Supercyclic Properties of the Sum of a Hypercyclic (Tuple of) Operator(s) and Identity Operator

Jun Wu*, Chong Yang*, Yi Lu

Abstract—We proved that the sum of a norm hypercyclic operator on a Banach space and identity operator on complex field $\mathbb{C}$ was norm supercyclic and the sum of a norm hypercyclic vector and any non-zero scalar was a norm supercyclic vector. These results also held in weak topology. We extended the above results to 2-tuples of operators.

Keywords—Norm hypercyclic operators; Norm supercyclic operators; Identity operator; Weak topology; 2-tuples of operators

I. Introduction

When studying the dynamic system of linear operators, it is important to understand how different types of operators interact with each other and the properties that arise from these interactions. One area of interest is the behavior of hypercyclic operators and supercyclic operators. In recent years, significant research has been conducted on hypercyclic operators and supercyclic operators, uncovering various properties and characteristics. However, there remains a gap in understanding how operators of different properties affect each other. This study aims to fill this gap by investigating the sum of hypercyclic (tuples of) operators and identity operators firstly.

Let X denote an arbitrary separable, infinite dimensional Banach space over the field $\mathbb{C}$ of complex numbers and $T: X \to X$ a bounded linear operator where T is called norm hypercyclic provided that there exists a vector $x \in X$ such that the set $\{T^n x: n = 0,1,2,\cdots\}$ is norm dense in X and x is norm hypercyclic for T. Under these assumptions, T is norm supercyclic provided there exists a vector $x \in X$ such that the set $\{\lambda T^n x: \lambda \in \mathbb{C}, n = 0,1,2,\cdots\}$ is norm dense in X where x is norm supercyclic for T. If weak topology is applied instead of norm topology, it can be similar to define weakly hypercyclic and weakly supercyclic operator. Hypercyclicity and supercyclicity actually described the density of operator orbits and scalar multiples of operator orbits respectively. In order to present them more intuitively, we have provided the following schematic diagram (Figure 1).

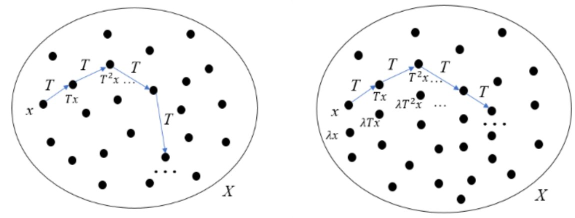

Figure 1 orbits of hypercyclic and supercyclic operators

Several research works have been performed on the results of hypercyclic and supercyclic operators. Montes and Salas [1] made several notable conclusions indicating that for a given supercyclic operator, the set comprising its supercyclic vectors along with zero vector could form subspaces. Bès and Peris [2] introduced the concept of disjoint hypercyclic operators. Chan and Sanders [3] developed an operator that exhibited weak hypercyclicity yet failed to display norm hypercyclicity. Rosa and Read [4] provided an illustrative example showing that there could be a hypercyclic operator for which the direct sum $T \oplus T$ did not possess hypercyclicity. Rezaei and Alvan [5] investigated hypercyclic operators within the context of vector-valued Hardy spaces. Recently, a generalized concept known as A-hypercyclicity has been introduced to systematically investigate different types of hypercyclic operators [6]. More recently, Chen et al. [7] introduced the notion of disjoint subspace-hypercyclic operators on separable Banach spaces. For further investigation in this domain, one could refer to the books [8, 9].

Feldman [10] was the first researcher to investigate n-tuples of operators in 2006. If $T_1, T_2, \ldots, T_n$ are commuting bounded linear operators on X, then the sequence $\mathcal{T} = (T_1, T_2, \ldots, T_n)$ is called an n-tuple of operators. Let $\mathcal{F}_{\mathcal{T}} = \{T_1^{k_1} T_2^{k_2} \cdots T_n^{k_n}: k_i \geq 0\}$ be the semigroup generated by $\mathcal{T}$. If there exists a vector $x \in X$, such that the set $\{Sx: S \in \mathcal{F}_{\mathcal{T}}\}$ is dense in X, then $\mathcal{T}$ is called hypercyclic and vector x is hypercyclic for $\mathcal{T}$. If there exists a vector $x \in X$ such that the set $\{\lambda Sx: \lambda \in \mathbb{C}, S \in \mathcal{F}_{\mathcal{T}}\}$ is dense in X, then $\mathcal{T}$ is supercyclic and the vector x is supercyclic for $\mathcal{T}$. When different topologies such as norm topology and weak topology are considered, it can be similar to define the norm hypercyclic, norm supercyclic, weakly hypercyclic and weakly supercyclic n-tuple of operators. For more information on the dynamics of tuples of operators, one can refer to survey articles [11-18].

Let X be a Banach space with the norm $\|\cdot\|$. For any $x \in X$ and $\alpha \in \mathbb{C}$, $\|x \oplus \alpha\| = \|x\| + |\alpha|$ is a norm and the space $X \oplus \mathbb{C}$ is a Banach space. Inspired by the results obtained by Herrero [19], González [20] and Sanders [21], in section 2, we proved that if T was a hypercyclic operator on X, the vector x was hypercyclic for T; then, for any non-zero scalar α, the vector $x \oplus \alpha$ was supercyclic for the sum operator $T \oplus I$. Also, $T \oplus I$ was supercyclic on $X \oplus \mathbb{C}$. These results also held true in weak topology. In section 3, we have extended these results to 2-tuples of operators.

Throughout this paper, $\mathbb{N}$ denotes the set of positive integers.

II. Supercyclic properties of $T \oplus I$

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In this section, we study the property of the direct sum of a single operator and an identity operator, which briefly generalize the conclusions that have been made with more hypercyclic vectors are available.

