Analysis of Atangana-Baleanu-Caputo Impulsive Fractional Delay Differential Equations

Wisdom Udogworen, Dodi Igobi, Uko Jim

Abstract—**This work uses the generalised boundedness condition on Schauder's fixed point and the generalised Lipschitz condition on Banach contraction principle to investigate sufficient conditions for the existence, uniqueness, and stability of solutions for a novel class of Atangana-Baleanu-Caputo impulsive fractional delay differential equations with the Caratheodory function. The developed notion is demonstrated with an example, and the outcomes validate its applicability.**

Index Terms—**Atangana-Baleanu-Caputo, fractional derivative, fractional delay equation, impulsive fractional equations, caratheodory function.**

I. Introduction

FRACTIONAL calculus has grown and garnered a lot of interest in the mathematical sciences in recent years. Of interest is the improvement on the modifications of the integral calculus of the fractional derivatives from the singular kernel types (the Riemann-Liouville and Caputo fractional differential equations) to the non-singular, non-local kernel types (the Caputo-Fabrizio fractional derivative and Atangana-Baleanu fractional derivative) introduced in [9] and [19], respectively. [1] defines the general classical Riemann-Liouville fractional integral of order ξ as

$$({}_{t_0}I_t^{\xi})f(t) = \frac{1}{\Gamma(\xi)}\int_{t_0}^{t}(t-s)^{\xi-1}f(s)ds, \qquad (1)$$

where

$$\Gamma(\xi) = \int_0^{\infty}(t)^{\xi-1}e^{-t}dt \qquad \xi > 0, \qquad (2)$$

is the gamma function, and the corresponding Riemann-Liouville derivative of order ξ is as follows:

$$\begin{aligned} &{}_{t_0}^{RL}D_t^{\xi}f(t) = \\ &\frac{d^n}{dt^n}{}_{t_0}I_t^{n-\xi}f(t) = \frac{1}{\Gamma(n-\xi)}\frac{d^n}{dt^n}\int_{t_0}^{t}(t-s)^{n-\xi-1}f(s)ds. \end{aligned} \qquad (3)$$

Equation (3) approach, however, results in initial conditions containing the limit values of the Riemann-Liouville fractional derivatives at the lower terminal $t = t_0$, such as $\lim_{t\to t_0}\ {}_{t_0}D_t^{\xi-1}f(t) = t_0$, $\lim_{t\to t_0}\ {}_{t_0}D_t^{\xi-2}f(t) = t_1$, $\lim_{t\to t_0}\ {}_{t_0}D_t^{\xi-n}f(t) = t_n$, where $t_k, k = 1, 2, \cdots n$ are given constants. Applied problems require definitions of fractional derivatives that allow the use of physically interpretable initial conditions, which contain $f(t_0), f^{'}(t_0)$.
With M. Caputo's modification of (3), initial conditions for initial value problems for fractional order differential equations can be formulated using only the limit values of integer order derivatives of unknown functions at the lower terminal (initial time) $t = t_0$, as in $f(t_0), f^{'}(t_0)$. This has a more physical interpretation, and the general Caputo fractional derivative of order ξ is defined in [6] as

$${}_{t_0}^{C}D_t^{\xi}f(t) = \frac{1}{\Gamma(n-\xi)}\int_{t_0}^{t}(t-s)^{n-\xi-1}D^nf(s)ds. \qquad (4)$$

An additional benefit of equations (4) over equations (3) is that a constant's derivative is zero (${}_0^CD_t^{\xi}C = 0$), The constant in equations (3) is not zero, but ${}_0^RD_t^{\xi}C = \frac{Ct^{-\xi}}{\Gamma(1-\xi)}$ (in contrast).
$\mathcal{L}({}_0I_t^{\xi}f)(t) = s^{-\xi}F(s)$,and $\mathcal{L}\ {}_0^{RL}D_t^{\xi}f(t)dt = s^{\xi}F(s) - \sum_{k=0}^{n-1}s_0^kD_t^{\xi-k-1}f(t)|_{t=0}$ are the Laplace transform of the equations (1) and (3) respectively and that of equations (4) is $\mathcal{L}\ {}_0^CD_t^{\xi}f(t)dt = s^{\xi}F(s) - \sum_{k=0}^{n-1}s^{\xi-k-1}f^k(0)$.
In order to address the singularity kernel issue with the fractional derivative discussed above, a new derivative with an exponential kernel known as the Caputo-Fabrizio fractional derivative presented in [9] as

$$({}_{t_0}^{CFC}D_t^{\xi})f(t) = \frac{\psi(\xi)}{1-\xi}\int_{t_0}^{t}f^{'}(s)e^{[-\xi\frac{(t-s)}{1-\xi}]}ds, \qquad (5)$$

was suggested. The goal of Caputo-Fabrizio's work was to improve the description of memory-effect systems' dynamics. Some disadvantages of the derivative were noted, though, such as the local nature of the kernel that was employed and the fact that the anti-derivative linked to their derivative is the average of the function and its integral rather than a fractional integral. Atangana-Baleanu suggested modifying (5) using the Mittag-Leffler function to solve the problems; as a result, this version of the fractional derivative is regarded as a non-singular, non-local fractional derivative that was first presented in [19] as

$$({}_{t_0}^{ABC}D_t^{\xi})z(t) = \frac{\psi(\xi)}{1-\xi}\int_{t_0}^{t}f^{'}(s)E_{\xi}\big[-\xi\frac{(t-s)^{\xi k}}{1-\xi}\big]ds, \qquad (6)$$

and in Riemann-Liouville sense as

$$({}_{t_0}^{ABR}D_t^{\xi})z(t) = \frac{\psi(\xi)}{1-\xi}\frac{df}{dt}\int_{t_0}^{t}f(s)E_{\xi}\big[-\xi\frac{(t-s)^{\xi k}}{1-\xi}\big]ds, \qquad (7)$$

The function $\psi(\xi)$ share the same characteristics as in the case of Caputo-Fabrizio fractional derivative, where the single parametric Mittag-Leffler function E_{ξ} is defined as

$$E_{\xi}(z) = \sum_{k=o}^{\infty}\frac{z^k}{\Gamma(k\xi+1)}. \qquad (8)$$

with its corresponding integral of equations (6) as

$$z(t) = \frac{1-\xi}{\psi(\xi)}f(t,z(t)) + \frac{\xi}{\psi(\xi)\Gamma(\xi)}\int_{t_0}^{t}f(s,z(s))(t-s)^{\xi-1}ds \qquad (9)$$

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(5) and (6) Laplace transform are $\mathcal{L}({}_0^{CFC}D_t^{\xi}f)(t) = \frac{\psi(\xi)(sF(s)-f(0))}{s(1-\xi)+\xi}$ and $\mathcal{L}({}_0^{ABC}D_t^{\xi}f)(t) = \frac{\psi(\xi)}{1-\xi}\frac{s^{\xi-1}(sF(s)-f(0))}{s^{\xi}+\frac{\xi}{1-\xi}}$ respectively.

Similar to other differential equations, functional differential equations are generalised to their fractional counterparts in [8]. Fractional delay differential equations are crucial for simulating the majority of real-world issues because the fractional rate of change dependent on the system's hereditary effects, which include a sequence of influences from previous states and future evolution processes.

