Interval Valued Fuzzy Ordered Almost (m,n)-Ideals and Almost (m,n)-Quasi-Ideals in Ordered Semigroups

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***Abstract*—An almost ideal in a semigroup is a generalization of the concept of an ideal with studied by Grosek and Stako in 1980. In 2019 S. Suebsung et al. studied almost (m,n)-ideals in semigroups. Later, in 2021, T. Gaketem. introduced interval valued fuzzy almost (m,n)-ideals in semigroups. This paper aims we extends interval valued fuzzy ordered almost (m,n)-ideals in semigroups to ordered semigroups. We prove some basic properties of interval valued fuzzy ordered almost (m,n)-ideals in ordered semigroups. And, we investigate a bridge between almost (m,n)-ideals and interval valued fuzzy ordered almost (m,n)-ideals in ordered semigroups. Finally, we studied properties interval valued fuzzy ordered almost (m,n)-quasi-ideals in ordered semigroups.**

***Index Terms*—Ordered Almost (m,n)-ideals, Interval valued fuzzy ordered almost (m,n)-ideals, minimal ordered almost (m,n)-ideals, Interval valued fuzzy ordered almost (m,n)-quasi-ideals**

I. Introduction

ORDERED semigroups are an algebraic structure in a binary operation consisting of the associative property and a partial order, which has been applied in the study of many fields of study, such as coding theory, automata, etc. In the late 20th century, the definition of fuzzy sets was studied by Zadeh 1965, [1]. He later developed the study into interval valued fuzzy sets in 1975, [2]. Both studies of fuzzy sets and interval valued fuzzy set models have been applied to many fields of study, such as medical science, theoretical physics, robotics, computer science, control engineering, information science, measure theory, logic, set theory, and topology. In 2006, A. L. Narayanan and T. Manikantan [3] developed the theory of interval valued fuzzy subsemigroup and studied types of interval valued fuzzy ideals in semigroups. In 1985, Satko and Grosek [4] discussed the concept of an almost-ideal (A-ideal) in semigroups. And S. Bogdanovic [5] gave the concept of almost bi-ideals in semigroups. In 2020, Chinram et al. [6] discussed almost interior ideals and weakly almost interior ideals in semigroups and studied the relationship between almost interior ideals and weakly almost interior ideals in semigroups. In 2019, S. Suebsung et al. [7] studied almost (m,n)-ideals in semigroups. Later, in 2022, S. Suebsung et al. [8] introduced almost ideals in ordered semigroups. This paper aims to define almost (m,n)-ideals and fuzzy almost (m,n)-ideals in ordered semigroups. Meanwhile T. Gaketem [9], [10], [11] studied types interval valued fuzzy almost (m,n)-ideals in semigroups.

In the same year, T. Gaketem and P. Khamrot [12] explored the concept of almost ideals within the framework of bipolar fuzzy sets, explicitly focusing on bipolar fuzzy almost bi-ideals in semigroups. In 2023, T. Gaketem and P. Khamrot [13] studied bipolar fuzzy almost interior ideals in semigroups. In 2024, T. Gaketem and P. Khamrot [14], [15] discussed bipolar fuzzy almost ideals and quasi ideals in semigroups. In addition, almost Ideal's work also has many studies, such as almost ideals in ordered semigroup [16], almost ideals in semirings [17], almost ideals in ternary semiring [18], etc. In 2025, P. Khamrot et al. [19] studied fuzzy (m,n)-ideals and n-interior ideals in ordered semigroups. In the same year, P. Khamrot et al. [20] studied concepts of ordered almost (m,n)-ideals in ordered semigroups.

In this paper, we extend the definition of interval valued almost (m,n)-ideals in semigroups to ordered semigroups. We discussed the properties of interval valued ordered almost (m,n)-ideals in ordered semigroups. We build a bridge between ordered almost (m,n)-ideals and interval valued fuzzy ordered almost (m,n)-ideals in ordered semigroups. Moreover, we studied interval valued fuzzy ordered almost (m,n)-quasi-ideals in ordered semigroups.

II. Preliminaries

Now, in this section, we will repeat the definitions such as ordered semigroups, fuzzy sets, interval valued fuzzy sets, almost ideals.

Definition 2.1. *[21]. Let $\ddot{\Omega}$ be a set with a binary opeation consisting of $\cdot$ and a binary operation relation $\leq$. Then $(\ddot{\Omega},\cdot,\leq)$ is called an* ordred semigroup *if*

(1) $(\ddot{\Omega},\cdot)$ *is a semigroup,*
(2) $(\ddot{\Omega},\leq)$ *is a partially ordered set,*
(3) *for all $\ddot{\mathfrak{x}},\ddot{\mathfrak{y}},\ddot{\mathfrak{z}}\in\ddot{\Omega}$, we have $\ddot{\mathfrak{x}}\leq\ddot{\mathfrak{y}}$ then $\ddot{\mathfrak{x}}\ddot{\mathfrak{z}}\leq\ddot{\mathfrak{y}}\ddot{\mathfrak{z}}$ and $\ddot{\mathfrak{x}}\ddot{\mathfrak{z}}\leq\ddot{\mathfrak{y}}\ddot{\mathfrak{z}}$.*

For a nonempty subset $\ddot{\Omega}_1$ and $\ddot{\Omega}_2$ of ordered semigroup $\ddot{\Omega}$, we write $\ddot{\Omega}_1 := \{\ddot{\mathfrak{x}}\in\ddot{\Omega}_1 \mid \ddot{\mathfrak{x}}\leq\ddot{\mathfrak{y}} \text{ for some } \ddot{\mathfrak{x}}\in\ddot{\Omega}\}$ and $\ddot{\Omega}_1\ddot{\Omega}_2 := \{\ddot{\mathfrak{x}}\ddot{\mathfrak{y}} \mid \ddot{\mathfrak{x}}\in\ddot{\Omega}_1 \text{ and } \ddot{\mathfrak{y}}\in\ddot{\Omega}_2\}$.

It is observed that

(1) $\ddot{\Omega}_1\subseteq(\ddot{\Omega}_1]$,

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(2) if $\ddot{\Omega}_1 \subseteq \ddot{\Omega}_2$, then $(\ddot{\Omega}_1] \subseteq (\ddot{\Omega}_2]$,
(3) $((\ddot{\Omega}_1]] = (\ddot{\Omega}_1]$,
(4) $(\ddot{\Omega}_1](\ddot{\Omega}_2] \subseteq (\ddot{\Omega}_1\ddot{\Omega}_2]$,
(5) $((\ddot{\Omega}_1](\ddot{\Omega}_2]] = (\ddot{\Omega}_1\ddot{\Omega}_2]$,
(6) $(\ddot{\Omega}_1 \cup \ddot{\Omega}_2] = (\ddot{\Omega}_1] \cup (\ddot{\Omega}_2]$,
(7) $(\ddot{\Omega}_1 \cap \ddot{\Omega}_2] = (\ddot{\Omega}_1] \cap (\ddot{\Omega}_2]$.

Let $(\emptyset \neq \ddot{\aleph} \subseteq \ddot{\Omega})$ is called a *subsemigroup* (SSG) such that $\ddot{\aleph}^2 \subseteq \ddot{\aleph}$ and $(\ddot{\aleph}]$. A *left (right) ideal* of a ordered semigroup $\ddot{\Omega}$ is a non-empty set $\ddot{\aleph}$ of $\ddot{\Omega}$ such that $\ddot{\aleph}\ddot{\Omega} \subseteq \ddot{\aleph}$ $(\ddot{\Omega}\ddot{\aleph} \subseteq \ddot{\aleph})$ and $(\ddot{\aleph}]$. By an *ideal* of an ordered semigroup $\ddot{\Omega}$, we mean a non-empty set of $\ddot{\Omega}$ which is both a left and a right ideal of $\ddot{\Omega}$.

Definition 2.2. *[21] A SSG $\ddot{\aleph}$ of an ordered semigroup $(\ddot{\Omega}, \cdot, \leq)$ is called an (m,n)-*ideal *of $\ddot{\Omega}$ if $\ddot{\aleph}$ satisfies the following conditions:*
(1) $\ddot{\aleph}^m\ddot{\Omega}\ddot{\aleph}^n \subseteq \ddot{\aleph}$.
(2) $\ddot{\aleph} = (\ddot{\aleph}]$, *that is for $\ddot{\mathfrak{x}} \in \ddot{\aleph}$ and $\ddot{\mathfrak{y}} \in \ddot{\Omega}$, $\ddot{\mathfrak{y}} \leq \ddot{\mathfrak{x}}$ implies $\ddot{\mathfrak{y}} \in \ddot{\aleph}$.*
where m, n are non-negative integers.

Definition 2.3. *[8] An non-empty subset $\ddot{\aleph}$ of an ordered semigroup $(\ddot{\Omega}, \cdot, \leq)$ is called an (m,n)-*quasi-ideal *of $\ddot{\Omega}$ if $\ddot{\aleph}$ satisfies the following conditions:*
(1) $(\ddot{\aleph}^m\ddot{\Omega}] \cap (\ddot{\Omega}\ddot{\aleph}^n] \subseteq \ddot{\aleph}$.
(2) $\ddot{\aleph} = (\ddot{\aleph}]$, *that is for $\ddot{\mathfrak{x}} \in \ddot{\aleph}$ and $\ddot{\mathfrak{y}} \in \ddot{\Omega}$, $\ddot{\mathfrak{y}} \leq \ddot{\mathfrak{x}}$ implies $\ddot{\mathfrak{y}} \in \ddot{\aleph}$.*
where m, n are non-negative integers.

Definition 2.4. *[8] A nonempty subset of $\ddot{\aleph}$ an ordered semigroup $\ddot{\Omega}$ is called*
(1) *a* left ordered almost ideal *(LOAI) of $\ddot{\Omega}$ if $(\ddot{\mathfrak{s}}\ddot{\aleph}] \cap \ddot{\aleph} \neq \emptyset$ for all $\ddot{\mathfrak{s}} \in \ddot{\Omega}$,*
(2) *a* right ordered almost ideal *(ROAI) of $\ddot{\Omega}$ if $(\ddot{\aleph}\ddot{\mathfrak{s}}] \cap \ddot{\aleph} \neq \emptyset$ for all $\ddot{\mathfrak{s}} \in \ddot{\Omega}$,*
(3) *an* ordered almost ideal *(OAI) of $\ddot{\Omega}$ if it is both LOAT and ROAI of $\ddot{\Omega}$.*

For any $\ddot{\mathfrak{h}}_i \in [0,1]$, $i \in \ddot{\mathcal{F}}$, define

$$\underset{i\in\ddot{\mathcal{F}}}{\vee}\ddot{\mathfrak{h}}_i := \sup_{i\in\ddot{\mathcal{F}}}\{\ddot{\mathfrak{h}}_i\} \quad \text{and} \quad \underset{i\in\ddot{\mathcal{F}}}{\wedge}\ddot{\mathfrak{h}}_i := \inf_{i\in\ddot{\mathcal{F}}}\{\ddot{\mathfrak{h}}_i\}.$$

We see that for any $\ddot{\mathfrak{h}}, \ddot{\mathfrak{r}} \in [0,1]$, we have

$$\ddot{\mathfrak{h}} \vee \ddot{\mathfrak{r}} = \max\{\ddot{\mathfrak{h}}, \ddot{\mathfrak{r}}\} \quad \text{and} \quad \ddot{\mathfrak{h}} \wedge \ddot{\mathfrak{r}} = \min\{\ddot{\mathfrak{h}}, \ddot{\mathfrak{r}}\}.$$

A *fuzzy set* $\ddot{\Upsilon}$ in a nonempty set $\ddot{\mathfrak{T}}$ is a function from $\ddot{\mathfrak{T}}$ into the unit closed interval $[0,1]$ of real numbers, i.e., $\ddot{\Upsilon} : \ddot{\mathfrak{T}} \to [0,1]$.

Let $\mathrm{CS}[0,1]$ be the set of all closed subintervals of $[0,1]$, i.e.,

$$\mathrm{CS}[0,1] = \{\tilde{\ddot{\Upsilon}} = [\ddot{\Upsilon}^-, \ddot{\Upsilon}^+] \mid 0 \leq \ddot{\Upsilon}^- \leq \ddot{\Upsilon}^+ \leq 1\}.$$

We note that $[\ddot{\Upsilon}, \ddot{\Upsilon}] = \{\ddot{\Upsilon}\}$ for all $\ddot{\Upsilon} \in [0,1]$. For $\ddot{\Upsilon} = 0$ or 1 we shall denote $[0,0]$ by $\tilde{0}$ and $[1,1]$ by $\tilde{1}$.

Let $\tilde{\ddot{\Upsilon}} = [\ddot{\Upsilon}^-, \ddot{\Upsilon}^+]$ and $\tilde{\ddot{\rho}} = [\ddot{\rho}^-, \ddot{\rho}^+] \in \mathrm{CS}[0,1]$. Define the operations $\preceq$, $=$, $\curlywedge$ and $\curlyvee$ as follows:
(1) $\tilde{\ddot{\Upsilon}} \preceq \tilde{\ddot{\rho}}$ if and only if $\ddot{\Upsilon}^- \leq \ddot{\rho}^-$ and $\ddot{\Upsilon}^+ \leq \ddot{\rho}^+$
(2) $\tilde{\ddot{\Upsilon}} = \tilde{\ddot{\rho}}$ if and only if $\ddot{\Upsilon}^- = \ddot{\rho}^-$ and $\ddot{\Upsilon}^+ = \ddot{\rho}^+$
(3) $\tilde{\ddot{\Upsilon}} \curlywedge \tilde{\ddot{\rho}} = [(\ddot{\Upsilon}^- \wedge \ddot{\rho}^-), (\ddot{\Upsilon}^+ \wedge \ddot{\rho}^+)]$
(4) $\tilde{\ddot{\Upsilon}} \curlyvee \tilde{\ddot{\rho}} = [(\ddot{\Upsilon}^- \vee \ddot{\rho}^-), (\ddot{\Upsilon}^+ \vee \ddot{\rho}^+)]$.

If $\tilde{\ddot{\Upsilon}} \succeq \tilde{\ddot{\rho}}$, we mean $\tilde{\ddot{\rho}} \preceq \tilde{\ddot{\Upsilon}}$.

For each interval $\tilde{\ddot{\Upsilon}}_i = [\ddot{\Upsilon}_i^-, \ddot{\Upsilon}_i^+] \in \mathrm{CS}[0,1]$, $i \in \ddot{\mathcal{A}}$ where $\ddot{\mathcal{A}}$ is an index set, we define

$$\underset{i\in\mathcal{A}}{\curlywedge}\tilde{\ddot{\Upsilon}}_i = [\underset{i\in\ddot{\mathcal{A}}}{\wedge}\ddot{\Upsilon}_i^-, \underset{i\in\ddot{\mathcal{A}}}{\wedge}\ddot{\Upsilon}_i^+] \quad \text{and} \quad \underset{i\in\ddot{\mathcal{A}}}{\curlyvee}\tilde{\ddot{\Upsilon}}_i = [\underset{i\in\ddot{\mathcal{A}}}{\vee}\ddot{\Upsilon}_i^-, \underset{i\in\ddot{\mathcal{A}}}{\vee}\ddot{\Upsilon}_i^+].$$

Definition 2.5. *[3] Let $\ddot{\mathfrak{T}}$ be a non-empty set. Then the function $\tilde{\ddot{\Upsilon}} : \ddot{\mathfrak{T}} \to CS[0,1]$ is called an* interval valued fuzzy set *(shortly, IVF set) of $\ddot{\mathfrak{T}}$.*

Definition 2.6. *[3] Let $\ddot{\mathfrak{L}}$ be a subset of a non-empty set $\ddot{\mathfrak{T}}$. An* interval valued characteristic function *of $\ddot{\mathfrak{L}}$ is defined to be a function $\tilde{\ddot{\chi}}_{\ddot{\mathfrak{L}}} : \ddot{\mathfrak{T}} \to CS[0,1]$ by*

$$\tilde{\ddot{\chi}}_{\ddot{\mathfrak{L}}}(\ddot{\mathfrak{e}}) = \begin{cases} \tilde{1} & \text{if} \quad \ddot{\mathfrak{e}} \in \ddot{\mathfrak{L}}, \\ \tilde{0} & \text{if} \quad \ddot{\mathfrak{e}} \notin \ddot{\mathfrak{L}} \end{cases}$$

for all $\ddot{\mathfrak{e}} \in \ddot{\mathfrak{T}}$.