Theorem 2.1. Let $T: X \to X$ be a hypercyclic operator, $I: \mathbb{C} \to \mathbb{C}$ be identity operator and $x \in X, \alpha \in \mathbb{C}$. The vector $x \oplus \alpha$ is supercyclic for $T \oplus I$ if and only if the vector x is hypercyclic for T and $\alpha \neq 0$.

Proof. (Forward Direction)⇒ If the vector $x \oplus \alpha$ is supercyclic for $T \oplus I$, then for any $y \in X$, $\varepsilon > 0$, there exist a scalar $\lambda \in \mathbb{C}$, $n \in \mathbb{N}$ satisfying $\left\|\lambda(T \oplus I)^n(x \oplus \alpha) - \frac{1}{\alpha}y \oplus 1\right\| < \varepsilon$. Selecting $M > 0$ satisfies $\|T^n x\| \leq M$. Then

$$\begin{aligned}\|T^n x - y\| &= \|T^n x - y + \lambda\alpha T^n x - \lambda\alpha T^n x\| \\ &\leq \|\lambda\alpha T^n x - y\| + \|(1 - \lambda\alpha)T^n x\| \\ &< |\alpha|\varepsilon + M\varepsilon \\ &= (|\alpha| + M)\varepsilon.\end{aligned}$$

Therefore, the vector x is hypercyclic for T.

(Reverse Direction) ⇐ If $\alpha \neq 0$ and the vector x is hypercyclic for T, then for any $\varepsilon > 0$, $y \oplus \gamma \in X \oplus \mathbb{C}$ is selected,

(1) $\gamma \neq 0$. Since x is hypercyclic for T, there exists $n \in \mathbb{N}$ satisfying $\left\|T^n x - \frac{\alpha}{\gamma}y\right\| < \left|\frac{\alpha}{\gamma}\right| \varepsilon$. Therefor,

$$\left\|\frac{\gamma}{\alpha}(T \oplus I)^n(x \oplus \alpha) - y \oplus \gamma\right\| = \left|\frac{\gamma}{\alpha}\right| \left\|T^n x - \frac{\alpha}{\gamma}y\right\| < \varepsilon.$$

(2) $\gamma = 0$. For above α and ε, there exists some $k_0 \in \mathbb{N}$ satisfying $\frac{|\alpha|}{k_0} < \frac{\varepsilon}{2}$. Since x is a hypercyclic vector for T, there is $n \in \mathbb{N}$ satisfying $\|T^n x - k_0 y\| < \frac{k_0\varepsilon}{2}$. Therefor,

$$\left\|\frac{1}{k_0}(T \oplus I)^n(x \oplus \alpha) - y \oplus 0\right\| = \frac{1}{k_0}\|T^n x - k_0 y\| + \frac{|\alpha|}{k_0} < \varepsilon.$$

Above all, for any $y \oplus \gamma \in X \oplus \mathbb{C}$, there exist $\lambda \in \mathbb{C}$ and an integer $n \in \mathbb{N}$ satisfying $\|\lambda(T \oplus I)^n(x \oplus \alpha) - y \oplus \gamma\| < \varepsilon$. Thus, $x \oplus \alpha$ is a supercyclic vector for $T \oplus I$.

Example 1. Let B be a backward shift operator defined as $B(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$ on sequence space $l_2(\mathbb{N})$, then $2B$ is hypercyclic. Obviously, $2B \oplus I$ satisfy Theorem 2.1. (The examples of the following theorems can be similarly constructed, and they will not be listed below.)

Theorem 2.2. Let $T: X \to X$ be a hypercyclic operator, $I: \mathbb{C} \to \mathbb{C}$ be the identity operator, and $x \in X, \alpha \in \mathbb{C}$. The sum $x \oplus \alpha$ is supercyclic for $T \oplus \beta I$ ($\beta \in \mathbb{C}$ and $\beta \neq 0$) if and only if the vector x is hypercyclic for $\beta^{-1}T$ and $\alpha \neq 0$.

Proof. (Forward Direction)⇒ If $x \oplus \alpha$ is a supercyclic vector for $T \oplus \beta I$, then for any $y \in X, \gamma \in \mathbb{C}$, and $\varepsilon > 0$, there exist a $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$ with $\|\lambda(T \oplus \beta I)^n(x \oplus \alpha) - y \oplus \gamma\| < \varepsilon$. Thus, $\|\lambda T^n x - y\| < \varepsilon$ and $|\lambda\beta^n\alpha - \gamma| < \varepsilon$.

$M > 0$ is selected to satisfy $\frac{\|T^n f\|}{|\beta^n\alpha|} \leq M$. Since

$$\begin{aligned}\left\|\frac{\gamma}{\alpha}(\beta^{-1}T)^n x - y\right\| &= \left\|\frac{\gamma}{\alpha}(\beta^{-1}T)^n x - y + \lambda T^n x - \lambda T^n x\right\| \\ &\leq \left\|\left(\frac{\gamma}{\alpha}\beta^{-n} - \lambda\right)T^n x\right\| + \|\lambda T^n x - y\| \\ &= \left\|\left(\frac{\gamma - \lambda\beta^n\alpha}{\beta^n\alpha}\right)T^n x\right\| + \|\lambda T^n x - y\| \\ &< (M + 1)\varepsilon.\end{aligned}$$

for any $y \oplus \gamma \in X \oplus \mathbb{C}$, there exist a scalar $\frac{\gamma}{\alpha} \in \mathbb{C}$ and an integer $n \in \mathbb{N}$ such that $\left\|\frac{\gamma}{\alpha}(\beta^{-1}T \oplus I)^n(x \oplus \alpha) - y \oplus \gamma\right\| < (M + 1)\varepsilon$. Therefore, the vector $x \oplus \alpha$ is hypercyclic for $\beta^{-1}T \oplus I$. According to Theorem 2.1, the vector x is hypercyclic for $\beta^{-1}T$.