For the Atangana-Baleanu-Caputo fractinoal delay differential equations, we take into consideration its general form, which is

$$({}_{t_0}^{ABC}D_t^{\xi})z(t) = f(t, z_t), \tag{10}$$

where the Atangana-Baleanu-Caputo derivative is represented by $({}_{t_0}^{ABC}D_t^{\xi})z(t)$ and the dynamics of the delay equations are described by $f(t, z_t)$. Many evolutionary processes undergo rapid changes in state at specific points in time, which may be caused by natural disasters, harvesting, or shocks [3]. This phenomenon characterises the state of dynamics of these processes. These processes are susceptible to brief disruptions, the length of which is insignificant when compared to the process's duration. These transient disruptions are frequently interpreted as having occurred instantly, that is, as impulses that are portrayed as

$$z(t_\tau^+) = z(t_\tau^-) + I_\tau(t_\tau, z(t_\tau^-)), \tag{11}$$

At some times t_τ, the system's state jumps due to abrupt changes introduced by an impulsive fractional differential delay equations. Typically, these jumps are explained as follows: $z(t_\tau^+)$ represents the system's state immediately following the impulse at time t_τ, $z(t_\tau^-)$ represents the system's state immediately before the impulse at time t_τ, $I_\tau(t_\tau, z(t_\tau^-))$ is the impulse function, which is dependent on the state $z(t_\tau^-)$ prior to the impulse and the time t_τ. Adding an initial term $z_{t_0} = \theta_0 \in PC'$ to equations (10) and (11) yields a novel class of Atangana-Baleanu impulsive fractional delay differential equations with fixed moments of impulse effect of the type

$$\begin{cases} {}_{t_0}^{ABC}D_t^{\xi}z(t) = f(t, z_t), & t \in Q = [t_0, q], \quad t \neq t_\tau \\ \Delta z(t_\tau) = I_\tau(t_\tau, z(t_\tau)), & t = t_\tau, \quad \tau = 1, 2, \cdots, p \\ z_{t_0} = \theta_0 \in PC' \end{cases} \tag{12}$$

where $\Delta z(t) = z(t_\tau^+) - z(t_\tau^-)$, $t_0 < t_1 <, \cdots < t_p < t_{p+1} = q$, $f : Q \times PC^1 \to \mathbb{R}^n$ is Lebesgue measurable with respect to $t \in Q$ and $f(t, \theta)$ is continuous with respect to $\theta \in PC^1$ and $I_\tau : Q \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous for $\tau = 1, \cdots, p$, where $PC^1 = PC^1[[-r, 0], \mathbb{R}^n]$ denotes the space of piecewise left continuous derivative functions $\theta[-r, 0] \to \mathbb{R}^n$ with the $\|\theta\|_1 = \sup_{-r\le s\le 0} \|\theta\|_1$, here $r > 0$, $\|.\|$ is a norm in $\mathbb{R}^n$. $z_t \in PC'$ is defined by $z_t(k) = z(t + k) = \theta(k)$ for $t_0 - r \le s \le t_0$. The set $PC^1(\eta) = \{\theta \in PC^1 : \|\theta\|_1 < \eta, \forall \eta > 0\}$

$PC'[[t_0 - r, q], \mathbb{R}^n]$ denote the Banach space of all continuous functions from Q into $\mathbb{R}^n$ with the norm $\|z\|_\infty = \sup_{t\in[t_0-r,q]} \|z(t)\|$.

A function $z \in PC'[[t_0 - r, q], \mathbb{R}^n]$ is said to be a solution of (12) if z satisfies the equations ${}_{t_0}^{AB}D_t^{\xi}z(t) = f(t, z_t)$ for almost everywhere $t \in Q$; the equations $z(t_\tau^+) = z(t_\tau^-) + I_\tau(t_\tau, z(t_\tau^-))$ and the condition $z_{t_0} = \theta_0$ such that

$$z(t) = \begin{cases} \theta_0(0) + \frac{1-\xi}{\psi(\xi)}f(t, z_t) + \\ \frac{\xi}{\psi(\xi)\Gamma(\xi)}\int_{t_0}^{t} f(s, z_s)(t-s)^{\xi-1}ds, & t \in [t_0, q], \\ \theta_0(0) + \frac{1-\xi}{\psi(\xi)}\sum_{i=1}^{\tau} f(t_i, z_t) + \\ \frac{\xi}{\psi(\xi)\Gamma(\xi)}\sum_{i=1}^{\tau}\int_{t_{i-1}}^{t_i}(t_i - s)^{\xi-1}f(s, z_s)ds \\ +\frac{\xi}{\psi(\xi)\Gamma(\xi)}\int_{t_\tau}^{t} f(s, z_s)(t-s)^{\xi-1}ds \\ +\sum_{i=1}^{\tau} I_i(t_i, z(t_i)), & t \in (t_\tau, t_{\tau+1}], \tau = 1\cdots, p \end{cases} \tag{13}$$

Examining how these model equations behave qualitatively is crucial, particularly when modelling real-world issues in the fields of biology, medicine, and economics.

It is discussed in [2] [5] [6] [7] [10] [12] [13] [18] that the initial value problem for nonlinear impulsive fractional differential equations involving Caputo, Caputo-Fabrizio, and Atangana-Baleanu has qualitative properties of existence, uniqueness and stability.

We extend the idea to examine the requirements for existence, uniqueness and stability of solutions for a novel class of Atangana-Baleanu impulsive fractional delay differential equations. This is motivated by the novel idea of generalising the boundedness condition and Lipschitz condition in [4] [6] [2] [15], to established results on the existence, uniqueness and stability of solutions of nonlinear fractional differential equations involving the Caputo derivative.

II. Preliminaries

This section presents the initial findings.

Let $0 < \xi < 1$ and $f : Q \times PC^1 \to \mathbb{R}^n$ be continuous, then the IVP

$$\begin{cases} {}_{t_0}^{AB}D_t^{\xi}z(t) = f(t, z_t), & t \in Q = [t_0, q], \\ z_{t_0} = \theta_0 \in PC' \end{cases} \tag{14}$$

is comparable to the following: Fractional integral Volterra with memory

$$\begin{cases} z_0 = \theta_0 \\ z(t) = \theta_0(0) + \frac{1-\xi}{\psi(\xi)}f(t, z_t) + \\ \frac{\xi}{\psi(\xi)\Gamma(\xi)}\int_{t_0}^{t} f(s, z_s)(t-s)^{\xi-1}ds, & t \in [t_0, q]. \end{cases} \tag{15}$$

In other words, all solutions of (14) are also solutions of (15), and vice versa.

Remark 1 Let $f : Q \times PC^1 \to \mathbb{R}^n$ be a Caratheodory function and $\zeta \in [0, \xi)$ be a constant, then a function $f(t, z_t)$ is a $L^{\frac{1}{\zeta}}$- Caratheodory if there exists a real-value function $h(t) \in L^{\frac{1}{\zeta}}(Q)$ such that

$$\|f(t, z_t)\| \le h(t) \qquad a.e \qquad t \in Q$$

Remark 2 Let $f : Q \times PC^1 \to \mathbb{R}^n$, a constant $\omega \in [0, \xi)$ and a real-value function $\nu(t) \in L^{\frac{1}{\omega}}(Q)$ then the function $f(t, z_t)$ is $L^{\frac{1}{\omega}}$- Lipschitz if

$$\|f(t, z_t) - f(t, y_t)\| \le \nu(t)\|z_t - y_t\| \qquad a.e \qquad t \in Q$$

Lemma 1. *Let $0 < \xi < 1$ and $f : Q \times PC^1 \to \mathbb{R}^n$ be Lebesgue measurable in t on Q and $f(t, \theta)$ is continuous in θ on PC^1. A function $z \in PC^1[[t_0 - r, q], \mathbb{R}^n]$, is a solution* *of the IVP (12) if and only if $z \in PC^1[[t_0-r,q],\mathbb{R}^n]$ satisfies the fractional integral equations* (13)

