Numerical Solution of Class of Third-Order Singularly Perturbed Delay Differential Equations via Quartic B-Spline Method

Shilpa Malge, and Ram Kishun Lodhi

Abstract—**This paper presents a quartic B-spline method to find the numerical solution of third-order singularly perturbed delay differential equations. The method is directly applied to the third-order singularly perturbed delay differential equation without reducing the order, thereby maintaining the original structure. Uniform mesh is used to generate the grid points. The quartic B-spline method converts the boundary value problem into a system of linear equations. This system is solved by matrix method to find the numerical solution. The convergence of the method is discussed via truncation error analysis. Three numerical examples are solved to test the efficiency of the proposed method. Maximum absolute error, uniform error and uniform convergence are determined. Further, the obtained results are compared with the existing results in the literature. Graphs are drawn to study the behavior of the solution.**

Index Terms—**Singular perturbation, Delay Differential Equations, Boundary Layer, Quartic B-spline Method, Convergence.**

I. Introduction

DELAY differential equations arise in the practical phenomenon where slow and advanced processes coexist. In these equations, the dependent variable or its derivatives depend on the past stages of the independent variable. Further, differential equations involving small parameters that cause abrupt changes in the solution in some part of the domain are referred as singularly perturbed differential equations (SPDE). The SPDEs that involve delays are called singularly perturbed delay differential equations (SPDDEs). SPDDEs occur in many practical applications, such as hydrodynamics and liquid helium [1], diffusion in polymers [2], and neural reflex mechanism [3].

Third and higher-order differential equations occur in the mathematical modelling of many real-life applications. Various models are presented in [4]. Different types of SPDEs and SPDDEs of third-order are seen in the literature. An SPDE arising in the theory of thin film flow is tackled by asymptotic expansions [5]. One such phenomenon of the mechanical robotics model with damping gives third-order delay differential equations [6]. Authors have studied the asymptotic stability properties of third-order delay differential equations. The boundedness and asymptotic stability of the solution of these equations are reported in [7], [8], and [9].

Manuscript received June 2, 2025; revised September, 21, 2025.

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Applications of SPDEs and SPDDEs of the third-order tempted many researchers to find the approximate solution of these equations and study different analytical properties. Various analytical and numerical methods are applied to solve SPDEs and SPDDEs. In [10], authors have applied analytical and numerical methods to find an approximate solution of third-order SPDEs with a discontinuous source term. A third-order SPDE of reaction-diffusion type is approximated by a numerical method comprised of shooting method and standard finite difference method (FDM)[11]. A parameterized third-order SPDE is discussed in [12]. Geetha et al. [13] have discussed the parameter uniform numerical method based on Shishkin mesh to find the numerical solution of a third-order SPDE with turning point exhibiting boundary layers.

Recently, third-order SPDDEs caught the attention of many researchers. Various numerical methods are used to find the solution of these equations. One common numerical method, FDM, is applied for different types of third-order SPDDEs [14], [15], [16], and [17]. A trigonometric B-spline collocation method is applied on a uniform mesh for numerical solution of third-order SPDDEs with discontinuous convection-diffusion term and source term [18].

In this article, the Quartic B-spline method (QRBSM) is employed to find the numerical solution of the third-order SPDDE with a large delay. Splines are piecewise continuous curves and these are defined by using the different basis functions, such as polynomial and non-polynomial functions. Spline methods can provide the solution at any point in the domain. Spline techniques provide a set of flexible methods and the order of the spline can be chosen as per the desired accuracy and order of convergence. Different types of boundary value problems are solved using various order spline-based methods. Ersoy [19] and Mittal et al. [20] have applied the cubic B-spline method to solve two-dimensional unsteady advection diffusion equations. In [21], authors have applied B-spline method to obtain the numerical solution of a differential equation in RLC closed series circuit with small inductance value. In [22], authors have applied two high-order numerical methods for solving a time-fractional Fokker–Planck equation based on Quintic B-splines and an improvised Quintic B-spline collocation method.

Spline methods are also applied for different SPDEs, such as: second-order SPDEs [23], [24], [25], SPDE with discontinuous source term [26], SPDDEs with turning point [27], linear and nonlinear SPDDEs [28], parabolic SPDDEs with large shifts and integral boundary condition [29]. QRBSM uses fourth-order polynomials, providing three times continuously differentiable solution. Thus, the application of QRBSM is seen for different BVPs. QRBSM is used for numerical solution of nonlinear SPDDEs [30]. A second-order nonlinear BVP with Neumann and Robin boundary conditions is solved using the Quartic B-spline method on a uniform mesh [31]. B-spline techniques are applied for Bratu-type and Lane–Emden problems [32]. The Quartic spline method is used for a fifth-order BVPs with a two-point boundary condition [33]. A Quartic trigonometric B-spline collocation method is applied for non-isothermal reaction diffusion model equations in a spherical catalyst and biocatalyst [34]. A comparison of Cubic and Quartic Hyperbolic B-splines for a coupled Navier Stokes equation is presented in [35].

This work motivated us to apply QRBSM for finding the numerical solution of third-order SPDDEs with large delay. The primary objective of the proposed work is to obtain a more accurate numerical solution and an enhanced order of convergence. The remaining part of the paper is organized into five sections. The problem statement is given in Section 2. Section 3 contains the detailed explanation of QRBSM. Convergence analysis and different error bounds are discussed in Section 4. Numerical illustrations and result discussion are included in Section 5. The summary of findings is presented in Section 6, the conclusion, at the end of the paper.

II. Statement of Problem

Consider the following third-order SPDDE defined on the interval $\Omega = \Omega^- \cup \Omega^+$, where $\Omega = (0,2)$, $\Omega^- = (0,1)$, and $\Omega^+ = (1,2)$:

$$\begin{gathered}-\varepsilon\psi'''(\upsilon) + a(\upsilon)\psi''(\upsilon) + b(\upsilon)\psi'(\upsilon) + c(\upsilon)\psi(\upsilon) \\ +d(\upsilon)\psi'(\upsilon-1) = f(\upsilon), \quad \upsilon \in \Omega^*,\end{gathered} \tag{1}$$

with the boundary conditions:

$$\begin{gathered}\psi(\upsilon) = \phi(\upsilon), \quad \upsilon \in [-1,0], \quad \psi'(2) = l, \\ \psi'(\upsilon) = \phi'(\upsilon), \quad \upsilon \in [-1,0],\end{gathered} \tag{2}$$

where, $0 < \varepsilon \ll 1$ is the perturbation parameter, and the functions $a(\upsilon) \geq \alpha_1 > \alpha + 2 > 3$, $\quad b(\upsilon) > \beta_0 \geq 0$, $\quad l_0 \leq c(\upsilon) \leq l \leq 0$, $\quad \eta_0 \leq d(\upsilon) \leq 0$ and $2\alpha + 24\gamma_0 + 5\eta_0 > 0$, further, $a(\upsilon), b(\upsilon), c(\upsilon), d(\upsilon), f(\upsilon)$ are assumed to be sufficiently smooth on Ω, and γ is a constant. With these conditions the above Equations (1)–(2) have solution [36].