(Reverse Direction)⇐ If x is hypercyclic for $\beta^{-1}T$, based on Theorem 2.1, $x \oplus \alpha$ is a supercyclic vector for $\beta^{-1}T \oplus I$. Then, for any $\varepsilon > 0$ and $y \in X, \gamma \in \mathbb{C}$, there exist a scalar $\lambda \in \mathbb{C}$ and an integer $n \in \mathbb{N}$ with $\|\lambda(\beta^{-1}T \oplus I)^n(x \oplus \alpha) - y \oplus \gamma\| < \varepsilon$. Thus, $\|\lambda(\beta^{-1}T)^n x - y\| < \varepsilon$ and $|\lambda\alpha - \gamma| < \varepsilon$.

$M > 0$ is selected such that $\frac{\|T^n x\|}{|\beta^n\alpha|} \leq M$. Then,

$$\begin{aligned}\left\|\frac{\gamma}{\alpha\beta^n}T^n x - y\right\| &= \left\|\frac{\gamma}{\alpha\beta^n}T^n x - y + \lambda(\beta^{-1}T)^n x - \lambda(\beta^{-1}T)^n x\right\| \\ &\leq \|\lambda(\beta^{-1}T)^n x - y\| + \left\|\left(\frac{\gamma}{\alpha} - \lambda\right)\beta^{-n}T^n x\right\| \\ &< \varepsilon + M\varepsilon \\ &= (M + 1)\varepsilon.\end{aligned}$$

Then, there exist $n \in \mathbb{N}$ and $\frac{\gamma}{\alpha\beta^n} \in \mathbb{C}$ such that $\left\|\frac{\gamma}{\alpha\beta^n}(T \oplus \beta I)^n(x \oplus \alpha) - y \oplus \gamma\right\| < (M + 1)\varepsilon$. Therefore, $x \oplus \alpha$ is a supercyclic vector for $T \oplus \beta I$.

These results also hold true for weak topology.

Theorem 2.3. Let $T: X \to X$ be a weakly hypercyclic operator, $I: \mathbb{C} \to \mathbb{C}$ be the identity operator, and $x \in X, \alpha \in \mathbb{C}$. The sum $x \oplus \alpha$ is weakly supercyclic for $T \oplus I$ if and only if the vector x is weakly hypercyclic for T and $\alpha \neq 0$.

Proof. (Forward Direction)⇒ For any $x_1^*, x_2^*, \ldots, x_m^* \in X^*$, and by defining $\overline{x}_i^*(z \oplus \alpha) = x_i^*(z)$ $(1 \leq i \leq m)$ and $\overline{x}_{m+1}^*(z \oplus \alpha) = \alpha$, then $\overline{x}_1^*, \overline{x}_2^*, \ldots, \overline{x}_{m+1}^* \in (X \oplus \mathbb{C})^*$. For any $y \in X$, $\varepsilon > 0$, if the vector$x \oplus \alpha$ is weakly supercyclic for $T \oplus I$, there exist a scalar $\lambda \in \mathbb{C}$ and an integer $n \in \mathbb{N}$ such that $\left|\overline{x}_i^*\left(\lambda(T \oplus I)^n(x \oplus \alpha) - \frac{1}{\alpha}y \oplus 1\right)\right| < \varepsilon$ $(1 \leq i \leq m + 1)$. Thus, $\left|x_i^*\left(\lambda T^n x - \frac{1}{\alpha}y\right)\right| < \varepsilon$ for $1 \leq i \leq m$ and $|1 - \lambda\alpha| < \varepsilon$. For $1 \leq i \leq m$, $M > 0$ is picked to satisfy $|x_i^*(T^n x)| \leq M$; then

$$\begin{aligned}|x_i^*(T^n x - y)| &= |x_i^*(T^n x - y + \lambda\alpha T^n x - \lambda\alpha T^n x)| \\ &\leq |x_i^*(\lambda\alpha T^n x - y)| + \left|x_i^*\left((1 - \lambda\alpha)T^n x\right)\right| \\ &< |\alpha|\varepsilon + M\varepsilon \\ &= (|\alpha| + M)\varepsilon.\end{aligned}$$

Thus, the vector x is weakly hypercyclic for T.

(Reverse Direction) ⇐ For any $\overline{x}_1^*, \overline{x}_2^*, \ldots, \overline{x}_m^* \in (X \oplus \mathbb{C})^*$, $x_i^*(z) = \overline{x}_i^*(z \oplus 0)$ $(1 \leq i \leq m)$ is defined. Then, $x_1^*, x_2^*, \ldots, x_m^* \in X^*$. $y \oplus \gamma \in X \oplus \mathbb{C}$ and $\varepsilon > 0$ are chosen,

(1) $\gamma \neq 0$. If the vector x is weakly hypercyclic for T, there exists $n \in \mathbb{N}$ satisfying $\left|x_i^*\left(T^n x - \frac{\alpha}{\gamma}y\right)\right| < \left|\frac{\alpha}{\gamma}\right| \varepsilon$ $(1 \leq i \leq m)$. Thus

$$\begin{aligned}\left|\overline{x}_i^*\left(\frac{\gamma}{\alpha}(T \oplus I)^n(x \oplus \alpha) - y \oplus \gamma\right)\right| &= \left|\overline{x}_i^*\left(\frac{\gamma}{\alpha}T^n x - y\right)\right| \\ &= \left|\frac{\gamma}{\alpha}\overline{x}_i^*\left(T^n x - \frac{\alpha}{\gamma}y\right)\right| \\ &< \varepsilon \ (1 \leq i \leq m).\end{aligned}$$