Proof: Assume z satisfies the IVP (15). for $t \in [t_0, t_1]$ such that

$$z(t) = \theta_0(0) + \frac{1-\xi}{\psi(\xi)} f(t, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_0}^{t} (t-s)^{\xi-1} f(s, z_s) ds.$$

Then

$$z(t_1^-) = \theta_0(0) + \frac{1-\xi}{\psi(\xi)} f(t, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_0}^{t_1} (t_1-s)^{\xi-1} f(s, z_s) ds,$$

after the impulse

$$z(t_1^+) = z(t_1^-) + I_1(t_1^-, z(t_1^-))$$

$$z(t_1^+) = \theta_0(0) + \frac{1-\xi}{\psi(\xi)} f(t_1, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_0}^{t_1} (t_1-s)^{\xi-1} f(s, z_s) ds + I_1(t_1^-, z(t_1^-)).$$

If $t \in (t_1, t_2]$

$$z(t) = z(t_1^+) + \frac{1-\xi}{\psi(\xi)} f(t, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_1}^{t} f(s, z_s)(t-s)^{\xi-1} ds,$$

$$z(t) = z(t_1^-) + I_1(t_1^-, z(t_1^-)) + \frac{1-\xi}{\psi(\xi)} f(t, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_1}^{t} (t-s)^{\xi-1} f(s, z_s) ds,$$

$$= \theta_0(0) + \frac{1-\xi}{\psi(\xi)} f(t_1, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_0}^{t_1} (t_1-s)^{\xi-1} f(s, z_s) ds + I_1(t_1^-, z(t_1^-)) + \frac{1-\xi}{\psi(\xi)} f(t, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_1}^{t} (t-s)^{\xi-1} f(s, z_s) ds.$$

Then

$$z(t_2^-) = \theta_0(0) + \frac{1-\xi}{\psi(\xi)} f(t_1, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_0}^{t_1} (t_1-s)^{\xi-1} f(s, z_s) ds + I_1(t_1^-, z(t_1^-)) + \frac{1-\xi}{\psi(\xi)} f(t_2, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_1}^{t_2} (t_2-s)^{\xi-1} f(s, z_s) ds,$$

after the impulse

$$z(t_2^+) = z(t_2^-) + I_2(t_2^-, z(t_2^-)).$$

$$z(t_2^+) = \theta_0(0) + \frac{1-\xi}{\psi(\xi)} f(t_1, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_0}^{t_1} (t_1-s)^{\xi-1} f(s, z_s) ds + I_1(t_1^-, z(t_1^-)) + \frac{1-\xi}{\psi(\xi)} f(t_2, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_1}^{t_2} (t_2-s)^{\xi-1} f(s, z_s) ds + I_2(t_2^-, z(t_2^-)).$$

If $t \in (t_2, t_3]$

$$z(t) = z(t_2^+) + \frac{1-\xi}{\psi(\xi)} f(t, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_2}^{t} (t_2-s)^{\xi-1} f(s, z_s) ds,$$

$$z(t) = \theta_0(0) + \frac{1-\xi}{\psi(\xi)} f(t_1, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_0}^{t_1} (t_1-s)^{\xi-1} f(s, z_s) ds + I_1(t_1^-, z(t_1^-)) + \frac{1-\xi}{\psi(\xi)} f(t_2, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_1}^{t_2} (t_2-s)^{\xi-1} f(s, z_s) ds + I_2(t_2^-, z(t_2^-)) + \frac{1-\xi}{\psi(\xi)} f(t, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_2}^{t} (t-s)^{\xi-1} f(s, z_s) ds.$$

Then

$$z(t_3^-) = \theta_0(0) + \frac{1-\xi}{\psi(\xi)} f(t_1, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_0}^{t_1} (t_1-s)^{\xi-1} f(s, z_s) ds + I_1(t_1^-, z(t_1^-)) + \frac{1-\xi}{\psi(\xi)} f(t_2, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_1}^{t_2} (t_2-s)^{\xi-1} f(s, z_s) ds + I_2(t_2^-, z(t_2^-)) + \frac{1-\xi}{\psi(\xi)} f(t, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_2}^{t_3} (t_3-s)^{\xi-1} f(s, z_s) ds.$$

This mean that after the impulse we have

$$z(t_3^+) = z(t_3^-) + I_3(t_3^-, z(t_3^-)).$$

$$z(t_3^+) = \theta_0(0) + \frac{1-\xi}{\psi(\xi)} f(t_1, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_0}^{t_1} (t_1-s)^{\xi-1} f(s, z_s) ds + I_1(t_1^-, z(t_1^-)) + \frac{1-\xi}{\psi(\xi)} f(t_2, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_1}^{t_2} (t_2-s)^{\xi-1} f(s, z_s) ds + I_2(t_2^-, z(t_2^-)) + \frac{1-\xi}{\psi(\xi)} f(t, z_t) + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_2}^{t_3} (t_3-s)^{\xi-1} f(s, z_s) ds + I_3(t_3^-, z(t_3^-)).$$

We assume that

$$z(t_\tau^+) = \theta_0(0) + \frac{1-\xi}{\psi(\xi)} \big[f(t_1, z_t) + f(t_2, z_t) + f(t_3, z_t) \cdots + f(t_\tau, z_t)\big] + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \big[\int_{t_0}^{t_1} (t_1-s)^{\xi-1} f(s, z_s) ds \cdots + \int_{t_{\tau-1}}^{t_\tau} (t_\tau-s)^{\xi-1} f(s, z_s) ds.\big]$$

$$+[I_1(t_1^-, z(t_1^-)) + I_2(t_2^-, z(t_2^-))$$
$$+I_3(t_3^-, z(t_3^-)) \cdots I_\tau(t_\tau^-, z(t_\tau^-))]$$

By induction, for $t \in (t_\tau, t_{\tau+1}], t = t_\tau, \tau = 1, \cdots, p$

$$z(t) = z(t_\tau^+) + \frac{1-\xi}{\psi(\xi)} f(t_\tau, z_t) +$$
$$\frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_\tau}^{t} (t-s)^{\xi-1} f(s, z_s) ds,$$

we obtained equations (15) for $z \in PC^1[[t_0 - r, q], \mathbb{R}^n]$. ■

III. Main Results

We convert the Atangana-Baleanu impulsive delay problem (12) into a fixed point problem in order to demonstrate the existence and uniqueness theorem. We defined a mapping as follows for the operator $T : PC^1(\eta) \to PC^1(\eta)$.

$$T(z_{t_0}) = \theta_0$$
$$T(z(t)) =$$
$$\begin{cases} \theta_0(0) + \frac{1-\xi}{\psi(\xi)} f(t, z_t) + \\ \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_0}^{t} (t-s)^{\xi-1} f(s, z_s) ds, \qquad t \in [t_0, q], \\ \theta_0(0) + \frac{1-\xi}{\psi(\xi)} \sum_{i=1}^{\tau} f(t_i, z_t) + \\ \frac{\xi}{\psi(\xi)\Gamma(\xi)} \sum_{i=1}^{\tau} \int_{t_{i-1}}^{t_i} (t-s)^{\xi-1} f(s, z_s) ds \\ + \frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_\tau}^{t} (t-s)^{\xi-1} f(s, z_s) ds \\ + \sum_{i=1}^{\tau} I_i(t_i, z(t_i)), \qquad t \in (t_\tau, t_{\tau+1}], \tau = 1 \cdots, p \end{cases} \tag{16}$$

A. Existence

Theorem 1. *Assume that the function $f : Q \times PC^1 \to \mathbb{R}^n$ satisfies the following conditions;*
(C_1) $f : Q \times PC^1 \to \mathbb{R}^n$ is Lebesgue measurable with respect to t on Q
(C_2) $f(t, \theta)$is continuous with respect to θ on PC^1
(C_3) there exists a real-valued function $h(t) \in L^{\frac{1}{\zeta}}(Q)$ such that $\|f(t, \theta)\| \le h(t)$, for almost every $t \in Q$ and all $\theta \in PC^1(\eta)$. Then there exists at least a solution to the IVP (12) on Q.