To handle the delay term, apply Taylor's expansion:

$$\psi'(\upsilon-1) \approx \psi'(\upsilon) - \psi''(\upsilon) + \frac{1}{2}\psi'''(\upsilon). \tag{3}$$

Substituting Equation (3) into Equation (1), we obtain the modified form:

$$\begin{gathered}\left(-\varepsilon + \tfrac{d(\upsilon)}{2}\right)\psi'''(\upsilon) + A(\upsilon)\psi''(\upsilon) + B(\upsilon)\psi'(\upsilon) \\ +C(\upsilon)\psi(\upsilon) = f(\upsilon),\end{gathered} \tag{4}$$

where $\upsilon \in \Omega^*$, subject to the boundary conditions:

$$\psi(0) = \phi(0), \quad \psi'(2) = l, \quad \psi'(0) = \phi'(0), \tag{5}$$

where

$$A(\upsilon) = a(\upsilon) - d(\upsilon), \quad B(\upsilon) = b(\upsilon) + d(\upsilon), \quad C(\upsilon) = c(\upsilon).$$

For continuity at $\upsilon = 1$, we assume:

$$\psi(1^-) = \psi(1^+), \quad \psi'(1^-) = \psi'(1^+).$$

TABLE I
VALUES OF $Q_{4,m}(\upsilon)$, $Q'_{4,m}(\upsilon)$, $Q''_{4,m}(\upsilon)$, AND $Q'''_{4,m}(\upsilon)$ AT NODAL POINTS

	υ_{m-3}	υ_{m-2}	υ_{m-1}	υ_m	υ_{m+1}	υ_{m+2}
$Q_{4,m}(\upsilon)$	0	$\frac{1}{24}$	$\frac{11}{24}$	$\frac{11}{24}$	$\frac{1}{24}$	0
$Q'_{4,m}(\upsilon)$	0	$\frac{1}{6h}$	$\frac{3}{2h}$	$-\frac{3}{2h}$	$-\frac{1}{6h}$	0
$Q''_{4,m}(\upsilon)$	0	$\frac{1}{2h^2}$	$-\frac{1}{2h^2}$	$-\frac{1}{2h^2}$	$\frac{1}{2h^2}$	0
$Q'''_{4,m}(\upsilon)$	0	$\frac{1}{h^3}$	$-\frac{3}{h^3}$	$\frac{3}{h^3}$	$-\frac{1}{h^3}$	0

III. Method Description

In this section, QRBSM is developed to find the numerical solution of Equations (4)–(5). The technique is applied on a uniform mesh. Let $\Pi_N = \{0 = \upsilon_0 < \upsilon_1 < \upsilon_2 < \cdots < \upsilon_N = 2\}$ be a partition of $[0,2]$, where $\upsilon_m = \upsilon_0 + mh$, $h = 2/N$. Define a set P_{4,Π_N} of all polynomials of degree ≤ 4 in the interval $[\upsilon_m, \upsilon_{m+1}]$ for $m = 0, 1, \ldots, N-1$ of Π_N. P_{4,Π_N} forms a linear space and the set P_4 of all functions $r(\upsilon) \in P_{4,\Pi_N} \cap C^3[0,2]$ is a subspace of P_{4,Π_N}. Add four extra knots on each side of the partition Π_N, as $\upsilon_{-3} < \upsilon_{-2} < \upsilon_{-1} < \upsilon_0$ and $\upsilon_N < \upsilon_{N+1} < \upsilon_{N+2} < \upsilon_{N+3}$. The quartic B-spline basis functions are defined as follows:

$$Q_{4,m}(\upsilon) = \frac{1}{24h^4}\begin{cases}(\upsilon-\upsilon_{m-2})^4, & \upsilon \in [\upsilon_{m-2}, \upsilon_{m-1}] \\ h^4 + 4h^3(\upsilon-\upsilon_{m-1}) + 6h^2(\upsilon-\upsilon_{m-1})^2 \\ +4h(\upsilon-\upsilon_{m-1})^3 - 4(\upsilon-\upsilon_{m-1})^4, \\ & \upsilon \in [\upsilon_{m-1}, \upsilon_m] \\ 11h^4 + 12h^3(\upsilon-\upsilon_m) - 6h^2(\upsilon-\upsilon_m)^2 \\ -12h(\upsilon-\upsilon_m)^3 + 6(\upsilon-\upsilon_m)^4, \\ & \upsilon \in [\upsilon_m, \upsilon_{m+1}] \\ h^4 + 4h^3(\upsilon_{m+2}-\upsilon) + 6h^2(\upsilon_{m+2}-\upsilon)^2 \\ +4h(\upsilon_{m+2}-\upsilon)^3 - 4(\upsilon_{m+2}-\upsilon)^4, \\ & \upsilon \in [\upsilon_{m+1}, \upsilon_{m+2}] \\ (\upsilon_{m+3}-\upsilon)^4, & \upsilon \in [\upsilon_{m+2}, \upsilon_{m+3}] \\ 0, & \text{otherwise, for } m = 0, 1, \ldots, N.\end{cases} \tag{6}$$

From the above Equation (6) it is clear that the basis functions $Q_{4,m}(\upsilon)$ are piecewise polynomials of fourth-degree having knots at Π_N. The set of all Quartic B-spline basis $Q_{4,N} = \{Q_{4,-2}, Q_{4,-1}, Q_{4,0}, ..., Q_{4,N}, Q_{4,N+1}\}$ is linearly independent on $[0,2]$. The linear span of functions in Q is a vector space $Q^*(\Pi_N)$ of dimension $(N+4)$ and $Q^*(\Pi_N) = P_4$. Table I gives the values of quartic B-splines and its derivative at the mesh points.

Lemma 1: The Quartic B-splines $(Q_{4,m})_{m=-2}^{N+1}$ satisfy the following inequality

$$\sum_{m=-2}^{N} |Q_{4,m}| \leq \frac{35}{24}. \tag{7}$$

Proof: Refer [37] for proof. ■

Let $S_N(\upsilon)$ be the Quartic B-spline approximation of the solution $\psi(\upsilon)$ defined as:

$$S_N(\upsilon) = \sum_{m=-1}^{N+1} \sigma_m Q_{4,m}(\upsilon). \tag{8}$$

where $Q_{4,m}(\upsilon)$ are the quartic B-spline basis functions defined by Equation (6) and $S_N(\upsilon)$ satisfies the interpolatory conditions

$$S_N(\upsilon_i) = \psi(\upsilon_i), \quad i = 0, 1, 2, ..., N, \tag{9}$$

$$S_N'''(\upsilon_i) = \psi'''(\upsilon_i) - \tfrac{1}{12}h^2\psi^{(5)}(\upsilon_i) + \tfrac{1}{240}h^4\psi^{(7)}(\upsilon_i), \quad i = 0, 1, 2, ..., N. \tag{10}$$

The values of $S_N(\upsilon_m)$ and its derivatives at the nodal points are given by:

$$S_N(\upsilon_m) = \frac{1}{24}\sigma_{m-2} + \frac{11}{24}\sigma_{m-1} + \frac{11}{24}\sigma_m + \frac{1}{24}\sigma_{m+1}, \tag{11}$$

$$S_N'(\upsilon_m) = -\frac{1}{6h}\sigma_{m-2} - \frac{1}{2h}\sigma_{m-1} + \frac{1}{2h}\sigma_m + \frac{1}{6h}\sigma_{m+1}, \tag{12}$$

$$S_N''(\upsilon_m) = \frac{1}{2h^2}\sigma_{m-2} - \frac{1}{2h^2}\sigma_{m-1} - \frac{1}{2h^2}\sigma_m + \frac{1}{2h^2}\sigma_{m+1}, \tag{13}$$

$$S_N'''(\upsilon_m) = -\frac{1}{h^3}\sigma_{m-2} + \frac{3}{h^3}\sigma_{m-1} - \frac{3}{h^3}\sigma_m + \frac{1}{h^3}\sigma_{m+1}. \tag{14}$$