(2) $\gamma = 0$. For $\alpha \in \mathbb{C}$ and $\varepsilon > 0$, there exist some $k_0 \in \mathbb{N}$ such that $\frac{|\alpha|}{k_0} < \frac{\varepsilon}{2}$. If the vector x is weakly hypercyclic for T, there exists an integer $n \in \mathbb{N}$ satisfying $|x_i^*(T^n x - k_0 y)| < \frac{k_0\varepsilon}{2}$ $(1 \leq i \leq m)$. Thus

$$\begin{aligned}&\left|\overline{x}_i^*\left(\frac{1}{k_0}(T \oplus I)^n(x \oplus \alpha) - y \oplus 0\right)\right| \\ &= \frac{1}{k_0}|x_i^*(T^n x - k_0 y)| + \frac{|\alpha|}{k_0} \\ &< \varepsilon \ (1 \leq i \leq m).\end{aligned}$$

Above all, for any $y \oplus \gamma \in X \oplus \mathbb{C}$, there exist a scalar $\lambda \in \mathbb{C}$ and an integer $n \in \mathbb{N}$ with $|\overline{x}_i^*(\lambda(T \oplus I)^n(x \oplus \alpha) - y \oplus \gamma)| < \varepsilon$ $(1 \leq i \leq m)$. Thus, the vector $x \oplus \alpha$ is weakly supercyclic for $T \oplus I$.

Theorem 2.4. Let $T: X \to X$ be a weakly hypercyclic operator, $I: \mathbb{C} \to \mathbb{C}$ be the identity operator, and $x \in X, \alpha \in \mathbb{C}$. The sum $x \oplus \alpha$ is weakly supercyclic for $T \oplus \beta I$ ($\beta \in \mathbb{C}, \beta \neq 0$) if and only if the vector x is weakly hypercyclic for $\beta^{-1}T$ and $\alpha \neq 0$.

Proof. Similar to the proof of Theorem 2.2, $x \oplus \alpha$ is weakly supercyclic for $T \oplus \beta I$ if and only if $x \oplus \alpha$ is weakly supercyclic for $\beta^{-1}T \oplus I$. Based on Theorem 2.3, $x \oplus \alpha$ is weakly supercyclic for $\beta^{-1}T \oplus I$ if and only if x is weakly hypercyclic for $\beta^{-1}T$.

III. Supercyclic properties of $\mathcal{T} \oplus I$

Influenced by the 2-tuple of operators related research, we want to generalize some of the dynamic properties to the 2-tuple of operators. In this section, the dynamics of the sum of a hypercyclic 2-tuple of operators and the identity operator are mainly studied which extended the results to the 2-tuple of operators.

Theorem 3.1. Let $\mathcal{T} = (T_1, T_2)$ be a hypercyclic 2-tuple of operators on X and $I: \mathbb{C} \to \mathbb{C}$ be the identity operator. Then, the sum $x \oplus 1 \in X \oplus \mathbb{C}$ is supercyclic for $(T_1 \oplus I, T_2 \oplus I)$ if and only if the vector x is hypercyclic for $\mathcal{T}$.

Proof. (Forward Direction) ⇒ If $x \oplus 1$ is supercyclic for $(T_1 \oplus I, T_2 \oplus I)$, then for any $y \in X$, $\varepsilon > 0$, there exist a scalar $\lambda \in \mathbb{C}$ and $n_1, n_2 \in \mathbb{N}$ such that

$$\left\|\lambda T_1^{n_1} T_2^{n_2} x - y\right\| < \varepsilon, \; |\lambda I^{n_1} I^{n_2} 1 - 1| = |\lambda - 1| < \varepsilon.$$

$M > 0$ is selected to satisfy $\left\|T_1^{n_1} T_2^{n_2} x\right\| \leq M$. Then

$$\begin{aligned}\left\|T_1^{n_1} T_2^{n_2} x - y\right\| &= \left\|T_1^{n_1} T_2^{n_2} x - y + \lambda T_1^{n_1} T_2^{n_2} x - \lambda T_1^{n_1} T_2^{n_2} x\right\| \\ &\leq \left\|\lambda T_1^{n_1} T_2^{n_2} x - y\right\| + \left\|(1-\lambda) T_1^{n_1} T_2^{n_2} x\right\| \\ &< \varepsilon + M\varepsilon.\end{aligned}$$

Therefore, the vector x is hypercyclic for $\mathcal{T} = (T_1, T_2)$.

(Reverse Direction)⇐ For any $y \oplus \beta \in X \oplus \mathbb{C}$ and $\varepsilon > 0$,

(1) $\beta \neq 0$. If the vector x is hypercyclic for $\mathcal{T}$, then there exist some $n_1, n_2 \in \mathbb{N}$ satisfying

$$\left\|T_1^{n_1} T_2^{n_2} x - \beta^{-1} y\right\| < \frac{\varepsilon}{|\beta|}.$$

Then

$$\left\|\beta T_1^{n_1} T_2^{n_2} x - y\right\| < \varepsilon.$$

That is $\|\beta (T_1 \oplus I)^{n_1} (T_2 \oplus I)^{n_2} (x \oplus 1) - y \oplus \beta\| < \varepsilon$.

(2) $\beta = 0$. For $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ satisfying $\frac{1}{k_0} < \frac{\varepsilon}{2}$. If x is hypercyclic for $\mathcal{T}$, there exist $n_1, n_2 \in \mathbb{N}$ such that

$$\left\|T_1^{n_1} T_2^{n_2} x - k_0 y\right\| < \frac{k_0 \varepsilon}{2}.$$

That is

$$\left\|\frac{1}{k_0} T_1^{n_1} T_2^{n_2} x - y\right\| < \frac{\varepsilon}{2}.$$

Since

$$\left|\frac{1}{k_0} I^{n_1} I^{n_2} 1 - 0\right| = \frac{1}{k_0} < \frac{\varepsilon}{2},$$

Therefore

$$\left\|\frac{1}{k} (T_1 \oplus I)^{n_1} (T_2 \oplus I)^{n_2} (x \oplus 1) - y \oplus 0\right\| < \varepsilon.$$

Above all, the vector $x \oplus 1$ is supercyclic for $(T_1 \oplus I, T_2 \oplus I)$.

