Existence and Asymptotic Behavior of the Time-Dependent Solution of a System Composed of Two Identical Units

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***Abstract*—In this study, we examined the existence and asymptotic behavior of the time-dependent solution of a system with two identical components whose lifetimes follow a bivariate exponential distribution. First, by using the strong continuous semigroup theory, we proved the existence and uniqueness of the nonnegative time-dependent solution of the system model. Next, by studying the spectral properties of the operator corresponding to the system model, we determined that zero is an eigenvalue of the operator and its adjoint operator with geometric multiplicity one, and all points on the imaginary axis except zero belong to the resolvent set of the operator. From the above results, we deduced that the time-dependent solution of the system model converges strongly to its steady-state solution.**

***Index Terms*—C_0-semigroup; Dispersive operator; Eigenvalue; Resolvent set; Geometric multiplicity.**

I. Introduction

TWO-unit repairable systems have been studied extensively to obtain measures of reliability performance. Epstein and Hosford [1] considered exponential failure distributions with exponential repair facilities. Gaver [2], [3] treated a similar situation with an arbitrary repair distribution, and Barlow [4] obtained results for a situation in which many inactive spare units are available. Gnedenko [5] examined the limiting distribution of the time to system failure as the repair time decreases to zero in a particular manner. However, in all of these cases, the lifetimes of the two active components were assumed to be independently exponentially distributed. A study [6] examined a two-unit system in which the lifetimes of the two units in service were interdependent in a particular way. This dependence was characterized by the bivariate exponential distribution reported by Marshall and Olkin [7]; this distribution has exponential marginal distributions and other physically motivating properties. In addition, two measures of reliability were determined by the author: first, the distribution and mean of the time to system failure. Second, by specifying a supplementary variable [8], [9], the author established a system model and obtained the steady-state probabilities of the number of working units. Some graphical results were obtained to illustrate the deviation of these quantities from the values obtained under the classical assumption of independent lifetimes. In addition, some authors investigated various two-unit repairable systems and obtained many performance measures. For example, Nakagawa and Osaki [10] studied the stochastic behavior of a two-unit parallel system with repair maintenance based on the Markov renewal theory. Shi [11] studied the reliability of a two-unit series repairable model to analyze the reliability indices of some automatic systems. Liu and Tang [12] dealt with a two-unit cold standby system using the separated repair rule and derived some steady-state reliability indices for the system. R. Liu and Z. Liu [13] examined a two-unit basic model with a repairman taking multiple vacations and derived some important reliability indices for the model. Mohamed and Zienab [14] calculated various reliability measures of an industrial system consisting of two nonidentical parallel units with one repairer using the supplementary variable technique and the theory of Markov processes. Many authors have used the Markov process to establish different mathematical models and obtain performance measures [15]-[17]. In this study, by using the C_0-semigroup theory in functional analysis, a dynamic analysis was performed for the two-unit system model obtained by Harris [6]. First, using the Hille–Yosida theorem, Phillips theorem, and Fattorini theorem in functional analysis, the existence and uniqueness of the nonnegative time-dependent solution of the system model were established. Next, the asymptotic behavior of the time-dependent solution of the system was examined. For this purpose, by studying the spectrum on the imaginary axis of the underlying operator corresponding to the two-unit system model, it was determined that zero is an eigenvalue of the operator and its adjoint operator with geometric multiplicity one and all points on the imaginary axis except zero belong to the resolvent set of the operator. By combining these above results, we deduced that the time-dependent solution of the system model converges strongly to its steady-state solution.

According to Harris [6], the system consists of two identical components whose lifetimes follow the bivariate exponential distribution described by the following set of integro-differential equations:

$$\frac{\mathrm{d}\pi_0(t)}{\mathrm{d}t} = -(2\lambda+\lambda_{12})\pi_0(t) + \int_0^\infty \pi_1(t,z)h(z)\mathrm{d}z, \quad \text{(I.1)}$$

$$\frac{\partial\pi_1(t,z)}{\partial t} + \frac{\partial\pi_1(t,z)}{\partial z} = -[\lambda+\lambda_{12}+h(z)]\pi_1(t,z), \quad \text{(I.2)}$$

$$\frac{\partial\pi_2(t,z)}{\partial t} + \frac{\partial\pi_2(t,z)}{\partial z} = -h(z)\pi_2(t,z) + (\lambda+\lambda_{12})\pi_1(t,z), \quad \text{(I.3)}$$

$$\pi_1(t,0) = 2\lambda\pi_0(t) + \int_0^\infty \pi_2(t,z)h(z)\mathrm{d}z, \quad \text{(I.4)}$$

$$\pi_2(t,0) = \lambda_{12}\pi_0(t), \quad \text{(I.5)}$$

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$$\pi_0(0) = 1, \pi_i(0, z) = 0, i = 1, 2. \tag{I.6}$$

Here, $(z, t) \in [0, \infty) \times [0, \infty)$; $\pi_0(t)$ is the probability that all components are working at time t and $\pi_i(z, t)\mathrm{d}z$ $(i = 1, 2)$ is the probability that i $(i = 1, 2)$ components failed at time t, with one component being repaired and the elapsed repair time lying between z and $z + \mathrm{d}z$. Further, $2\lambda\Delta + o(\Delta)$ represents the probability that one of the components fails in time Δ under the condition that both components were working before that. $\lambda_{12}\Delta + o(\Delta)$ represents the probability that both components fail in time Δ under the condition that both were working before this time. Finally, $(\lambda + \lambda_{12})\Delta + o(\Delta)$ denotes the probability that one of the components failed in time Δ under the condition that just one of the components was working before this time. $h(z)\mathrm{d}z$ represents the probability that the repair of the unit will be completed in the time interval $(z, z + \mathrm{d}z)$, given that the same has not been completed until time z.

II. Problem Formulation

In the section, we first reformulate the system represented in (I.1)–(I.6) as an abstract Cauchy problem in a suitable state space. For convenience, we introduce the following:

$$\Psi = \begin{pmatrix} e^{-z} & 0 & 0 \\ 2\lambda e^{-z} & 0 & h(z) \\ \lambda_{12} & 0 & 0 \end{pmatrix}$$

If we choose the state space as follows:

$$Z = \left\{ \pi \in R \times \left(L^1[0, \infty)\right)^2 \,\middle|\, \|\pi\| = |\pi_0| + \sum_{i=1}^{2} \|\pi_i\|_{L^1[0,\infty)} \right\},$$

then it is obvious that Z is a Banach space. Now, we define operators and their domains.

$$A \begin{pmatrix} \pi_0 \\ \pi_1(z) \\ \pi_2(z) \end{pmatrix} = \begin{pmatrix} -(2\lambda + \lambda_{12}) & 0 & 0 \\ 0 & -\frac{\mathrm{d}}{\mathrm{d}z} - (\lambda + \lambda_{12} + h(z)) & 0 \\ 0 & 0 & -\frac{\mathrm{d}}{\mathrm{d}z} - h(z) \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1(z) \\ \pi_2(z) \end{pmatrix},$$

$$D(A) = \left\{ \pi \in Z \,\middle|\, \frac{\mathrm{d}\pi_i(z)}{\mathrm{d}z} \in L^1[0, \infty), \pi_i(z) (i = 1, 2) \text{ are absolutely continuous functions and } \pi(0) = \int_0^\infty \Psi\pi(z)\mathrm{d}z \right\},$$

$$B \begin{pmatrix} \pi_0 \\ \pi_1(z) \\ \pi_2(z) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \lambda + \lambda_{12} & 0 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1(z) \\ \pi_2(z) \end{pmatrix},$$

$$C \begin{pmatrix} \pi_0 \\ \pi_1(z) \\ \pi_2(z) \end{pmatrix} = \begin{pmatrix} \int_0^\infty \pi_1(z)h(z)\mathrm{d}z \\ 0 \\ 0 \end{pmatrix},$$

$$D(B) = D(C) = Z.$$

Then, the previous system of equations (I.1)–(I.6) can be expressed as an abstract Cauchy problem in the Banach space Z

$$\begin{cases} \frac{\mathrm{d}\pi(t)}{\mathrm{d}t} = (A + B + C)\pi(t), \quad t \in [0, \infty), \\ \pi(0) = (1, 0, 0)^T. \end{cases} \tag{II.1}$$

III. Existence Of A Nonnegative Time-Dependent Solution Of The System Given in (II.1)

Theorem III.1 If $\bar{h} = \sup_{z \in [0,\infty)} h(z) < \infty$, then $A + B + C$ generates a positive contraction C_0-semigroup $T(t)$.

Proof The proof of the theorem is divided into four steps. First, we estimate the norm of $(\gamma I - A)^{-1}$. Second, we show that $D(A)$ is dense in Z. Third, we verify that B and C are bounded linear operators. Finally, we check that $A + B + C$ is dispersive and thus deduce the desired result.