Substituting Equation (8) in Equations (4)–(5), at each node υ_m, we get:

$$\left(-\varepsilon + \tfrac{d(\upsilon_m)}{2}\right) S_N'''(\upsilon_m) + A(\upsilon_m)S_N''(\upsilon_m) + B(\upsilon_m)S_N'(\upsilon_m) + C(\upsilon_m)S_N(\upsilon_m) = f(\upsilon_m), \quad m = 0, 1, \ldots, N, \tag{15}$$

along with the boundary conditions:

$$S_N(0) = \phi(0), \quad S_N'(2) = l, \quad S_N'(0) = \phi'(0). \tag{16}$$

Using Equations (11)–(14) in Equations (15)–(16), we have

$$\begin{aligned}
&\left(-\varepsilon + \frac{d_m}{2}\right)\left(-\frac{1}{h^3}\sigma_{m-2} + \frac{3}{h^3}\sigma_{m-1} - \frac{3}{h^3}\sigma_m + \frac{1}{h^3}\sigma_{+1}\right) \\
&+ A_m\left(\frac{1}{2h^2}\sigma_{m-2} - \frac{1}{2h^2}\sigma_{m-1} - \frac{1}{2h^2}\sigma_+ \frac{1}{2h^2}\sigma_{m+1}\right) \\
&+ B_m\left(-\frac{1}{6h}\sigma_{m-2} - \frac{1}{2h}\sigma_{m-1} + \frac{1}{2h}\sigma_m + \frac{1}{6h}\sigma_{+1}\right) \\
&+ C_m\left(\frac{1}{24}\sigma_{m-2} + \frac{11}{24}\sigma_{m-1} + \frac{11}{24}\sigma_m + \frac{1}{24}\sigma_{m+1}\right) \\
&= f_m, \qquad m = 0, 1, \ldots, N,
\end{aligned} \tag{17}$$

where $A_m = A(\upsilon_m)$, $B_m = B(\upsilon_m)$, $C_m = C(\upsilon_m)$, $d_m = d(\upsilon_m)$ and $f_m = f(\upsilon_m)$. Simplifying Equation (17), we obtain a five-point recurrence relation:

$$\mu_1(\upsilon_m)\sigma_{m-2} + \mu_2(\upsilon_m)\sigma_{m-1} + \mu_3(\upsilon_m)\sigma_m + \mu_4(\upsilon_m)\sigma_{m+1} = 24hf_m, \quad m = 0, 1, \ldots, N, \tag{18}$$

where the coefficients μ_k are defined as:

$$\mu_1(\upsilon_m) = 24\left(-\varepsilon + \frac{d_m}{2}\right) + 12hA_m + 4h^2B_m + h^3C_m,$$

$$\mu_2(\upsilon_m) = 72\left(\varepsilon - \frac{d_m}{2}\right) - 12hA_m - 124h^2B_m + 11h^3C_m,$$

$$\mu_3(\upsilon_m) = 72\left(-\varepsilon + \frac{d_m}{2}\right) - 12hA_m + 124h^2B_m + 11h^3C_m,$$

$$\mu_4(\upsilon_m) = 24\left(\varepsilon - \frac{d_m}{2}\right) + 12hA_m + 4h^2B_m + h^3C_m,$$

and the boundary conditions (16) becomes:

$$\begin{aligned}
\sigma_2 + 11\sigma_{-1} + 11\sigma_0 + \sigma_1 &= 24\phi_0, \\
-\sigma_2 - 3\sigma_{-1} + 3\sigma_0 + \sigma_1 &= 6h\,\phi_0', \\
-\sigma_{N-2} - 3\sigma_{N-1} + 3\sigma_N + \sigma_{N+1} &= 6h\,l.
\end{aligned} \tag{19}$$

Equations (18) along with the boundary conditions (19) gives a system of $(N+4)$ equations in $(N+4)$ unknowns $\sigma_{-2}, \sigma_{-1}, \sigma_0, ..., \sigma_{N+1}$. Further, we eliminate σ_{-2}, σ_{-1} and σ_{N+1} from this system. Substitute the value of σ_{-2} from the first boundary condition in the equation obtained putting $m = 0$ in Equation (18), we get

$$\begin{aligned}
&\left[-336\left(-\varepsilon + \frac{d_0}{2}\right) - 144hA_0 - 168h^2B_0\right]\sigma_{-1} \\
&+ \left[-192\left(-\varepsilon + \frac{d_0}{2}\right) - 144hA_0 + 80h^2B_0\right]\sigma_0 \\
&- 48\left(-\varepsilon + \frac{d_0}{2}\right)\sigma_1 = 24h^2f_0 \\
&- 24\phi_0\left[24\left(-\varepsilon + \frac{d_0}{2}\right) + 12hA_0 + 4h^2B_0 + h^3C_0\right].
\end{aligned} \tag{20}$$

Using the second boundary condition, eliminate σ_{-1} from the equation obtained by putting $m = 1$ in Equations (18) and (20), we obtain

$$\begin{aligned}
&144\left(-\varepsilon + \frac{d_0}{2}\right) + 248h^2B_0\sigma_0 + 48\left(\varepsilon - \frac{d_0}{2}\right)\sigma_1 \\
&= 24h^2f_0 - 24\phi_0\left[24\left(-\varepsilon + \frac{d_0}{2}\right) + 12hA_0\right. \\
&\left.+4h^2B_0 + h^3C_0\right] + 42\left(-\varepsilon + \frac{d_0}{2}\right) + 18hA_0 \\
&+ 21h^2B_0\left(6h\phi_0' + 24\phi_0\right),
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
&\left(-96\left(-\varepsilon + \frac{d_1}{2}\right) - 24hA_1 - 128h^2B_1 + 10h^3C_1\right)\sigma_0 \\
&+ \left(72\left(-\varepsilon + \frac{d_1}{2}\right) - 12hA_1 + 124h^2B_1 + 11h^3C_1\right)\sigma_1 \\
&+ \left(-24\left(-\varepsilon + \frac{d_1}{2}\right) + 12hA_1 + 4h^2B_1 + h^3C_1\right)\sigma_2 \\
&= 192h^2f_1 - (6h\phi_0' + 24\phi_0)\left(24\left(-\varepsilon + \frac{d_0}{2}\right) + 12hA_0\right. \\
&\left.+4h^2B_0 + h^3C_0\right).
\end{aligned} \tag{22}$$

Further, using the third boundary condition and the equation obtained by putting $m = N$ in Equation (18), we obtain

$$\left(24hA_N + 8h^2B_N + 2h^3C_N)\right)\sigma_{N-2} + \left(-144(-\varepsilon + \frac{d_N}{2}) + 24hA_N - 112h^2B_N + 14h^3C_N)\right)\sigma_{N-1} + \left(144(-\varepsilon + \frac{d_N}{2}) - 48hA_N + 112h^2B_N + 8h^3C_N)\right)\sigma_N = 24h^3f_N - 6hl\left(-24(-\varepsilon + \frac{d_N}{2}) + 12hA_N + 4h^2B_N\right. \left. h^3C_N\right) \tag{23}$$