Proof: We demonstrate that T defined by (16) has a fixed point by applying the Schauder fixed point theorem. The evidence will be presented in steps.

Step 1: $T : PC^1(\eta) \to PC^1(\eta)$
Using Holder inequality;

$$\int_{t_0}^{t} \|(t-s)^{\xi-1} f(s, z_s)\| ds \le$$
$$\Big(\int_{t_0}^{t} \big((t-s)^{\xi-1}\big)^{\frac{1}{1-\zeta}} ds\Big)^{1-\zeta} \Big(\int_{t_0}^{t} \big(h(s)\big)^{\frac{1}{\zeta}} ds\Big)^{\zeta} \tag{17}$$

and conditions $(C_1) - (C_3)$ for $t \in [t_0, t_1]$ we have

$$\|T(z(t))\| \le \|\theta_0(0)\| + \frac{1-\xi}{\psi(\xi)} \|f(t, z_t)\| +$$
$$\frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_0}^{t} \|(t-s)^{\xi-1} f(s, z_s)\| ds,$$
$$\le \|\theta_0(0)\| + \frac{1-\xi}{\psi(\xi)} \|f(t, z_t)\| +$$
$$\frac{\xi}{\psi(\xi)\Gamma(\xi)} \Big(\int_{t_0}^{t} \big((t-s)^{\xi-1}\big)^{\frac{1}{1-\zeta}} ds\Big)^{1-\zeta} \Big(\int_{t_0}^{t} (h(s))^{\frac{1}{\zeta}} ds\Big)^{\zeta}$$
$$\le \|\theta_0(0)\| + \frac{1-\xi}{\psi(\xi)} h(t) +$$
$$\frac{\xi}{\psi(\xi)\Gamma(\xi)} \Big(\frac{1-\zeta}{\xi-\zeta}\Big)^{1-\zeta} (t-t_0)^{\xi-\zeta} \Big(\int_{t_0}^{t} (h(s))^{\frac{1}{\zeta}} ds\Big)^{\zeta}$$
$$\le \|\theta_0(0)\| + \frac{1-\xi}{\psi(\xi)} h(t) +$$
$$\frac{\xi}{\psi(\xi)\Gamma(\xi)} \Big(\frac{1-\zeta}{\xi-\zeta}\Big)^{1-\zeta} (t-t_0)^{\xi-\zeta} H = \eta$$

Similarly $t \in (t_\tau, t_{\tau+1}]$

$$\|T(z(t))\| \le \|\theta_0(0)\| + \frac{1-\xi}{\psi(\xi)} \sum_{i=1}^{\tau} \|f(t_i, z_t)\| +$$
$$\frac{\xi}{\psi(\xi)\Gamma(\xi)} \sum_{i=1}^{\tau} \int_{t_{i-1}}^{t_i} (t_i - s)^{\xi-1} f(s, z_s) ds +$$
$$\frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_\tau}^{t} (t-s)^{\xi-1} f(s, z_s) ds +$$
$$\sum_{i=1}^{\tau} \|I_i(t_i, z(t_i))\|, \le \|\theta_0(0)\| + \frac{1-\xi}{\psi(\xi)} \sum_{i=1}^{\tau} h(t)$$
$$+ \frac{\xi}{\psi(\xi)\Gamma(\xi)} \sum_{i=1}^{\tau} \int_{t_{i-1}}^{t_i} (t_i - s)^{\xi-1} h(s) ds +$$
$$\frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_\tau}^{t} (t-s)^{\xi-1} h(s) ds +$$
$$\sum_{i=1}^{\tau} \|I_i(t_i, z(t_i))\|, \le \|\theta_0(0)\| + \frac{1-\xi}{\psi(\xi)} \sum_{i=1}^{\tau} h(t)$$
$$+ \frac{\xi}{\psi(\xi)\Gamma(\xi)} \sum_{i=1}^{\tau} \Big(\int_{t_{i-1}}^{t_i} \big((t_i - s)^{\xi-1}\big)^{\frac{1}{1-\zeta}} ds\Big)^{1-\zeta}$$
$$\Big(\int_{t_{i-1}}^{t_i} (h(s))^{\frac{1}{\zeta}} ds\Big)^{\zeta} +$$
$$\frac{\xi}{\psi(\xi)\Gamma(\xi)} \Big(\int_{t_\tau}^{t} \big((t-s)^{\xi-1}\big)^{\frac{1}{1-\zeta}} ds\Big)^{1-\zeta}$$
$$\Big(\int_{t_\tau}^{t} (h(s))^{\frac{1}{\zeta}} ds\Big)^{\zeta} + \sum_{i=1}^{\tau} \|I_i(t_i, z(t_i))\|$$
$$\le \|\theta_0(0)\| + \frac{1-\xi}{\psi(\xi)} h(t)$$
$$+ \frac{\xi}{\psi(\xi)\Gamma(\xi)} \Big(\int_{t_0}^{t} \big((t-s)^{\xi-1}\big)^{\frac{1}{1-\zeta}} ds\Big)^{1-\zeta}$$
$$\Big(\int_{t_0}^{t} (h(s))^{\frac{1}{\zeta}} ds\Big)^{\zeta} + \sum_{i=1}^{\tau} \|I_i(t_i, z(t_i))\|$$
$$\|T(z(t))\|_\infty \le \|\theta(0)\| + \frac{1-\xi}{\psi(\xi)} h(t)$$
$$+ \frac{\xi}{\psi(\xi)\Gamma(\xi)} \Big(\frac{1-\zeta}{\xi-\zeta}\Big)^{1-\zeta} q^{\xi-\zeta} H + pH^*$$
$$T : PC^1(\eta) \to PC^1(\eta)$$

Step 2: Let $\{\theta_n\}$ be a sequence such that $\theta_n \to \theta$ on $PC^1(\eta)$. and $f(s,\theta_n) \to f(s,\theta)$ and $I_\tau(t_\tau, \theta_n(0)) \to I_\tau(t_\tau, \theta(0))$ on $PC^1(\eta)$ as $n \to \infty$, $\tau = 1, \cdots, p$ for each $t \in [t_0, q]$ then

$$\|T(z_n(t)) - T(z(t))\| = \frac{1-\xi}{\psi(\xi)}\|f(t,\theta_n) - f(t,\theta)\| +$$

$$\frac{\xi}{\psi(\xi)\Gamma(\xi)}\Big\| \int_{t_0}^{t} (t-s)^{\xi-1}(f(s,\theta_n) - f(s,\theta))ds, \Big\|$$

$$\leq +\frac{1-\xi}{\psi(\xi)}\|f(t,\theta_n) - f(t,\theta)\| +$$

$$\frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_0}^{t} (t-s)^{\xi-1}\big\|f(s,\theta_n) - f(s,\theta), \big\|ds$$

$$\leq +\frac{1-\xi}{\psi(\xi)}\|f(t,\theta_n) - f(t,\theta)\|_\infty +$$

$$\frac{\xi}{\psi(\xi)\Gamma(\xi+1)} q^\xi \big\|f(.,\theta_n) - f(.,\theta), \big\|_\infty$$