For any given $j = (j_0, j_1, j_2) \in Z$, we consider the equation $(\gamma I - A)\pi = j$, i.e.,

$$(\gamma + 2\lambda + \lambda_{12})\,\pi_0 = j_0, \tag{III.1}$$

$$\frac{\mathrm{d}\pi_1(z)}{\mathrm{d}z} + (\gamma + \lambda + \lambda_{12} + h(z))\,\pi_1(z) = j_1(z), \tag{III.2}$$

$$\frac{\mathrm{d}\pi_2(z)}{\mathrm{d}z} + (\gamma + h(z))\pi_2(z) = j_2(z), \tag{III.3}$$

$$\pi_1(0) = 2\lambda\pi_0 + \int_0^\infty \pi_2(z)h(z)\mathrm{d}z, \tag{III.4}$$

$$\pi_2(0) = \lambda_{12}\pi_0. \tag{III.5}$$

By solving (III.1)–(III.3), we have

$$\pi_0 = \frac{1}{\gamma + 2\lambda + \lambda_{12}} j_0, \tag{III.6}$$

$$\pi_1(z) = d_1 e^{-(\gamma+\lambda+\lambda_{12})z - \int_0^z h(\tau)\mathrm{d}\tau} + e^{-(\gamma+\lambda+\lambda_{12})z - \int_0^z h(\tau)\mathrm{d}\tau} \times \int_0^z j_1(\zeta) e^{(\gamma+\lambda+\lambda_{12})\zeta + \int_0^\zeta h(\tau)\mathrm{d}\tau}\mathrm{d}\zeta, \tag{III.7}$$

$$\pi_2(z) = d_2 e^{-\gamma z - \int_0^z h(\tau)\mathrm{d}\tau} + e^{-\gamma z - \int_0^z h(\tau)\mathrm{d}\tau} \times \int_0^z j_2(\zeta) e^{\gamma\zeta + \int_0^\zeta h(\tau)\mathrm{d}\tau}\mathrm{d}\zeta. \tag{III.8}$$

Combining (III.4) and (III.5) with (III.7) and (III.8), we derive

$$d_1 = \pi_1(0) = 2\lambda\pi_0 + \int_0^\infty \pi_2(z)h(z)\mathrm{d}z$$

$$\Rightarrow$$

$$|d_1| \le 2\lambda|\pi_0| + \int_0^\infty |\pi_2(z)|\,h(z)\mathrm{d}z \le \frac{2\lambda}{\gamma + 2\lambda + \lambda_{12}}|j_0| + \bar{h}\|\pi_2\|_{L^1[0,\infty)}, \tag{III.9}$$

$$|d_2| = |\pi_2(0)| = |\lambda_{12}\pi_0| = \frac{\lambda_{12}}{\gamma + 2\lambda + \lambda_{12}}|j_0|\,. \tag{III.10}$$

From (III.8), (III.10), and the Fubini theorem, we have

$$\|\pi_2\|_{L^1[0,\infty)} = \int_0^\infty |\pi_2(z)|\,\mathrm{d}z \le |d_2| \int_0^\infty e^{-\gamma z - \int_0^z h(\tau)\mathrm{d}\tau}\mathrm{d}z + \int_0^\infty e^{-\gamma z - \int_0^z h(\tau)\mathrm{d}\tau} \int_0^z |j_2(\zeta)| e^{\gamma\zeta + \int_0^\zeta h(\tau)\mathrm{d}\tau}\mathrm{d}\zeta\mathrm{d}z$$

$$
\begin{aligned}
&\leq |d_2| \int_0^\infty e^{-\gamma z} \mathrm{d}z \\
&\quad + \int_0^\infty e^{-\gamma z} \int_0^z |j_2(\zeta)|\, e^{rz - \int_\zeta^z h(\tau)\mathrm{d}\tau} \mathrm{d}\zeta \mathrm{d}z \\
&\leq \frac{1}{\gamma} |d_2| + \int_0^\infty e^{-\gamma z} \int_0^z |j_2(\zeta)|\, e^{\gamma\zeta} \mathrm{d}\zeta \mathrm{d}z \\
&= \frac{1}{\gamma} |d_2| + \int_0^\infty |j_2(\zeta)|\, e^{\gamma\zeta} \int_\zeta^\infty e^{-\gamma z} \mathrm{d}z \mathrm{d}\zeta \\
&= \frac{1}{\gamma} |d_2| + \frac{1}{\gamma} \|j_2\|_{L^1[0,\infty)} \\
&= \frac{\lambda_{12}}{\gamma(\gamma + 2\lambda + \lambda_{12})} |j_0| + \frac{1}{\gamma} \|j_2\|_{L^1[0,\infty)}. \qquad \text{(III.11)}
\end{aligned}
$$

By (III.7), (III.9), and the Fubini theorem, we obtain

$$
\begin{aligned}
&\|\pi_1\|_{L^1[0,\infty)} = \int_0^\infty |\pi_1(z)|\, \mathrm{d}z \\
&\leq |d_1| \int_0^\infty e^{-(\gamma+\lambda+\lambda_{12})z - \int_0^z h(\tau)\mathrm{d}\tau} \mathrm{d}z \\
&\quad + \int_0^\infty e^{-(\gamma+\lambda+\lambda_{12})z - \int_0^z h(\tau)\mathrm{d}\tau} \\
&\quad \int_0^z |j_1(\zeta)| e^{(\gamma+\lambda+\lambda_{12})\zeta + \int_0^\zeta h(\tau)\mathrm{d}\tau} \mathrm{d}\zeta \mathrm{d}z \\
&\leq |d_1| \int_0^\infty e^{-(\gamma+\lambda+\lambda_{12})z} \mathrm{d}z + \int_0^\infty e^{-(\gamma+\lambda+\lambda_{12})z} \\
&\quad \int_0^z |j_1(\zeta)|\, e^{(\gamma+\lambda+\lambda_{12})\zeta - \int_\zeta^z h(\tau)\mathrm{d}\tau} \mathrm{d}\zeta \mathrm{d}z \\
&\leq \frac{1}{\gamma+\lambda+\lambda_{12}} |d_1| \\
&\quad + \int_0^\infty e^{-(\gamma+\lambda+\lambda_{12})z} \int_0^z |j_1(\zeta)|\, e^{(\gamma+\lambda+\lambda_{12})\zeta} \mathrm{d}\zeta \mathrm{d}z \\
&= \frac{1}{\gamma+\lambda+\lambda_{12}} |d_1| \\
&\quad + \int_0^\infty |j_1(\zeta)|\, e^{(\gamma+\lambda+\lambda_{12})\zeta} \int_\zeta^\infty e^{-(\gamma+\lambda+\lambda_{12})z} \mathrm{d}z \mathrm{d}\zeta \\
&= \frac{1}{\gamma+\lambda+\lambda_{12}} |d_1| + \frac{1}{\gamma+\lambda+\lambda_{12}} \|j_1\|_{L^1[0,\infty)} \\
&\leq \frac{1}{\gamma+\lambda+\lambda_{12}} \left[\frac{2\lambda}{\gamma+2\lambda+\lambda_{12}} |j_0| + \overline{h} \|\pi_2\|_{L^1[0,\infty)} \right] \\
&\quad + \frac{1}{\gamma+\lambda+\lambda_{12}} \|j_1\|_{L^1[0,\infty)} \\
&\leq \frac{1}{\gamma+\lambda+\lambda_{12}} \left[\frac{2\lambda}{\gamma+2\lambda+\lambda_{12}} |j_0| + \frac{\lambda_{12}\overline{h}}{\gamma(\gamma+2\lambda+\lambda_{12})} |j_0| \right. \\
&\quad \left. + \frac{\overline{h}}{\gamma} \|j_2\|_{L^1[0,\infty)} \right] + \frac{1}{\gamma+\lambda+\lambda_{12}} \|j_1\|_{L^1[0,\infty)} \\
&= \frac{2\lambda\gamma + \lambda_{12}\overline{h}}{\gamma(\gamma+\lambda+\lambda_{12})(\gamma+2\lambda+\lambda_{12})} |j_0| \\
&\quad + \frac{1}{\gamma+\lambda+\lambda_{12}} \|j_1\|_{L^1[0,\infty)} \\
&\quad + \frac{\overline{h}}{\gamma(\gamma+\lambda+\lambda_{12})} \|j_2\|_{L^1[0,\infty)}. \qquad \text{(III.12)}
\end{aligned}
$$