Equations (18) for $m = 2, 3, ..., N-1$ along with Equations (21) – (23) form a $(N+1)\times(N+1)$ system of linear equations. This system can be expressed in matrix form as follows:

$$PX = F, \tag{24}$$

where P is the coefficient matrix given below. The entries in the matrix P are given by:
$\gamma_0 = -792(-\varepsilon + \frac{d_0}{2}) + 264hA_0 - 51h^2B_0$,
$\gamma_1 = -72(-\varepsilon + \frac{d_0}{2}) - 72hA_0 - 9h^2B_0$
$\gamma_2 = 120(-\varepsilon + \frac{d_1}{2}) - 132hA_1 + 62h^2B_1 + 43h^3C_1$,
$\gamma_3 = -264(-\varepsilon + \frac{d_1}{2}) - 60hA_1 + 50h^2B_1 + 43h^3C_1$,
$\gamma_4 = 96(-\varepsilon + \frac{d_1}{2}) + 48hA_1 + 8h^2B_1 + 4h^3C_1$,
$\gamma_5 = 24hA_N + 2h^3C_N$,
$\gamma_6 = 144(-\varepsilon + \frac{d_N}{2}) + 24hA_N - 6h^2B_N + 14h^3C_N$,
$\gamma_7 = -144(-\varepsilon + \frac{d_N}{2}) - 48hA_N + 6h^2B_N + 8h^3C_N$,
$F = [F_0 \quad F_1 \quad 24h^3f_0 \quad 24h^3f_1 \quad \cdots \quad 24h^3f_N \quad F_2]^T$ is the column matrix of right-hand sides, with:
$F_0 = 48h^3f_0 - 48\phi_0(24(-\epsilon + \frac{d_0}{2}) + 12hA_0 - 2h^2B_0 + h^3C_0) - (168(-\varepsilon + \frac{d_0}{2}) - 72hA_0 + 17h^2B_0)(12\phi_0 - 3h\phi_0')$,
$F_1 = 96h^3f_1 - (12\phi_0 - 3h\phi_0')(24(-\varepsilon + \frac{d_0}{2}) + 12hA_0 - 2h^2B_0 + h^3C_0$,
$F_2 = 24h^3f_N - 6hl(24(-\varepsilon + \frac{d_N}{2}) + 12hA_N + 2h^2B_N + h^3C_N$,
and $X = [\sigma_{-2} \quad \sigma_{-1} \quad \sigma_0 \quad \cdots \quad \sigma_N \quad \sigma_{N+1}]^T$.

This system is solved to find the $\sigma_m's$ and hence the solution of the SPDDE.

IV. Convergence Analysis

This section discusses various identities for $S_N(\upsilon)$ and it's derivatives $S_N^{(r)}(\upsilon)$ up to fifth order at the nodal points υ_m, $m = 0, 1, \ldots, N$. Truncation error analysis is conducted to prove the convergence of the proposed method. Following consistency relation for Quartic B-splines can be obtained using [38]:

$$\gamma S_N'(\upsilon_m) = \tfrac{4}{h}(-S_N(\upsilon_{m-2}) - 3S_N(\upsilon_{m-1}) + 3S_N(\upsilon_m) + S_N(\upsilon_{m+1})), m = 2, 3, ..., N-1, \tag{25}$$

$$\gamma S_N''(\upsilon_m) = \tfrac{12}{h^2}(S_N(\upsilon_{m-2}) - S_N(\upsilon_{m-1}) - S_N(\upsilon_m) + S_N(\upsilon_{m+1})), m = 2, 3, ..., N-1, \tag{26}$$

$$\gamma S_N'''(\upsilon_m) = \tfrac{24}{h^3}(-S_N(\upsilon_{m-2}) + 3S_N(\upsilon_{m-1}) - 3S_N(\upsilon_m) + S_N(\upsilon_{m+1})), m = 2, 3, ..., N-1, \tag{27}$$

where γ is the discrete operator defined by
$\gamma y(\upsilon_m) = y(\upsilon_{m-2}) + 11y(\upsilon_{m-1}) + 11y(\upsilon_m) + y(\upsilon_{m+1})$

Now we discuss the following two lemmas to prove the convergence of the method.

Lemma 2: Let $\psi(\upsilon) \in C^8[0, 2]$ and S_N be the Quartic B-spline approximations of ψ, then

$$\gamma S_N'(\upsilon_m) = 24\psi'(\upsilon_m) - 12h\psi''(\upsilon_m) + 8h^2\psi'''(\upsilon_m) - 3h^3\psi^{(4)}(\upsilon_m) + \tfrac{6}{5}h^4\psi^{(5)}(\upsilon_m) - \tfrac{11}{30}h^5\psi^{(6)}(\upsilon_m) + O(h^6), \tag{28}$$

$$\gamma S_N''(\upsilon_m) = 24\psi''(\upsilon_m) - 12h\psi'''(\upsilon_m) + 8h^2\psi^{(4)}(\upsilon_m) - 3h^3\psi^{(5)}(\upsilon_m) + \tfrac{16}{15}h^4\psi^{(6)}(\upsilon_m) - \tfrac{3}{10}h^5\psi^{(7)}(\upsilon_m) + O(h^6), \tag{29}$$

$$\gamma S_N'''(\upsilon_m) = 24\psi'''(\upsilon_m) - 12h\psi^{(4)}(\upsilon_m) + 6h^2\psi^{(5)}(\upsilon_m) - 2h^3\psi^{(6)}(\upsilon_m) + \tfrac{3}{5}h^4\psi^{(7)}(\upsilon_m) - \tfrac{3}{20}h^5\psi^{(8)}(\upsilon_m) + O(h^6), \tag{30}$$

Proof: Using the interpolatory condition (9) in Equation (25), we get

$$\gamma S_N'(\upsilon_m) = \tfrac{4}{h}(-\psi(\upsilon_{m-2}) - 3\psi(\upsilon_{m-1}) + 3\psi(\upsilon_m) \psi(\upsilon_{m+1})).$$

Expanding $\psi(\upsilon_{m-2}), \psi(\upsilon_{m-1})$ and $\psi(\upsilon_{m+1})$ using Taylor's series expansion, we obtain

$$\gamma S_N'(\upsilon_m) = \tfrac{4}{h}(6h\psi'(\upsilon_m) - 3h^2\psi''(\upsilon_m) + 2h^3\psi'''(\upsilon_m) - \tfrac{6}{8}h^4\psi^{(4)}(\upsilon_m) + \tfrac{3}{10}h^5\psi^{(5)}(\upsilon_m) - \tfrac{11}{120}h^6\psi^{(6)}(\upsilon_m) + O(h^7)).$$