Lemma 2.7. *If $\ddot{\mathfrak{M}}$ and $\ddot{\mathfrak{L}}$ are nonempty subsets of an oredred semigroup $\mathfrak{T}$, then the following are true:*
(1) $\tilde{\ddot{\chi}}_{\ddot{\mathfrak{M}}} \curlywedge \tilde{\ddot{\chi}}_{\ddot{\mathfrak{L}}} = \tilde{\ddot{\chi}}_{\ddot{\mathfrak{M}}\cap\ddot{\mathfrak{L}}}$.
(2) *If $\ddot{\mathfrak{M}} \subseteq \ddot{\mathfrak{L}}$, then $\tilde{\ddot{\chi}}_{\ddot{\mathfrak{M}}} \preceq \tilde{\ddot{\chi}}_{\ddot{\mathfrak{L}}}$.*
(3) $\tilde{\ddot{\chi}}_{\ddot{\mathfrak{M}}} \circ \tilde{\ddot{\chi}}_{\ddot{\mathfrak{L}}} = \tilde{\ddot{\chi}}_{\ddot{\mathfrak{M}}\ddot{\mathfrak{L}}}$.

Definition 2.8. *Let $\mathfrak{T}$ be an ordered semigroup and F be a non-empty subset of $\ddot{\Omega}$, we define the set $F_{\ddot{\mathfrak{u}}}$ by*

$$F_{\ddot{\mathfrak{u}}} := \{(\ddot{\mathfrak{x}}, \ddot{\mathfrak{y}}) \in \ddot{\Omega} \times \ddot{\Omega} \mid \ddot{\mathfrak{u}} \leq \ddot{\mathfrak{x}}\ddot{\mathfrak{y}}\}.$$

Definition 2.9. *Let $\tilde{\ddot{\Upsilon}}$ and $\tilde{\ddot{\eta}}$ be IVF sets of an ordered semigroup $\ddot{\Omega}$. The product of IVF sets $\tilde{\ddot{\Upsilon}}$ and $\tilde{\ddot{\eta}}$ of $\ddot{\Omega}$ is defined as follow, for all $\ddot{\mathfrak{x}} \in \ddot{\Omega}$*

$$(\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\eta}})(\ddot{\mathfrak{u}}) = \begin{cases} \bigvee\limits_{(\ddot{\mathfrak{x}},\ddot{\mathfrak{y}})\in F_{\ddot{\mathfrak{u}}}} \{\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{x}}) \wedge \tilde{\ddot{\eta}}(\ddot{\mathfrak{y}})\} & \text{if} \quad F_{\ddot{\mathfrak{u}}} \neq \emptyset, \\ 0 & \text{if} \quad F_{\ddot{\mathfrak{u}}} = \emptyset. \end{cases}$$

For $\ddot{\mathfrak{k}} \in \mathbb{N}$, let $\tilde{\ddot{\Upsilon}}^n := \underbrace{\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\Upsilon}} \circ \cdots \circ \tilde{\ddot{\Upsilon}}}_{n\text{-times}}$.

The support of IVF set $\tilde{\ddot{\Upsilon}}$ of a set $\ddot{\Omega}$ is defined by $\mathrm{supp}(\tilde{\ddot{\Upsilon}}) = \{\ddot{\mathfrak{u}} \in \ddot{\Omega} | \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{u}}) \neq \tilde{0}\}$.

Lemma 2.10. *[16] If $\tilde{\ddot{\Upsilon}}$, $\tilde{\ddot{\rho}}$ and $\tilde{\ddot{\xi}}$ are IVF sets of an ordered semigroup $\ddot{\Omega}$, then the following are true:*
(1) *If $\tilde{\ddot{\Upsilon}} \preceq \tilde{\ddot{\rho}}$, then $\tilde{\ddot{\Upsilon}}^n \preceq \tilde{\ddot{\rho}}^n$*
(2) *If $\tilde{\ddot{\Upsilon}} \preceq \tilde{\ddot{\rho}}$, then $\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\xi}} \preceq \tilde{\ddot{\rho}} \circ \tilde{\ddot{\xi}}$.*
(3) *If $\tilde{\ddot{\Upsilon}} \preceq \tilde{\ddot{\rho}}$, then $\tilde{\ddot{\Upsilon}} \curlyvee \tilde{\ddot{\xi}} \preceq \tilde{\ddot{\rho}} \vee \tilde{\ddot{\xi}}$.*
(4) *If $\tilde{\ddot{\Upsilon}} \preceq \tilde{\ddot{\rho}}$, then $\tilde{\ddot{\Upsilon}} \curlywedge \tilde{\ddot{\xi}} \preceq \tilde{\ddot{\rho}} \curlywedge \tilde{\ddot{\xi}}$.*
(5) *If $\tilde{\ddot{\Upsilon}} \preceq \tilde{\ddot{\rho}}$, then* $\mathrm{supp}(\tilde{\ddot{\Upsilon}}) \subseteq \mathrm{supp}(\tilde{\ddot{\rho}})$.

For an IVF set $\tilde{\ddot{\Upsilon}}$ of an ordered semigroup $\ddot{\Omega}$, we define $(\tilde{\ddot{\Upsilon}}] : \ddot{\Omega} \to \mathrm{CS}[0,1]$ by $(\tilde{\ddot{\Upsilon}}] := \sup\limits_{\ddot{\mathfrak{a}}\leq\ddot{\aleph}} \tilde{\ddot{\Upsilon}}(\ddot{\aleph})$ for all $\ddot{\mathfrak{a}} \in \ddot{\Omega}$.

Lemma 2.11. *[16] If $\tilde{\ddot{\Upsilon}}$, $\tilde{\ddot{\rho}}$ and $\tilde{\ddot{\xi}}$ are IVF sets of an ordered semigroup $\ddot{\Omega}$, then the following are true:*
(1) $\tilde{\ddot{\Upsilon}} \preceq (\tilde{\ddot{\Upsilon}}]$.
(2) *If $\tilde{\ddot{\Upsilon}} \preceq \tilde{\ddot{\rho}}$, then $(\tilde{\ddot{\Upsilon}}] \preceq (\tilde{\ddot{\xi}}]$.*
(3) *If $\tilde{\ddot{\Upsilon}} \preceq \tilde{\ddot{\rho}}$, then $(\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\xi}}] \preceq (\tilde{\ddot{\rho}} \circ \tilde{\ddot{\xi}}]$ and $(\tilde{\ddot{\xi}} \circ \tilde{\ddot{\Upsilon}}] \preceq (\tilde{\ddot{\xi}} \circ \tilde{\ddot{\rho}}]$.*

Lemma 2.12. *[16] If $\tilde{\ddot{\Upsilon}}$ is an IVF set of an ordered semigroup $\ddot{\Omega}$, then the following are equivalent.*

(1) *If* $\ddot{\mathfrak{a}} \leq \ddot{\mathfrak{b}}$, *then* $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{a}}) \succeq \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{b}})$.
(2) $(\tilde{\ddot{\Upsilon}}] = \tilde{\ddot{\Upsilon}}$.

Definition 2.13. *An IVF set* $\tilde{\ddot{\Upsilon}}$ *of an ordered semigroup* $\ddot{\Omega}$ *is called*

(1) *an* IVF subsemigroup *(IVFSG) of* $\ddot{\Omega}$ *if* $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{a}}\ddot{\mathfrak{b}}) \preceq \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{a}}) \curlywedge \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{b}})$ *for all* $\ddot{\mathfrak{a}}, \ddot{\mathfrak{b}} \in \ddot{\Omega}$,
(2) *an* IVF left ideal *(IVFLI) of* $\ddot{\Omega}$ *if* $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{a}}\ddot{\mathfrak{b}}) \preceq \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{b}})$ *and if* $\ddot{\mathfrak{a}} \leq \ddot{\mathfrak{b}}$, *then* $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{a}}) \succeq \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{b}})$ *for all* $\ddot{\mathfrak{a}}, \ddot{\mathfrak{b}} \in \ddot{\Omega}$,
(3) *an* IVF right ideal *(IVFRI) of* $\ddot{\Omega}$ *if* $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{a}}\ddot{\mathfrak{b}}) \preceq \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{a}})$ *and if* $\ddot{\mathfrak{a}} \leq \ddot{\mathfrak{b}}$, *then* $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{a}}) \succeq \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{b}})$ *for all* $\ddot{\mathfrak{a}}, \ddot{\mathfrak{b}} \in \ddot{\Omega}$,
(4) *an* TVF ideal *(IVFI) of* $\ddot{\Omega}$ *if it is both an IVFLI and IVFRI of* $\ddot{\Omega}$,
(5) *a* IVF left ordered almost ideal *(IVFLOAI) of* $\ddot{\Omega}$ *if* $(\tilde{\chi}_{\ddot{\Omega}} \circ \tilde{\ddot{\Upsilon}}] \curlywedge \tilde{\ddot{\Upsilon}} \neq \tilde{0}$ *where* $\tilde{\chi}_{\ddot{\Omega}}$ *is an IVF set of* $\ddot{\Omega}$ *mapping every element to* $\tilde{1}$,
(6) *an* IVF right ordered almost ideal *(IVFROAI) of* $\ddot{\Omega}$ *if* $(\tilde{\ddot{\Upsilon}} \circ \tilde{\chi}_{\ddot{\Omega}}] \curlywedge \tilde{\ddot{\Upsilon}} \neq \tilde{0}$,
(7) *an* IVF ordered almost ideal *(IVFOAI) of* $\ddot{\Omega}$ *if it is both an IVFLOAI and IVFROAI of* $\ddot{\Omega}$.

Definition 2.14. *[19] An IVF subsemigroup* $\tilde{\ddot{\Upsilon}}$ *of a ordered semigroup* $\ddot{\Omega}$ *is said to be an* IVF (m,n)-ideal *of* $\mathfrak{T}$ *if*

(1) $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{u}}_1\ddot{\mathfrak{u}}_2\cdots\ddot{\mathfrak{u}}_m\ddot{\mathfrak{z}}\ddot{\mathfrak{v}}_1\ddot{\mathfrak{v}}_2\cdots\ddot{\mathfrak{v}}_n) \succeq \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{u}}_1) \curlywedge \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{u}}_2) \curlywedge \ldots \curlywedge \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{u}}_m) \curlywedge \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{v}}_1) \wedge \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{v}}_2) \curlywedge \ldots \curlywedge \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{v}}_n)$ *for all* $\ddot{\mathfrak{u}}_1, \ddot{\mathfrak{u}}_2, \ldots, \ddot{\mathfrak{u}}_m, \ddot{\mathfrak{v}}_1, \ddot{\mathfrak{v}}_2, \ldots, \ddot{\mathfrak{v}}_n, \ddot{\mathfrak{z}} \in \mathfrak{T}$ *and* $m, n \in \mathbb{N}$.
(2) *If* $\ddot{\mathfrak{u}}_1 \leq \ddot{\mathfrak{u}}_2$, *then* $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{u}}_1) \succeq \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{u}}_2)$, *for all* $\ddot{\mathfrak{u}}_1, \ddot{\mathfrak{u}}_2 \in \ddot{\Omega}$.

III. Interval Valued Fuzzy Almost (m,n)-ideals

In this section, we define an IVFOA-(m,n)-ideal in an ordered semigroup. We prove $\tilde{\ddot{\Upsilon}} \curlyvee \tilde{\ddot{\xi}}$ is an IVFOA-(m,n)-ideal where $\tilde{\ddot{\Upsilon}}$ and $\tilde{\ddot{\xi}}$ are IVFOA-(m,n)-ideals. And we create a bridge between almost (m,n)-ideals and IVFOA-(m,n)-ideals.

Definition 3.1. *A non-empty subset* $\ddot{\aleph}$ *on an ordered semigroup* $\ddot{\Omega}$ *is called an* almost (m,n)-ideal *(OA-(m,n)-I) of* $\ddot{\Omega}$ *if* $(\ddot{\aleph}^m\ddot{\mathfrak{t}}\ddot{\aleph}^n] \cap \ddot{\aleph} \neq \emptyset$ *for all* $\ddot{\mathfrak{t}} \in \ddot{\Omega}$ *where* $m, n \in \{1, 2, \ldots, n\}$.

Example 3.2. (1) *An OA-$(1,0)$-I of an ordered semigroup* $\ddot{\Omega}$ *is an ORAI of* $\ddot{\Omega}$.
(2) *An OA-$(0,1)$-I of an ordered semigroup* $\ddot{\Omega}$ *is an OLAI of* $\ddot{\Omega}$.
(3) *Consider the ordered semigroup* $\mathbb{Z}_6$ *under the usual addition and the partial ordered* $\leq := \{(\ddot{\mathfrak{a}}, \ddot{\mathfrak{a}}) \mid \ddot{\mathfrak{a}} \in \mathbb{Z}_6\}$. *We have* $\mathfrak{A} = \{\overline{1}, \overline{4}, \overline{5}\}$ *is an OA-$(1,0)$-I of* $\mathbb{Z}_6$.

Definition 3.3. *An IVF set* $\tilde{\ddot{\Upsilon}}$ *on an ordered semigroup* $\ddot{\Omega}$ *is called an* interval valued fuzzy ordered almost (m,n)-ideal *(IVFOA-(m,n)-I) of* $\ddot{\Omega}$ *if* $(\tilde{\ddot{\Upsilon}}^m \circ \tilde{\chi}_{\ddot{\Omega}} \circ \tilde{\ddot{\Upsilon}}^n] \curlywedge \tilde{\ddot{\Upsilon}} \neq \tilde{0}$ *where* $m, n \in \{1, 2, \ldots, n\}$.

Theorem 3.4. *If* $\tilde{\ddot{\Upsilon}}$ *is an IVFOA-(m,n)-I of an ordered semigroup* $\ddot{\Omega}$ *and* $\tilde{\ddot{\xi}}$ *is an IVF set of* $\ddot{\Omega}$ *such that* $\tilde{\ddot{\Upsilon}} \preceq \tilde{\ddot{\xi}}$, *then* $\tilde{\ddot{\xi}}$ *is an IVFOA-(m,n)-I of* $\ddot{\Omega}$.