Theorem 3.2. A vector $x \oplus \mu$ is supercyclic for $(T_1 \oplus \alpha_1 I, T_2 \oplus \alpha_2 I)$ ($\alpha_1, \alpha_2 \in \mathbb{C}$ and $\alpha_1, \alpha_2 \neq 0$) if and only the vector x is hypercyclic for $(\alpha_1^{-1} T_1, \alpha_2^{-1} T_2)$ and $\mu \neq 0$.

Proof. (Forward Direction)⇒For any $y \in X, \gamma \in \mathbb{C}$ and $\varepsilon > 0$, since $x \oplus \mu$ is a supercyclic vector for $(T_1 \oplus \alpha_1 I, T_2 \oplus \alpha_2 I)$, there exist a scalar $\lambda \in \mathbb{C}$ and $n_1, n_2 \in \mathbb{N}$ such that $\|\lambda (T_1 \oplus \alpha_1 I)^{n_1} (T_2 \oplus \alpha_2 I)^{n_2} (x \oplus \mu) - y \oplus \gamma\| < \varepsilon$. Thus, $\left\|\lambda T_1^{n_1} T_2^{n_2} x - y\right\| < \varepsilon$ and $\left|\lambda \alpha_1^{n_1} \alpha_2^{n_2} \mu - \gamma\right| < \varepsilon$.

$M > 0$ is selected to satisfy $\frac{\|T_1^{n_1} T_2^{n_2} x\|}{|\alpha_1^{n_1} \alpha_2^{n_2} \mu|} \leq M$, then

$$\begin{aligned}&\left\|\frac{\gamma}{\mu} (\alpha_1^{-1} T_1)^{n_1} (\alpha_2^{-1} T_2)^{n_2} x - y\right\| \\ &= \left\|\frac{\gamma}{\mu} (\alpha_1^{-1} T_1)^{n_1} (\alpha_2^{-1} T_2)^{n_2} x - y + \lambda T_1^{n_1} T_2^{n_2} x - \lambda T_1^{n_1} T_2^{n_2} x\right\| \\ &\leq \left\|\lambda T_1^{n_1} T_2^{n_2} x - y\right\| + \left\|\left(\frac{\gamma}{\mu} \alpha_1^{-n_1} \alpha_2^{-n_2} - \lambda\right) T_1^{n_1} T_2^{n_2} x\right\| \\ &< \varepsilon + M\varepsilon \\ &= (M+1)\varepsilon.\end{aligned}$$

Then, $\left\|\frac{\gamma}{\mu} (\alpha_1^{-1} T_1 \oplus I)^{n_1} (\alpha_2^{-1} T_2 \oplus I)^{n_2} (x \oplus \mu) - y \oplus \gamma\right\| < (M+1)\varepsilon$. Thus, the vector $x \oplus \mu$ is supercyclic for $(\alpha_1^{-1} T_1 \oplus I, \alpha_2^{-1} T_2 \oplus I)$. Based on Theorem 3.1, x is hypercyclic for $(\alpha_1^{-1} T_1, \alpha_2^{-1} T_2)$.

(Reverse Direction) ⇐ If the vector x is hypercyclic for $(\alpha_1^{-1} T_1, \alpha_2^{-1} T_2)$, according to Theorem 3.1, $x \oplus \mu$ is supercyclic for $(\alpha_1^{-1} T_1 \oplus I, \alpha_2^{-1} T_2 \oplus I)$. For any $y \in X, \gamma \in \mathbb{C}, \varepsilon > 0$, there exist a scalar $\lambda \in \mathbb{C}$ and $n_1, n_2 \in \mathbb{N}$, such that $\|\lambda (\alpha_1^{-1} T_1 \oplus I)^{n_1} (\alpha_2^{-1} T_2 \oplus I)^{n_2} (x \oplus \mu) - y \oplus \gamma\| < \varepsilon$. Thus, $\|\lambda (\alpha_1^{-1} T_1)^{n_1} (\alpha_2^{-1} T_2)^{n_2} x - y\| < \varepsilon$ and $|\lambda \mu - \gamma| < \varepsilon$.

$M > 0$ is selected to satisfy $\frac{\|T_1^{n_1} T_2^{n_2} x\|}{|\alpha_1^{n_1} \alpha_2^{n_2} \mu|} \leq M$, then

$$\begin{aligned}&\left\|\frac{\gamma}{\mu \alpha_1^{n_1} \alpha_2^{n_2}} T_1^{n_1} T_2^{n_2} x - y\right\| \\ &\leq \|\lambda (\alpha_1^{-1} T_1)^{n_1} (\alpha_2^{-1} T_2)^{n_2} x - y\| + \left\|\left(\frac{\gamma}{\mu} - \lambda\right) \alpha_1^{-n_1} \alpha_2^{-n_2} T_1^{n_1} T_2^{n_2} x\right\| \\ &< \varepsilon + M\varepsilon \\ &= (M+1)\varepsilon,\end{aligned}$$

That is $\left\|\frac{\gamma}{\mu \alpha_1^{n_1} \alpha_2^{n_2}} (T_1 \oplus \alpha_1 I)^{n_1} (T_2 \oplus \alpha_2 I)^{n_2} (x \oplus \mu) - y \oplus \gamma\right\| < (M+1)\varepsilon$. Therefore, $x \oplus \mu$ is a supercyclic vector for $(T_1 \oplus \alpha_1 I, T_2 \oplus \alpha_2 I)$.