This proves (28). One can prove (29) and (30) with similar arguments. ∎

Lemma 3: Let S_N be approximations of ψ given by the QRBSM which satisfy the required smoothness conditions, then the following holds:

$$S_N'(\upsilon_m) = \psi'(\upsilon_m) + \tfrac{1}{720}h^4\psi^{(5)}(\upsilon_m) - \tfrac{1}{2016}h^6\psi^{(7)}(\upsilon_m) + \mathcal{O}(h^8), \tag{31}$$

$$S_N''(\upsilon_m) = \psi''(\upsilon_m) - \tfrac{1}{240}h^4\psi^{(6)}(\upsilon_m) + \tfrac{1}{6080}h^6\psi^{(8)}(\upsilon_m) + \mathcal{O}(h^8), \tag{32}$$

$$S_N'''(\upsilon_m) = \psi'''(\upsilon_m) - \tfrac{1}{12}h^2\psi^{(5)}(\upsilon_m) + \tfrac{1}{240}h^4\psi^{(7)}(\upsilon_m) + \mathcal{O}(h^6), \tag{33}$$

$$\bar{S}_N^{(4)}(\upsilon_m) = \psi^{(4)}(\upsilon_m) + \tfrac{1}{12}h^2\psi^{(6)}(\upsilon_m) - \tfrac{1}{720}h^4\psi^{(8)}(\upsilon_m) + \mathcal{O}(h^6), \tag{34}$$

$$\bar{S}_N^{(5)}(\upsilon_m) = \psi^{(5)}(\upsilon_m) + \tfrac{211}{151200}h^6\psi^{(11)}(\upsilon_m) - \tfrac{421}{1814400}h^8\psi^{(13)}(\upsilon_m) + \mathcal{O}(h^{10}), \tag{35}$$

where,

$$\bar{S}_N^{(4)}(\upsilon_m) = \frac{\bar{S}_N^{(4)}(\upsilon_{m-}) + \bar{S}_N^{(4)}(\upsilon_{m+})}{2}, \tag{36}$$

$$\bar{S}_N^{(5)}(\upsilon_m) = \frac{\bar{S}_N^{(5)}(\upsilon_{m-}) + \bar{S}_N^{(5)}(\upsilon_{m+})}{2}. \tag{37}$$

Proof: Using Equations (11)-(14), we obtain

$$h[S_N'(\upsilon_{m-2}) + 11S_N'(\upsilon_{m-1}) + 11S_N'(\upsilon_m) + S_N'(\upsilon_{m+1})] = 4[S_N(\upsilon_{m+1}) + 3S_N(\upsilon_m) - 3S_N(\upsilon_{m-1}) - S_N(\upsilon_{m-2})], \tag{38}$$

$$h^2[S_N''(\upsilon_m) = 2[S_N(\upsilon_{m+1}) - 2S_N(\upsilon_m) + S_N(\upsilon_{m-1})] - \tfrac{h}{2}[S_N'(\upsilon_{m+1}) - S_N'(\upsilon_{m-1})], \tag{39}$$

$$P=\begin{pmatrix}\gamma_0 & \gamma_1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0\\ \gamma_2 & \gamma_3 & \gamma_4 & 0 & 0 & 0 & \dots & 0 & 0 & 0\\ \mu_1(\upsilon_1) & \mu_2(\upsilon_1) & \mu_3(\upsilon_1) & \mu_4(\upsilon_1) & 0 & 0 & \dots & 0 & 0 & 0\\ 0 & \mu_1(\upsilon_2) & \mu_2(\upsilon_2) & \mu_3(\upsilon_2) & \mu_4(\upsilon_2) & 0 & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \dots & \mu_1(\upsilon_{N-1}) & \mu_2(\upsilon_{N-1}) & \mu_3(\upsilon_{N-1}) & \mu_4(\upsilon_{N-1})\\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \gamma_5 & \gamma_6 & \gamma_7\end{pmatrix}$$

$$\begin{aligned}h^3[S_N^{'''}(\upsilon_m) &= 2[S_N(\upsilon_{m+1})-S_N(\upsilon_{m-1})]\\ &-3h[S_N^{'}(\upsilon_{m+1})+6S_N^{'}(\upsilon_m)-S_N^{'}(\upsilon_{m-1})],\end{aligned}\tag{40}$$

Apply operator notation in Equation (38), we get

$$\begin{aligned}&h[E^{-2}+11E^{-1}+11I+E]S_N^{'}(\upsilon_m)\\ &=4[E+3I-3E^{-1}-E^{-2}]S_N(\upsilon_m).\end{aligned}\tag{41}$$

where $E(S_N(\upsilon_m))=S_N(\upsilon_{m+1})$.
Since $E=e^{hD}$, where $D=\frac{d}{d\upsilon}$, we get

$$\begin{aligned}&h[e^{-2hD}+11e^{-hD}+11I+e^{hD}]S_N^{'}(\upsilon_m)\\ &=4[e^{hD}+3I-3e^{-hD}-e^{-2hD}]S_N(\upsilon_m).\end{aligned}\tag{42}$$

Expanding $E=e^{hD}$ in Taylor's series, we get

$$\begin{aligned}&24h(1-\tfrac{1}{2}hD+\tfrac{1}{3}h^2D^2-\tfrac{1}{8}h^3D^3+\tfrac{7}{144}h^4D^4\\ &+...)S_N^{'}(\upsilon_m)=24h\left(D-\tfrac{1}{2}hD^2+\tfrac{1}{3}h^2D^3\right.\\ &\left.-\tfrac{1}{8}h^3D^4+...\right)\psi(\upsilon_m).\end{aligned}\tag{43}$$

Simplifying this, we get

$$\begin{aligned}S_N^{'}(\upsilon_m)&=\left(D-\tfrac{1}{2}hD^2+\tfrac{1}{3}h^2D^3-\tfrac{1}{8}h^3D^4+...\right)\\ &\Big\{1+(-\tfrac{1}{2}hD+\tfrac{1}{3}h^2D^2-\tfrac{1}{8}h^3D^3\\ &+...)\Big\}^{-1}\psi(\upsilon_m).\end{aligned}\tag{44}$$

Further Simplifying this, we obtain

$$\begin{aligned}S_N^{'}(\upsilon_m)&=(D+\tfrac{1}{720}h^4D^5-\tfrac{1}{2016}h^6D^7\\ &+\tfrac{1}{17280}h^8D^9+...)\psi(\upsilon_m).\end{aligned}\tag{45}$$

This proves Equation (31). Similarly, we can prove the identities (32) and (33). Now to prove Eq. (34), we consider the central difference approximation for $\bar{S}_N^{(4)}(\upsilon_m)$ as:

$$\bar{S}_N^{(4)}(\upsilon_m)=\frac{S_N^{'''}(\upsilon_{m+1})-S_N^{'''}(\upsilon_{m-1})}{2h}.\tag{46}$$

Substitute Equation (33) into Equation (46), we get

$$\begin{aligned}\bar{S}_N^{(4)}(\upsilon_m)&=\tfrac{1}{2h}\Big\{[\psi^{'''}(\upsilon_{m+1})-\tfrac{1}{12}h^2\psi^{(5)}(\upsilon_{m+1})\\ &+\tfrac{1}{240}h^4\psi^{(7)}(\upsilon_{m+1})]\\ &-[\psi^{'''}(\upsilon_{m-1})-\tfrac{1}{12}h^2\psi^{(5)}(\upsilon_{m-1})\\ &+\tfrac{1}{240}h^4\psi^{(7)}(\upsilon_{m-1})]\Big\}.\end{aligned}\tag{47}$$

Expanding the terms in above Equation (47) by Taylor's series about $\upsilon=\upsilon_m$, and simplify. This yields Equation (34).