Further, (III.6), (III.11), and (III.12) give

$$
\begin{aligned}
\|\pi\| &= |\pi_0| + \|\pi_1\|_{L^1[0,\infty)} + \|\pi_2\|_{L^1[0,\infty)} \\
&\leq \frac{1}{\gamma+2\lambda+\lambda_{12}} |j_0| \\
&\quad + \frac{2\lambda\gamma + \lambda_{12}\overline{h}}{\gamma\,(\gamma+\lambda+\lambda_{12})\,(\gamma+2\lambda+\lambda_{12})} |j_0| \\
&\quad + \frac{1}{\gamma+\lambda+\lambda_{12}} \|j_1\|_{L^1(0,\infty]} \\
&\quad + \frac{\overline{h}}{\gamma\,(\gamma+\lambda+\lambda_{12})} \|j_2\|_{L^1(0,\infty]} \\
&\quad + \frac{\lambda_{12}}{\gamma\,(\gamma+2\lambda+\lambda_{12})} |j_0| + \frac{1}{\gamma} \|j_2\|_{L^1(0,\infty]} \\
&= \frac{(\gamma+\lambda+\lambda_{12})\,(\gamma+\lambda_{12}) + 2\lambda\gamma + \lambda_{12}\overline{h}}{\gamma\,(\gamma+\lambda+\lambda_{12})\,(\gamma+2\lambda+\lambda_{12})} |j_0| \\
&\quad + \frac{1}{\gamma+\lambda+\lambda_{12}} \|j_1\|_{L^1[0,\infty)} \\
&\quad + \frac{\gamma+\lambda+\lambda_{12}+\overline{h}}{\gamma\,(\gamma+\lambda+\lambda_{12})} \|j_2\|_{L^1[0,\infty)} \\
&\leq \frac{1}{\gamma-\overline{h}} \|j\|_{L^1[0,\infty)}. \qquad \text{(III.13)}
\end{aligned}
$$

Equation (III.13) means that $(\gamma I - A)^{-1}$ exists for $\gamma > \overline{h}$ and

$$
(\gamma I - A)^{-1}:\ Z \to D(A),\ \left\|(\gamma I - A)^{-1}\right\| \leq \frac{1}{\gamma - \overline{h}}.
$$

Second, we prove that $D(A)$ is dense in Z. If we set

$$
W = \left\{ \pi(z) = (\pi_0, \pi_1, \pi_2) \;\middle|\; \begin{array}{l} \pi_i \in C_0^\infty[0,\infty), \exists a_i > 0, \\ \text{such that } \pi_i(z) = 0, \\ \forall z \in [0, a_i], i = 1, 2. \end{array} \right\},
$$

then, according to Adams results [18], we know that W is dense in Z. Hence, it sufficient to prove that $D(A)$ is dense in W. If we take $\forall \pi = (\pi_0, \pi_1(z), \pi_2(z)) \in W$, then there exists $a_i > 0$ such that $\pi_i(z) = 0$ for $z \in [0, a_i]$, $i = 1, 2$. It follows that $\pi_i(z) = 0$ $(i = 1, 2)$ for $z \in [0, a]$, here $a = \min\{a_1, a_2\}$.

Next, set

$$
\begin{aligned}
f^a(0) &= (\pi_0, f_1^a(0), f_2^a(0)) \\
&= \left(\pi_0, 2\lambda\pi_0 + \int_a^\infty \pi_2(z)h(z)\mathrm{d}z, \lambda_{12}\pi_0 \right) \\
f^a(z) &= (\pi_0, f_1^a(z), f_2^a(z)),
\end{aligned}
$$

where

$$
\begin{aligned}
f_1^a(z) &= \begin{cases} f_1^a(0)\left(1 - \frac{z}{a}\right)^2, & z \in [0, a), \\ \pi_1(z), & z \in [a, \infty), \end{cases} \\
f_2^a(z) &= \pi_2(z).
\end{aligned}
$$

Then, it is easy to check that $f^a(z) \in D(A)$. Moreover,

$$
\begin{aligned}
\|\pi - f^a\| &= \sum_{i=1,2} \int_0^\infty |\pi_i(z) - f_i^a(z)| \mathrm{d}z \\
&= \int_0^a |f_1^a(0)| \left(1 - \frac{z}{a}\right)^2 \mathrm{d}z = \frac{a}{3} |f_1^a(0)| \to 0, \text{as } a \to 0.
\end{aligned}
$$

This shows that $D(A)$ is dense in W. Hence, $D(A)$ is dense in Z. From the previous two steps and the Hille–Yosida theorem [19], we know that A generates a C_0- semigroup. Next, we verify that operators B and C are bounded and linear. By the definition of B and C, for any $\pi \in Z$, we derive

$$
\|B\pi\| = \|(0, 0, (\lambda_1 + \lambda_2)\pi_1)\| \leq (\lambda_1 + \lambda_2)\|\pi\|,\ \|C\pi\| \leq \overline{h}\|\pi\|.
$$

The previous two formulas imply that B and C are bounded operators. It is easy to show that B and C are linear operators. Thus, from the perturbation theorem of C_0-semigroup

[19], we obtain that $A+B+C$ generates a C_0-semigroup $T(t)$.

Finally, we prove that $A+B+C$ is a dispersive operator. For any given $\pi \in D(A)$ we take $\Gamma(z)$ as

$$\Gamma(z)=\left(\frac{[\pi_0]^+}{\pi_0},\frac{[\pi_1(z)]^+}{\pi_1(z)},\frac{[\pi_2(z)]^+}{\pi_2(z)}\right),$$

where

$$[\pi_0]^+=\begin{cases}\pi_0 & \text{if} \quad \pi_0>0\\ 0 & \text{if} \quad \pi_0\le 0\end{cases},$$

$$[\pi_i(z)]^+=\begin{cases}\pi_i(z) & \text{if} \quad \pi_i(z)>0\\ 0 & \text{if} \quad \pi_i(z)\le 0\end{cases}, i=1,2.$$

If we define $V_i^+=\{z\in[0,\infty)\mid \pi_i(z)>0\}$ and $V_i^-=\{z\in[0,\infty)\mid \pi_i(z)\le 0\}(i=1,2)$, then by a short argument we obtain

$$\begin{aligned}&\int_0^\infty \frac{\mathrm{d}\pi_i(z)}{\mathrm{d}z}\frac{[\pi_i(z)]^+}{\pi_i(z)}\mathrm{d}z\\
&=\int_{V_i^+}\frac{\mathrm{d}\pi_i(z)}{\mathrm{d}z}\frac{[\pi_i(z)]^+}{\pi_i(z)}\mathrm{d}z+\int_{V_i^-}\frac{\mathrm{d}\pi_i(z)}{\mathrm{d}z}\frac{[\pi_i(z)]^+}{\pi_i(z)}\mathrm{d}z\\
&=\int_{V_i^+}\frac{\mathrm{d}\pi_i(z)}{\mathrm{d}z}\frac{[\pi_i(z)]^+}{\pi_i(z)}\mathrm{d}z=\int_{V_i^+}\frac{\mathrm{d}\pi_i(z)}{\mathrm{d}z}\mathrm{d}z\\
&=\int_0^\infty \frac{\mathrm{d}[\pi_i(z)]^+}{\mathrm{d}z}\mathrm{d}z=-[\pi_i(0)]^+, \quad i=1,2. \qquad \text{(III.14)}\end{aligned}$$

Using boundary conditions on $\pi\in D(A)$, (III.14), and the inequality

$$\int_0^\infty \pi_i(z)h(z)\mathrm{d}z\le\int_0^\infty[\pi_i(z)]^+h(z)\mathrm{d}z,\ i=1,2,$$
$$\int_0^\infty \pi_i(z)\frac{[\pi_j(z)]^+}{\pi_j(z)}\mathrm{d}z\le\int_0^\infty[\pi_i(z)]^+\mathrm{d}z,\ i,j=1,2;i\ne j.$$

For above $\Gamma(z)$, we deduce

$$\begin{aligned}&\langle (A+B+C)\pi,\Gamma\rangle\\
&=\left\{-(2\lambda+\lambda_{12})\pi_0+\int_0^\infty\pi_1(z)h(z)\mathrm{d}z\right\}\frac{[\pi_0]^+}{\pi_0}\\
&+\int_0^\infty\left\{-\frac{\mathrm{d}\pi_1(z)}{\mathrm{d}z}-[\lambda+\lambda_{12}+h(z)]\pi_1(z)\right\}\frac{[\pi_1(z)]^+}{\pi_1(z)}\mathrm{d}z\\
&+\int_0^\infty\left\{-\frac{\mathrm{d}\pi_2(z)}{\mathrm{d}z}-h(z)\pi_2(z)+[\lambda+\lambda_{12}]\pi_1(z)\right\}\\
&\times\frac{[\pi_2(z)]^+}{\pi_2(z)}\mathrm{d}z\\
&=-(2\lambda+\lambda_{12})[\pi_0]^++\frac{[\pi_0]^+}{\pi_0}\int_0^\infty\pi_1(z)h(z)\mathrm{d}z\\
&-\sum_{i=1}^2\int_0^\infty\frac{\mathrm{d}\pi_i(z)}{\mathrm{d}z}\frac{[\pi_i(z)]^+}{\pi_i(z)}\mathrm{d}z\\
&-\int_0^\infty[\lambda+\lambda_{12}+h(z)][\pi_1(z)]^+\mathrm{d}z-\int_0^\infty h(z)[\pi_2(z)]^+\mathrm{d}z\\
&+\int_0^\infty[\lambda+\lambda_{12}]\pi_1(z)\frac{[\pi_2(z)]^+}{\pi_2(z)}\mathrm{d}z\\
&\le-(2\lambda+\lambda_{12})[\pi_0]^++\frac{[\pi_0]^+}{\pi_0}\int_0^\infty[\pi_1(z)]^+h(z)\mathrm{d}z\\
&+\sum_{i=1}^2[\pi_i(0)]^+-(\lambda+\lambda_{12})\int_0^\infty[\pi_1(z)]^+\mathrm{d}z\\
&-\sum_{i=1}^2\int_0^\infty[\pi_i(z)]^+h(z)\mathrm{d}z\\
&+(\lambda+\lambda_{12})\int_0^\infty\pi_1(z)\frac{[\pi_2(z)]^+}{\pi_2(z)}\mathrm{d}z\\
&\le-(2\lambda+\lambda_{12})[\pi_0]^++\frac{[\pi_0]^+}{\pi_0}\int_0^\infty[\pi_1(z)]^+h(z)\mathrm{d}z\\
&+2\lambda[\pi_0]^++\int_0^\infty[\pi_2(z)]^+h(z)\mathrm{d}z+\lambda_{12}[\pi_0]^+\\
&-(\lambda+\lambda_{12})\int_0^\infty[\pi_1(z)]^+\mathrm{d}z-\int_0^\infty[\pi_1(z)]^+h(z)\mathrm{d}z\\
&-\int_0^\infty[\pi_2(z)]^+h(z)\mathrm{d}z+(\lambda+\lambda_{12})\int_0^\infty[\pi_1(z)]^+\mathrm{d}z\\
&=\left(\frac{[\pi_0]^+}{\pi_0}-1\right)\int_0^\infty[\pi_1(z)]^+h(z)\mathrm{d}z\le 0.\end{aligned}$$