Proof: Suppose that $\tilde{\ddot{\Upsilon}}$ is an IVFOA-(m,n)-I of $\ddot{\Omega}$ and $\tilde{\ddot{\xi}}$ is an IVF set of $\ddot{\Omega}$ such that $\tilde{\ddot{\Upsilon}} \preceq \tilde{\ddot{\xi}}$. Then we obtain that $(\tilde{\ddot{\Upsilon}}^m \circ \tilde{\chi}_{\ddot{\Omega}} \circ \tilde{\ddot{\Upsilon}}^n] \curlywedge \tilde{\ddot{\Upsilon}} \neq \tilde{0}$. Thus,

$$(\tilde{\ddot{\Upsilon}}^m \circ \tilde{\chi}_{\ddot{\Omega}} \circ \tilde{\ddot{\Upsilon}}^n] \curlywedge \tilde{\ddot{\Upsilon}} \preceq (\tilde{\ddot{\xi}}^m \circ \tilde{\chi}_{\ddot{\Omega}} \circ \tilde{\ddot{\xi}}^n] \curlywedge \tilde{\ddot{\xi}} \neq \tilde{0}.$$

Hence, $(\tilde{\ddot{\xi}}^m \circ \tilde{\chi}_{\ddot{\Omega}} \circ \tilde{\ddot{\xi}}^n] \curlywedge \tilde{\ddot{\xi}} \neq \tilde{0}$. Therefore, $\tilde{\ddot{\xi}}$ is an IVFOA-(m,n)-I of $\ddot{\Omega}$. ∎

The following results are an obvious consequence of Theorem 3.4.

Theorem 3.5. *Let* $\tilde{\ddot{\Upsilon}}$ *and* $\tilde{\ddot{\xi}}$ *be IVFOA-(m,n)-Is of an ordered semigroup* $\ddot{\Omega}$. *Then* $\tilde{\ddot{\Upsilon}} \curlyvee \tilde{\ddot{\xi}}$ *is also an IVFOA-(m,n)-I of* $\ddot{\Omega}$.

Proof: Since $\tilde{\ddot{\Upsilon}} \preceq \tilde{\ddot{\Upsilon}} \vee \tilde{\ddot{\xi}}$, by Theorem 3.4, $\tilde{\ddot{\Upsilon}} \vee \tilde{\ddot{\xi}}$ is an IVFOA-(m,n)-I of $\ddot{\Omega}$. ∎

Theorem 3.6. *If* $\tilde{\ddot{\Upsilon}}$ *is an IVFOA-(m,n)-I of an ordered semigroup* $\ddot{\Omega}$. *and* $\tilde{\ddot{\xi}}$ *is an IVF set, then* $\tilde{\ddot{\Upsilon}} \curlyvee \tilde{\ddot{\xi}}$ *is an IVFOA-(m,n)-Ideal of* $\ddot{\Omega}$.

Proof: By Theorem 3.4, and $\tilde{\ddot{\Upsilon}} \preceq \tilde{\ddot{\Upsilon}} \vee \tilde{\ddot{\xi}}$. Thus, $\tilde{\ddot{\Upsilon}} \vee \tilde{\ddot{\xi}}$ is an IVFOA-(m,n)-I of $\ddot{\Omega}$. ∎

Corollary 3.7. *Let* $\tilde{\ddot{\Upsilon}}_i$ *be an IVFOA-(m,n)-I in an ordered semigroup* $\ddot{\Omega}$. *Then* $\underset{i \in \mathcal{F}}{\curlyvee} \tilde{\ddot{\Upsilon}}_i$ *is an IVFOA-(m,n)-I of* $\ddot{\Omega}$.

Example 3.8. *Consider* $n = 1, m = 0$ *and the ordered semigroup* $\mathbb{Z}_6$ *under the usual addition and the partial ordered* $\leq := \{(\ddot{\mathfrak{a}}, \ddot{\mathfrak{a}}) \mid \ddot{\mathfrak{a}} \in \mathbb{Z}_6\}$. $\tilde{\ddot{\Upsilon}} : \mathbb{Z}_6 \to \mathrm{CS}[0,1]$ *is defined by* $\tilde{\ddot{\Upsilon}}(\overline{0}) = \tilde{0}$, $\tilde{\ddot{\Upsilon}}(\overline{1}) = [0.1, 0.2]$, $\tilde{\ddot{\Upsilon}}(\overline{2}) = \tilde{0}$, $\tilde{\ddot{\Upsilon}}(\overline{3}) = \tilde{0}$, $\tilde{\ddot{\Upsilon}}(\overline{4}) = [0.3, 0.4]$, $\tilde{\ddot{\Upsilon}}(\overline{5}) = [0.5, 0.6]$ *and* $\tilde{\ddot{\rho}} : \mathbb{Z}_6 \to \mathrm{CS}[0,1]$ *is defined by* $\tilde{\ddot{\rho}}(\overline{0}) = \tilde{0}$, $\tilde{\ddot{\rho}}(\overline{1}) = [0.7, 0.8]$, $\tilde{\ddot{\rho}}(\overline{2}) = [0.3, 0.4]$, $\tilde{\ddot{\rho}}(\overline{3}) = [0.1, 0.2]$, $\tilde{\ddot{\rho}}(\overline{4}) = \tilde{0}$, $\tilde{\ddot{\rho}}(\overline{5}) = [0.3, 0.4]$. *We have* $\tilde{\ddot{\Upsilon}}$ *and* $\tilde{\ddot{\rho}}$ *are IVFOA-$(1,0)$-Is of* $\mathbb{Z}_6$ *but* $\tilde{\ddot{\Upsilon}} \curlywedge \tilde{\ddot{\rho}}$ *is not a n IVFOA-$(1,0)$-I of* $\mathbb{Z}_6$.

Lemma 3.9. *Let* $\mathfrak{A}$ *be a subset of a set* $\mathfrak{T}$ *and* $n \in \mathbb{N} \cup \{0\}$. *Then* $(\tilde{\chi}_{\mathfrak{A}})^n = \tilde{\chi}_{\mathfrak{A}}^n$

Theorem 3.10. *Let* $\ddot{\aleph}$ *be a nonempty subset of an ordered semigroup* $\ddot{\Omega}$. *Then* $\ddot{\aleph}$ *is an OA-(m,n)-I of* $\ddot{\Omega}$ *if and only if* $\tilde{\chi}_{\ddot{\aleph}}$ *is an IVFOA-(m,n)-I of* $\ddot{\Omega}$.

Proof: Suppose that $\ddot{\aleph}$ is an OA-(m,n)-I of $\ddot{\Omega}$. Then $(\ddot{\aleph}^m\ddot{\mathfrak{t}}\ddot{\aleph}^n] \cap \ddot{\aleph} \neq \emptyset$ for all $\ddot{\mathfrak{t}} \in \ddot{\Omega}$ and $m, n \in \{1, 2, \ldots, n\}$. Thus, there exists $\ddot{\mathfrak{c}} \in \ddot{\Omega}$ such that $\ddot{\mathfrak{c}} \in (\ddot{\aleph}^m\ddot{\mathfrak{t}}\ddot{\aleph}^n]$ and $\ddot{\mathfrak{c}} \in \ddot{\aleph}$. It implies that $((\tilde{\chi}_{\ddot{\aleph}}^m \circ \tilde{\chi}_{\ddot{\Omega}} \circ \tilde{\chi}_{\ddot{\aleph}}^n])(\ddot{\mathfrak{c}}) \neq \tilde{0}$ and $\tilde{\chi}_{\ddot{\aleph}}(\ddot{\mathfrak{c}}) = \tilde{1}$ and $m, n \in \{1, 2, \ldots, n\}$. Thus, $(\tilde{\chi}_{\ddot{\aleph}}^m \circ \tilde{\chi}_{\ddot{\Omega}} \circ \tilde{\chi}_{\ddot{\aleph}}^n] \curlywedge \tilde{\chi}_{\ddot{\aleph}}(\ddot{\mathfrak{c}}) = ((\tilde{\chi}_{\ddot{\aleph}})^m \circ \tilde{\chi}_{\ddot{\Omega}} \circ (\tilde{\chi}_{\ddot{\aleph}})^n] \curlywedge \tilde{\chi}_{\ddot{\aleph}}(\ddot{\mathfrak{c}}) \neq \tilde{0}$ and $m, n \in \{1, 2, \ldots, n\}$. So $(\tilde{\chi}_{\ddot{\aleph}}^m \circ \tilde{\chi}_{\ddot{\Omega}} \circ \tilde{\chi}_{\ddot{\aleph}}^n] \curlywedge \tilde{\chi}_{\ddot{\aleph}} \neq \tilde{0}$ and $m, n \in \{1, 2, \ldots, n\}$. Hence, $\tilde{\chi}_{\ddot{\aleph}}$ is an IVFOA-(m,n)-I of $\ddot{\Omega}$.

Conversely, suppose that $\tilde{\chi}_{\ddot{\aleph}}$ is an IVFOA-(m,n)-I of $\ddot{\Omega}$ and let $m, n \in \{1, 2, \ldots, n\}$. Then $(\tilde{\chi}_{\ddot{\aleph}}^m \circ \tilde{\chi}_{\ddot{\Omega}} \circ \tilde{\chi}_{\ddot{\aleph}}^n] \curlywedge \tilde{\chi}_{\ddot{\aleph}} \neq \tilde{0}$. Thus, there exists $\mathfrak{c} \in \ddot{\aleph}$ such that $(\tilde{\chi}_{\ddot{\aleph}}^m \circ \tilde{\chi}_{\ddot{\Omega}} \circ \tilde{\chi}_{\ddot{\aleph}}^n] \curlywedge \tilde{\chi}_{\ddot{\aleph}}(\ddot{\mathfrak{c}}) \neq \tilde{0}$. It implies that $((\tilde{\chi}_{\ddot{\aleph}}^m \circ \tilde{\chi}_{\ddot{\Omega}} \circ \tilde{\chi}_{\ddot{\aleph}}^n])(\ddot{\mathfrak{c}}) \neq \tilde{0}$ and $\tilde{\chi}_{\ddot{\aleph}}(\ddot{\mathfrak{c}}) = \tilde{1}$. Hence $\ddot{\mathfrak{c}} \in (\ddot{\aleph}^m\ddot{\mathfrak{t}}\ddot{\aleph}^n]$ and $\ddot{\mathfrak{c}} \in \ddot{\aleph}$. So $(\ddot{\aleph}^m\mathfrak{t}\ddot{\aleph}^n] \cap \ddot{\aleph} \neq \emptyset$. We conclude that $\ddot{\aleph}$ is an OA-(m,n)-I of $\ddot{\Omega}$. ∎

Theorem 3.11. *Let* $\tilde{\ddot{\Upsilon}}$ *be an IVF set of an ordered semigroup* $\ddot{\Omega}$. *Then* $\tilde{\ddot{\Upsilon}}$ *is an IVFOA-(m,n)-I of* $\ddot{\Omega}$ *if and only if* $\mathrm{supp}(\tilde{\ddot{\Upsilon}})$ *is an OA-(m,n)-I of* $\ddot{\Omega}$.

Proof: Assume that $\tilde{\ddot{\Upsilon}}$ is an IVFOA-(m,n)-I of $\ddot{\Omega}$. Then $(\tilde{\ddot{\Upsilon}}^m \circ \tilde{\chi}_{\ddot{\Omega}} \circ \tilde{\ddot{\Upsilon}}^n] \curlywedge \tilde{\ddot{\Upsilon}} \neq \tilde{0}$ for all $m, n \in \{1, 2, \ldots, n\}$. Thus, there exists $\ddot{\mathfrak{z}} \in \ddot{\Omega}$ such that $((\tilde{\ddot{\Upsilon}}^m \circ \tilde{\chi}_{\ddot{\Omega}} \circ \tilde{\ddot{\Upsilon}}^n] \curlywedge \tilde{\ddot{\Upsilon}})(\ddot{\mathfrak{z}}) \neq 0$. So $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{z}}) \neq 0$ and $\ddot{\mathfrak{z}} = \ddot{\mathfrak{a}}_1\ddot{\mathfrak{a}}_2\cdots\ddot{\mathfrak{a}}_m\ddot{\mathfrak{r}}\ddot{\mathfrak{b}}_1\ddot{\mathfrak{b}}_2\cdots\ddot{\mathfrak{b}}_n$ for some $\ddot{\mathfrak{a}}_1, \ddot{\mathfrak{a}}_2, \cdots, \ddot{\mathfrak{a}}_m, \ddot{\mathfrak{r}}, \ddot{\mathfrak{b}}_1, \ddot{\mathfrak{b}}_2, \cdots \ddot{\mathfrak{b}}_n \in \ddot{\Omega}$ such that $\tilde{\ddot{\Upsilon}}(\mathfrak{a}_1) \neq 0$, $\tilde{\ddot{\Upsilon}}(\mathfrak{a}_2) \neq 0, \cdots, \tilde{\ddot{\Upsilon}}(\mathfrak{a}_m) \neq 0$ $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{b}}_1) \neq 0$, $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{b}}_2) \neq 0, \cdots, \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{b}}_n) \neq 0$. It implies that, $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_m, \ddot{\mathfrak{b}}_1, \ddot{\mathfrak{b}}_2, \cdots, \ddot{\mathfrak{b}}_n \in \operatorname{supp}(\tilde{\ddot{\Upsilon}})$. Thus, $((\tilde{\ddot{\chi}}_{\operatorname{supp}(\tilde{\ddot{\Upsilon}})}{}^m \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\chi}}_{\operatorname{supp}(\tilde{\ddot{\Upsilon}})}{}^n] \curlywedge \tilde{\ddot{\chi}}_{\operatorname{supp}(\tilde{\ddot{\Upsilon}})})(\ddot{\mathfrak{z}}) \neq 0$. Hence, $(\tilde{\ddot{\chi}}_{\operatorname{supp}(\tilde{\ddot{\Upsilon}})}{}^m \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\chi}}_{\operatorname{supp}(\tilde{\ddot{\Upsilon}})}{}^n] \curlywedge \tilde{\ddot{\chi}}_{\operatorname{supp}(\tilde{\ddot{\Upsilon}})} \neq 0$ for all $m, n \in \{1, 2, ..., n\}$. Therefore, $\tilde{\ddot{\chi}}_{\operatorname{supp}(\tilde{\ddot{\Upsilon}})}$ is an IVFOA-(m,n)-I of $\ddot{\Omega}$. By Theorem 3.10, $\operatorname{supp}(\tilde{\ddot{\Upsilon}})$ is an OA-(m,n)-I of $\ddot{\Omega}$.