To prove Equation (35), use:

$$\bar{S}_N^{(5)}(\upsilon_m)=\frac{S_N^{'''}(\upsilon_{m+1})-2S_N^{'''}(\upsilon_m)+S_N^{'''}(\upsilon_{m-1})}{h^2}.\tag{48}$$

Substituting from Equation (33) and expanding via Taylor series about $\upsilon=\upsilon_m$, we can get the estimate for $\bar{S}_N^{(5)}(\upsilon_m)$ as Equation (35). ∎

Theorem 1: Let S_N be the QRBSM approximation of ψ, and define the error as $e(\upsilon)=S_N(\upsilon)-\psi(\upsilon)$. Then for $0\le\theta\le1$:

$$\begin{aligned}e(\upsilon_m+\theta h)=&-\frac{(1-10\theta^2)\theta}{720}h^5\psi^{(5)}(\upsilon_m)\\ &+\frac{(5\theta^2-3)\theta^2}{1440}h^6\psi^{(6)}(\upsilon_m)\\ &+\frac{(7\theta^2-5)\theta^2}{10080}h^7\psi^{(7)}(\upsilon_m)+\mathcal{O}(h^8).\end{aligned}\tag{49}$$

Proof: Expanding $e(\upsilon_m+\theta h)$ in Taylor's series and using Equation (31)–(35), we obtain (49) upon simplification. ∎

This shows that the truncation error is of order $O(h^5)$ and hence the order of convergence of the proposed method is $O(h^2)$.

V. Numerical Illustrations and Discussion

In this section, we have implemented the proposed method on three numerical examples to examine the efficiency and accuracy. Maximum absolute error (MAE) and rate of convergence (RCGT) are determined for different values of ϵ and N. The obtained results are compared with [15] and [17]. Graphs are plotted to see the behavior of the solution for different values of the parameter. As the exact solutions are not available, the double mesh principle is employed to determine the MAE at the nodal points, which is given by:

$$E_N=max_{0\le m\le N}|S_N(\upsilon_m)-S_N(\upsilon_{2m})|.\tag{50}$$

Further, the RCGT is determined using the following formula:

$$RCGT=\frac{ln(E_N/E_{2N})}{ln2}.\tag{51}$$

Example 1: Consider the following SPDDE:
$-\varepsilon\psi'''(\upsilon)+(16+\upsilon)\psi''(\upsilon)-\psi(\upsilon)-\frac{1}{2}\psi'(\upsilon-1)=\upsilon$,
$\upsilon\in\Omega^*$,

TABLE II
MAE FOR EXAMPLE 1 WITH DIFFERENT VALUES OF ϵ

ε	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
2^{-1}	3.5834E-03	7.2699E-04	1.7327E-04	4.2813E-05	1.0671E-05	2.6670E-06
2^{-2}	8.3757E-03	1.1920E-03	2.6661E-04	6.4907E-05	1.6120E-05	4.0231E-06
2^{-3}	3.3202E-02	1.8067E-03	3.6664E-04	8.7387E-05	2.1593E-05	5.3816E-06
2^{-4}	6.4733E-02	2.4934E-03	4.5345E-04	1.0574E-04	2.5994E-05	6.4708E-06
2^{-5}	2.7639E-02	3.1153E-03	5.1564E-04	1.1820E-04	2.9439E-05	7.1972E-06
2^{-6}	2.2141E-02	3.5744E-03	5.5414E-04	1.2563E-04	3.0679E-05	7.6253E-06
2^{-7}	2.0366E-02	3.8637E-03	5.7581E-04	1.2971E-04	3.1629E-05	7.8584E-06
2^{-8}	1.9650E-02	4.0280E-03	5.8734E-04	1.3185E-04	3.2127E-05	7.9810E-06
2^{-9}	1.9329E-02	4.1159E-03	5.9330E-04	1.3295E-04	3.2382E-05	8.0436E-06
2^{-10}	1.9177E-02	4.1614E-03	5.9632E-04	1.3351E-04	3.2512E-05	8.0752E-06
2^{-11}	1.9103E-02	4.1846E-03	5.9785E-04	1.3379E-04	3.2576E-05	8.0906E-06
2^{-12}	1.9067E-02	4.1962E-03	5.9861E-04	1.3393E-04	3.2608E-05	8.0990E-06
2^{-16}	1.9033E-02	4.2073E-03	5.9933E-04	1.3406E-04	3.2639E-05	8.1064E-06
2^{-20}	1.9031E-02	4.2079E-03	5.9938E-04	1.3407E-04	3.2641E-05	8.1068E-06
2^{-24}	1.9031E-02	4.2080E-03	5.9938E-04	1.3407E-04	3.2641E-05	8.1069E-06

TABLE III
RCGT FOR EXAMPLE 1

ε	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
2^{-1}	2.0689E+00	2.0169E+00	2.0043E+00	2.0005E+00	2.0005E+00
2^{-2}	2.1606E+00	2.0383E+00	2.0095E+00	2.0025E+00	2.0025E+00
2^{-3}	2.3009E+00	2.0689E+00	2.0169E+00	2.0044E+00	2.0044E+00
2^{-4}	2.4591E+00	2.1004E+00	2.0244E+00	2.0061E+00	2.0061E+00
2^{-5}	2.5949E+00	2.1251E+00	2.0301E+00	2.0075E+00	2.0075E+00
2^{-6}	2.6894E+00	2.1411E+00	2.0338E+00	2.0084E+00	2.0084E+00
2^{-7}	2.7463E+00	2.1503E+00	2.0359E+00	2.0090E+00	2.0090E+00
2^{-8}	2.7778E+00	2.1553E+00	2.0371E+00	2.0091E+00	2.0091E+00
2^{-9}	2.7944E+00	2.1578E+00	2.0377E+00	2.0093E+00	2.0093E+00
2^{-10}	2.8029E+00	2.1591E+00	2.0380E+00	2.0093E+00	2.0093E+00
2^{-11}	2.8072E+00	2.1598E+00	2.0381E+00	2.0095E+00	2.0095E+00
2^{-12}	2.8094E+00	2.1601E+00	2.0382E+00	2.0094E+00	2.0094E+00
2^{-16}	2.8114E+00	2.1604E+00	2.0383E+00	2.0095E+00	2.0095E+00
2^{-20}	2.8116E+00	2.1605E+00	2.0383E+00	2.0095E+00	2.0095E+00
2^{-24}	2.8116E+00	2.1605E+00	2.0383E+00	2.0094E+00	2.0094E+00

TABLE IV
COMPARISON OF UNIFORM MAE FOR EXAMPLE 1

	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
QRBSM	1.9031E-02	4.2080E-03	5.9938E-04	1.3407E-04	3.2641E-05	8.1069E-06
Method in [15]	2.6596E-02	1.2806E-02	6.3131E-03	3.1369E-03	1.5637E-03	7.8070E-04

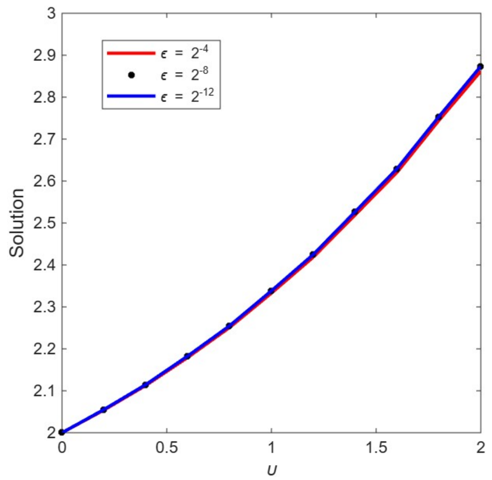

Fig. 1. Numerical solution of Example 1 for $N = 10$, and different values of ε.