Combining the above expression with the definition of the dispersive operator, we know the operator $A+B+C$ is a dispersive operator. From the previous steps and the Phillips theorem, we know that $A+B+C$ generates a positive C_0-semigroup of contraction. Because a C_0-semigroup is unique [19], we know that this semigroup is just $T(t)$.

By the same proof steps of Theorem III.1, we easily get the following result:

Corollary III.1 If $\bar{h}=\sup_{z\in[0,\infty)} h(z)<\infty$, then $A+B$ generates a positive contraction C_0-semigroup $Q(t)$.

It is not difficult to see that Z^*, the dual space of Z, is

$$\begin{aligned}Z^*=\Big\{&\pi^*\in\mathbb{R}\times(L^\infty[0,\infty))^2\mid |||\pi^*|||=\\
&\sup\left\{|\pi_0|,\ \|\pi_1^*\|_{L^\infty[0,\infty)},\ \|\pi_2^*\|_{L^\infty[0,\infty)}\right\}\Big\},\end{aligned}$$

obviously, Z^* is a Banach space. If we take a set

$$\begin{aligned}\Omega=\{&\pi\in Z|\pi(z)=(\pi_0,\pi_1(z),\pi_2(z)),\\
&\pi_0\ge 0,\ \pi_i(z)\ge 0,\ i=1,2\},\end{aligned}$$

then Ω is a cone in Z. For $\pi\in D(A)\cap\Omega$, if we take $\pi^*=\|\pi\|(1,1,1)$, then we derive

$$\begin{aligned}&\langle (A+B+C)\pi,\pi*\rangle\\
&=\left\{-(2\lambda+\lambda_{12})\pi_0+\int_0^\infty\pi_1(z)h(z)\mathrm{d}z\right\}\|\pi\|\\
&+\int_0^\infty\left\{-\frac{\mathrm{d}\pi_1(z)}{\mathrm{d}z}-[\lambda+\lambda_{12}+h(z)]\pi_1(z)\right\}\|\pi\|\mathrm{d}z\\
&+\int_0^\infty\left\{\left(-\frac{\mathrm{d}}{\mathrm{d}z}-h(z)\right)\pi_2(z)+[\lambda+\lambda_{12}]\pi_1(z)\right\}\|\pi\|\mathrm{d}z\\
&=-(2\lambda+\lambda_{12})\pi_0\|\pi\|+\|\pi\|\int_0^\infty\pi_1(z)h(z)\mathrm{d}z\\
&+\sum_{i=1}^2\|\pi\|\pi_i(0)-(\lambda+\lambda_{12})\|\pi\|\int_0^\infty\pi_1(z)\mathrm{d}z\\
&-\|\pi\|\int_0^\infty\pi_1(z)h(z)\mathrm{d}z-\|\pi\|\int_0^\infty\pi_2(z)h(z)\mathrm{d}z\\
&+(\lambda+\lambda_{12})\|\pi\|\int_0^\infty\pi_1(z)\mathrm{d}z\\
&=-(2\lambda+\lambda_{12})\pi_0\|\pi\|+\|\pi\|\int_0^\infty\pi_1(z)h(z)\mathrm{d}z\\
&+2\lambda\pi_0\|\pi\|+\|\pi\|\int_0^\infty\pi_2(z)h(z)\mathrm{d}z\end{aligned}$$

$$- (\lambda + \lambda_{12})\|\pi\| \int_0^\infty \pi_1(z)\mathrm{d}z - \|\pi\| \int_0^\infty \pi_1(z)h(z)\mathrm{d}z$$
$$+ \lambda_{12}\pi_0\|\pi\| - \|\pi\| \int_0^\infty \pi_2(z)h(z)\mathrm{d}z$$
$$+ (\lambda + \lambda_{12})\|\pi\| \int_0^\infty \pi_1(z)\mathrm{d}z = 0.$$

The above result shows that $A + B + C$ is a conservative operator. Hence, we can use the Fattorini theorem [20] and state it as follows:

Theorem III.2 $T(t)$ is isometric for the initial value of the system (2.1), that is,

$$\|T(t)\pi(0)\| = \|\pi(0)\|, \quad \forall t \in [0, \infty).$$

From Theorem III.1 and Theorem III.2, we obtain the main result in this section.

Theorem III.3 If $\overline{h} = \sup_{z\in[0,\infty)} h(z) < \infty$, then the system (II.1) has a unique nonnegative time-dependent solution $\pi(t, z)$, which satisfies

$$\|\pi(t, \cdot)\| = \|(\pi_0, \pi_1, \pi_2)(t, \cdot)\| = 1, \ \ \forall t \in [0, \infty).$$

Proof Because $\pi(0) \in D(A^2) \cap \Omega$, by Theorem III.1 and Theorem 11 of Gupur et al. [19], we know that the system (II.1) has a unique nonnegative time-dependent solution $\pi(t, z)$, and it can be expressed as

$$\pi(t, z) = T(t)\pi(0), \ \ \forall t \in [0, \infty),$$

Theorem III.2 and the above equation lead to

$$\begin{aligned} \|(\pi_0, \pi_1, \pi_2)(t, \cdot)\| &= \|T(t)(1, 0, 0)\| \\ &= \|(1, 0, 0)\| = 1, \ \ \forall t \in [0, \infty). \end{aligned} \tag{III.15}$$

Expression (III.15) just reflects the physical background of $\pi(t, x)$.

IV. Asymptotic Behavior Of The Time-Dependent Solution Of The System (II.1)

In this section, we describe the asymptotic behavior of the time-dependent solution of the two-unit system model. According to Theorem 14 in Gupur et al. [19], the asymptotic behavior of the time-dependent solution of an abstract Cauchy problem is decided by the spectral properties of the underlying operator. Hence, we examine the spectral properties of the underlying operator corresponding to this two-unit system model. First, we prove that zero is an eigenvalue of the underlying operator with geometric multiplicity one. Next, we determine the expression of the adjoint operator of the underlying operator of the two-unit system and verify that zero is an eigenvalue of the aforementioned adjoint operator. Lastly, we discuss the resolvent set of the underlying operator by applying the concept reported by Haji and Radl [21] and a result reported by Nagel [22] to prove that all points on the imaginary axis except zero belong to the resolvent set of the underlying operator. Thus, from the above results and Gupur et al. [19], we obtain the desired result.

Lemma IV.1 Zero is an eigenvalue of $A + B + C$ with geometric multiplicity one.