Similarly $t \in (t_\tau, t_{\tau+1}]$

$$\|T(z_n(t)) - T(z(t))\| = \frac{1-\xi}{\psi(\xi)} \sum_{i=1}^{\tau} \|f(t_i,\theta_n) - f(t_i,\theta)\| +$$

$$\frac{\xi}{\psi(\xi)\Gamma(\xi)} \sum_{i=1}^{\tau} \int_{t_{i-1}}^{t_i} (t_i - s)^{\xi-1}\|f(s,\theta_n) - f(s,\theta)\|ds$$

$$+\frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_\tau}^{t} (t-s)^{\xi-1}\|f(s,\theta_n) - f(s,\theta)\|ds$$

$$+\sum_{i=1}^{\tau} \|I_i(t_i, \theta_n(0)) - I_i(t_i, \theta(0))\|,$$

$$\leq \frac{1-\xi}{\psi(\xi)}\|f(t_i,\theta_n) - f(t_i,\theta)\| +$$

$$+\frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_0}^{t} (t-s)^{\xi-1}\|f(s,\theta_n) - f(s,\theta)\|ds$$

$$+\sum_{i=1}^{\tau} \|I_i(t_i, \theta_n(0)) - I_i(t_i, \theta(0))\|,$$

$$\leq \frac{1-\xi}{\psi(\xi)}\|f(t_i,\theta_n) - f(t_i,\theta)\| +$$

$$+\frac{\xi}{\psi(\xi)\Gamma(\xi+1)} q^\xi \|f(.,\theta_n) - f(.,\theta)\|ds$$

$$+\sum_{i=1}^{\tau} \|I_i(.,\theta_n(0)) - I_i(.,\theta(0))\|,$$

$$\leq \frac{1-\xi}{\psi(\xi)}\|f(t_i,\theta_n) - f(t_i,\theta)\|_\infty +$$

$$\frac{\xi}{\psi(\xi)\Gamma(\xi+1)} q^\xi \|f(.,\theta_n) - f(.,\theta)\|_\infty$$

$$+p\|I_i(.,\theta_n(0)) - I_i(.,\theta(0))\|_\infty,$$

as $n \to \infty$, $f(.,\theta_n) \to f(.,\theta)$ and $I_i(t_i, \theta_n(0)) \to I_i(t_i, \theta(0))$ therefore $\|T(z_n(t)) - T(z(t))\|_\infty \to 0$ as $n \to \infty$ This show that T is continuous

Step 3: T maps bounded sets into equicontinuous sets on Q Let $q_1, q_2 \in [t_0, t_1]$ and $q_1 <, q_2$,$PC^1(\eta)$ be a bounded set as in Step 1 and Step 2 and $z \in PC^1(\eta)$. based on the Holder inequality (17) we obtain

$$\|T(z)q_2 - T(z)q_1\| = \frac{1-\xi}{\psi(\xi)}\|f(q_2, z_{q_2}) - f(q_1, z_{q_1})\| +$$

$$\frac{\xi}{\psi(\xi)\Gamma(\xi)}\Big\| \int_{t_0}^{q_1} ((q_1 - s)^{\xi-1} - (q_2 - s)^{\xi-1}) f(s, z_s)ds$$

$$+\int_{q_1}^{q_2} (q_2 - s)^{\xi-1} f(s, z_s)\big\|ds$$

$$\leq \frac{1-\xi}{\psi(\xi)}\|f(q_2, z_{q_2}) - f(q_1, z_{q_1})\|$$

$$+\frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_0}^{q_1} \big\|((q_1 - s)^{\xi-1} - (q_2 - s)^{\xi-1}) f(s, z_s)\big\|ds$$

$$+\frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{q_1}^{q_2} \big\|(q_2 - s)^{\xi-1} f(s, x_s)\big\|ds$$

$$\leq \frac{1-\xi}{\psi(\xi)}\|f(q_2, z_{q_2}) - f(q_1, z_{q_1})\| +$$

$$\frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{t_0}^{q_1} ((q_1 - s)^{\xi-1} - (q_2 - s)^{\xi-1}) h(s)ds$$

$$+\frac{\xi}{\psi(\xi)\Gamma(\xi)} \int_{q_1}^{q_2} (q_2 - s)^{\xi-1} h(s)ds$$

$$\leq \frac{1-\xi}{\psi(\xi)}\|f(q_2, z_{q_2}) - f(q_1, z_{q_1})\| +$$

$$\frac{\xi}{\psi(\xi)\Gamma(\xi)}\Big(\int_{t_0}^{q_1} ((q_1 - s)^{\xi-1} - (q_2 - s)^{\xi-1})^{\frac{1}{1-\zeta}} ds\Big)^{1-\zeta}$$

$$\Big(\int_{t_0}^{q_1} \big(h(s)\big)^{\frac{1}{\zeta}} ds\Big)^{\zeta} +$$

$$\frac{\xi}{\psi(\xi)\Gamma(\xi)}\Big(\int_{q_1}^{q_2} ((q_2 - s)^{\alpha-1})^{\frac{1}{1-\zeta}} ds\Big)^{1-\zeta}$$

$$\Big(\int_{q_1}^{q_2} \big(h(s)\big)^{\frac{1}{\zeta}} ds\Big)^{\zeta}$$

$$\leq \frac{1-\xi}{\psi(\xi)}\|f(q_2, z_{q_2}) - f(q_1, z_{q_1})\| +$$

$$\frac{\xi}{\psi(\xi)\Gamma(\xi)}\Big(\int_{t_0}^{q_1} \big((q_1 - s)^{\frac{\xi-1}{1-\zeta}} - (q_2 - s)^{\frac{\xi-1}{1-\zeta}} ds)\Big)^{1-\zeta}$$

$$\Big(\int_{t_0}^{q} \big(h(s)\big)^{\frac{1}{\zeta}} ds\Big)^{\zeta} +$$

$$\frac{\xi}{\psi(\xi)\Gamma(\xi)}\Big(\int_{q_1}^{q_2} (q_2 - s)^{\frac{\xi-1}{1-\zeta}} ds\Big)^{1-\zeta}\Big(\int_{q_1}^{q} \big(h(s)\big)^{\frac{1}{\zeta}} ds\Big)^{\zeta}$$

$$\leq \frac{1-\xi}{\psi(\xi)}\|f(q_2, z_{q_2}) - f(q_1, z_{q_1})\|$$

$$+\frac{\xi H}{\psi(\xi)\Gamma(\xi)}\Big(\frac{1-\zeta}{\xi-\zeta}\Big)^{1-\zeta}(q_1 - t_0)^{\frac{\xi-1}{1-\zeta}+1} +$$

$$\Big((q_2 - q_1)^{\frac{\xi-1}{1-\zeta}+1} - (q_2 - t_0)^{\frac{\xi-1}{1-\zeta}+1}\Big)^{1-\zeta}$$

$$+\frac{\xi H}{\psi(\xi)\Gamma(\xi)}\Big(\frac{1-\zeta}{\xi-\zeta}\Big)^{1-\zeta}\Big((q_2 - q_1)^{\frac{\xi-1}{1-\zeta}+1}\Big)^{1-\zeta}$$

$$\leq \frac{1-\xi}{\psi(\xi)}\|f(q_2, z_{q_2}) - f(q_1, z_{q_1})\|$$

$$+\frac{2H\xi}{\psi(\xi)\Gamma(\xi)}(\frac{1-\zeta}{\xi-\zeta})^{1-\zeta}(q_2-q_1)^{\xi-\zeta}$$

