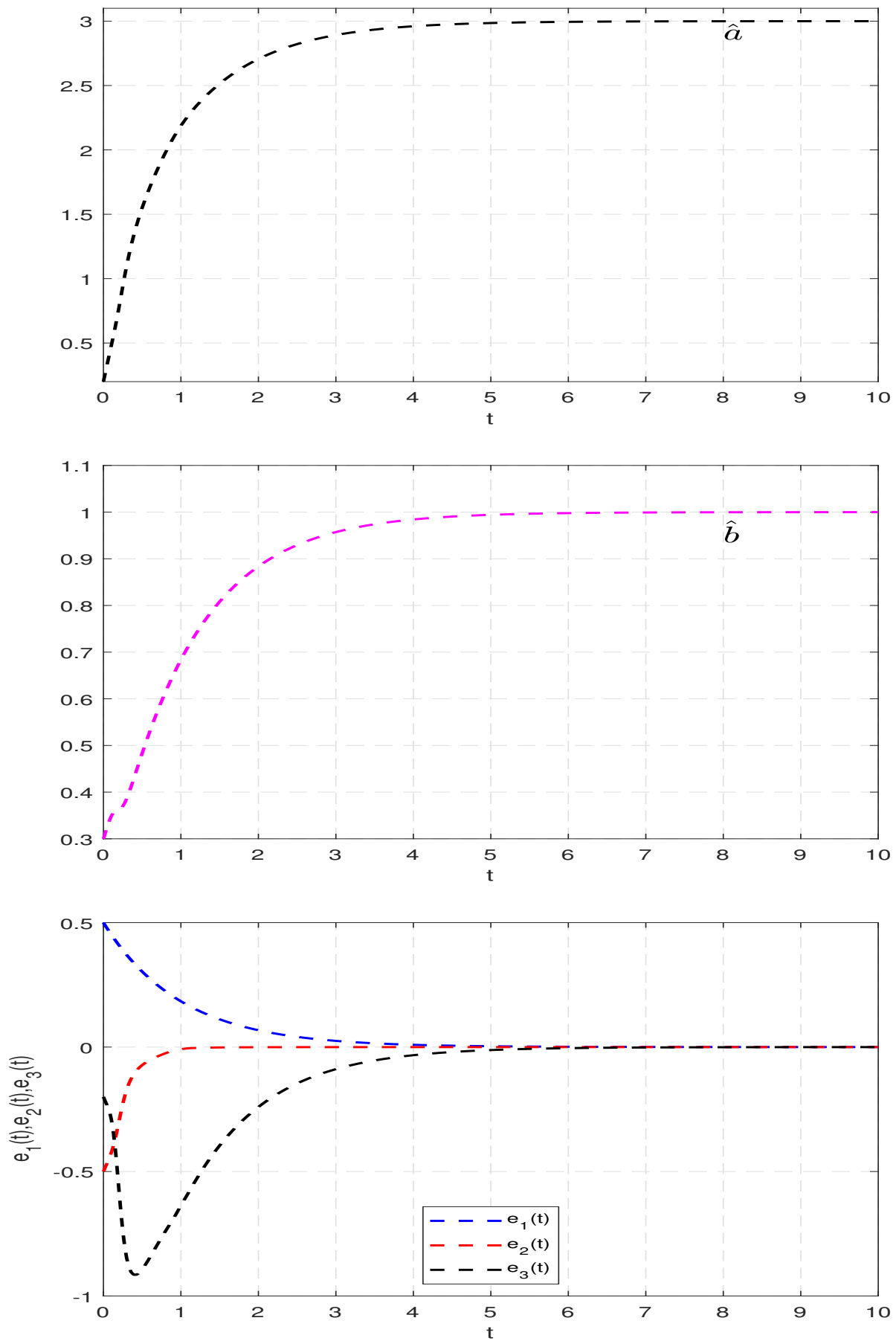


Fig. 7. Parameter estimations of $\hat{a}$, $\hat{b}$ and the time evolution of error state $e_1(t), e_2(t), e_3(t)$ between identical chaotic systems.