Proof Consider the equation $(A + B + C)\pi = 0$, that is,

$$(2\lambda + \lambda_{12})\pi_0 = \int_0^\infty \pi_1(z)h(z)\mathrm{d}z, \tag{IV.1}$$

$$\frac{\mathrm{d}\pi_1(z)}{\mathrm{d}z} = -[\lambda + \lambda_{12} + h(z)]\pi_1(z), \tag{IV.2}$$

$$\frac{\mathrm{d}\pi_2(z)}{\mathrm{d}z} = -h(z)\pi_2(z) + (\lambda + \lambda_{12})\pi_1(z), \tag{IV.3}$$

$$\pi_1(0) = 2\lambda\pi_0 + \int_0^\infty \pi_2(z)h(z)\mathrm{d}z, \tag{IV.4}$$

$$\pi_2(0) = \lambda_{12}\pi_0. \tag{IV.5}$$

By solving (IV.2) and (IV.3), we get

$$\pi_1(z) = d_1 e^{-(\lambda+\lambda_{12})z - \int_0^z h(\tau)\mathrm{d}\tau}, \tag{IV.6}$$

$$\begin{aligned} \pi_2(z) &= d_2 e^{-\int_0^z h(\tau)\mathrm{d}\tau} \\ &+ (\lambda + \lambda_{12}) e^{-\int_0^z h(\tau)\mathrm{d}\tau} \int_0^z \pi_1(\zeta) e^{\int_0^\zeta h(\tau)\mathrm{d}\tau}\mathrm{d}\zeta. \end{aligned} \tag{IV.7}$$

By combining (IV.6) with (IV.1), we obtain

$$d_1 = \frac{2\lambda + \lambda_{12}}{\int_0^\infty e^{-(\lambda+\lambda_{12})z - \int_0^z h(\tau)\mathrm{d}\tau} h(z)\mathrm{d}z}\pi_0. \tag{IV.8}$$

(4.5) means

$$d_2 = \pi_2(0) = \lambda_{12}\pi_0. \tag{IV.9}$$

Inserting (IV.9) into (IV.7), we obtain

$$\begin{aligned} \pi_2(z) =& \lambda_{12}\pi_0 e^{-\int_0^z h(\tau)\mathrm{d}\tau} + (\lambda + \lambda_{12}) e^{-\int_0^z h(\tau)\mathrm{d}\tau} \\ & \times \int_0^z \pi_1(\zeta) e^{\int_0^\zeta h(\tau)\mathrm{d}\tau}\mathrm{d}\zeta. \end{aligned} \tag{IV.10}$$

By inserting (IV.6) into (IV.10) and using the expression (IV.8) at the same time, we get

$$\begin{aligned} \pi_2(z) &= \lambda_{12}\pi_0 e^{-\int_0^z h(\tau)\mathrm{d}\tau} + (\lambda + \lambda_{12})\, e^{-\int_0^z h(\tau)\mathrm{d}\tau} \\ &\quad \times \int_0^z d_1 e^{-(\lambda+\lambda_{12})\zeta - \int_0^\zeta h(\tau)\mathrm{d}\tau} e^{\int_0^\zeta h(\tau)\mathrm{d}\tau}\mathrm{d}\zeta \\ &= e^{-\int_0^z h(\tau)d\tau} \\ &\times \left(\lambda_{12}\pi_0 + (\lambda + \lambda_{12})\, d_1 \int_0^z e^{-(\lambda+\lambda_{12})\zeta}\ \mathrm{d}\zeta\right) \\ &= e^{-\int_0^z h(\tau)d\tau} \left(\lambda_{12}\pi_0 - d_1\left(1 - e^{-(\lambda+\lambda_{12})z}\right)\right) \\ &= e^{-\int_0^z h(\tau)\mathrm{d}\tau} \\ &\times \left(\lambda_{12} - \frac{(2\lambda + \lambda_{12})\left(1 - e^{-(\lambda+\lambda_{12})z}\right)}{\int_0^\infty e^{-(\lambda+\lambda_{12})z - \int_0^z h(\tau)\mathrm{d}\tau} h(z)\mathrm{d}z}\right)\pi_0. \end{aligned} \tag{IV.11}$$

Here, (IV.6), (IV.8), and (IV.11) imply that

$$\begin{aligned} \|\pi\| &= |\pi_0| + \sum_{i=1}^{2} \|\pi_i\|_{L^1[0,\infty)} \\ &\leqslant |\pi_0| + |d_1| \int_0^\infty e^{-(\lambda+\lambda_{12})z - \int_0^z h(\tau)\mathrm{d}\,\tau}\mathrm{d}\ z \\ &\quad + \int_0^\infty \left| e^{-\int_0^z h(\tau)\mathrm{d}\tau} \right. \\ &\quad \times \left.\left(\lambda_{12} + \frac{(2\lambda + \lambda_{12})\left(1 - e^{-(\lambda+\lambda_{12})z}\right)}{\int_0^\infty e^{-(\lambda+\lambda_{12})z - \int_0^z h(\tau)\mathrm{d}\tau} h(z)\mathrm{d}z}\right)\pi_0 \right| \mathrm{d}z \\ &\leq \left(1 + \lambda_{12}\int_0^\infty e^{-\int_0^z h(\tau)\mathrm{d}\tau}\mathrm{d}z \right. \\ &\quad \left. + \frac{(2\lambda + \lambda_{12})\int_0^\infty e^{-\int_0^z h(\tau)\mathrm{d}\tau}\mathrm{d}z}{\int_0^\infty e^{-(\lambda+\lambda_{12})z - \int_0^z h(\tau)\mathrm{d}\tau} h(z)\mathrm{d}z}\right)|\pi_0| < \infty. \end{aligned}$$

This shows 0 is an eigenvalue of $A + B + C$. Moreover, (IV.6), (IV.8), and (IV.11) indicate that the eigenvectors corresponding to 0 span one-dimensional linear space; that is, the geometric multiplicity of 0 is one.

Lemma IV.2 $(A+B+C)^*$, the adjoint operator of $A+B+C$, is as follows:

$$(A+B+C)^*\pi^*$$
$$= \begin{pmatrix} -2\lambda+\lambda_{12} & 0 & 0 \\ h(z) & \frac{d}{dz}-[\lambda+\lambda_{12}+h(z)] & \lambda+\lambda_{12} \\ 0 & 0 & \frac{d}{dz}-h(z) \end{pmatrix}$$
$$\begin{pmatrix} \pi_0^* \\ \pi_1^*(z) \\ \pi_2^*(z) \end{pmatrix} + \begin{pmatrix} 0 & 2\lambda & \lambda_{12} \\ 0 & 0 & 0 \\ 0 & h(z) & 0 \end{pmatrix} \begin{pmatrix} \pi_0^* \\ \pi_1^*(0) \\ \pi_2^*(0) \end{pmatrix},$$
$$D((A+B+C))$$
$$= \left\{ \pi^* \in Z^* \left| \frac{d\pi_i^*(z)}{dz} \text{ exists and } \pi_i^*(\infty) = \mathbb{M},\ i=1,2 \right.\right\}.$$

(where $\mathbb{M}$ is a constant which irrelevant to i $(i=1,2)$).

Proof Using integration by parts and the boundary conditions on $\pi \in D(A+B+C)$, we have, for $\pi^* \in D((A+B+C)^*)$,

$$\langle (A+B+C)\pi, \pi^* \rangle$$
$$= [-(2\lambda+\lambda_{12})\pi_0 + \int_0^\infty \pi_1(z)h(z)dz]\pi_0^*$$
$$+ \int_0^\infty \left\{ -\frac{d\pi_1(z)}{dz} - [\lambda+\lambda_{12}+h(z)]\pi_1(z) \right\} \pi_1^*(z)dz$$
$$+ \int_0^\infty \left\{ -\frac{d\pi_2(z)}{dz} - h(z)\pi_2(z) + (\lambda+\lambda_{12})\pi_1(z) \right\} \pi_1^*(z)dz$$
$$= -(2\lambda+\lambda_{12})\pi_0\pi_0^* + \int_0^\infty \pi_1(z)\pi_0^* h(z)dz$$
$$- \int_0^\infty \frac{d\pi_1(z)}{dz}\pi_1^*(z)dz$$
$$- \int_0^\infty [\lambda+\lambda_{12}+h(z)]\pi_1(z)\pi_1^*(z)dz$$
$$- \int_0^\infty \frac{d\pi_2(z)}{dz}\pi_2^*(z)dz - \int_0^\infty h(z)\pi_2(z)\pi_2^*(z)dz$$
$$+ (\lambda+\lambda_{12}) \int_0^\infty \pi_1(z)\pi_2^*(z)dz$$
$$= -(2\lambda+\lambda_{12})\pi_0\pi_0^* + \int_0^\infty \pi_1(z)h(z)\pi_0^* dz$$
$$+ \pi_1(0)\pi_1^*(0) + \int_0^\infty \pi_1(z)\frac{d\pi_1^*(z)}{dz}dz$$
$$- \int_0^\infty \pi_1(z)[\lambda+\lambda_{12}+h(z)]\pi_1^*(z)dz + \pi_2(0)\pi_2^*(0)$$
$$+ \int_0^\infty \pi_2(z)\frac{d\pi_2^*(z)}{dz}dz - \int_0^\infty \pi_2(z)h(z)\pi_2^*(z)dz$$
$$+ (\lambda+\lambda_{12}) \int_0^\infty \pi_1(z)\pi_2^*(z)dz$$
$$= \pi_0[-(2\lambda+\lambda_{12})\pi_0^* + 2\lambda\pi_1^*(0) + \lambda_{12}\pi_2^*(0)]$$
$$+ \int_0^\infty \pi_1(z)[h(z)\pi_0^* + \frac{d\pi_1^*(z)}{dz}$$
$$- (\lambda+\lambda_{12}+h(z))\pi_1^*(z) + (\lambda+\lambda_{12})\pi_2^*(z)]dz$$
$$+ \int_0^\infty \pi_2(z) \left[h(z)\pi_1^*(0) + \frac{d\pi_2^*(z)}{dz} - h(z)\pi_2^*(z) \right] dz$$
$$= \langle \pi, (A+B+C)^*\pi^* \rangle.$$

From this and with the definition of adjoint operator, we know that the result of the lemma is right.