Conversely, suppose that $\operatorname{supp}(\tilde{\ddot{\Upsilon}})$ is an OA-(m,n)-I of $\ddot{\Omega}$. By Theorem 3.10, $\tilde{\ddot{\chi}}_{\operatorname{supp}(\tilde{\ddot{\Upsilon}})}$ is an IVFOA-(m,n)-I of $\ddot{\Omega}$. Then $m, n \in \{1, 2, ..., n\}$, we have $(\tilde{\ddot{\chi}}_{\operatorname{supp}(\tilde{\ddot{\Upsilon}})}{}^m \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\chi}}_{\operatorname{supp}(\tilde{\ddot{\Upsilon}})}{}^n] \curlywedge \tilde{\ddot{\chi}}_{\operatorname{supp}(\tilde{\ddot{\Upsilon}})} \neq 0$. Thus, there exists $\ddot{\mathfrak{z}} \in \ddot{\Omega}$ such that $((\tilde{\ddot{\chi}}_{\operatorname{supp}(\tilde{\ddot{\Upsilon}})}{}^m \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\chi}}_{\operatorname{supp}(\tilde{\ddot{\Upsilon}})}{}^n] \curlywedge \tilde{\ddot{\chi}}_{\operatorname{supp}(\tilde{\ddot{\Upsilon}})})(\ddot{\mathfrak{z}}) \neq 0$. Hence, $(\tilde{\ddot{\Upsilon}}^m \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\Upsilon}}^n](\ddot{\mathfrak{z}}) \neq 0$. So, $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{z}}) \neq 0$ and $\ddot{\mathfrak{z}} = \ddot{\mathfrak{a}}_1\ddot{\mathfrak{a}}_2 \cdots \ddot{\mathfrak{a}}_m \ddot{\mathfrak{r}} \ddot{\mathfrak{b}}_1 \ddot{\mathfrak{b}}_2 \cdots \ddot{\mathfrak{b}}_n$ for some $\ddot{\mathfrak{a}}_1, \ddot{\mathfrak{a}}_2, \cdots, \ddot{\mathfrak{a}}_m, \ddot{\mathfrak{r}}, \ddot{\mathfrak{b}}_1, \ddot{\mathfrak{b}}_2, \cdots \mathfrak{b}_n \in \ddot{\Omega}$ such that $\tilde{\ddot{\Upsilon}}(\mathfrak{a}_1) \neq 0$, $\tilde{\ddot{\Upsilon}}(\mathfrak{a}_2) \neq 0, \cdots, \tilde{\ddot{\Upsilon}}(\mathfrak{a}_m) \neq 0$ $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{b}}_1) \neq 0$, $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{b}}_2) \neq 0, \cdots, \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{b}}_n) \neq 0$. Thus, $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_m, \ddot{\mathfrak{b}}_1, \ddot{\mathfrak{b}}_2, \cdots, \ddot{\mathfrak{b}}_n \in \operatorname{supp}(\tilde{\ddot{\Upsilon}})$. Hence, $((\tilde{\ddot{\Upsilon}}^m \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\Upsilon}}^n] \curlywedge \tilde{\ddot{\Upsilon}})(\ddot{\mathfrak{z}}) \neq 0$. So $(\tilde{\ddot{\Upsilon}}^m \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\Upsilon}}^n] \curlywedge \tilde{\ddot{\Upsilon}} \neq 0$. Therefore, $\tilde{\ddot{\Upsilon}}$ is an IVFOA-(m,n)-I of $\ddot{\Omega}$. ■

Next, we investigate the connection between minimal, prime, semiprime, strongly prime OA-(m,n)-ideals and minimal, prime, semiprime, strongly prime IVFOA-(m,n)-Is of ordered semigroups.

Definition 3.12. *An OA-(m,n)-I $\ddot{\aleph}$ of an ordered semigroup $\ddot{\Omega}$ is called*

(1) minimal *if for any OA-(m,n)-I $\ddot{\aleph}_1$ of $\ddot{\Omega}$ if whenever $\ddot{\aleph}_1 \subseteq \ddot{\aleph}$, then $\ddot{\aleph}_1 = \ddot{\aleph}$,*
(2) prime *if for any two OA-(m,n)-Is $\ddot{\aleph}_1$ and $\ddot{\aleph}_2$ of $\ddot{\Omega}$ such that $\ddot{\aleph}_1\ddot{\aleph}_2 \subseteq \ddot{\aleph}$ implies that $\ddot{\aleph}_1 \subseteq \ddot{\aleph}$ or $\ddot{\aleph}_2 \subseteq \ddot{\aleph}$.*
(3) semiprime *if for any OA-(m,n)-I $\ddot{\aleph}_1$ of $\ddot{\Omega}$ such that $\ddot{\aleph}_1^2 \subseteq \ddot{\aleph}$ implies that $\ddot{\aleph}_1 \subseteq \ddot{\aleph}$.*
(4) strongly prime *if for any OA-(m,n)-Is $\ddot{\aleph}_1$ and $\ddot{\aleph}_2$ of $\ddot{\Omega}$ such that $\ddot{\aleph}_1\ddot{\aleph}_2 \cap \ddot{\aleph}_2\ddot{\aleph}_1 \subseteq \ddot{\aleph}$ implies that $\ddot{\aleph}_1 \subseteq \ddot{\aleph}$ or $\ddot{\aleph}_2 \subseteq \ddot{\aleph}$.*

Definition 3.13. *An IVFOA-(m,n)-I $\tilde{\ddot{\Upsilon}}$ of an ordered semigroup $\ddot{\Omega}$ is called*

(1) minimal *if for any IVFOA-(m,n)-I $\tilde{\ddot{\xi}}$ of Ω if whenever $\tilde{\ddot{\xi}} \preceq \tilde{\ddot{\Upsilon}}$, then $\operatorname{supp}(\tilde{\ddot{\xi}}) = \operatorname{supp}(\tilde{\ddot{\Upsilon}})$,*
(2) prime *if for any two IVFOA-(m,n)-Is $\tilde{\ddot{\xi}}$ and $\tilde{\ddot{\rho}}$ of $\dot{\Omega}$ such that $\tilde{\ddot{\xi}} \circ \tilde{\ddot{\rho}} \preceq \tilde{\ddot{\Upsilon}}$ implies that $\tilde{\ddot{\xi}} \preceq \tilde{\ddot{\Upsilon}}$ or $\tilde{\ddot{\rho}} \preceq \tilde{\ddot{\Upsilon}}$.*
(3) semiprime *if for any IVFOA-(m,n)-I $\tilde{\ddot{\xi}}$ of $\ddot{\Omega}$ such that $\tilde{\ddot{\xi}} \circ \tilde{\ddot{\xi}} \preceq \tilde{\ddot{\xi}}$ implies that $\tilde{\ddot{\xi}} \preceq \tilde{\ddot{\Upsilon}}$.*
(4) strongly prime *if for any two IVFOA-(m,n)-Is $\tilde{\ddot{\xi}}$ and $\tilde{\ddot{\rho}}$ of $\ddot{\Omega}$ such that $(\tilde{\ddot{\xi}} \circ \tilde{\ddot{\rho}}) \curlywedge (\tilde{\ddot{\rho}} \circ \tilde{\ddot{\xi}}) \preceq \tilde{\ddot{\Upsilon}}$ implies that $\tilde{\ddot{\xi}} \preceq \tilde{\ddot{\Upsilon}}$ or $\tilde{\ddot{\rho}} \preceq \tilde{\ddot{\Upsilon}}$.*

It is clear that every strongly prime IVFOA-(m,n)-I of an ordered semigroup is a prime IVFOA-(m,n)-I, and every prime IVFOA-(m,n)-I of an ordered semigroup is a semiprime IVFOA-(m,n)-I.

Theorem 3.14. *Let $\ddot{\aleph}$ be a nonempty subset of an ordered semigroup $\ddot{\Omega}$. Then*

(1) *$\ddot{\aleph}$ is a minimal OA-(m,n)-I of $\ddot{\Omega}$ if and only if $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ is a minimal IVFOA-(m,n)-I of $\ddot{\Omega}$.*
(2) *$\ddot{\aleph}$ is a prime OA-(m,n)-I of $\ddot{\Omega}$ if and only if $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ is a prime IVFOA-(m,n)-I of $\ddot{\Omega}$.*
(3) *$\ddot{\aleph}$ is a semiprime OA-(m,n)-I of $\ddot{\Omega}$ if and only if $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ is an semiprime IVFOA-(m,n)-I of $\ddot{\Omega}$.*
(4) *$\ddot{\aleph}$ is a strongly prime OA-(m,n)-I of $\mathfrak{T}$ if and only if $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ is a strongly prime IVFOA-(m,n)-I of $\ddot{\Omega}$.*

Proof:

(1) Assume that $\ddot{\aleph}$ is a minimal OA-(m,n)-I of $\ddot{\Omega}$. Then $\ddot{\aleph}$ is an OA-(m,n)-I of $\ddot{\Omega}$. Thus by Theorem 3.10, $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ is an IVFOA-(m,n)-I of Ω. Let $\tilde{\ddot{\xi}}$ be an IVFOA-(m,n)-I of Ω such that $\tilde{\ddot{\xi}} \preceq \tilde{\ddot{\chi}}_{\ddot{\aleph}}$. Then by Theorem 3.11, $\operatorname{supp}(\tilde{\ddot{\xi}})$ is an OA-(m,n)-I of Ω such that $\operatorname{supp}(\tilde{\ddot{\xi}}) \subseteq \operatorname{supp}(\tilde{\ddot{\chi}}_{\ddot{\aleph}}) = \ddot{\aleph}$. By assumption, $\operatorname{supp}(\tilde{\ddot{\xi}}) = \operatorname{supp}(\tilde{\ddot{\chi}}_{\ddot{\aleph}})$. Therefore, $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ is a minimal IVFOA-(m,n)-I of $\ddot{\Omega}$.
Conversely, suppose that $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ is a minimal IVFOA-(m,n)-I of $\ddot{\Omega}$. Then $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ is an IVFOA-(m,n)-I of $\ddot{\Omega}$. Thus by Theorem 3.10, $\ddot{\aleph}$ is an OA-(m,n)-I of Ω. Let $\ddot{\aleph}_1$ be an OA-(m,n)-I of Ω such that $\ddot{\aleph}_1 \subseteq \ddot{\aleph}$. Then $\tilde{\ddot{\chi}}_{\ddot{\aleph}_1}$ is an IVFOA-(m,n)-I of Ω such that $\tilde{\ddot{\chi}}_{\ddot{\aleph}_1} \preceq \tilde{\ddot{\chi}}_{\ddot{\aleph}}$. Thus, $\operatorname{supp}(\tilde{\ddot{\chi}}_{\ddot{\aleph}_1}) \subseteq \operatorname{supp}(\tilde{\ddot{\chi}}_{\ddot{\aleph}})$. By assumption, $\ddot{\aleph}_1 = \operatorname{supp}(\tilde{\ddot{\chi}}_{\ddot{\aleph}_1}) = \operatorname{supp}(\tilde{\ddot{\chi}}_{\ddot{\aleph}}) = \ddot{\aleph}$. Therefore, $\ddot{\aleph}$ is a minimal OA-(m,n)-I of $\ddot{\Omega}$.
(2) Suppose that $\ddot{\aleph}$ is a prime OA-(m,n)-I of $\ddot{\Omega}$. Then $\ddot{\aleph}$ is an OA-(m,n)-I of $\ddot{\Omega}$. Thus by Theorem 3.10, $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ is an IVFOA-(m,n)-I of Ω. Let $\tilde{\ddot{\Upsilon}}$ and $\tilde{\ddot{\xi}}$ be IVFOA-(m,n)-Is such that $\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\xi}} \preceq \tilde{\ddot{\chi}}_{\ddot{\aleph}}$. Assume that $\tilde{\ddot{\Upsilon}} \npreceq \tilde{\ddot{\chi}}_{\ddot{\aleph}}$ and $\tilde{\ddot{\xi}} \npreceq \tilde{\ddot{\chi}}_{\ddot{\aleph}}$. Then there exist $\ddot{\mathfrak{h}}, \ddot{\mathfrak{r}} \in \ddot{\Omega}$ such that $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{h}}) \neq \tilde{0}$ and $\tilde{\ddot{\xi}}(\mathfrak{r}) \neq \tilde{0}$. While $\tilde{\ddot{\chi}}_{\ddot{\aleph}}(\ddot{\mathfrak{h}}) = \tilde{0}$ and $\tilde{\ddot{\chi}}_{\ddot{\aleph}}(\ddot{\mathfrak{r}}) = 0$. Thus, $\ddot{\mathfrak{h}} \in \operatorname{supp}(\tilde{\ddot{\Upsilon}})$ and $\ddot{\mathfrak{r}} \in \operatorname{supp}(\tilde{\ddot{\xi}})$, but $\ddot{\mathfrak{h}}, \ddot{\mathfrak{r}} \notin \ddot{\aleph}$. So $\operatorname{supp}(\tilde{\ddot{\Upsilon}}) \nsubseteq \ddot{\aleph}$ and $\operatorname{supp}(\tilde{\ddot{\xi}}) \nsubseteq \ddot{\aleph}$. Since $\operatorname{supp}(\tilde{\ddot{\Upsilon}})$ and $\operatorname{supp}(\tilde{\ddot{\xi}})$ are OA-(m,n)-Is of $\mathfrak{T}$ we have $\operatorname{supp}(\tilde{\ddot{\Upsilon}})\operatorname{supp}(\tilde{\ddot{\xi}}) \nsubseteq \ddot{\aleph}$. Thus, there exists $\ddot{\mathfrak{m}} = \ddot{\mathfrak{p}}\ddot{\mathfrak{q}}$ for some $\ddot{\mathfrak{p}} \in \operatorname{supp}(\tilde{\ddot{\Upsilon}})$ and $\ddot{\mathfrak{q}} \in \operatorname{supp}(\tilde{\ddot{\xi}})$ such that $\mathfrak{m} \in \ddot{\aleph}$. Hence, $\tilde{\ddot{\chi}}_{\ddot{\aleph}}(\ddot{\mathfrak{m}}) = \tilde{0}$ implies that $(\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\xi}})(\ddot{\mathfrak{m}}) = \tilde{0}$. Since $\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\xi}} \leq \tilde{\ddot{\chi}}_{\ddot{\aleph}}$. we have $\ddot{\mathfrak{p}} \in \operatorname{supp}(\tilde{\ddot{\Upsilon}})$ and $\ddot{\mathfrak{q}} \in \operatorname{supp}(\tilde{\ddot{\xi}})$. Thus, $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{p}}) \neq \tilde{0}$, and $\tilde{\ddot{\xi}}(\ddot{\mathfrak{q}}) \neq 0$. It implies that
$$(\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\xi}})(\ddot{\mathfrak{m}}) = \curlyvee_{(\ddot{\mathfrak{p}},\ddot{\mathfrak{q}}) \in F_{\ddot{\mathfrak{m}}}} \{\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{p}}) \curlywedge \tilde{\ddot{\xi}}(\ddot{\mathfrak{q}})\} \neq \tilde{0}.$$
It is a contradiction so $\tilde{\ddot{\Upsilon}} \preceq \tilde{\ddot{\chi}}_{\ddot{\aleph}}$ or $\tilde{\ddot{\xi}} \preceq \tilde{\ddot{\chi}}_{\ddot{\aleph}}$. Therefore, $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ is a prime IVFOA-(m,n)-I of $\ddot{\Omega}$.
Conversely, suppose that $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ is a prime IVFOA-(m,n)-I of $\ddot{\Omega}$. Then $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ is an IVFOA-(m,n)-I of $\ddot{\Omega}$. Thus by Theorem 3.10, $\ddot{\aleph}$ is an OA-(m,n)-I of $\ddot{\Omega}$. Let $\ddot{\aleph}_1$ and $\ddot{\aleph}_2$ be OA-(m,n)-Is of $\ddot{\Omega}$ such that $\ddot{\aleph}_1\ddot{\aleph}_2 \subseteq \ddot{\aleph}$. Then $\tilde{\ddot{\chi}}_{\ddot{\aleph}_1}$ and $\tilde{\ddot{\chi}}_{\ddot{\aleph}_2}$ are IVFOA-(m,n)-Is of $\ddot{\Omega}$. By Lemma 2.7 $\tilde{\ddot{\chi}}_{\ddot{\aleph}_1} \circ \tilde{\ddot{\chi}}_{\ddot{\aleph}_2} = \tilde{\ddot{\chi}}_{\ddot{\aleph}_1\ddot{\aleph}_2} \preceq \tilde{\ddot{\chi}}_{\ddot{\aleph}}$. By assumption, $\tilde{\ddot{\chi}}_{\ddot{\aleph}_1} \preceq \tilde{\ddot{\chi}}_{\ddot{\aleph}}$ or $\tilde{\ddot{\chi}}_{\ddot{\aleph}_2} \preceq \tilde{\ddot{\chi}}_{\ddot{\aleph}}$. Thus, $\ddot{\aleph}_1 \subseteq \ddot{\aleph}$ or $\ddot{\aleph}_2 \subseteq \ddot{\aleph}$. We conclude that $\ddot{\aleph}$ is a prime OA-(m,n)-I of $\ddot{\Omega}$.
(3) Suppose that $\ddot{\aleph}$ is a semiprime OA-(m,n)-I of $\ddot{\Omega}$. Then $\ddot{\aleph}$ is an OA-(m,n)-I of $\mathfrak{T}$. Thus by Theorem 3.10, $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ is an IVFOA-(m,n)-I of $\ddot{\Omega}$. Let $\tilde{\ddot{\Upsilon}}$ be an IVFOA-(m,n)-I of $\ddot{\Omega}$ such that $\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\Upsilon}} \preceq \tilde{\ddot{\chi}}_{\ddot{\aleph}}$. Assume that $\tilde{\ddot{\Upsilon}} \npreceq \tilde{\ddot{\chi}}_{\ddot{\aleph}}$. Then