As $q_2 \to q_1$ the right-hand side of the above inequality tends to zero, then $(Tz)(t)$ is equicontinuous on $t \in [t_0, t_1]$ the evidence can be presented similarly on $t \in (t_\tau, t_{\tau+1}]$, hence $(Tz)(t)$ is equicontinuous on $t \in (t_\tau, t_{\tau+1}]$ therefore $(Tz)(t)$ is piecewise equicontinuous on Q. Thus, using the Arzelà-Ascoli theorem for equicontinuous functions, we conclude that $(Tz)(t)$ is relatively compact, and hence T is completely continuous on $PC^1(\eta)$. By Schauder Fixed point theorem, T has a fixed point in $PC^1(\eta)$ which is a solution of the IVP (12) on Q ■

B. Uniqueness

Theorem 2. *Assume that the function $f : Q \times PC^1 \to \mathbb{R}^n$ is continuous and bounded, and if it satisfied the following conditions:*
(C_4) there exists a constant $\omega \in (0, \xi)$ and a real-valued function $\nu(t) \in L^{\frac{1}{\omega}}(Q)$ such that $\|f(t,\theta) - f(t,\phi)\| \le \nu(t)\|\theta - \phi\|$, for almost every $t \in Q$ and all $\theta, \phi \in PC^1(\eta)$
(C_5) there exists a positive constant μ^* such that $\|I_\tau(t_\tau, \theta(0)) - I_\tau(t_\tau, \phi(0))\| \le \mu^*\|\theta(0) - \phi(0)\|$, for each $\theta, \phi \in PC^1(\eta)$ and $\tau = 1, 2, \cdots, p$
then the IVP (12) has a unique solution on Q provided

$$0 < [\frac{1-\xi}{\psi(\xi)}\nu(t)+ \frac{\xi}{\psi(\xi)\Gamma(\xi)}(\frac{1-\omega}{\xi-\omega})^{1-\omega}(q-t_0)^{\xi-\omega}\nu^* + p\mu^*] < 1 \qquad (18)$$

where $\nu^* = \big(\int_{t_0}^{q}(\nu(s))^{\frac{1}{\omega}}ds\big)^{\omega}$,

Proof: The existence of the solution to (12) is established by the second theorem. Next, we demonstrate that T has a unique fixed point by applying the Banach contraction mapping principle.

$$\|T(z_1)(t) - T(z_2)(t)\| = \frac{1-\xi}{\psi(\xi)}\|f(t,\theta) - f(t,\phi)\|+ \frac{\xi}{\psi(\xi)\Gamma(\xi)}\|\int_{t_0}^{t}(t-s)^{\xi-1}[f(s,\theta)-f(s,\phi)]ds$$

$$\le \frac{1-\xi}{\psi(\xi)}\nu(t)\|\theta-\phi\| + \frac{\xi}{\psi(\xi)\Gamma(\xi)}\int_{t_0}^{t}(t-s)^{\xi-1}\nu(s)ds\|\theta-\phi\|$$

$$\le \frac{1-\xi}{\psi(\xi)}\nu(t)\|z_1-z_2\|_\infty + \frac{\xi}{\psi(\xi)\Gamma(\xi)}\int_{t_0}^{t}(t-s)^{\xi-1}\nu(s)ds\|z_1-z_2\|_\infty$$

$$\le \frac{1-\xi}{\psi(\xi)}\nu(t)\|z_1-z_2\|_\infty + \frac{\xi}{\psi(\xi)\Gamma(\xi)}\big(\int_{t_0}^{t}\big((t-s)^{\xi-1}\big)^{\frac{1}{1-\omega}}ds\big)^{1-\omega} \big(\int_{t_0}^{t}\big(\nu(s)\big)^{\frac{1}{\omega}}ds\big)^{\omega}\|z_1-z_2\|_\infty$$

$$\le \frac{1-\xi}{\psi(\xi)}\nu(t)\|z_1-z_2\|_\infty + \frac{\xi\nu^*}{\psi(\xi)\Gamma(\xi)}(\frac{1-\omega}{\xi-\omega})^{1-\omega}(q-t_0)^{\xi-\omega}\|z_1-z_2\|_\infty$$

$$\le \frac{1-\xi}{\psi(\xi)}\nu(t)+ \frac{\xi\nu^*}{\psi(\xi)\Gamma(\xi)}(\frac{1-\omega}{\xi-\omega})^{1-\omega}\big((q-t_0)^{\xi-\omega}\big)\|z_1-z_2\|_\infty.$$

Where

$$\frac{1-\xi}{\psi(\xi)}\nu(t) + \frac{\xi\nu^*}{\psi(\xi)\Gamma(\xi)}(\frac{1-\omega}{\xi-\omega})^{1-\omega}(q-t_0)^{\xi-\omega} < 1.$$

Similarly $t \in (t_\tau, t_{\tau+1}]$

$$\|T(z_1)(t) - T(z_2)(t)\| = \frac{1-\xi}{\psi(\xi)}\sum_{i=1}^{\tau}\|f(t_i,\theta) - f(t_i,\phi)\|+ \frac{\xi}{\psi(\xi)\Gamma(\xi)}\sum_{i=1}^{\tau}\|\int_{t_{i-1}}^{t_i}(t_i-s)^{\xi-1}[f(s,\theta)-f(s,\phi)]ds+ \frac{\xi}{\psi(\xi)\Gamma(\xi)}\|\int_{t_\tau}^{t}(t-s)^{\xi-1}[f(s,\theta)-f(s,\phi)]ds + \sum_{i=1}^{\tau}\|I_i(z_1(t_i),\theta(0)) - I_i(z_2(t_i,\phi(0)))\|$$

$$\le \frac{1-\xi}{\psi(\xi)}\sum_{i=1}^{\tau}\nu(t)\|\theta-\phi\|+ \frac{\xi}{\psi(\xi)\Gamma(\xi)}\sum_{i=1}^{\tau}\int_{t_{i-1}}^{t_i}(t_i-s)^{\xi-1}\nu(s)\|\theta-\phi\|ds+ \frac{\xi}{\psi(\xi)\Gamma(\xi)}\int_{t_\tau}^{t}(t-s)^{\xi-1}\nu(s)\|\theta-\phi\|ds + \sum_{i=1}^{\tau}\mu^*\max_{t\in Q}\|\theta(0)-\phi(0)\|$$

$$\le \frac{1-\xi}{\psi(\xi)}\sum_{i=1}^{\tau}\nu(t)\|z_1-z_2\|+ \frac{\xi}{\psi(\xi)\Gamma(\xi)}\sum_{i=1}^{\tau}\big(\int_{t_{i-1}}^{t_i}\big((t_i-s)^{\xi-1}\big)^{\frac{1}{1-\omega}}ds\big)^{1-\omega} \big(\int_{t_{i=1}}^{t_i}\big(\nu(s)\big)^{\frac{1}{\omega}}ds\big)^{\omega}\|z_1-z_2\|_\infty + \frac{\xi}{\psi(\xi)\Gamma(\xi)}\big(\int_{t_\tau}^{t}\big((t-s)^{\xi-1}\big)^{\frac{1}{1-\omega}}ds\big)^{1-\omega} \big(\int_{t_\tau}^{t}\big(\nu(s)\big)^{\frac{1}{\omega}}ds\big)^{\omega}\|z_1-z_2\|_\infty + \sum_{i=1}^{\tau}\mu^*\|z_1-z_2\|_\infty$$

$$\le \frac{1-\xi}{\psi(\xi)}\nu(t)\|z_1-z_2\|+ + \frac{\xi}{\psi(\xi)\Gamma(\xi)}\big(\int_{t_0}^{t}\big((t-s)^{\xi-1}\big)^{\frac{1}{1-\omega}}ds\big)^{1-\omega} \big(\int_{t_0}^{t}\big(\nu(s)\big)^{\frac{1}{\omega}}ds\big)^{\omega}\|z_1-z_2\|_\infty + p\mu^*\|z_1-z_2\|_\infty$$

$$\leq (\frac{1-\xi}{\psi(\xi)}\nu(t)+$$

$$\frac{\xi\nu^*}{\psi(\xi)\Gamma(\xi)}(\frac{1-\omega}{\xi-\omega})^{1-\omega}q^{\xi-\omega}+p\mu^*)\|z_1-z_2\|_\infty.$$