Lemma IV.3 Zero is an eigenvalue of $(A+B+C)^*$ with geometric multiplicity one.

Proof We consider the equation $(A+B+C)^*\pi^* = 0$, which is equivalent to

$$-(2\lambda+\lambda_{12})\pi_0^* + 2\lambda\pi_1^*(0) + \lambda_{12}\pi_2^*(0) = 0, \tag{IV.12}$$
$$\frac{d\pi_1^*(z)}{dz} = -h(z)\pi_0^* + [\lambda+\lambda_{12}+h(z)]\pi_1^*(z) - (\lambda+\lambda_{12})\pi_2^*(z), \tag{IV.13}$$
$$\frac{d\pi_2^*(z)}{dz} = -h(z)\pi_1^*(0) + h(z)\pi_2^*(z), \tag{IV.14}$$
$$\pi_i^*(\infty) = \mathbb{M},\ i=1,2. \tag{IV.15}$$

By solving (IV.13) and (IV.14), we get

$$\pi_1^*(z) = c_1 e^{(\lambda+\lambda_{12})z+\int_0^z h(\tau)d\tau} - e^{(\lambda+\lambda_{12})z+\int_0^z h(\tau)d\tau}$$
$$\times \int_0^z [h(\zeta)\pi_0^* + (\lambda+\lambda_{12})\pi_2^*(\zeta)]e^{-(\lambda+\lambda_{12})\zeta-\int_0^\zeta h(\tau)d\tau}d\zeta, \tag{IV.16}$$
$$\pi_2^*(z) = c_2 e^{\int_0^z h(\tau)d\tau}$$
$$- e^{\int_0^z h(\tau)d\tau} \int_0^z h(\zeta)\pi_1^*(0)e^{-\int_0^\zeta h(\tau)d\tau}d\zeta. \tag{IV.17}$$

By multiplying $e^{-(\lambda+\lambda_{12})z-\int_0^z h(\tau)d\tau}$ and $e^{-\int_0^z h(\tau)d\tau}$ two sides of (IV.16), (IV.17), and using (IV.15), we deduce

$$c_1 = \int_0^\infty [h(\zeta)\pi_0^* + (\lambda+\lambda_{12})\pi_2^*(\zeta)] \times e^{-(\lambda+\lambda_{12})\zeta-\int_0^\zeta h(\tau)d\tau}d\zeta, \tag{IV.18}$$
$$c_2 = \int_0^\infty h(\zeta)\pi_1^*(0)e^{-\int_0^\zeta h(\tau)d\tau}d\zeta. \tag{IV.19}$$

Inserting (IV.19) into (IV.17), we obtain

$$\pi_2^*(z) = e^{\int_0^z h(\tau)d\tau} \int_0^\infty h(\zeta)\pi_1^*(0)e^{-\int_0^\zeta h(\tau)d\tau}d\zeta$$
$$- e^{\int_0^z h(\tau)d\tau} \int_0^z h(\zeta)\pi_1^*(0)e^{-\int_0^\zeta h(\tau)d\tau}d\zeta$$
$$= e^{\int_0^z h(\tau)d\tau} \int_z^\infty h(\zeta)\pi_1^*(0)e^{-\int_0^\zeta h(\tau)d\tau}d\zeta = \pi_1^*(0). \tag{IV.20}$$

From (IV.20) and (IV.12), we calculate

$$-(2\lambda+\lambda_{12})\pi_0^* + 2\lambda\pi_1^*(0) + \lambda_{12}\pi_2^*(0) = 0$$
$$\Rightarrow \pi_1^*(0) = \pi_2^*(0) = \pi_2^*(z) = \pi_0^*. \tag{IV.21}$$

Substituting (IV.18) and (IV.21) into (IV.16), we get

$$\pi_1^*(z) = e^{(\lambda+\lambda_{12})z+\int_0^z h(\tau)d\tau}$$
$$\times \int_0^\infty [h(\zeta)\pi_0^* + (\lambda+\lambda_{12})\pi_2^*(\zeta)]e^{-(\lambda+\lambda_{12})\zeta-\int_0^\zeta h(\tau)d\tau}d\zeta$$
$$- e^{(\lambda+\lambda_{12})z+\int_0^z h(\tau)d\tau}$$
$$\times \int_0^z [h(\zeta)\pi_0^* + (\lambda+\lambda_{12})\pi_2^*(\zeta)]e^{-(\lambda+\lambda_{12})\zeta-\int_0^\zeta h(\tau)d\tau}d\zeta$$
$$= \pi_0^* e^{(\lambda+\lambda_{12})z+\int_0^z h(\tau)d\tau} \int_z^\infty [h(\zeta) + (\lambda+\lambda_{12})]$$
$$\times e^{-(\lambda+\lambda_{12})\zeta-\int_0^\zeta h(\tau)d\tau}d\zeta = \pi_0^*. \tag{IV.22}$$

Further, (IV.21) and (IV.22) give

$$|||\pi^*||| = \sup\left\{ |\pi_0^*|, \|\pi_1^*\|_{L^\infty[0,\infty)}, \|\pi_2^*\|_{L^\infty[0,\infty)} \right\} = |\pi_0^*| < \infty.$$

This shows that 0 is an eigenvalue of the adjoint operator $(A+B+C)^*$. Moreover, from (IV.21) and (IV.22), we see that the eigenvector space corresponding to 0 is one dimensional; that is, the geometric multiplicity of 0 is one.

In the following section, we discuss the resolvent set of the underlying operator. To do this, we first introduce the maximal operator A_m, the boundary operator G, Φ, and their domains. Next, we define the operator A_0 and its domain and discuss the resolvent set of A_0. Finally, we study the kernel of $\gamma I - A_m$, based on which we introduce the Dirichlet operator D_γ and determine the expression of ΦD_γ. By considering the spectral radius of ΦD_γ and using the result of Haji and Radl [21], we obtain the resolvent set of $A+B+C$.

The maximal operator $(A_m, D(A_m)$(see [23]) is defined as

$$A_m\pi = \begin{pmatrix} -(2\lambda+\lambda_{12}) & H & 0 \\ 0 & -(\lambda+\lambda_{12})+F & 0 \\ 0 & \lambda+\lambda_{12} & F \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1(z) \\ \pi_2(z) \end{pmatrix},$$

$$D(A_m) = \left\{ \pi \in Z \,\middle|\, \begin{array}{l} \frac{d\pi_i(z)}{dz} \in L^1[0,\infty), \pi_i(z) \text{ are absolutely} \\ \text{continuous functions}, i=1,2. \end{array} \right\},$$

where

$$Ff(z) = -\frac{df(z)}{dz} - h(z)f(z), \quad f \in W^{1,1}[0,\infty),$$
$$Hg(z) = \int_0^\infty g(z)h(z)dz, \; g \in L^1[0,\infty).$$

We take the boundary space ∂Z of Z and introduce the boundary operators

$$G : D(A_m) \to \partial Z, \;\; \Phi : D(A_m) \to \partial Z,$$

as

$$G\begin{pmatrix} \pi_0 \\ \pi_1(z) \\ \pi_2(z) \end{pmatrix} = \begin{pmatrix} \pi_1(0) \\ \pi_2(0) \end{pmatrix}$$
$$\Phi\begin{pmatrix} \pi_0 \\ \pi_1(z) \\ \pi_2(z) \end{pmatrix} = \begin{pmatrix} 2\lambda & 0 & H \\ \lambda_{12} & 0 & 0 \end{pmatrix}\begin{pmatrix} \pi_0 \\ \pi_1(z) \\ \pi_2(z) \end{pmatrix}.$$

Now, if we define the underlying operator $((A+B+C), D(A+B+C))$ as

$$(A+B+C)\pi = A_m\pi,$$
$$D(A+B+C) = \{\pi \in D(A_m) \,|\, G\pi = \Phi\pi\},$$

then the system of equations (I.1)–(I.6) can be written as the abstract Cauchy problem (II.1).