The above theorem can be extended to weak topology.

Theorem 3.3. Let $\mathcal{T} = (T_1, T_2)$ be a weakly hypercyclic 2-tuple of operators acting on X and $I: \mathbb{C} \to \mathbb{C}$ be the identity operator. Then, the sum $x \oplus 1$ is weakly supercyclic for $(T_1 \oplus I, T_2 \oplus I)$ if and only if x is weakly hypercyclic for $\mathcal{T}$.

Proof. (Forward Direction)⇒ For any given $x_1^*, x_2^*, \dots, x_m^* \in X^*$, $\overline{x}_i^*(z \oplus \alpha) = x_i^*(z)$ $(1 \leq i \leq m)$ and $\overline{x}_{m+1}^*(z \oplus \alpha) = \alpha$ are defined; then, $\overline{x}_1^*, \overline{x}_2^*, \dots, \overline{x}_{m+1}^* \in (X \oplus \mathbb{C})^*$. $y \in X$, $\varepsilon > 0$ is chosen, if the vector $x \oplus 1$ is weakly supercyclic for $(T_1 \oplus I, T_2 \oplus I)$, there exist a scalar $\lambda \in \mathbb{C}$ and $n_1, n_2 \in \mathbb{N}$ satisfying $\left|\overline{x}_i^*(\lambda (T_1 \oplus I)^{n_1} (T_2 \oplus I)^{n_2} (x \oplus 1) - y \oplus 1)\right| < \varepsilon$ $(1 \leq i \leq m+1)$. $M > 0$ is selected to satisfy $|x_i^*(T_1^{n_1} T_2^{n_2} x)| \leq M$ $(1 \leq i \leq m)$, then for any $1 \leq i \leq m$,

$$\begin{aligned}&\left|x_i^*\left(T_1^{n_1} T_2^{n_2} x - y\right)\right| \\ &= \left|x_i^*\left(T_1^{n_1} T_2^{n_2} - y + \lambda T_1^{n_1} T_2^{n_2} x - \lambda T_1^{n_1} T_2^{n_2} x\right)\right| \\ &\leq \left|x_i^*\left(\lambda T_1^{n_1} T_2^{n_2} x - y\right)\right| + \left|x_i^*\left((1-\lambda) T_1^{n_1} T_2^{n_2} x\right)\right| \\ &< \varepsilon + M\varepsilon \\ &= (M+1)\varepsilon.\end{aligned}$$

Then, the vector x is weakly hypercyclic for $\mathcal{T}$.

(Reverse Direction) ⇐ For any $\overline{x}_1^*, \overline{x}_2^*, \dots, \overline{x}_m^* \in (X \oplus \mathbb{C})^*$, $x_i^*(z) = \overline{x}_i^*(z \oplus 0)$ $(1 \leq \mathrm{i} \leq \mathrm{m})$ is defined. Then, $x_1^*, x_2^*, \dots, x_m^* \in X^*$. Therefore, $y \oplus \gamma \in X \oplus \mathbb{C}$ and $\varepsilon > 0$ are selected.

(1)$\gamma \neq 0$. If the vector x is weakly hypercyclic for $\mathcal{T}$, there exists $n_1, n_2 \in \mathbb{N}$ satisfying $\left|x_i^*\left(T_1^{n_1}T_2^{n_2}x - \frac{1}{\gamma}y\right)\right| < \left|\frac{1}{\gamma}\right| \varepsilon\ (1 \leq i \leq m)$. Then

$$\left|\overline{x}_i^*(\gamma(T_1 \oplus I)^{n_1}(T_2 \oplus I)^{n_2}(x \oplus 1) - y \oplus \gamma)\right| < \varepsilon\ (1 \leq i \leq m).$$

(2) $\gamma = 0$. For $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ satisfying $\frac{1}{k_0} < \frac{\varepsilon}{2}$. If x is weakly hypercyclic for $\mathcal{T}$, there exist $n_1, n_2 \in \mathbb{N}$ such that $\left|x_i^*\left(T_1^{n_1}T_2^{n_2}x - k_0 y\right)\right| < \frac{k_0\varepsilon}{2}$ $(1 \leq i \leq m)$. Then

$$\left|\overline{x}_i^*\left(\frac{1}{k_0}(T_1 \oplus I)^{n_1}(T_2 \oplus I)^{n_2}(x \oplus 1) - y \oplus 0\right)\right| < \varepsilon.$$

Above all, for any $y \oplus \gamma \in X \oplus \mathbb{C}$, there exist $\lambda \in \mathbb{C}$ and $n_1, n_2 \in \mathbb{N}$ such that $\left|\overline{x}_i^*(\lambda(T_1 \oplus I)^{n_1}(T_2 \oplus I)^{n_2}(x \oplus 1) - y \oplus \gamma)\right| < \varepsilon\ (1 \leq i \leq m)$.

Theorem 3.4. The vector $x \oplus \mu \in X \oplus \mathbb{C}$ is weakly supercyclic for $(T_1 \oplus \alpha_1 I, T_2 \oplus \alpha_2 I)$ ($\alpha_1, \alpha_2 \in \mathbb{C}$ and $\alpha_1, \alpha_2 \neq 0$) if and only if the vector x is weakly hypercyclic for $(\alpha_1^{-1}T_1, \alpha_2^{-1}T_2)$ and $\mu \neq 0$.

The proof of Theorem 3.4 is similar to that of Theorem 2.4 and we did not discuss it here.

IV. The sum of a hypercyclic operator and a supercyclic operator

From the above research, we posed the following question.