there exist $\ddot{\mathfrak{h}} \in \ddot{\Omega}$ such that $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{h}}) \neq \tilde{0}$. While $\tilde{\chi}_{\ddot{\aleph}}(\ddot{\mathfrak{h}}) = \tilde{0}$. Thus, there exists $\ddot{\mathfrak{m}} = \ddot{\mathfrak{p}}\ddot{\mathfrak{q}}$ for some $\ddot{\mathfrak{p}} \in \operatorname{supp}(\tilde{\ddot{\Upsilon}})$ and $\ddot{\mathfrak{q}} \in \operatorname{supp}(\tilde{\ddot{\Upsilon}})$ such that $\ddot{\mathfrak{m}} \in \ddot{\aleph}$. Hence, $\tilde{\chi}_{\ddot{\aleph}}(\ddot{\mathfrak{m}}) = \tilde{0}$ implies that $(\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\Upsilon}})(\ddot{\mathfrak{m}}) = \tilde{0}$. Since $\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\Upsilon}} \leq \tilde{\chi}_{\ddot{\aleph}}$ we have $\ddot{\mathfrak{p}} \in \operatorname{supp}(\tilde{\ddot{\Upsilon}})$ and $\ddot{\mathfrak{q}} \in \operatorname{supp}(\tilde{\ddot{\Upsilon}})$. Thus, $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{p}}) \neq \tilde{0}$ and $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{q}}) \neq \tilde{0}$. It implies that

$$(\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\Upsilon}})(\ddot{\mathfrak{m}}) = \curlyvee_{(\ddot{\mathfrak{p}},\ddot{\mathfrak{q}}) \in F_{\ddot{\mathfrak{m}}}} \{\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{p}}) \curlywedge \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{q}})\} \neq \tilde{0}.$$

It is a contradiction so $\tilde{\ddot{\Upsilon}} \preceq \tilde{\chi}_{\ddot{\aleph}}$. Therefore, $\tilde{\chi}_{\ddot{\aleph}}$ is a semiprime IVFOA-(m,n)-I of $\ddot{\Omega}$.

Conversely, suppose that $\tilde{\chi}_{\ddot{\aleph}}$ is a semiprime IVFOA-(m,n)-I of $\ddot{\Omega}$. Then $\tilde{\chi}_{\ddot{\aleph}}$ is an IVFOA-(m,n)-I of $\ddot{\Omega}$. Thus by Theorem 3.10, $\ddot{\aleph}$ is an OA-(m,n)-I of $\ddot{\Omega}$. Let $\ddot{\aleph}_1$ be an OA-(m,n)-I of $\ddot{\Omega}$ such that $\ddot{\aleph}_1^2 \subseteq \ddot{\aleph}$. Then $\tilde{\chi}_{\ddot{\aleph}_1}$ is an IVFOA-(m,n)-I of $\ddot{\Omega}$. By Lemma 2.7 $\tilde{\chi}_{\ddot{\aleph}_1} \circ \tilde{\chi}_{\ddot{\aleph}_1} = \tilde{\chi}_{\ddot{\aleph}_1^2} \preceq \tilde{\chi}_{\ddot{\aleph}}$. By assumption, $\tilde{\chi}_{\ddot{\aleph}_1} \preceq \tilde{\chi}_{\ddot{\aleph}}$. Thus, $\ddot{\aleph}_1 \subseteq \ddot{\aleph}$. We conclude that $\ddot{\aleph}$ is a semiprime OA-(m,n)-I of $\ddot{\Omega}$.

(4) Suppose that $\ddot{\aleph}$ is a strongly prime OA-(m,n)-I of $\ddot{\Omega}$. Then $\ddot{\aleph}$ is an OA-(m,n)-I of $\ddot{\Omega}$. Thus by Theorem 3.10, $\tilde{\chi}_{\ddot{\aleph}}$ is an IVFOA-(m,n)-I of $\ddot{\Omega}$. Let $\tilde{\ddot{\Upsilon}}$ and $\tilde{\ddot{\xi}}$ be IVFOA-(m,n)-Is of $\ddot{\Omega}$ such that $(\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\xi}}) \curlywedge (\tilde{\ddot{\xi}} \circ \tilde{\ddot{\Upsilon}}) \preceq \tilde{\chi}_{\ddot{\aleph}}$. Assume that $\tilde{\ddot{\Upsilon}} \npreceq \tilde{\chi}_{\ddot{\aleph}}$ and $\tilde{\ddot{\xi}} \npreceq \tilde{\chi}_{\ddot{\aleph}}$. Then there exist $\ddot{\mathfrak{h}}, \ddot{\mathfrak{r}} \in \ddot{\Omega}$ such that $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{h}}) \neq \tilde{0}$ and $\tilde{\ddot{\xi}}(\ddot{\mathfrak{r}}) \neq \tilde{0}$. While $\tilde{\chi}_{\ddot{\aleph}}(\ddot{\mathfrak{h}}) = \tilde{0}$ and $\tilde{\chi}_{\ddot{\aleph}}(\ddot{\mathfrak{r}}) = \tilde{0}$. Thus, $\ddot{\mathfrak{h}} \in \operatorname{supp}(\tilde{\ddot{\Upsilon}})$ and $\ddot{\mathfrak{r}} \in \operatorname{supp}(\tilde{\ddot{\xi}})$, but $\ddot{\mathfrak{h}}, \ddot{\mathfrak{r}} \notin \ddot{\aleph}$. So, $\operatorname{supp}(\tilde{\ddot{\Upsilon}}) \nsubseteq \ddot{\aleph}$ and $\operatorname{supp}(\tilde{\ddot{\xi}}) \nsubseteq \ddot{\aleph}$. Hence, there exists $\ddot{\mathfrak{m}} \in (\operatorname{supp}(\tilde{\ddot{\Upsilon}}) \operatorname{supp}(\tilde{\ddot{\xi}})) \cap (\operatorname{supp}(\tilde{\ddot{\xi}}) \operatorname{supp}(\tilde{\ddot{\Upsilon}}))$ such that $\ddot{\mathfrak{m}} \notin \ddot{\aleph}$. Thus, $\chi_{\ddot{\aleph}}(\mathfrak{m}) = \tilde{0}$ such that $(\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\xi}})(\mathfrak{m}) \curlywedge (\tilde{\ddot{\xi}} \circ \tilde{\ddot{\Upsilon}})(\mathfrak{m}) = 0$. Since $\ddot{\mathfrak{m}} \in \operatorname{supp}(\tilde{\ddot{\Upsilon}}) \operatorname{supp}(\tilde{\ddot{\xi}})$ and $\ddot{\mathfrak{m}} \in \operatorname{supp}(\tilde{\ddot{\xi}}) \operatorname{supp}(\tilde{\ddot{\Upsilon}})$ we have $\ddot{\mathfrak{m}} = \ddot{\mathfrak{d}}\ddot{\mathfrak{k}}$ and $\mathfrak{m} = \mathfrak{g}\ddot{\mathfrak{q}}$ for some $\ddot{\mathfrak{d}}, \ddot{\mathfrak{q}} \in \operatorname{supp}(\tilde{\ddot{\Upsilon}})$, and for some $\ddot{\mathfrak{k}}, \ddot{\mathfrak{g}} \in \operatorname{supp}(\tilde{\ddot{\xi}})$. we have

$$(\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\xi}})(\ddot{\mathfrak{m}}) = \curlyvee_{(\ddot{\mathfrak{d}},\ddot{\mathfrak{r}}) \in F_{\ddot{\mathfrak{m}}}} \{\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{d}}) \curlywedge \tilde{\ddot{\xi}}(\ddot{\mathfrak{k}})\}.$$

Similarly

$$(\tilde{\ddot{\xi}} \circ \tilde{\ddot{\Upsilon}})(\ddot{\mathfrak{m}}) = \curlyvee_{(\ddot{\mathfrak{g}},\ddot{\mathfrak{q}}) \in F_{\ddot{\mathfrak{m}}}} \{\tilde{\ddot{\xi}}(\ddot{\mathfrak{g}}) \curlywedge \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{q}})\}.$$

So, $(\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\xi}})(\ddot{\mathfrak{m}}) \curlywedge (\tilde{\ddot{\xi}} \circ \tilde{\ddot{\Upsilon}})(\ddot{\mathfrak{m}}) \neq \tilde{0}$. It is a contradiction Hence, $\tilde{\ddot{\Upsilon}} \preceq \tilde{\chi}_{\ddot{\aleph}}$ or $\tilde{\ddot{\xi}} \preceq \tilde{\chi}_{\ddot{\aleph}}$. Therefore, $\tilde{\chi}_{\ddot{\aleph}}$ is a strongly prime IVFOA-(m,n)-I of $\ddot{\Omega}$.

Conversely, suppose that $\tilde{\chi}_{\ddot{\aleph}}$ is a strongly prime IVFOA-(m,n)-I of $\ddot{\Omega}$. Then $\tilde{\chi}_{\ddot{\aleph}}$ is an IVFOA (m,n)-I of $\ddot{\Omega}$. Thus, by Theorem 3.10, $\ddot{\aleph}$ is an OA-(m,n)-I of $\mathfrak{T}$. Let $\ddot{\aleph}_1$ and $\ddot{\aleph}_2$ be OA-(m,n)-Is of $\ddot{\Omega}$ such that $\ddot{\aleph}_1\ddot{\aleph}_2 \cap \ddot{\aleph}_2\ddot{\aleph}_1 \subseteq \ddot{\aleph}$. Then $\tilde{\chi}_{\ddot{\aleph}_1}$ and $\tilde{\chi}_{\ddot{\aleph}_2}$ are IVFOA (m,n)-Is of $\ddot{\Omega}$. By Lemma 2.7 $\tilde{\chi}_{\ddot{\aleph}_1\ddot{\aleph}_2} = \tilde{\chi}_{\ddot{\aleph}_1} \circ \tilde{\chi}_{\ddot{\aleph}_2}$ and $\tilde{\chi}_{\ddot{\aleph}_2\ddot{\aleph}_1} = \tilde{\chi}_{\ddot{\aleph}_2} \circ \tilde{\chi}_{\ddot{\aleph}_1}$. Thus, $(\tilde{\chi}_{\ddot{\aleph}_1} \circ \tilde{\chi}_{\ddot{\aleph}_2}) \curlywedge (\tilde{\chi}_{\ddot{\aleph}_1} \circ \tilde{\chi}_{\ddot{\aleph}_2}) = \tilde{\chi}_{\ddot{\aleph}_1\ddot{\aleph}_2} \curlywedge \tilde{\chi}_{\ddot{\aleph}_2\ddot{\aleph}_1} = \tilde{\chi}_{\ddot{\aleph}_1\ddot{\aleph}_2 \cap \ddot{\aleph}_2\ddot{\aleph}_1} \preceq \tilde{\chi}_{\ddot{\aleph}}$. By assumption, $\tilde{\chi}_{\ddot{\aleph}_1} \preceq \tilde{\chi}_{\ddot{\aleph}}$ and $\tilde{\chi}_{\ddot{\aleph}_2} \preceq \tilde{\chi}_{\ddot{\aleph}}$. Thus, $\ddot{\aleph}_1 \subseteq \ddot{\aleph}$ or $\ddot{\aleph}_2 \subseteq \ddot{\aleph}$. We conclude that $\ddot{\aleph}$ is a strongly prime OA-(m,n)-I of $\ddot{\Omega}$. ■

Corollary 3.15. *Let $\ddot{\Omega}$ be an ordered semigroup. Then $\ddot{\Omega}$ has no proper OA-(m,n)-I if and only if $\operatorname{supp}(\tilde{\ddot{\Upsilon}}) = \ddot{\Omega}$ for every IVFOA-(m,n)-I $\tilde{\ddot{\Upsilon}}$ of $\ddot{\Omega}$.*

IV. Interval Valued Fuzzy Almost (m,n)-Quasi-ideals

In this section, we define IVFOA-(m,n)-quasi-deal in an ordered semigroup. We prove $\tilde{\ddot{\Upsilon}} \curlyvee \tilde{\ddot{\xi}}$ is an IVFOA- (m,n)-qausi-ideal where $\tilde{\ddot{\Upsilon}}$ and $\tilde{\ddot{\xi}}$ are IVFOA- (m,n)-Quasi-ideals. And we create a bridge between almost (m,n)-ideals and IVFOA- (m,n)-ideals.

Definition 4.1. *[22] A non-empty subset $\ddot{\aleph}$ on an ordered semigroup $\ddot{\Omega}$ is called an* almost (m,n)-quasi-ideal *(OA-(m,n)-QI) of $\ddot{\Omega}$ if $(\ddot{\aleph}^m\ddot{\mathfrak{t}}] \cap (\ddot{\mathfrak{t}}\ddot{\aleph}^n] \cap \ddot{\aleph} \neq \emptyset$ for all $\ddot{\mathfrak{t}} \in \ddot{\Omega}$ where $m,n \in \{1,2,...,n\}$.*

Example 4.2. (1) *An almost $(1,0)$-ideal of an ordered semigroup $\ddot{\Omega}$ is a right almost ideal of $\ddot{\Omega}$.*

(2) *An almost $(0,1)$-ideal of an ordered semigroup $\ddot{\Omega}$ is a left almost ideal of $\mathfrak{T}$.*

(3) *Consider the ordered semigroup $\mathbb{Z}_6$ under the usual addition and the partial ordered $\leq := \{(\mathfrak{a},\mathfrak{a}) \mid \mathfrak{a} \in \mathbb{Z}_6\}$. We have $\mathfrak{A} = \{\overline{1},\overline{4},\overline{5}\}$ is an OA-$(1,0)$-QI of $\mathbb{Z}_6$.*

Definition 4.3. *An IVF set $\tilde{\ddot{\Upsilon}}$ on an ordered semigroup $\ddot{\Omega}$ is called an* interval valued fuzzy ordered almost (m,n)-quasi-ideal *(IVFOA-(m,n)-QI) of $\ddot{\Omega}$ if $(\tilde{\ddot{\Upsilon}}^m \circ \tilde{\chi}_{\ddot{\Omega}}] \curlywedge (\tilde{\chi}_{\ddot{\Omega}} \circ \tilde{\ddot{\Upsilon}}^n] \curlywedge \tilde{\ddot{\Upsilon}} \neq \tilde{0}$. where $m,n \in \{1,2,...,n\}$.*

Theorem 4.4. *If $\tilde{\ddot{\Upsilon}}$ is an IVFOA-(m,n)-QI of an ordered semigroup $\ddot{\Omega}$ and $\tilde{\ddot{\xi}}$ is an IVF set of $\ddot{\Omega}$ such that $\tilde{\ddot{\Upsilon}} \preceq \tilde{\ddot{\xi}}$, then $\tilde{\ddot{\xi}}$ is an IVFOA-(m,n)-QI of $\ddot{\Omega}$.*

Proof: Suppose that $\tilde{\ddot{\Upsilon}}$ is an IVFOA-(m,n)-QI of $\ddot{\Omega}$ and $\tilde{\ddot{\xi}}$ is an IVF set of $\ddot{\Omega}$ such that $\tilde{\ddot{\Upsilon}} \preceq \tilde{\ddot{\xi}}$. Then $(\tilde{\ddot{\Upsilon}}^m \circ \tilde{\chi}_{\ddot{\Omega}}] \curlywedge (\tilde{\chi}_{\ddot{\Omega}} \circ \tilde{\ddot{\Upsilon}}^n] \curlywedge \tilde{\ddot{\Upsilon}} \neq \tilde{0}$. Thus,

$$(\tilde{\ddot{\Upsilon}}^m \circ \tilde{\chi}_{\ddot{\Omega}}] \curlywedge (\tilde{\chi}_{\ddot{\Omega}} \circ \tilde{\ddot{\Upsilon}}^n] \curlywedge \tilde{\ddot{\Upsilon}} \preceq (\tilde{\ddot{\xi}}^m \circ \tilde{\chi}_{\ddot{\Omega}}] \curlywedge (\tilde{\chi}_{\ddot{\Omega}} \circ \tilde{\ddot{\xi}}^n] \curlywedge \tilde{\ddot{\xi}} \neq \tilde{0}.$$

Hence, $(\tilde{\ddot{\xi}}^m \circ \tilde{\chi}_{\ddot{\Omega}}] \curlywedge (\tilde{\chi}_{\ddot{\Omega}} \circ \tilde{\ddot{\xi}}^n] \curlywedge \tilde{\ddot{\xi}} \neq \tilde{0}$. Therefore, $\tilde{\ddot{\xi}}$ is an IVFOA-(m,n)-QI of $\ddot{\Omega}$. ■

The following results are obvious of Theorem 4.4.