with the boundary conditions:
$\psi(\upsilon) = 2 + \upsilon, \upsilon \in [-1, 0], \psi'(2) = 2.$

Example 2: Consider the following SPDDE:
$-\varepsilon\psi'''(\upsilon) + (12 + \upsilon^2)\psi''(\upsilon) + \psi'(\upsilon) - \psi(\upsilon) - \psi'(\upsilon - 1) = e^{-\upsilon^2}, \upsilon \in \Omega^*,$
with the boundary conditions:
$\psi(\upsilon) = 1 + \upsilon, \upsilon \in [-1, 0], \psi'(2) = 2.$

Example 3: Consider the following SPDDE:
$-\varepsilon\psi'''(\upsilon) + 16\psi''(\upsilon) - \psi(\upsilon) - \psi'(\upsilon - 1) = 0, \upsilon \in \Omega^*,$
with the boundary conditions:
$\psi(\upsilon) = 1 + \upsilon, \upsilon \in [-1, 0], \psi'(2) = 1.$

Tables II, V and IX provide the MAE for Examples 1, 2 and 3 respectively. The consistency of the method is evident as the MAE decreases with an increasing number of mesh points. It can also be seen that the method provides a good numerical solution even for small values of N. Tables III, VI, and X represents the RCGT for Examples 1, 2 and 3 respectively. A uniform rate of convergence is observed as two for all values of ϵ and N. A comparison of results with [15] for Examples 1 and 2 are provided in Tables IV, VII and VIII for the maximum MAE values for $\epsilon \in \{2^{-4}, 2^{-8}, ..., 2^{-32}\}$; MAE for Example 3 are compared with the values of MAE in [17] and presented in Table IX. This comparison highlights the efficiency of the proposed method. Figures 1, 2 and 3 show the behavior of the solution for Examples 1, 2 and 3 respectively for different values of ε and it is seen that the considered problem have a boundary

TABLE V
MAE for Example 2

ε	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
2^{-1}	3.1557E-03	7.1708E-04	1.7521E-04	4.3554E-05	1.0873E-05	2.7183E-06
2^{-2}	4.6258E-03	9.6951E-04	2.3266E-04	5.7584E-05	1.4360E-05	3.5885E-06
2^{-3}	6.1873E-03	1.1862E-03	2.7947E-04	6.8863E-05	1.7154E-05	4.2845E-06
2^{-4}	7.5319E-03	1.3399E-03	3.1116E-04	7.6414E-05	1.9019E-05	4.7501E-06
2^{-5}	8.4852E-03	1.4343E-03	3.3002E-04	8.0869E-05	2.0117E-05	5.0227E-06
2^{-6}	9.0698E-03	1.4872E-03	3.4037E-04	8.3302E-05	2.0716E-05	5.1721E-06
2^{-7}	9.3965E-03	1.5153E-03	3.4581E-04	8.4577E-05	2.1030E-05	5.2504E-06
2^{-8}	9.5697E-03	1.5298E-03	3.4860E-04	8.5229E-05	2.1190E-05	5.2901E-06
2^{-9}	9.6589E-03	1.5372E-03	3.5001E-04	8.5559E-05	2.1271E-05	5.3104E-06
2^{-10}	9.7042E-03	1.5409E-03	3.5072E-04	8.5725E-05	2.1312E-05	5.3216E-06
2^{-11}	9.7270E-03	1.5427E-03	3.5107E-04	8.5809E-05	2.1333E-05	5.3259E-06
2^{-12}	9.7385E-03	1.5437E-03	3.5125E-04	8.5850E-05	2.1343E-05	5.3273E-06
2^{-16}	9.7493E-03	1.5445E-03	3.5142E-04	8.5889E-05	2.1352E-05	5.3305E-06
2^{-20}	9.7499E-03	1.5446E-03	3.5143E-04	8.5892E-05	2.1353E-05	5.3299E-06
2^{-24}	9.7500E-03	1.5446E-03	3.5143E-04	8.5892E-05	2.1353E-05	5.3300E-06

TABLE VI
RCGT for Example 2

ε	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
2^{-1}	2.1384E+00	2.0332E+00	2.0082E+00	2.0021E+00	2.0014E+00
2^{-2}	2.2550E+00	2.0592E+00	2.0145E+00	2.0036E+00	2.0014E+00
2^{-3}	2.3836E+00	2.0858E+00	2.0209E+00	2.0052E+00	2.0012E+00
2^{-4}	2.4916E+00	2.1065E+00	2.0258E+00	2.0064E+00	2.0019E+00
2^{-5}	2.5652E+00	2.1199E+00	2.0289E+00	2.0072E+00	2.0016E+00
2^{-6}	2.6091E+00	2.1276E+00	2.0307E+00	2.0076E+00	2.0018E+00
2^{-7}	2.6331E+00	2.1318E+00	2.0317E+00	2.0079E+00	2.0020E+00
2^{-8}	2.6458E+00	2.1339E+00	2.0322E+00	2.0080E+00	2.0019E+00
2^{-9}	2.6522E+00	2.1350E+00	2.0324E+00	2.0081E+00	2.0020E+00
2^{-10}	2.6555E+00	2.1355E+00	2.0326E+00	2.0081E+00	2.0025E+00
2^{-11}	2.6572E+00	2.1358E+00	2.0326E+00	2.0081E+00	2.0021E+00
2^{-12}	2.6580E+00	2.1360E+00	2.0327E+00	2.0081E+00	2.0015E+00
2^{-16}	2.6588E+00	2.1361E+00	2.0327E+00	2.0081E+00	2.0019E+00
2^{-20}	2.6588E+00	2.1361E+00	2.0327E+00	2.0081E+00	2.0016E+00
2^{-24}	2.6588E+00	2.1361E+00	2.0327E+00	2.0081E+00	2.0016E+00

TABLE VII
Comparison of uniform MAE for Example 2

	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
QRBSM	1.5446E-03	3.5143E-04	8.5892E-05	2.1353E-05	5.3300E-06
Method in [15]	6.7293E-03	3.3391E-03	1.6750E-03	8.4238E-04	4.2398E-04

TABLE VIII
Comparison of uniform MAE for Example 3

	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
Quartic B-Spline	4.4214E-03	7.9221E-04	1.8424E-04	4.5258E-05	1.1265E-05	2.8140E-06
Method in [15]	1.2232E+00	3.4213E+00	7.8455E-01	3.8738E-01	1.9251E-01	9.5967E-02

layer on the right side of the domain.

VI. Conclusion

This paper presents QRBSM for third-order SPDDEs with a large delay. The method is directly applied to the third-order SPDDEs without reducing it to lower-order BVPs to preserve the original problem structure.

The proposed approach is applied to three numerical examples, and the obtained results demonstrate high accuracy. The effectiveness of the QRBSM is evaluated through a comparative analysis with existing numerical methods. The convergence of the method is discussed using truncation error analysis. The method is uniformly convergent, and the second-order convergence is verified theoretically and numerically.