Open Problem 1. *Is the sum of a hypercyclic operator and a supercyclic operator supercyclic? If not, are there any counterexamples?*

Regarding this problem, we give a counter example, which solve this problem. We first give the following lemma.

Lemma 4.1. Let x be a hypercyclic vector of T acting on X, and y be a supercyclic vector of S acting on Y. If $T \oplus S$ is supercyclic, then $x \oplus y$ is a supercyclic vector of $T \oplus S$ acting on $X \oplus Y$.

Proof. If $x \oplus y$ is not a supercyclic vector of $T \oplus S$, then there exist $\alpha \oplus \beta \in X \oplus Y$, for all $\lambda \in \mathbb{C}$, $n \in \mathbb{N}$,

$$||\lambda(T \oplus S)^n(x \oplus y) - \alpha \oplus \beta|| > \varepsilon,$$

obviously contradicts x is a hypercyclic vector of T and y is a supercyclic vector of S.

Example 2. Let T be the translation operator $Tf(z) = f(z+1)$ on $H(\mathbb{C})$, and B be the unilateral backward operator $B(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$ on $c_0(\mathbb{N})$, obviously T is hypercyclic, B is supercyclic. Let f be a hypercyclic vector of T and x be a supercyclic vector of B, but $f \oplus x$ is not a supercyclic vector of $T \oplus B$.

Proof. Let $U = \{f \in H(\mathbb{C}): 1 < \|f\| < 2\}$, $V = \{x = (x_i)_{i \in \mathbb{N}} \in c_0(\mathbb{N}): |\frac{x_{i+1}}{x_i}| < \frac{1}{2^3}\}$. According to the density of hypercyclic vector set and supercyclic vector set, there exist a hypercyclic vector of T with $f \in U$, a supercyclic vector of B with $x \in V$, then $f \oplus x$ is a supercyclic vector of $T \oplus B$ which can be obtained by Lemma 4.1. For arbitrary $g \in H(\mathbb{C})$, $y, 2y \in c_0(\mathbb{N})$, there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, $n_1, n_2 \in \mathbb{N}$ make

$$\lambda_1 T^{n_1} f \to g,$$
$$\lambda_1 T^{n_1} f \to g,$$
$$\lambda_2 T^{n_2} f \to g,$$
$$\lambda_2 B^{n_2} x \to 2y.$$

(1) $n_1 = n_2$

Obviously, the above formula cannot be established at the same time.

(2) $n_1 \neq n_2$

$$\lambda_2 T^{n_2} f \to \lambda_1 T^{n_1} f,$$
$$\lambda_2 B^{n_2} x \to 2\lambda_1 B^{n_1} x,$$

namely

$$\lambda_2 f(z + n_2) \to \lambda_1 f(z + n_1),$$
$$\lambda_2\left(x_{1+n_2}, x_{2+n_2}, \ldots\right) \to 2\lambda_1\left(x_{1+n_1}, x_{2+n_1}, \ldots\right),$$

then

$$|\frac{\lambda_2}{\lambda_1}| \to |\frac{f(z+n_1)}{f(z+n_2)}|,$$
$$\frac{1}{2} < |\frac{\lambda_2}{\lambda_1}| < 2,$$

contradicts

$$|\frac{\lambda_2}{\lambda_1}| \to 2||\frac{(x_{1+n_2}, x_{2+n_2}, \ldots)}{(x_{1+n_1}, x_{2+n_1}, \ldots)}||,$$
$$|\frac{\lambda_2}{\lambda_1}| > 2^4 \text{ or } |\frac{\lambda_2}{\lambda_1}| < \frac{1}{2^2}.$$

So $f \oplus x$ is not a supercyclic vector of $T \oplus B$.

V. Conclusion

This research has focused on whether the sum of hypercyclic and identity operator were supercyclic or not. The research extended the previous results and generalized the results to 2-tuples of operators through more complex calculations. Similar results could be obtained on n-tuples and 2-tuples of operators.

We summarize the main results as shown in the table 1 below.

Table 1 Main Results

Operator	Topology	Condition	Result
Operators	norm topology	T is hypercyclic	$T \oplus I$ is supercyclic
	norm topology	T is hypercyclic	$T \oplus \beta I (\beta \in \mathbb{C}$ and $\beta \neq 0)$ is supercyclic
	weak topology	T is weakly hypercyclic	$T \oplus I$ is weakly supercyclic
	weak topology	T is weakly hypercyclic	$T \oplus \beta I (\beta \in \mathbb{C}$ and $\beta \neq 0)$ is weakly supercyclic
2-tuples of operators	norm topology	(T_1, T_2) is hypercyclic	$(T_1 \oplus I, T_2 \oplus I)$ is supercyclic
	norm topology	(T_1, T_2) is hypercyclic	$(T_1 \oplus \alpha_1 I, T_2 \oplus \alpha_2 I)$ $(\alpha_1, \alpha_2 \in \mathbb{C}$ and $\alpha_1, \alpha_2 \neq 0)$ is supercyclic
	weak topology	(T_1, T_2) is weakly hypercyclic	$(T_1 \oplus I, T_2 \oplus I)$ is weakly supercyclic
	weak topology	(T_1, T_2) is weakly hypercyclic	$(T_1 \oplus \alpha_1 I, T_2 \oplus \alpha_2 I)$ $(\alpha_1, \alpha_2 \in \mathbb{C}$ and $\alpha_1, \alpha_2 \neq 0)$ is weakly supercyclic

Ps: T is an operator on Banach space X, (T_1, T_2) is a 2-tuple of operators on Banach space X, I is the identity operator on $\mathbb{C}$.

To enrich the study, we present a result on N-tuples of operators, and the rest of the results can be obtained by combining the methods in the previous section.