Theorem 4.5. *Let $\tilde{\ddot{\Upsilon}}$ and $\tilde{\ddot{\xi}}$ be IVFOA-(m,n)-QIs of an ordered semigroup $\ddot{\Omega}$. Then $\tilde{\ddot{\Upsilon}} \curlyvee \tilde{\ddot{\xi}}$ is also an IVFOA-(m,n)-QI of $\ddot{\Omega}$.*

Proof: Since $\tilde{\ddot{\Upsilon}} \preceq \tilde{\ddot{\Upsilon}} \curlyvee \tilde{\ddot{\xi}}$, by Theorem 4.4, $\tilde{\ddot{\Upsilon}} \curlyvee \tilde{\ddot{\xi}}$ is an IVFOA-(m,n)-QI of $\ddot{\Omega}$. ■

Theorem 4.6. *If $\tilde{\ddot{\Upsilon}}$ is an IVFOA-(m,n)-QI of an ordered semigroup $\ddot{\Omega}$ and $\tilde{\ddot{\xi}}$ is an IVF set, then $\tilde{\ddot{\Upsilon}} \curlyvee \tilde{\ddot{\xi}}$ is also an IVFOA-(m,n)-QI of $\ddot{\Omega}$.*

Proof: By Theorem 4.4, and $\tilde{\ddot{\Upsilon}} \preceq \tilde{\ddot{\Upsilon}} \curlyvee \tilde{\ddot{\xi}}$. Thus, $\tilde{\ddot{\Upsilon}} \curlyvee \tilde{\ddot{\xi}}$ is also an IVFOA-(m,n)-QI of $\ddot{\Omega}$. ■

Corollary 4.7. *Let $\tilde{\ddot{\Upsilon}}_i$ be an IVFOA-(m,n)-QI in an ordered semigroup $\ddot{\Omega}$. Then $\curlyvee_{i \in \mathcal{F}} \tilde{\ddot{\Upsilon}}_i$ is an IVFOA-(m,n)-QI of $\ddot{\Omega}$.*

Example 4.8. *Consider the ordered semigroup $\mathbb{Z}_6$ under the usual addition and he partial ordered $\leq := \{(\mathfrak{a},\mathfrak{a}) \mid \mathfrak{a} \in \mathbb{Z}_6\}$.*

$\tilde{\ddot{\vartheta}}: \mathbb{Z}_6 \to \mathrm{CS}[0,1]$ *is defined by* $\tilde{\ddot{\vartheta}}(\overline{0}) = \tilde{0}$, $\tilde{\ddot{\vartheta}}(\overline{1}) = [0.2, 0.3]$, $\tilde{\ddot{\vartheta}}(\overline{2}) = \tilde{0}$, $\tilde{\ddot{\vartheta}}(\overline{3}) = \tilde{0}$, $\tilde{\ddot{\vartheta}}(\overline{4}) = [0.5, 0.6]$, $\tilde{\ddot{\vartheta}}(\overline{5}) = [0.4, 0.6]$ *and* $\tilde{\ddot{\nu}} : \mathbb{Z}_6 \to \mathrm{CS}[0,1]$ *is defined by* $\tilde{\ddot{\nu}}(\overline{0}) = \tilde{0}$, $\tilde{\ddot{\nu}}(\overline{1}) = [0.8, 0.9]$, $\tilde{\ddot{\nu}}(\overline{2}) = [0.4, 0.5]$, $\tilde{\ddot{\nu}}(\overline{3}) = [0.3, 0.4]$, $\tilde{\ddot{\nu}}(\overline{4}) = \tilde{0}$, $\tilde{\ddot{\nu}}(\overline{5}) = [0.3, 0.4]$. *We have* $\tilde{\ddot{\vartheta}}$ *and* $\tilde{\ddot{\nu}}$ *are IVFOA-*$(1,1)$*-QIs of* $\mathbb{Z}_6$ *but* $(\tilde{\ddot{\vartheta}} \curlywedge \tilde{\ddot{\nu}})$ *is not an IVFOA-*$(1,1)$*-QI of* $\mathbb{Z}_6$.

Theorem 4.9. *Let* $\ddot{\aleph}$ *be a nonempty subset of an ordered semigroup* $\ddot{\Omega}$*. Then* $\ddot{\aleph}$ *is an OA-*(m,n)*-QI of* $\ddot{\Omega}$ *if and only if* $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ *is an IVFOA-*(m,n)*-QI of* $\ddot{\Omega}$.

Proof: Suppose that $\ddot{\aleph}$ is an OA-(m,n)-QI of $\ddot{\Omega}$. Then $(\ddot{\aleph}^m\ddot{\mathfrak{t}}] \cap (\ddot{\mathfrak{t}}\ddot{\aleph}^n] \cap \ddot{\aleph} \neq \emptyset$ for all $\ddot{\mathfrak{t}} \in \ddot{\Omega}$ and $m,n \in \{1,2,...,n\}$. Thus there exists $\ddot{\mathfrak{c}} \in \ddot{\Omega}$ such that $\ddot{\mathfrak{c}} \in (\ddot{\aleph}^m\ddot{\mathfrak{t}}] \cap (\ddot{\mathfrak{t}}\ddot{\aleph}^n]$ and $\ddot{\mathfrak{c}} \in \ddot{\aleph}$. So $((\tilde{\ddot{\chi}}^m_{\ddot{\aleph}} \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}}] \circ \curlywedge(\tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\chi}}^n_{\ddot{\aleph}}])(\ddot{\mathfrak{c}}) \neq \tilde{0}$ and $\tilde{\ddot{\chi}}_{\ddot{\aleph}}(\ddot{\mathfrak{c}}) \neq \tilde{0}$ where $m,n \in \{1,2,...,n\}$. Thus, $((\tilde{\ddot{\chi}}^m_{\ddot{\aleph}} \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}}] \circ \curlywedge(\tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\chi}}^n_{\ddot{\aleph}}] \curlywedge \tilde{\ddot{\chi}}_{\ddot{\aleph}})(\ddot{\mathfrak{c}}) = (((\tilde{\ddot{\chi}}_{\ddot{\aleph}^m} \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}}] \circ \curlywedge(\tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\chi}}_{\ddot{\aleph}^n}] \curlywedge \tilde{\ddot{\chi}}_{\ddot{\aleph}})(\ddot{\mathfrak{c}}) \neq \tilde{0}$ and $m,n \in \{1,2,...,n\}$. So, $(\tilde{\ddot{\chi}}^m_{\ddot{\aleph}} \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}}] \circ \curlywedge(\tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\chi}}^n_{\ddot{\aleph}}] \curlywedge \tilde{\ddot{\chi}}_{\ddot{\aleph}}) \neq \tilde{0}$. Hence, $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ is an IVFOA-(m,n)-QI of $\ddot{\Omega}$.

Conversely, suppose that $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ is an IVFOA-(m,n)-QI of $\ddot{\Omega}$ and let $m,n \in \{1,2,...,n\}$. Then $(\tilde{\ddot{\chi}}^m_{\ddot{\aleph}} \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}}] \circ \curlywedge(\tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\chi}}^n_{\ddot{\aleph}}] \curlywedge \tilde{\ddot{\chi}}_{\ddot{\aleph}}) \neq \tilde{0}$. Thus, there exists $\ddot{\mathfrak{c}} \in \ddot{\aleph}$ such that $((\tilde{\ddot{\chi}}^m_{\ddot{\aleph}} \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}}] \circ \curlywedge(\tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\chi}}^n_{\ddot{\aleph}}] \curlywedge \tilde{\ddot{\chi}}_{\ddot{\aleph}})(\ddot{\mathfrak{c}}) \neq \tilde{0}$. It implies that $((\tilde{\ddot{\chi}}^m_{\ddot{\aleph}} \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}}] \circ \curlywedge(\tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\chi}}^n_{\ddot{\aleph}}])(\ddot{\mathfrak{c}}) \neq \tilde{0}$ and $\tilde{\ddot{\chi}}_{\ddot{\aleph}}(\ddot{\mathfrak{c}}) \neq \tilde{0}$. Hence, $\ddot{\mathfrak{c}} \in (\ddot{\aleph}^m\ddot{\mathfrak{t}}] \cap (\ddot{\mathfrak{t}}\ddot{\aleph}^n]$ and $\ddot{\mathfrak{c}} \in \ddot{\aleph}$. So $(\ddot{\aleph}^m\ddot{\mathfrak{t}}] \cap (\ddot{\mathfrak{t}}\ddot{\aleph}^n] \cap \ddot{\aleph} \neq \emptyset$ for all $\ddot{\mathfrak{t}} \in \ddot{\Omega}$. We conclude that $\ddot{\aleph}$ is an OA-(m,n)-QI of $\ddot{\Omega}$. ■

Theorem 4.10. *Let* $\tilde{\ddot{\Upsilon}}$ *be an IVF set of an ordered semigroup* $\ddot{\Omega}$*. Then* $\tilde{\ddot{\Upsilon}}$ *is an IVFOA-*(m,n)*-QI of* $\ddot{\Omega}$ *if and only if* $\mathrm{supp}(\tilde{\ddot{\Upsilon}})$ *is an OA-*(m,n)*-QI of* $\ddot{\Omega}$.

Proof: Assume that $\tilde{\ddot{\Upsilon}}$ is an IVFOA-(m,n)-QI of $\ddot{\Omega}$ and let $m,n \in \{1,2,...,n\}$. Then $(\tilde{\ddot{\Upsilon}}^m \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}}] \curlywedge (\tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\Upsilon}}^n] \curlywedge \tilde{\ddot{\Upsilon}} \neq \tilde{0}$. Thus, there exists $\ddot{\mathfrak{z}} \in \ddot{\Omega}$ such that $((\tilde{\ddot{\Upsilon}}^m \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}}] \curlywedge (\tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\Upsilon}}^n] \curlywedge \tilde{\ddot{\Upsilon}} \neq \tilde{0}(\ddot{\mathfrak{z}}) \neq \tilde{0}$. So $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{z}}) \neq \tilde{0}$ and $\ddot{\mathfrak{z}} = \ddot{\mathfrak{a}}_1\ddot{\mathfrak{a}}_2 \cdots \ddot{\mathfrak{a}}_m\ddot{\mathfrak{r}}\ddot{\mathfrak{b}}_1\ddot{\mathfrak{b}}_2 \cdots \ddot{\mathfrak{b}}_n$ for some $\ddot{\mathfrak{a}}_1\ddot{\mathfrak{a}}_2 \cdots \ddot{\mathfrak{a}}_m, \ddot{\mathfrak{b}}_1\ddot{\mathfrak{b}}_2 \cdots \ddot{\mathfrak{b}}_n \in \ddot{\Omega}$ such that $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{a}}_1) \neq \tilde{0}$, $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{a}}_2) \neq \tilde{0}, \cdots, \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{a}}_m) \neq \tilde{0}$ and $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{b}}_1) \neq \tilde{0}$, $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{b}}_2) \neq \tilde{0}, \cdots \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{b}}_n) \neq \tilde{0}$. Thus, $\ddot{\mathfrak{a}}_1\ddot{\mathfrak{a}}_2, \ldots, \ddot{\mathfrak{a}}_m, \ddot{\mathfrak{b}}_1\ddot{\mathfrak{b}}_2, \ldots, \ddot{\mathfrak{b}}_n \in \mathrm{supp}(\tilde{\ddot{\Upsilon}})$. It implies that $((\tilde{\ddot{\chi}}_{\mathrm{supp}(\tilde{\ddot{\Upsilon}})^m} \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}}] \curlywedge (\tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\chi}}_{\mathrm{supp}(\tilde{\ddot{\Upsilon}})^n}] \curlywedge \tilde{\ddot{\chi}}_{\mathrm{supp}(\tilde{\ddot{\Upsilon}})})(\ddot{\mathfrak{z}}) \neq \tilde{0}$. Hence, $((\tilde{\ddot{\chi}}_{\mathrm{supp}(\tilde{\ddot{\Upsilon}})^m} \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}}] \curlywedge (\tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\chi}}_{\mathrm{supp}(\tilde{\ddot{\Upsilon}})^n}] \curlywedge \tilde{\ddot{\chi}}_{\mathrm{supp}(\tilde{\ddot{\Upsilon}})}) \neq \tilde{0}$. Therefore, $\tilde{\ddot{\chi}}_{\mathrm{supp}(\tilde{\ddot{\Upsilon}})}$ is an IVFOA-(m,n)-QI of $\ddot{\Omega}$. By Theorem 4.9, $\mathrm{supp}(\tilde{\ddot{\Upsilon}})$ is an OA-(m,n)-QI of $\ddot{\Omega}$.