Where

$$\frac{1-\xi}{\psi(\xi)}\nu(t)+\frac{\xi\nu^*}{\psi(\xi)\Gamma(\xi)}(\frac{1-\omega}{\xi-\omega})^{1-\omega}q^{\xi-\omega}+p\mu^*)<1.$$

T is therefore a rigorous contraction. As a result, there is a single fixed point that is a unique solution for (12). ■

C. Stability

In this section, the Lipschitz stability of the solution is examined.

Definition 1. *If a constant $K>0$ exists such that for any two solutions $z(t)$ and $y(t)$ with initial conditions $z_{t_0}=\theta_0$, where $\theta_0\in PC^1$ and $y_{t_0}=\phi_0$, where $\phi_0\in PC^1$ respectively, we have*

$$\|z(t)-y(t)\|\leq K\|\theta-\phi\|$$

for all $t\geq 0$. This indicates that equations (12) is Lipschitz stable.

Theorem 3. *If all the conditions of theorem 2 are satisfied, then the solution of the initial value problem (12) is Lipschitz stable if*

$$\|z-y\|_\infty\leq\frac{1}{1-\lambda}\|\theta_0-\phi_0\|$$

Proof: Let $z(t)$ be a solution of (12), and $y(t)$ be a solution of (12) satisfying the initial value condition $y_{t_0}=\phi_0$, where $\phi_0\in PC^1$ then

$$(z(t)-y(t))=$$

$$\begin{cases}(\theta_0(0)-\phi_0(0))+\frac{1-\xi}{\psi(\xi)}(f(t,z_t)-f(t,y_t))\\ +\frac{\xi}{\psi(\xi)\Gamma(\xi)}\int_{t_0}^t(t-s)^{\xi-1}(f(s,z_s)-f(s,y_s))ds, t\in[t_0,q],\\ (\theta_0(0)-\phi_0(0))+\frac{1-\xi}{\psi(\xi)}\sum_{i=1}^{\tau}(f(t_i,z_t)-f(t_i,y_t))+\\ \frac{\xi}{\psi(\xi)\Gamma(\xi)}\sum_{i=1}^{\tau}\int_{t_{i-1}}^{t_i}(t-s)^{\xi-1}(f(s,z_s)-f(s,y_s))ds\\ +\frac{\xi}{\psi(\xi)\Gamma(\xi)}\int_{t_\tau}^t(t-s)^{\xi-1}(f(s,z_s)-f(s,y_s))ds\\ +\sum_{i=1}^{\tau}I_i((t_i,z(t_i))-(t_i,y(t_i))), t\in(t_\tau,t_{\tau+1}], \tau=1..,p.\end{cases} \quad (19)$$

For $t\in[t_0,q]$

$$\|z(t)-y(t)\|=\|\theta_0(0)-\phi_0(0)\|+\frac{1-\xi}{\psi(\xi)}\|f(t,z_t)-f(t,y_t)\|+$$

$$\frac{\xi}{\psi(\xi)\Gamma(\xi)}\|\int_{t_0}^t((t-s)^{\xi-1}[f(s,z_s)-f(s,y_s)]ds$$

$$\leq\|\theta_0(0)-\phi_0(0)\|+\frac{1-\xi}{\psi(\xi)}\nu(t)\|\theta_0(0)-\phi_0(0)\|+$$

$$\frac{\xi}{\psi(\xi)\Gamma(\xi)}\int_{t_0}^t((t-s)^{\xi-1}\nu(s)ds\|\theta-\phi\|$$

$$\leq\|\theta_0(0)-\phi_0(0)\|+\frac{1-\xi}{\psi(\xi)}\nu(t)\|z-y\|_\infty+$$

$$(\int_{t_0}^t((t-s)^{\xi-1})^{\frac{1}{1-\omega}}ds)^{1-\omega}(\int_{t_0}^t(\nu(s))^{\frac{1}{\omega}}ds)^{\omega}\|z-y\|_\infty$$

$$\leq\|\theta_0(0)-\phi_0(0)\|+\frac{1-\xi}{\psi(\xi)}\nu(t)\|z-y\|_\infty+$$

$$\frac{\xi\nu^*}{\psi(\xi)\Gamma(\xi)}((\frac{1-\omega}{\xi-\omega})^{1-\omega}((q-t_0)^{\xi-\omega}\|z-y\|_\infty$$

$$\leq\|\theta_0(0)-\phi_0(0)\|+$$

$$(\frac{1-\xi}{\psi(\xi)}\nu(t)+\frac{\xi\nu^*}{\psi(\xi)\Gamma(\xi)}(\frac{1-\omega}{\xi-\omega})^{1-\omega}((q-t_0)^{\xi-\omega})\|z-y\|_\infty$$

$$\leq\|\theta_0(0)-\phi_0(0)\|+\frac{1-\xi}{\psi(\xi)}\nu(t)\|z-y\|_\infty+$$

$$\frac{\xi\nu^*}{\psi(\xi)\Gamma(\xi)}((\frac{1-\omega}{\xi-\omega})^{1-\omega}((q-t_0)^{\xi-\omega}\|z-y\|_\infty$$

$$\leq\|\theta_0(0)-\phi_0(0)\|+\lambda\|z-y\|_\infty$$

Similarly $t\in(t_\tau,t_{\tau+1}]$

$$\|z(t)-y(t)\|=\|\theta_0(0)-\phi_0(0)\|+$$

$$\frac{1-\xi}{\psi(\xi)}\sum_{i=1}^{\tau}\|f(t,z_t)-f(t,y_t)\|+$$

$$\frac{\xi}{\psi(\xi)\Gamma(\xi)}\sum_{i=1}^{\tau}\|\int_{t_{i-1}}^{t_i}(t_i-s)^{\xi-1}[f(s,z_s)-f(s,y_s)]\|ds$$

$$+\frac{\xi}{\psi(\xi)\Gamma(\xi)}\|\int_{t_\tau}^t(t-s)^{\xi-1}[f(s,z_s)-f(s,y_s)]\|ds+$$

$$\sum_{i=1}^{\tau}\|I_i(z(t_i),\theta(0))-I_i(y(t_i),\phi(0))\|$$

$$\leq\|\theta_0(0)-\phi_0(0)\|+\frac{1-\xi}{\psi(\xi)}\sum_{i=1}^{\tau}\nu(t)\|\theta-\phi\|+$$

$$\frac{\xi}{\psi(\xi)\Gamma(\xi)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-s)^{\xi-1}ds\nu(s)\|\theta-\phi\|+$$

$$\frac{\xi}{\psi(\xi)\Gamma(\xi)}\int_{t_\tau}^t(t-s)^{\xi-1}ds\nu(s)\|\theta-\phi\|$$