Theorem 4.1. Let $\mathcal{T} = (T_1, T_2, \ldots, T_N)$ be a hypercyclic N-tuple of operators on X and $I: \mathbb{C} \to \mathbb{C}$ be the identity operator. Then, the sum $x \oplus 1 \in X \oplus \mathbb{C}$ is supercyclic for $(T_1 \oplus I, T_2 \oplus I, \ldots, T_N \oplus I)$ if and only if the vector x is hypercyclic for $\mathcal{T}$.

Proof. (Forward Direction) $\Rightarrow$ If $x \oplus 1$ is supercyclic for $(T_1 \oplus I, T_2 \oplus I, \ldots, T_N \oplus I)$, then for any $y \in X$, $\varepsilon > 0$, there exist a scalar $\lambda \in \mathbb{C}$ and $n_1, n_2, \ldots, n_N \in \mathbb{N}$ such that $\left\|\lambda T_1^{n_1}T_2^{n_2}\cdots T_N^{n_N}x - y\right\| < \varepsilon$, $|\lambda I^{n_1}I^{n_2}\cdots I^{n_N}1 - 1| = |\lambda - 1| < \varepsilon$.

$M > 0$ is selected to satisfy $\left\|T_1^{n_1}T_2^{n_2}\cdots T_N^{n_N}x\right\| \le M$. Then

$$\begin{aligned}&\left\|T_1^{n_1}T_2^{n_2}\cdots T_N^{n_N}x-y\right\|\\ &=\left\|T_1^{n_1}T_2^{n_2}\cdots T_N^{n_N}x-y+\lambda T_1^{n_1}T_2^{n_2}\cdots T_N^{n_N}x-\lambda T_1^{n_1}T_2^{n_2}\cdots T_N^{n_N}x\right\|\\ &\le\left\|\lambda T_1^{n_1}T_2^{n_2}\cdots T_N^{n_N}x-y\right\|+\left\|(1-\lambda)T_1^{n_1}T_2^{n_2}\cdots T_N^{n_N}x\right\|\\ &<\varepsilon+M\varepsilon.\end{aligned}$$

Therefore, the vector x is hypercyclic for $\mathcal{T}=(T_1,T_2,\ldots,T_N)$.

(Reverse Direction)$\Leftarrow$ For any $y\oplus\beta\in X\oplus\mathbb{C}$ and $\varepsilon>0$,

(1) $\beta\neq 0$. If the vector x is hypercyclic for $\mathcal{T}$, then there exist some $n_1,n_2,\ldots,n_N\in\mathbb{N}$ satisfying

$$\left\|T_1^{n_1}T_2^{n_2}\cdots T_N^{n_N}x-\beta^{-1}y\right\|<\frac{\varepsilon}{|\beta|}.$$

Then

$$\left\|\beta T_1^{n_1}T_2^{n_2}\cdots T_N^{n_N}x-y\right\|<\varepsilon.$$

That is $\|\beta(T_1\oplus I)^{n_1}(T_2\oplus I)^{n_2}\cdots(T_N\oplus I)^{n_N}(x\oplus 1)-y\oplus\beta\|<\varepsilon$.

(2) $\beta=0$. For $\varepsilon>0$, there exists $k_0\in\mathbb{N}$ satisfying $\frac{1}{k_0}<\frac{\varepsilon}{2}$. If x is hypercyclic for $\mathcal{T}$, there exist $n_1,n_2,\ldots,n_N\in\mathbb{N}$ such that

$$\left\|T_1^{n_1}T_2^{n_2}\cdots T_N^{n_N}x-k_0y\right\|<\frac{k_0\varepsilon}{2}.$$

That is

$$\left\|\frac{1}{k_0}T_1^{n_1}T_2^{n_2}\cdots T_N^{n_N}x-y\right\|<\frac{\varepsilon}{2}.$$

Since

$$\left|\frac{1}{k_0}I^{n_1}I^{n_2}\cdots I^{n_N}1-0\right|=\frac{1}{k_0}<\frac{\varepsilon}{2},$$

Therefore

$$\left\|\frac{1}{k}(T_1\oplus I)^{n_1}(T_2\oplus I)^{n_2}\cdots(T_N\oplus I)^{n_N}(x\oplus 1)-y\oplus 0\right\|<\varepsilon.$$

Above all, the vector $x\oplus 1$ is supercyclic for $(T_1\oplus I,T_2\oplus I,\ldots,T_N\oplus I)$.

To better understand the theorem, we give an example here.

Example 3. Let $\mathcal{T}=(T_1,T_2,T_3)=(2I,\frac{1}{3}I,e^{i\pi\theta}I)$ acting on $\mathbb{C}$, θ is an irrational number, then $x\oplus 1\in\mathbb{C}\oplus\mathbb{C}$ is supercyclic for $(T_1\oplus I,T_2\oplus I,T_3\oplus I)$ if and only if the vector x is hypercyclic for $\mathcal{T}$.

Proof. Obviously $\mathcal{T}$ is hypercyclic, and the subsequent proof follows the proof of the theorem 4.1.

The most significant contribution of this research was the generalization of some conclusions in dynamical systems to more general cases and proved the sum of a hypercyclic operator and a supercyclic operator could not be supercyclic. We focus on the updated settings like the tuple of operators. Actually, many dynamic properties can be generalized to the 2-tuple of operators and the N-tuple of operators. follow-up research can focus on whether there is a single operator property that cannot be generalized to the 2-tuple of operators or the N-tuple of operators, or there are other interesting forms of generalization. However, this research had certain limitations. There was no major breakthrough in the main research methods. Future research could focus on the sum of a hypercyclic operator and a common supercyclic operator. Further more, the dynamic properties of the dierct sum of broader landscape of operator dynamics such as disjoint hypercyclicity or A-hypercyclicity deserves our further research.

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