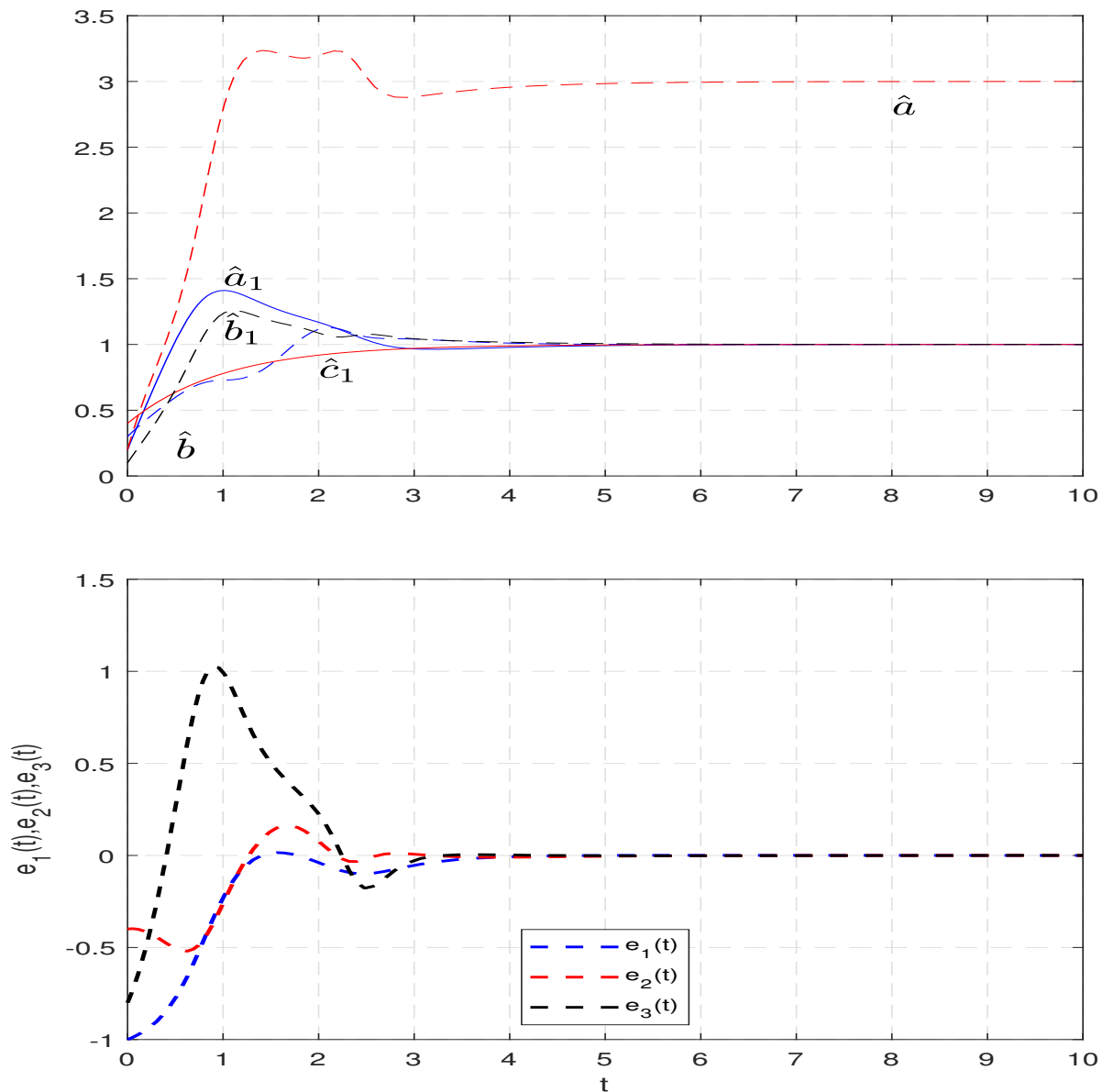


Fig. 8. Parameter estimations of $\hat{a}$, $\hat{b}$, $\hat{a}_1$, $\hat{b}_1$, $\hat{c}_1$ and the time evolution of error state $e_1(t)$, $e_2(t)$, $e_3(t)$ between non-identical chaotic systems.

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