Overall, the QRBSM provides a stable and consistent solution for third-order SPDDEs. In the future, QRBSM can be applied to further higher-order SPDEs and SPDDEs, or it can be combined with a piecewise uniform mesh to enhance accuracy.

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TABLE IX
COMPARISON OF MAE FOR EXAMPLE 3 FOR DIFFERENT VALUES ε AND GRID SIZES N.

ε		$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-4}	QRBSM	4.4214E-03	7.9221E-04	1.8424E-04	4.5259E-05	1.1266E-05	2.8140E-06
	Method in [17]	1.9004E-02	8.9587E-03	4.4032E-03	2.1906E-03	1.0943E-03	5.4757E-04
2^{-5}	QRBSM	4.9783E-03	8.4826E-04	1.9547E-04	4.7915E-05	1.1920E-05	2.9763E-06
	Method in [17]	1.8997E-02	8.9332E-03	4.3840E-03	2.1781E-03	1.0866E-03	5.4295E-04
2^{-6}	QRBSM	5.3192E-03	8.7965E-04	2.0163E-04	4.9365E-05	1.2278E-05	3.0658E-06
	Method in [17]	1.8995E-02	8.9223E-03	4.3760E-03	2.1730E-03	1.0834E-03	5.4110E-04
2^{-7}	QRBSM	5.5097E-03	8.9631E-04	2.0487E-04	5.0125E-05	1.2465E-05	3.1121E-06
	Method in [17]	1.8995E-02	8.9174E-03	4.3723E-03	2.1707E-03	1.0821E-03	5.4030E-04
2^{-8}	QRBSM	5.6106E-03	9.0490E-04	2.0653E-04	5.0514E-05	1.2560E-05	3.1358E-06
	Method in [17]	1.8995E-02	8.9151E-03	4.3706E-03	2.1696E-03	1.0814E-03	5.3993E-04
2^{-9}	QRBSM	5.6626E-03	9.0927E-04	2.0737E-04	5.0711E-05	1.2609E-05	3.1479E-06
	Method in [17]	1.8995E-02	8.9139E-03	4.3698E-03	2.1691E-03	1.0811E-03	5.3975E-04
2^{-10}	QRBSM	5.6889E-03	9.1147E-04	2.0779E-04	5.0810E-05	1.2633E-05	3.1539E-06
	Method in [17]	1.8995E-02	8.9134E-03	4.3694E-03	2.1688E-03	1.0810E-03	5.3967E-04
2^{-11}	QRBSM	5.7022E-03	9.1257E-04	2.0801E-04	5.0859E-05	1.2645E-05	3.1562E-06
	Method in [17]	1.8995E-02	8.9131E-03	4.3691E-03	2.1687E-03	1.0809E-03	5.3962E-04
2^{-12}	QRBSM	5.7089E-03	9.1312E-04	2.0811E-04	5.0884E-05	1.2651E-05	3.1587E-06
	Method in [17]	1.8995E-02	8.9130E-03	4.3690E-03	2.1686E-03	1.0808E-03	5.3960E-04
2^{-13}	QRBSM	5.7123E-03	9.1340E-04	2.0816E-04	5.0897E-05	1.2654E-05	3.1599E-06
	Method in [17]	1.8995E-02	8.9129E-03	4.3690E-03	2.1686E-03	1.0808E-03	5.3959E-04
2^{-14}	QRBSM	5.7139E-03	9.1354E-04	2.0819E-04	5.0903E-05	1.2656E-05	3.1593E-06
	Method in [17]	1.8995E-02	8.9129E-03	4.3690E-03	2.1686E-03	1.0808E-03	5.3959E-04
2^{-15}	QRBSM	5.7148E-03	9.1361E-04	2.0820E-04	5.0906E-05	1.2657E-05	3.1594E-06
	Method in [17]	1.8995E-02	8.9128E-03	4.3690E-03	2.1686E-03	1.0808E-03	5.3958E-04
2^{-16}	QRBSM	5.7152E-03	9.1364E-04	2.0821E-04	5.0907E-05	1.2657E-05	3.1600E-06
	Method in [17]	1.8995E-02	8.9128E-03	4.3689E-03	2.1686E-03	1.0808E-03	5.3958E-04
2^{-23}	QRBSM	5.7156E-03	9.1368E-04	2.0822E-04	5.0909E-05	1.2657E-05	3.1604E-06
	Method in [17]	1.8995E-02	8.9128E-03	4.3689E-03	2.1686E-03	1.0808E-03	5.3958E-04

TABLE X
RCGT FOR EXAMPLE 3

ε	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
2^{-4}	2.4806E+00	2.1043E+00	2.0253E+00	2.0063E+00	2.0012E+00
2^{-8}	2.6323E+00	2.1315E+00	2.0317E+00	2.0077E+00	2.0020E+00
2^{-12}	2.6444E+00	2.1334E+00	2.0321E+00	2.0079E+00	2.0019E+00
2^{-16}	2.6451E+00	2.1336E+00	2.0321E+00	2.0080E+00	2.0019E+00
2^{-20}	2.6452E+00	2.1336E+00	2.0321E+00	2.0079E+00	2.0019E+00
2^{-24}	2.6452E+00	2.1336E+00	2.0321E+00	2.0079E+00	2.0019E+00

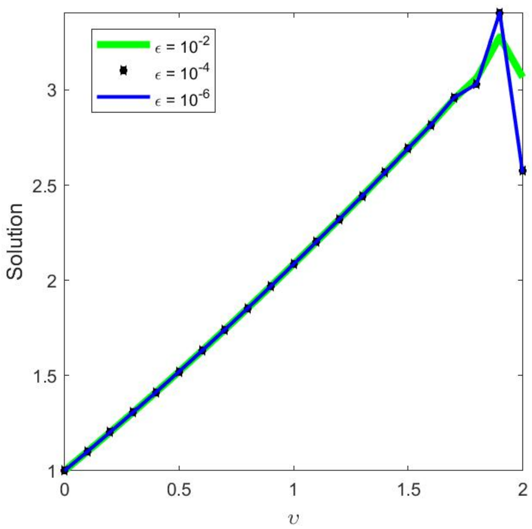

Fig. 2. Numerical solution of Example 2 for $N = 20$, and different values of ϵ.

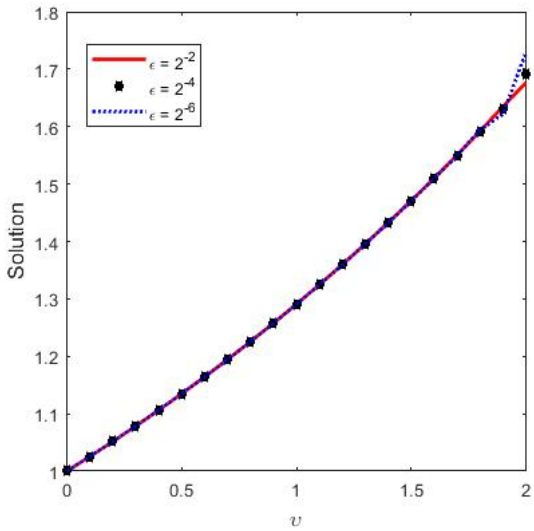

Fig. 3. Numerical solution of Example 3 for $N = 20$, and different values of ϵ.

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