$$+\sum_{i=1}^{\tau}\mu^*\max_{t\in Q}\|\theta(0)-\phi(0)\|$$

$$\leq\|\theta_0(0)-\phi_0(0)\|+\frac{1-\xi}{\psi(\xi)}\sum_{i=1}^{\tau}\nu(t)\|z-y\|+$$

$$\frac{\xi}{\psi(\xi)\Gamma(\xi)}\sum_{i=1}^{\tau}\int_{t_{i-1}}^{t_i}(t_i-s)^{\xi-1}ds$$

$$(\int_{t_{i-1}}^t(\nu(s))^{\frac{1}{\omega}}ds)^{\omega}\|z-y\|+$$

$$\frac{\xi}{\psi(\xi)\Gamma(\xi)}(\int_{t_\tau}^t((t-s)^{\xi-1})^{\frac{1}{1-\omega}}ds)^{1-\omega}$$

$$(\int_{t_\tau}^t(\nu(s))^{\frac{1}{\omega}}ds)^{\omega}\|z-y\|_\infty+p\mu^*\|z-y\|_\infty$$

$$\leq\|\theta_0(0)-\phi_0(0)\|+\frac{1-\xi}{\psi(\xi)}\nu(t)\|z-y\|+$$

$$\frac{\xi}{\psi(\xi)\Gamma(\xi)}(\int_{t_0}^t((t-s)^{\xi-1})^{\frac{1}{1-\omega}}ds)^{1-\omega}$$

$$(\int_{t_0}^t(\nu(s))^{\frac{1}{\omega}}ds)^{\omega}\|z-y\|_\infty+p\mu^*\|z-y\|_\infty$$

$$\leq \|\theta_0(0)-\phi_0(0)\| + \frac{1-\xi}{\psi(\xi)}\nu(t)\big\|z-y\big\|_\infty +$$

$$\frac{\xi\nu^*}{\psi(\xi)\Gamma(\xi)}\big(\frac{1-\omega}{\xi-\omega}\big)^{1-\omega}\big(q-t_0\big)^{\xi-\omega}\big\|z-y\big\|_\infty + p\mu^*\big\|z-y\big\|_\infty$$

$$\leq \|\theta_0(0)-\phi_0(0)\| + \big(\frac{1-\xi}{\psi(\xi)}\nu(t)+$$

$$\frac{\xi\nu^*}{\psi(\xi)\Gamma(\xi)}\big(\frac{1-\omega}{\xi-\omega}\big)^{1-\omega}\big(q-t_0\big)^{\xi-\omega} + p\mu^*\big)\big\|z-y\big\|_\infty$$

$$\leq \|\theta_0(0)-\phi_0(0)\| + \lambda\big\|z-y\big\|_\infty,$$

since $\lambda < 1$ we have

$$\|z-y\|_\infty \leq \|\theta_0-\phi_0\| + \lambda\big\|z-y\big\|_\infty$$

$$\|z-y\|_\infty \leq \frac{1}{1-\lambda}\|\theta_0-\phi_0\|,$$

where

$$\lambda = \frac{1-\xi}{\psi(\xi)}\nu(t)+$$

$$\frac{\xi\nu^*}{\psi(\xi)\Gamma(\xi)}\big(\frac{1-\omega}{\xi-\omega}\big)^{1-\omega}\big(q-t_0\big)^{\xi-\omega} + p\mu^*$$

accounts for the fractional growth.
$p\mu^*$ represents impulse contributions.

$$K = \frac{1}{1-\lambda}.$$

As a result, Lipschitz stability is valid. To make Lipschitz stability significant, we need $0 < \lambda < 1$, which guarantees that k is finite. ■

Example 1. *Consider the Atangana-Baleanu impulsive fractional differential equations with constant delay.*

$$\begin{cases} {}^{ABC}D^{\frac{1}{2}}z(t) = \frac{\sin(z(t-\frac{\pi}{2}))}{7}, & t \in [0, \frac{7\pi}{2}], \quad t \neq \frac{\pi}{2} \\ \Delta z(\frac{\tau\pi}{2}) = \frac{1}{7}, & t = \frac{\tau\pi}{2}, \quad \tau = 1,\cdots,6 \\ z(t) = \sin t & t \in [-\frac{\pi}{2}, 0], \end{cases} \tag{20}$$

We have that,

$$\|f(t,z_t)\| = \|\frac{\sin(z(t-\frac{\pi}{2}))}{7}\| \leq \frac{1}{7},$$

where $h(t) = \frac{1}{7}, \zeta = \frac{1}{4}$, $H = \big(\int_{t_0}^{q}(h(s)^{\frac{1}{\zeta}})\big)^{\zeta} = \big(\int_0^{\frac{7\pi}{2}}(\frac{1}{7})^4\big)^{\frac{1}{4}} = \frac{1}{7}(\frac{7\pi}{2})^{\frac{1}{4}}, \|\theta_0(o)\| = 0, p = 6, H^* = \frac{1}{7}, \psi(\xi) = 1$. *Thus, on the interval* $[0, \frac{7\pi}{2}]$, *there is at least one solution to the IVP (20) based on theorem 1.*
We then confirm that the solution is unique with the following parameters,

$$\|\frac{\sin(z(t-\frac{\pi}{2}))}{7} - \frac{\sin(y(t-\frac{\pi}{2}))}{7}\|$$

$$= \frac{1}{7}\big\|\sin(z(t-\frac{\pi}{2})) - \sin(y(t-\frac{\pi}{2}))\big\|$$

$$\leq \frac{1}{7}\big\|z(t-\frac{\pi}{2}) - y(t-\frac{\pi}{2})\big\|$$

where $\Delta z(\frac{k\pi}{2})$ *=constant,* $\nu(t) = \frac{1}{7}, \omega = \frac{1}{4}$, $\nu^* = \big(\int_{t_0}^{q}(\nu(s)^{\frac{1}{\omega}})\big)^{\omega} = \big(\int_0^{\frac{7\pi}{2}}(\frac{1}{7})^4\big)^{\frac{1}{4}} = \frac{1}{7}(\frac{7\pi}{2})^{\frac{1}{4}}, \|\theta_0(o)\| = 0, p = 6, \mu^* = 0, \psi(\xi) = 1.$

$$\big(\frac{1-\xi}{\psi(\xi)}\nu(t) + \frac{\xi\nu^*}{\psi(\xi)\Gamma(\xi)}\big(\frac{1-\omega}{\xi-\omega}\big)^{1-\omega}q^{\xi-\omega} + p\mu^*\big) < 1$$

$$= \frac{1}{2}\times\frac{1}{7}\times(\frac{7\pi}{2})^{\frac{1}{4}} + \frac{1}{2\sqrt{\pi}}\times\big(3\big)^{\frac{3}{4}}\times\frac{1}{7}\times(\frac{7\pi}{2})^{\frac{1}{2}}$$

$$= 0.4 < 1.$$

According to theorem 2 (20) have a unique solution on the interval $t \in [0, \frac{7\pi}{2}]$
Since $0 < \lambda < 1$ *and* $1-\lambda > 0$, *which guarantees that* k *is finite, equation (20) is Lipschitz stable.*

IV. Conclusion

Impulsive fractional delay differential equations were formulated in the context of the Atangana-Baleanu-Caputo derivative. Using the generalised boundedness condition on Schauder's point and the generalised Lipschitz condition on Banach contraction principle, conditions for existence, uniqueness, and stability were analyzed. The developed notion was demonstrated using examples, and the outcomes validated its applicability.

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