Next, we discuss the resolvent set of $A+B+C$. To do this, we define $(A_0, D(A_0))$ as

$$A_0\pi = A_m\pi, \quad D(A_0) = \{\pi \in D(A_m) | G\pi = 0\},$$

and study the inverse of A_0. For any given $y \in Z$, consider the equation $(\gamma I - A_0)\pi = y$, that is,

$$(\gamma+2\lambda+\lambda_{12})\pi_0 = \int_0^\infty \pi_1(z)h(z)dz + y_0, \tag{IV.23}$$

$$\frac{d\pi_1(z)}{dz} = -(\gamma+\lambda+\lambda_{12}+h(z))\pi_1(z) + y_1(z), \tag{IV.24}$$

$$\frac{d\pi_2(z)}{dz} = -(\gamma+h(z))\pi_2(z) + (\lambda+\lambda_{12})\pi_1(z) + y_2(z), \tag{IV.25}$$

$$\pi_i(0) = 0. \; i=1,2. \tag{IV.26}$$

From (IV.23)–(IV.25), it is easy to obtain the following:

$$\pi_0 = \frac{1}{\gamma+2\lambda+\lambda_{12}}y_0 + \frac{1}{\gamma+2\lambda+\lambda_{12}}\int_0^\infty \pi_1(z)h(z)dz, \tag{IV.27}$$

$$\pi_1(z) = e^{-(\gamma+\lambda+\lambda_{12})z - \int_0^z h(\zeta)d\zeta} \times \int_0^z y_1(\zeta)e^{(\gamma+\lambda+\lambda_{12})\zeta + \int_0^\zeta h(\tau)d\tau}d\zeta, \tag{IV.28}$$

$$\pi_2(z) = e^{-\gamma z - \int_0^z h(\zeta)d\zeta}\int_0^z y_2(\zeta)e^{\gamma\zeta + \int_0^\zeta h(\tau)d\tau}d\zeta + (\lambda+\lambda_{12})e^{-\gamma z - \int_0^z h(\zeta)d\zeta} \times \int_0^z \pi_1(\zeta)e^{\gamma\zeta + \int_0^\zeta h(\tau)d\tau}d\zeta. \tag{IV.29}$$

For $\forall g \in L^1[0,\infty)$, if we define

$$K_1g(z) = e^{-(\gamma+\lambda+\lambda_{12})z - \int_0^z h(\zeta)d\zeta} \times \int_0^z g(\zeta)e^{(\gamma+\lambda+\lambda_{12})\zeta + \int_0^\zeta h(\tau)d\tau}d\zeta,$$

$$K_2g(z) = e^{-\gamma z - \int_0^z h(\zeta)d\zeta}\int_0^z g(\zeta)e^{\gamma\zeta + \int_0^\zeta h(\tau)d\tau}d\zeta,$$

then (IV.27)–(IV.29) lead to

$$\pi_0 = \frac{1}{\gamma+2\lambda+\lambda_{12}}(y_0 + HK_1y_1(z)),$$
$$\pi_1(z) = K_1y_1(z),$$
$$\pi_2(z) = K_2y_2(z) + (\lambda+\lambda_{12})K_2\pi_1(z) = (\lambda+\lambda_{12})K_2K_1y_1(z) + K_2y_2(z).$$

That is,

$$(\gamma I - A_0)^{-1}y = \begin{pmatrix} \frac{1}{\gamma+2\lambda+\lambda_{12}} & \frac{1}{\gamma+2\lambda+\lambda_{12}}HK_1 & 0 \\ 0 & K_1 & 0 \\ 0 & (\lambda+\lambda_{12})K_2K_1 & K_2 \end{pmatrix}\begin{pmatrix} y_0 \\ y_1(z) \\ y_2(z) \end{pmatrix}.$$

From the above expression and the definition of the resolvent set, we have the following result:

Lemma IV.4 Let $h(z) : [0,\infty) \to [0,\infty)$ be measurable and

$$0 < \underline{h} = \inf_{z\in[0,\infty)} h(z) \le \sup_{z\in[0,\infty)} h(z) = \overline{h} < \infty.$$

Then,

$$\{\gamma \in \mathbb{C} \,|\, \mathrm{Re}\gamma + \lambda + \lambda_{12} > 0, \; \mathrm{Re}\gamma + \underline{h} > 0\} \subset \rho(A_0).$$

Proof By the definition of K_i $(i=1,2)$ and using integration by parts, we estimate, for any $g \in L^1[0,\infty)$,

$$\|K_1g\|_{L^1[0,\infty)} = \int_0^\infty |K_1g(z)|dz$$
$$= \int_0^\infty \left| e^{-(\gamma+\lambda+\lambda_{12})z - \int_0^z h(\zeta)d\zeta} \int_0^z g(\zeta)e^{(\gamma+\lambda+\lambda_{12})\zeta + \int_0^\zeta h(\tau)d\tau}d\zeta \right| dz$$
$$\le \int_0^\infty \left(e^{-(\mathrm{Re}\gamma+\lambda+\lambda_{12})z - \int_0^z h(\zeta)d\zeta} \int_0^z |g(\zeta)|e^{(\mathrm{Re}\gamma+\lambda+\lambda_{12})\zeta + \int_0^\zeta h(\tau)d\tau}d\zeta \right) dz$$

$$= \int_0^\infty \frac{-1}{\mathrm{Re}\gamma+\lambda+\lambda_{12}+h(z)} \left(\int_0^z |g(\zeta)| e^{(\mathrm{Re}\gamma+\lambda+\lambda_{12})\zeta+\int_0^\zeta h(\tau)\mathrm{d}\tau}\mathrm{d}\zeta\right) \times \mathrm{d}e^{-(\mathrm{Re}\gamma+\lambda+\lambda_{12})z-\int_0^z h(\zeta)\mathrm{d}\zeta}$$

$$\leq \frac{-1}{\mathrm{Re}\gamma+\lambda+\lambda_{12}+\underline{h}}\times \Big\{ e^{-(\mathrm{Re}\gamma+\lambda+\lambda_{12})-\int_0^z h(\zeta)\mathrm{d}\zeta} \times \left(\int_0^z |g(\zeta)| e^{(\mathrm{Re}\gamma+\lambda+\lambda_{12})\zeta+\int_0^\zeta h(\tau)\mathrm{d}\tau}\mathrm{d}\zeta\right)\Big|_{z=0}^{z=\infty} - \int_0^\infty |g(z)| e^{(\mathrm{Re}\gamma+\lambda+\lambda_{12})z+\int_0^z h(\zeta)\mathrm{d}\zeta} e^{-(\mathrm{Re}\gamma+\lambda+\lambda_{12})z-\int_0^z h(\zeta)\mathrm{d}\zeta}\mathrm{d}z\Big\}$$

$$= \frac{1}{\mathrm{Re}\gamma+\lambda+\lambda_{12}+\underline{h}}\|g\|_{L^1[0,\infty)}$$

$$\Rightarrow$$

$$\|K_1\| \leq \frac{1}{\mathrm{Re}\gamma+\lambda+\lambda_{12}+\underline{h}}. \tag{IV.30}$$

Similarly, we get

$$\|K_2\| \leq \frac{1}{\mathrm{Re}\gamma+\underline{h}}. \tag{IV.31}$$

For any $y \in Z$, using (IV.30) and (IV.31), the expression of $(\gamma I - A_0)^{-1}$, $\|H\| \leq \overline{h}$, and the conditions of the lemma, we estimate

$$\begin{aligned}
&\|(\gamma I - A_0)^{-1}y\| \\
&= \left|\frac{1}{\gamma+2\lambda+\lambda_{12}}y_0 + \frac{1}{\gamma+2\lambda+\lambda_{12}}HK_1y_1\right| \\
&\quad + \|K_1y_1\|_{L^1[0,\infty)} + \|(\lambda+\lambda_{12})K_2K_1y_1 + K_2y_2\|_{L^1[0,\infty)} \\
&\leq \frac{1}{|\gamma+2\lambda+\lambda_{12}|}|y_0| + \frac{1}{|\gamma+2\lambda+\lambda_{12}|}\|HK_1y_1\|_{L^1[0,\infty)} \\
&\quad + \|K_1y_1\|_{L^1[0,\infty)} + \|(\lambda+\lambda_{12})K_2K_1y_1\|_{L^1[0,\infty)} \\
&\quad + \|K_2y_2\|_{L^1[0,\infty)} \\
&\leq \frac{1}{|\gamma+2\lambda+\lambda_{12}|}|y_0| \\
&\quad + \frac{1}{|\gamma+2\lambda+\lambda_{12}|}\|H\|\|K_1\|\|y_1\|_{L^1[0,\infty)} \\
&\quad + \|K_1\|\|y_1\|_{L^1[0,\infty)} + (\lambda+\lambda_{12})\|K_2\|\|K_1\|\|y_1\|_{L^1[0,\infty)} \\
&\quad + \|K_2\|\|y_2\|_{L^1[0,\infty)} \\
&\leq \frac{1}{|\gamma+2\lambda+\lambda_{12}||}|y_0| \\
&\quad + \frac{\overline{h}}{|\gamma+2\lambda+\lambda_{12}|}\frac{1}{\mathrm{Re}\gamma+\lambda+\lambda_{12}+\underline{h}}\|y_1\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\mathrm{Re}\gamma+\lambda+\lambda_{12}+\underline{h}}\|y_1\|_{L^1[0,\infty)} \\
&\quad + \frac{\lambda+\lambda_{12}}{\mathrm{Re}\gamma+\lambda+\lambda_{12}+\underline{h}}\frac{1}{\mathrm{Re}\gamma+\underline{h}}\|y_1\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\mathrm{Re}\gamma+\underline{h}}\|y_2\|_{L^1[0,\infty)} \\
&\leq \frac{1}{\mathrm{Re}\gamma+2\lambda+\lambda_{12}}|y_0| \\
&\quad + \frac{\overline{h}}{\mathrm{Re}\gamma+2\lambda+\lambda_{12}}\frac{1}{\mathrm{Re}\gamma+\lambda+\lambda_{12}+\underline{h}}\|y_1\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\mathrm{Re}\gamma+\lambda+\lambda_{12}+\underline{h}}\|y_1\|_{L^1[0,\infty)} \\
&\quad + \frac{\lambda+\lambda_{12}}{\mathrm{Re}\gamma+\lambda+\lambda_{12}+\underline{h}}\frac{1}{\mathrm{Re}\gamma+\underline{h}}\|y_1\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\mathrm{Re}\gamma+\underline{h}}\|y_2\|_{L^1[0,\infty)} \\
&\leq \sup\left\{\frac{1}{\mathrm{Re}\gamma+2\lambda+\lambda_{12}}, \frac{1}{\mathrm{Re}\gamma+\lambda+\lambda_{12}+\underline{h}} \times \left(\frac{\overline{h}}{\mathrm{Re}\gamma+2\lambda+\lambda_{12}} + \frac{\lambda+\lambda_{12}}{\mathrm{Re}\gamma+\underline{h}}+1\right), \frac{1}{\mathrm{Re}\gamma+\underline{h}}\right\}\|y\| < \infty.
\end{aligned}$$