Conversely, suppose that $\mathrm{supp}(\tilde{\ddot{\Upsilon}})$ is an OA-(m,n)-QI of $\ddot{\Omega}$. By Theorem 4.9, $\tilde{\ddot{\chi}}_{\mathrm{supp}(\tilde{\ddot{\Upsilon}})}$ is an IVFOA-(m,n)-QI of $\ddot{\Omega}$. Then for any $m,n \in \{1,2,...,n\}$, we have $((\tilde{\ddot{\chi}}_{\mathrm{supp}(\tilde{\ddot{\Upsilon}})^m} \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}}] \curlywedge (\tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\chi}}_{\mathrm{supp}(\tilde{\ddot{\Upsilon}})^n}] \curlywedge \tilde{\ddot{\chi}}_{\mathrm{supp}(\tilde{\ddot{\Upsilon}})}) \neq \tilde{0}$. Thus, there exists $\ddot{\mathfrak{z}} \in \ddot{\aleph}$ such that $((\tilde{\ddot{\chi}}_{\mathrm{supp}(\tilde{\ddot{\Upsilon}})^m} \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}}] \curlywedge (\tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\chi}}_{\mathrm{supp}(\tilde{\ddot{\Upsilon}})^n}] \curlywedge \tilde{\ddot{\chi}}_{\mathrm{supp}(\tilde{\ddot{\Upsilon}})})(\ddot{\mathfrak{z}}) \neq \tilde{0}$. It implies that $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{z}}) \neq \tilde{0}$ and $\ddot{\mathfrak{z}} = \ddot{\mathfrak{a}}_1\ddot{\mathfrak{a}}_2 \cdots \ddot{\mathfrak{a}}_m\ddot{\mathfrak{r}}\ddot{\mathfrak{b}}_1\ddot{\mathfrak{b}}_2 \cdots \ddot{\mathfrak{b}}_n$ for some $\ddot{\mathfrak{a}}_1\ddot{\mathfrak{a}}_2, \cdots, \ddot{\mathfrak{a}}_m, \ddot{\mathfrak{b}}_1\ddot{\mathfrak{b}}_2, \cdots, \ddot{\mathfrak{b}}_n \in \ddot{\Omega}$ such that $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{a}}_1) \neq \tilde{0}$, $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{a}}_2) \neq \tilde{0}, \cdots, \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{a}}_m) \neq \tilde{0}$ and $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{b}}_1) \neq \tilde{0}$, $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{b}}_2) \neq \tilde{0}, \cdots \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{b}}_n) \neq \tilde{0}$. Thus, $\ddot{\mathfrak{a}}_1\ddot{\mathfrak{a}}_2, \ldots, \ddot{\mathfrak{a}}_m, \ddot{\mathfrak{b}}_1\ddot{\mathfrak{b}}_2, \ldots, \ddot{\mathfrak{b}}_n \in \mathrm{supp}(\tilde{\ddot{\Upsilon}})$. So, there exists $\ddot{\mathfrak{z}} \in \ddot{\Omega}$ such that $((\tilde{\ddot{\Upsilon}}^m \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}}] \curlywedge (\tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\Upsilon}}^n] \curlywedge \tilde{\ddot{\Upsilon}} \neq \tilde{0}(\ddot{\mathfrak{z}}) \neq \tilde{0}$. Hence, $(\tilde{\ddot{\Upsilon}}^m \circ \tilde{\ddot{\chi}}_{\ddot{\Omega}}] \curlywedge (\tilde{\ddot{\chi}}_{\ddot{\Omega}} \circ \tilde{\ddot{\Upsilon}}^n] \curlywedge \tilde{\ddot{\Upsilon}} \neq \tilde{0}$. Therefore, $\tilde{\ddot{\Upsilon}}$ is an IVFOA-(m,n)-QI of $\ddot{\Omega}$. ■

Next, we investigate the connection between minimal, prime, semiprime, strongly prime OA-(m,n)-QIs and minimal, prime, semiprime, strongly prime IVFOA-(m,n)-QIs of ordered semigroups.

Definition 4.11. *An OA-*(m,n)*-IQ* $\ddot{\aleph}$ *of an ordered semigroup* $\ddot{\Omega}$ *is called*

(1) minimal *if for any OA-*(m,n)*-IQ* $\ddot{\aleph}_1$ *of* $\ddot{\Omega}$ *if whenever* $\ddot{\aleph}_1 \subseteq \ddot{\aleph}$*, then* $\ddot{\aleph}_1 = \ddot{\aleph}$*,*
(2) prime *if for any two OA-*(m,n)*-IQs* $\ddot{\aleph}_1$ *and* $\ddot{\aleph}_2$ *of* $\ddot{\Omega}$ *such that* $\ddot{\aleph}_1\ddot{\aleph}_2 \subseteq \ddot{\aleph}$ *implies that* $\ddot{\aleph}_1 \subseteq \ddot{\aleph}$ *or* $\ddot{\aleph}_2 \subseteq \ddot{\aleph}$*.*
(3) semiprime *if for any OA-*(m,n)*-IQ* $\ddot{\aleph}_1$ *of* $\ddot{\Omega}$ *such that* $\ddot{\aleph}_1^2 \subseteq \ddot{\aleph}$ *implies that* $\ddot{\aleph}_1 \subseteq \ddot{\aleph}$*.*
(4) strongly prime *if for any OA-*(m,n)*-IQs* $\ddot{\aleph}_1$ *and* $\ddot{\aleph}_2$ *of* $\ddot{\Omega}$ *such that* $\ddot{\aleph}_1\ddot{\aleph}_2 \cap \ddot{\aleph}_2\ddot{\aleph}_1 \subseteq \ddot{\aleph}$ *implies that* $\ddot{\aleph}_1 \subseteq \ddot{\aleph}$ *or* $\ddot{\aleph}_2 \subseteq \ddot{\aleph}$*.*

Definition 4.12. *an IVFOA-*(m,n)*-IQ* $\tilde{\ddot{\Upsilon}}$ *of an ordered semigroup* $\ddot{\Omega}$ *is called*

(1) minimal *if for any IVFOA-*(m,n)*-IQ* $\tilde{\ddot{\xi}}$ *of* Ω *if whenever* $\tilde{\ddot{\xi}} \preceq \tilde{\ddot{\Upsilon}}$*, then* $\mathrm{supp}(\tilde{\ddot{\xi}}) = \mathrm{supp}(\tilde{\ddot{\Upsilon}})$*,*
(2) prime *if for any two IVFOA-*(m,n)*-IQs* $\tilde{\ddot{\xi}}$ *and* $\tilde{\ddot{\rho}}$ *of* $\ddot{\Omega}$ *such that* $\tilde{\ddot{\xi}} \circ \tilde{\ddot{\rho}} \preceq \tilde{\ddot{\Upsilon}}$ *implies that* $\tilde{\ddot{\xi}} \preceq \tilde{\ddot{\Upsilon}}$ *or* $\tilde{\ddot{\rho}} \preceq \tilde{\ddot{\Upsilon}}$*.*
(3) semiprime *if for any IVFOA-*(m,n)*-IQ* $\tilde{\ddot{\xi}}$ *of* $\ddot{\Omega}$ *such that* $\tilde{\ddot{\xi}} \circ \tilde{\ddot{\xi}} \preceq \tilde{\ddot{\xi}}$ *implies that* $\tilde{\ddot{\xi}} \preceq \tilde{\ddot{\Upsilon}}$*.*
(4) strongly prime *if for any two IVFOA-*(m,n)*-IQs* $\tilde{\ddot{\xi}}$ *and* $\tilde{\ddot{\rho}}$ *of* $\ddot{\Omega}$ *such that* $(\tilde{\ddot{\xi}} \circ \tilde{\ddot{\rho}}) \curlywedge (\tilde{\ddot{\rho}} \circ \tilde{\ddot{\xi}}) \preceq \tilde{\ddot{\Upsilon}}$ *implies that* $\tilde{\ddot{\xi}} \preceq \tilde{\ddot{\Upsilon}}$ *or* $\tilde{\ddot{\rho}} \preceq \tilde{\ddot{\Upsilon}}$*.*

It is clearly, every IVF strongly prime OA-(m,n)-IQ of an ordered semigroup is a prime IVFOA-(m,n)-IQ, and every prime IVFOA-(m,n)-IQ of an ordered semigroup is a semiprime IVFOA-(m,n)-IQ.

Theorem 4.13. *Let* $\ddot{\aleph}$ *be a nonempty subset of an ordered semigroup* $\ddot{\Omega}$*. Then*

(1) $\ddot{\aleph}$ *is a minimal OA-*(m,n)*-IQ of* $\ddot{\Omega}$ *if and only if* $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ *is a minimal IVFOA-*(m,n)*-IQ of* $\ddot{\Omega}$*.*
(2) $\ddot{\aleph}$ *is a prime OA-*(m,n)*-IQ of* $\ddot{\Omega}$ *if and only if* $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ *is an prime IVFOA-*(m,n)*-IQ of* $\ddot{\Omega}$*.*
(3) $\ddot{\aleph}$ *is a semiprime OA-*(m,n)*-IQ of* $\ddot{\Omega}$ *if and only if* $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ *is an semiprime IVFOA-*(m,n)*-IQ of* $\ddot{\Omega}$*.*
(4) $\ddot{\aleph}$ *is a strongly prime OA-*(m,n)*-IQ of* $\mathfrak{T}$ *if and only if* $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ *is an strongly prime IVFOA-*(m,n)*-IQ of* $\ddot{\Omega}$*.*

Proof:

(1) Assume that $\ddot{\aleph}$ is a minimal OA-(m,n)-IQ of $\ddot{\Omega}$. Then $\ddot{\aleph}$ is an OA-(m,n)-IQ of $\ddot{\Omega}$. Thus by Theorem 4.9, $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ is an IVFOA-(m,n)-IQ of Ω. Let $\tilde{\ddot{\xi}}$ be an IVFOA-(m,n)-IQ of Ω such that $\tilde{\ddot{\xi}} \preceq \tilde{\ddot{\chi}}_{\ddot{\aleph}}$. Then by Theorem 4.10, $\mathrm{supp}(\tilde{\ddot{\xi}})$ is an OA-(m,n)-IQ of Ω such that $\mathrm{supp}(\tilde{\ddot{\xi}}) \subseteq \mathrm{supp}(\tilde{\ddot{\chi}}_{\ddot{\aleph}}) = \ddot{\aleph}$. By assumption, $\mathrm{supp}(\tilde{\ddot{\xi}}) = \mathrm{supp}(\tilde{\ddot{\chi}}_{\ddot{\aleph}})$. Therefore, $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ is a minimal IVFOA-(m,n)-IQ of $\ddot{\Omega}$.
Conversely, suppose that $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ is a minimal IVFOA-(m,n)-IQ of $\ddot{\Omega}$. Then $\tilde{\ddot{\chi}}_{\ddot{\aleph}}$ is an IVFOA-(m,n)-IQ of $\ddot{\Omega}$. Thus by Theorem 4.9, $\ddot{\aleph}$ is an OA-(m,n)-IQ of Ω. Let $\ddot{\aleph}_1$ be an OA-(m,n)-IQ of Ω such that $\ddot{\aleph}_1 \subseteq \ddot{\aleph}$. Then $\tilde{\ddot{\chi}}_{\ddot{\aleph}_1}$ is an IVFOA-(m,n)-IQ of Ω such that $\tilde{\ddot{\chi}}_{\ddot{\aleph}_1} \preceq \tilde{\ddot{\chi}}_{\ddot{\aleph}}$. Thus, $\mathrm{supp}(\tilde{\ddot{\chi}}_{\ddot{\aleph}_1}) \subseteq \mathrm{supp}(\tilde{\ddot{\chi}}_{\ddot{\aleph}})$. By assumption, $\ddot{\aleph}_1 =$

$\mathrm{supp}(\tilde{\chi}_{\ddot{\aleph}_1}) = \mathrm{supp}(\tilde{\chi}_{\ddot{\aleph}}) = \ddot{\aleph}$. Therefore, $\ddot{\aleph}$ is a minimal OA-(m,n)-IQ of $\ddot{\Omega}$.

(2) Suppose that $\ddot{\aleph}$ is a prime OA-(m,n)-IQ of $\ddot{\Omega}$. Then $\ddot{\aleph}$ is an OA-(m,n)-IQ of $\ddot{\Omega}$. Thus by Theorem 4.9, $\tilde{\chi}_{\ddot{\aleph}}$ is an IVFOA-(m,n)-IQ of Ω. Let $\tilde{\ddot{\Upsilon}}$ and $\tilde{\ddot{\xi}}$ be IVFOA-(m,n)-IQs such that $\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\xi}} \preceq \tilde{\chi}_{\ddot{\aleph}}$. Assume that $\tilde{\ddot{\Upsilon}} \not\preceq \tilde{\chi}_{\ddot{\aleph}}$ and $\tilde{\ddot{\xi}} \not\preceq \tilde{\chi}_{\ddot{\aleph}}$. Then there exist $\ddot{\mathfrak{h}}, \ddot{\mathfrak{r}} \in \ddot{\Omega}$ such that $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{h}}) \neq \tilde{0}$ and $\tilde{\ddot{\xi}}(\mathfrak{r}) \neq \tilde{0}$. While $\tilde{\chi}_{\ddot{\aleph}}(\ddot{\mathfrak{h}}) = \tilde{0}$ and $\tilde{\chi}_{\ddot{\aleph}}(\ddot{\mathfrak{r}}) = 0$. Thus, $\ddot{\mathfrak{h}} \in \mathrm{supp}(\tilde{\ddot{\Upsilon}})$ and $\ddot{\mathfrak{r}} \in \mathrm{supp}(\tilde{\ddot{\xi}})$, but $\ddot{\mathfrak{h}}, \ddot{\mathfrak{r}} \notin \ddot{\aleph}$. So $\mathrm{supp}(\tilde{\ddot{\Upsilon}}) \not\subseteq \ddot{\aleph}$ and $\mathrm{supp}(\tilde{\ddot{\xi}}) \not\subseteq \ddot{\aleph}$. Since $\mathrm{supp}(\tilde{\ddot{\Upsilon}})$ and $\mathrm{supp}(\tilde{\ddot{\xi}})$ are OA-(m,n)-IQs of $\ddot{\Omega}$ we have $\mathrm{supp}(\tilde{\ddot{\Upsilon}})\,\mathrm{supp}(\tilde{\ddot{\xi}}) \not\subseteq \ddot{\aleph}$. Thus, there exists $\ddot{\mathfrak{m}} = \ddot{\mathfrak{p}}\ddot{\mathfrak{q}}$ for some $\ddot{\mathfrak{p}} \in \mathrm{supp}(\tilde{\ddot{\Upsilon}})$ and $\ddot{\mathfrak{q}} \in \mathrm{supp}(\tilde{\ddot{\xi}})$ such that $\ddot{\mathfrak{m}} \notin \ddot{\aleph}$. Hence, $\tilde{\chi}_{\ddot{\aleph}}(\ddot{\mathfrak{m}}) = \tilde{0}$ implies that $(\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\xi}})(\ddot{\mathfrak{m}}) = \tilde{0}$. Since $\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\xi}} \preceq \tilde{\chi}_{\ddot{\aleph}}$ we have $\ddot{\mathfrak{p}} \in \mathrm{supp}(\tilde{\ddot{\Upsilon}})$ and $\ddot{\mathfrak{q}} \in \mathrm{supp}(\tilde{\ddot{\xi}})$. Thus, $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{p}}) \neq 0$ and $\tilde{\ddot{\xi}}(\ddot{\mathfrak{q}}) \neq \tilde{0}$. It implies that

$$(\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\xi}})(\ddot{\mathfrak{m}}) = \bigvee_{(\ddot{\mathfrak{p}},\ddot{\mathfrak{q}}) \in F_{\ddot{\mathfrak{m}}}} \{\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{p}}) \wedge \tilde{\ddot{\xi}}(\ddot{\mathfrak{q}})\} \neq \tilde{0}.$$

It is a contradiction so $\tilde{\ddot{\Upsilon}} \preceq \tilde{\chi}_{\ddot{\aleph}}$ or $\tilde{\ddot{\xi}} \preceq \tilde{\chi}_{\ddot{\aleph}}$. Therefore, $\tilde{\chi}_{\ddot{\aleph}}$ is a prime IVFOA-(m,n)-IQ of Ω.

Conversely, suppose that $\tilde{\chi}_{\ddot{\aleph}}$ is a prime IVFOA-(m,n)-IQ of $\ddot{\Omega}$. Then $\tilde{\chi}_{\ddot{\aleph}}$ is an IVFOA-(m,n)-IQ of $\ddot{\Omega}$. Thus by Theorem 4.9, $\ddot{\aleph}$ is an OA-(m,n)-IQ of $\ddot{\Omega}$. Let $\ddot{\aleph}_1$ and $\ddot{\aleph}_2$ be OA-(m,n)-IQs of $\ddot{\Omega}$ such that $\ddot{\aleph}_1\ddot{\aleph}_2 \subseteq \ddot{\aleph}$. Then $\tilde{\chi}_{\ddot{\aleph}_1}$ and $\tilde{\chi}_{\ddot{\aleph}_2}$ are IVFOA-(m,n)-IQs of $\ddot{\Omega}$. By Lemma 2.7 $\tilde{\chi}_{\ddot{\aleph}_1} \circ \tilde{\chi}_{\ddot{\aleph}_2} = \tilde{\chi}_{\ddot{\aleph}_1\ddot{\aleph}_2} \preceq \tilde{\chi}_{\ddot{\aleph}}$. By assumption, $\tilde{\chi}_{\ddot{\aleph}_1} \preceq \tilde{\chi}_{\ddot{\aleph}}$ or $\tilde{\chi}_{\ddot{\aleph}_2} \preceq \tilde{\chi}_{\ddot{\aleph}}$. Thus, $\ddot{\aleph}_1 \subseteq \ddot{\aleph}$ or $\ddot{\aleph}_2 \subseteq \ddot{\aleph}$. We conclude that $\ddot{\aleph}$ is a prime OA-(m,n)-IQ of $\ddot{\Omega}$.

(3) Suppose that $\ddot{\aleph}$ is a semiprime OA-(m,n)-IQ of $\ddot{\Omega}$. Then $\ddot{\aleph}$ is an OA-(m,n)-IQ of $\mathfrak{T}$. Thus by Theorem 4.9, $\tilde{\chi}_{\ddot{\aleph}}$ is an IVFOA-(m,n)-IQ of $\ddot{\Omega}$. Let $\tilde{\ddot{\Upsilon}}$ be an IVFOA-(m,n)-IQ of $\ddot{\Omega}$ such that $\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\Upsilon}} \preceq \tilde{\chi}_{\ddot{\aleph}}$. Assume that $\tilde{\ddot{\Upsilon}} \not\preceq \tilde{\chi}_{\ddot{\aleph}}$. Then there exist $\ddot{\mathfrak{h}} \in \ddot{\Omega}$ such that $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{h}}) \neq \tilde{0}$. While $\tilde{\chi}_{\ddot{\aleph}}(\ddot{\mathfrak{h}}) = \tilde{0}$. Thus, $\ddot{\mathfrak{h}} \in \mathrm{supp}(\tilde{\ddot{\Upsilon}})$, but $\ddot{\mathfrak{h}} \notin \ddot{\aleph}$. So $\mathrm{supp}(\tilde{\ddot{\Upsilon}}) \not\subseteq \ddot{\aleph}$. Since $\mathrm{supp}(\tilde{\ddot{\Upsilon}})$ is an OA-(m,n)-IQ of $\ddot{\Omega}$ we have $\mathrm{supp}(\tilde{\ddot{\Upsilon}})\,\mathrm{supp}(\tilde{\ddot{\Upsilon}}) \not\subseteq \ddot{\aleph}$. Thus, there exists $\ddot{\mathfrak{m}} = \ddot{\mathfrak{p}}\ddot{\mathfrak{q}}$ for some $\ddot{\mathfrak{p}} \in \mathrm{supp}(\tilde{\ddot{\Upsilon}})$ and $\ddot{\mathfrak{q}} \in \mathrm{supp}(\tilde{\ddot{\Upsilon}})$ such that $\ddot{\mathfrak{m}} \in \ddot{\aleph}$. Hence, $\tilde{\chi}_{\ddot{\aleph}}(\ddot{\mathfrak{m}}) = \tilde{0}$ implies that $(\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\Upsilon}})(\ddot{\mathfrak{m}}) = \tilde{0}$. Since $\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\Upsilon}} \preceq \tilde{\chi}_{\ddot{\aleph}}$ we have $\ddot{\mathfrak{p}} \in \mathrm{supp}(\tilde{\ddot{\Upsilon}})$ and $\ddot{\mathfrak{q}} \in \mathrm{supp}(\tilde{\ddot{\Upsilon}})$. Thus, $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{p}}) \neq 0$ and $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{q}}) \neq \tilde{0}$. It implies that

$$(\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\Upsilon}})(\ddot{\mathfrak{m}}) = \bigvee_{(\ddot{\mathfrak{p}},\ddot{\mathfrak{q}}) \in F_{\ddot{\mathfrak{m}}}} \{\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{p}}) \wedge \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{q}})\} \neq \tilde{0}.$$

It is a contradiction so $\tilde{\ddot{\Upsilon}} \preceq \tilde{\chi}_{\ddot{\aleph}}$. Therefore, $\tilde{\chi}_{\ddot{\aleph}}$ is a semiprime IVFOA-(m,n)-IQ of $\ddot{\Omega}$.

Conversely, suppose that $\tilde{\chi}_{\ddot{\aleph}}$ is a semiprime IVFOA-(m,n)-IQ of $\ddot{\Omega}$. Then $\tilde{\chi}_{\ddot{\aleph}}$ is an IVFOA-(m,n)-IQ of $\ddot{\Omega}$. Thus by Theorem 3.10, $\ddot{\aleph}$ is an OA-(m,n)-IQ of $\ddot{\Omega}$. Let $\ddot{\aleph}_1$ be an OA-(m,n)-IQ of $\ddot{\Omega}$ such that $\ddot{\aleph}_1^2 \subseteq \ddot{\aleph}$. Then $\tilde{\chi}_{\ddot{\aleph}_1}$ is an IVFOA-(m,n)-IQ of $\ddot{\Omega}$. By Lemma 2.7 $\tilde{\chi}_{\ddot{\aleph}_1} \circ \tilde{\chi}_{\ddot{\aleph}_1} = \tilde{\chi}_{\ddot{\aleph}_1^2} \preceq \tilde{\chi}_{\ddot{\aleph}}$. By assumption, $\tilde{\chi}_{\ddot{\aleph}_1} \preceq \tilde{\chi}_{\ddot{\aleph}}$. Thus, $\ddot{\aleph}_1 \subseteq \ddot{\aleph}$. We conclude that $\ddot{\aleph}$ is a semiprime OA-(m,n)-IQ of $\ddot{\Omega}$.

(4) Suppose that $\ddot{\aleph}$ is a strongly prime OA-(m,n)-IQ of $\ddot{\Omega}$. Then $\ddot{\aleph}$ is an OA-(m,n)-IQ of $\ddot{\Omega}$. Thus by Theorem 3.10, $\tilde{\chi}_{\ddot{\aleph}}$ is an IVFOA-(m,n)-IQ of $\ddot{\Omega}$. Let $\tilde{\ddot{\Upsilon}}$ and $\tilde{\ddot{\xi}}$ be IVFOA-(m,n)-IQs of $\ddot{\Omega}$ such that $(\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\xi}}) \curlywedge (\tilde{\ddot{\xi}} \circ \tilde{\ddot{\Upsilon}}) \preceq \tilde{\chi}_{\ddot{\aleph}}$. Assume that $\tilde{\ddot{\Upsilon}} \not\preceq \tilde{\chi}_{\ddot{\aleph}}$ and $\tilde{\ddot{\xi}} \not\preceq \tilde{\chi}_{\ddot{\aleph}}$. Then there exist $\ddot{\mathfrak{h}}, \ddot{\mathfrak{r}} \in \ddot{\Omega}$ such that $\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{h}}) \neq \tilde{0}$ and $\tilde{\ddot{\xi}}(\ddot{\mathfrak{r}}) \neq \tilde{0}$. While $\tilde{\chi}_{\ddot{\aleph}}(\ddot{\mathfrak{h}}) = \tilde{0}$ and $\tilde{\chi}_{\ddot{\aleph}}(\ddot{\mathfrak{r}}) = \tilde{0}$. Thus, $\ddot{\mathfrak{h}} \in \mathrm{supp}(\tilde{\ddot{\Upsilon}})$ and $\ddot{\mathfrak{r}} \in \mathrm{supp}(\tilde{\ddot{\xi}})$, but $\ddot{\mathfrak{h}}, \ddot{\mathfrak{r}} \notin \ddot{\aleph}$. So, $\mathrm{supp}(\tilde{\ddot{\Upsilon}}) \not\subseteq \ddot{\aleph}$ and $\mathrm{supp}(\tilde{\ddot{\xi}}) \not\subseteq \ddot{\aleph}$. Hence, there exists $\ddot{\mathfrak{m}} \in [\mathrm{supp}(\tilde{\ddot{\Upsilon}})\,\mathrm{supp}(\tilde{\ddot{\xi}})] \cap (\mathrm{supp}(\tilde{\ddot{\xi}})\,\mathrm{supp}(\tilde{\ddot{\Upsilon}}))$ such that $\ddot{\mathfrak{m}} \notin \ddot{\aleph}$. Thus, $\chi_{\ddot{\aleph}}(\mathfrak{m}) = \tilde{0}$ such that so, $(\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\xi}})(\mathfrak{m}) \curlywedge (\tilde{\ddot{\xi}} \circ \tilde{\ddot{\Upsilon}})(\mathfrak{m}) = 0$. Since $\ddot{\mathfrak{m}} \in \mathrm{supp}(\tilde{\ddot{\Upsilon}})\,\mathrm{supp}(\tilde{\ddot{\xi}})$ and $\ddot{\mathfrak{m}} \in \mathrm{supp}(\tilde{\ddot{\xi}})\,\mathrm{supp}(\tilde{\ddot{\Upsilon}})$ we have $\ddot{\mathfrak{m}} = \ddot{\mathfrak{d}}\ddot{\mathfrak{k}}$ and $\mathfrak{m} = \mathfrak{g}\ddot{\mathfrak{q}}$ for some $\ddot{\mathfrak{d}}, \ddot{\mathfrak{q}} \in \mathrm{supp}(\tilde{\ddot{\Upsilon}})$, and for some $\ddot{\mathfrak{k}}, \ddot{\mathfrak{g}} \in \mathrm{supp}(\tilde{\ddot{\xi}})$. we have

$$(\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\xi}})(\ddot{\mathfrak{m}}) = \curlyvee_{(\ddot{\mathfrak{d}},\ddot{\mathfrak{r}}) \in F_{\ddot{\mathfrak{m}}}} \{\tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{d}}) \curlywedge \tilde{\ddot{\xi}}(\ddot{\mathfrak{k}})\}.$$

Similarly

$$(\tilde{\ddot{\xi}} \circ \tilde{\ddot{\Upsilon}})(\ddot{\mathfrak{m}}) = \curlyvee_{(\ddot{\mathfrak{g}},\ddot{\mathfrak{q}}) \in F_{\ddot{\mathfrak{m}}}} \{\tilde{\ddot{\xi}}(\ddot{\mathfrak{g}}) \curlywedge \tilde{\ddot{\Upsilon}}(\ddot{\mathfrak{q}})\}.$$

So, $(\tilde{\ddot{\Upsilon}} \circ \tilde{\ddot{\xi}})(\ddot{\mathfrak{m}}) \curlywedge (\tilde{\ddot{\xi}} \circ \tilde{\ddot{\Upsilon}})(\ddot{\mathfrak{m}}) \neq \tilde{0}$. It is a contradiction. Hence, $\tilde{\ddot{\Upsilon}} \preceq \tilde{\chi}_{\ddot{\aleph}}$ or $\tilde{\ddot{\xi}} \preceq \tilde{\chi}_{\ddot{\aleph}}$. Therefore, $\tilde{\chi}_{\ddot{\aleph}}$ is a strongly prime IVFOA-(m,n)-IQ of $\ddot{\Omega}$.

Conversely, suppose that $\tilde{\chi}_{\ddot{\aleph}}$ is a strongly prime IVFOA-(m,n)-IQ of $\ddot{\Omega}$. Then $\tilde{\chi}_{\ddot{\aleph}}$ is an IVFOA (m,n)-IQ of $\ddot{\Omega}$. Thus, by Theorem 3.10, $\ddot{\aleph}$ is an OA-(m,n)-IQ of $\mathfrak{T}$. Let $\ddot{\aleph}_1$ and $\ddot{\aleph}_2$ be OA-(m,n)-Is of $\ddot{\Omega}$ such that $\ddot{\aleph}_1\ddot{\aleph}_2 \cap \ddot{\aleph}_2\ddot{\aleph}_1 \subseteq \ddot{\aleph}$. Then $\tilde{\chi}_{\ddot{\aleph}_1}$ and $\tilde{\chi}_{\ddot{\aleph}_2}$ are IVFOA (m,n)-IQs of $\ddot{\Omega}$. By Lemma 2.7 $\tilde{\chi}_{\ddot{\aleph}_1\ddot{\aleph}_2} = \tilde{\chi}_{\ddot{\aleph}_1} \circ \tilde{\chi}_{\ddot{\aleph}_2}$ and $\tilde{\chi}_{\ddot{\aleph}_2\ddot{\aleph}_1} = \tilde{\chi}_{\ddot{\aleph}_2} \circ \tilde{\chi}_{\ddot{\aleph}_1}$. Thus, $(\tilde{\chi}_{\ddot{\aleph}_1} \circ \tilde{\chi}_{\ddot{\aleph}_2}) \curlywedge (\tilde{\chi}_{\ddot{\aleph}_1} \circ \tilde{\chi}_{\ddot{\aleph}_2}) = \tilde{\chi}_{\ddot{\aleph}_1\ddot{\aleph}_2} \curlywedge \tilde{\chi}_{\ddot{\aleph}_2\ddot{\aleph}_1} = \tilde{\chi}_{\ddot{\aleph}_1\ddot{\aleph}_2 \cap \ddot{\aleph}_2\ddot{\aleph}_1} \preceq \tilde{\chi}_{\ddot{\aleph}}$. By assumption, $\tilde{\chi}_{\ddot{\aleph}_1} \preceq \tilde{\chi}_{\ddot{\aleph}}$ and $\tilde{\chi}_{\ddot{\aleph}_2} \preceq \tilde{\chi}_{\ddot{\aleph}}$. Thus, $\ddot{\aleph}_1 \subseteq \ddot{\aleph}$ or $\ddot{\aleph}_2 \subseteq \ddot{\aleph}$. We conclude that $\ddot{\aleph}$ is a strongly prime OA-(m,n)-IQ of $\ddot{\Omega}$.

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Corollary 4.14. *Let $\ddot{\Omega}$ be an ordered semigroup. Then $\ddot{\Omega}$ has no proper OA-(m,n)-IQ if and only if $\mathrm{supp}(\tilde{\ddot{\Upsilon}}) = \ddot{\Omega}$ for every IVFOA-(m,n)-IQ $\tilde{\ddot{\Upsilon}}$ of $\ddot{\Omega}$.*

V. Conclusion

The aim of the paper is to give the concept of IVFOA-(m,n)-Is and IVFOA-(m,n)-IQs in ordered semigroups. We prove properties IVFOA-(m,n)-Is and IVFOA-(m,n)-IQs. In Theorems 3.10, 3.11, 4.9, 4.10, 3.14, and 4.13, we prove the relationship between OA-(m,n)-Is, OA-(m,n)-QIs and class fuzzifications. In future work, we can study other kinds of almost ideals and their fuzzifications in an ordered ternary semigroup.

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