This shows that the assertion of this lemma is right.

Lemma IV.5 Let $h(z)$ be measurable and

$$0 < \underline{h} = \inf_{z\,\in\,[0,\,\infty)} h(z) \leq \sup_{z\,\in\,[0,\,\infty)} h(z) = \overline{h} < \infty.$$

If $\gamma \in \rho(A_0)$, then

$$\pi \in \ker(\gamma I - A_m) \Longleftrightarrow$$

$$\pi_0 = \frac{a_1}{\gamma+2\lambda+\lambda_{12}} \times \int_0^\infty e^{-(\gamma+\lambda+\lambda_{12})z-\int_0^z h(\tau)\mathrm{d}\tau}h(z)\mathrm{d}z, \tag{IV.32}$$

$$\pi_1(z) = a_1e^{-(\gamma+\lambda+\lambda_{12})z-\int_0^z h(\tau)\mathrm{d}\tau},\ |a_1| < \infty, \tag{IV.33}$$

$$\pi_2(z) = \sum_{i=1}^{2} a_ie^{-\gamma z-\int_0^z h(\tau)\mathrm{d}\tau},\ |a_2| < \infty. \tag{IV.34}$$

Proof If $\pi \in \ker(\gamma I - A_m)$, then $(\gamma I - A_m)\pi = 0$, which is equivalent to

$$(\gamma+2\lambda+\lambda_{12})\pi_0 = \int_0^\infty \pi_1(z)h(z)\mathrm{d}z, \tag{IV.35}$$

$$\frac{\mathrm{d}\pi_1(z)}{\mathrm{d}z} = -[\gamma+2\lambda+\lambda_{12}+h(z)]\pi_1(z), \tag{IV.36}$$

$$\frac{\mathrm{d}\pi_2(z)}{\mathrm{d}z} = -[\gamma+h(z)]\pi_2(z) + (\lambda+\lambda_{12})\pi_1(z), \tag{IV.37}$$

By solving (IV.36) and (IV.37), we obtain

$$\pi_1(z) = a_1e^{-(\gamma+\lambda+\lambda_{12})z-\int_0^z h(\tau)\mathrm{d}\tau}. \tag{IV.38}$$

$$\pi_2(z) = a_2e^{-\gamma z-\int_0^z h(\tau)\mathrm{d}\tau} + (\lambda+\lambda_{12})e^{-\gamma z-\int_0^z h(\tau)\mathrm{d}\tau} \times \int_0^z \pi_1(\zeta)e^{\gamma\zeta+\int_0^\zeta h(\tau)\mathrm{d}\tau}\mathrm{d}z. \tag{IV.39}$$

By inserting (IV.38) into (IV.35) and (IV.39), we obtain

$$\pi_0 = \frac{a_1}{\gamma+2\lambda+\lambda_{12}} \times \int_0^\infty e^{-(\gamma+\lambda+\lambda_{12})z-\int_0^z h(\tau)\mathrm{d}\tau}h(z)\mathrm{d}z, \tag{IV.40}$$

$$\begin{aligned}
\pi_2(z) &= a_2e^{-\gamma z-\int_0^z h(\tau)\mathrm{d}\tau} + (\lambda+\lambda_{12})a_1 \\
&\quad \times e^{-\gamma z-\int_0^z h(\tau)\mathrm{d}\tau}\int_0^\infty e^{-(\lambda+\lambda_{12})\zeta}\mathrm{d}\zeta, \\
&= \sum_{i=1}^{2} a_ie^{-\gamma z-\int_0^z h(\tau)\mathrm{d}\tau}.
\end{aligned} \tag{IV.41}$$

Since $\pi \in \ker(\gamma I - A_m)$, $\pi \in D(A_m)$, the following is obtained based on the imbedding theorem reported by Adams [18]:

$$|a_i| \leq \sum_{i\,=\,1,\,2} |a_i| = \sum_{i\,=\,1,\,2} |\pi_i(0)|$$

$$\leq \sum_{i=1,2} \|\pi_i\|_{L^\infty[0,\infty)} \leq \sum_{i=1,2} \|\pi_i\|_{L^1[0,\infty)} + \sum_{i=1,2} \left\|\frac{\mathrm{d}\pi_i}{\mathrm{d}z}\right\|_{L^1[0,\infty)} < \infty,\ i=1,2. \tag{IV.42}$$

Further, (IV.38) and (IV.40)–(IV.42) imply that (IV.32)–(IV.34) hold.

Conversely, if (IV.32)–(IV.34) are true, then by direct calculation, we obtain

$$\|\pi_1\|_{L^1[0,\infty)} \leq |a_1| \int_0^\infty e^{-(\mathrm{Re}\gamma+\lambda+\lambda_{12}+\underline{h})z}\mathrm{d}z = \frac{|a_1|}{\mathrm{Re}\gamma+\lambda+\lambda_{12}+\underline{h}}, \tag{IV.43}$$

$$\|\pi_2\|_{L^1[0,\infty)} \leq \sum_{i=1}^{2} a_i \int_0^\infty e^{-(\mathrm{Re}\gamma+\underline{h})z}\mathrm{d}z = \frac{|a_1|+|a_2|}{\mathrm{Re}\gamma+\underline{h}}, \tag{IV.44}$$

$$\begin{aligned}|\pi_0| &\leq \frac{|a_1|}{|\gamma+2\lambda+\lambda_{12}|} \times \int_0^\infty e^{-(\mathrm{Re}\gamma+\lambda+\lambda_{12})z-\int_0^z h(\tau)\mathrm{d}\tau} h(z)\mathrm{d}z \\ &\leq \frac{-|a_1|}{|\gamma+2\lambda+\lambda_{12}|} \times \int_0^\infty e^{-\int_0^z h(\tau))\mathrm{d}\tau}\mathrm{d}\left(-\int_0^z h(\tau)\mathrm{d}\tau\right) \\ &= \frac{|a_1|}{|\gamma+2\lambda+\lambda_{12}|}.\end{aligned} \tag{IV.45}$$

From (IV.43)–(IV.45), we obtain

$$\begin{aligned}\|\pi\| &= |\pi_0| + \|\pi_1\|_{L^1[0,\infty)} + \|\pi_2\|_{L^1[0,\infty)} \\ &\leq \frac{1}{|\gamma+2\lambda+\lambda_{12}|}|a_1| + \frac{1}{\mathrm{Re}\gamma+\lambda+\lambda_{12}+\underline{h}}|a_1| + \frac{1}{\mathrm{Re}+\underline{h}}|a_1| + \frac{1}{\mathrm{Re}+\underline{h}}|a_2| \\ &\leq \left(\frac{1}{|\gamma+2\lambda+\lambda_{12}|} + \frac{1}{\mathrm{Re}\gamma+\lambda+\lambda_{12}+\underline{h}} + \frac{1}{\mathrm{Re}+\underline{h}}\right)\sum_{i=1}^{2}|a_i| < \infty.\end{aligned} \tag{IV.46}$$

From (IV.33) and (IV.34), we get

$$\frac{\mathrm{d}\pi_1(z)}{\mathrm{d}z} = -a_1(\gamma+\lambda+\lambda_{12}+h(z))e^{-(\gamma+\lambda+\lambda_{12})z-\int_0^z h(\tau)\mathrm{d}\tau} = -(\gamma+\lambda+\lambda_{12}+h(z))\pi_1(z), \tag{IV.47}$$

$$\frac{\mathrm{d}\pi_2(z)}{\mathrm{d}z} = -(a_1+a_2)(\gamma+h(z))e^{-\gamma z-\int_0^z h(\tau)\mathrm{d}\tau} = -(\gamma+h(z))\pi_2(z). \tag{IV.48}$$

Further, (IV.47) and (IV.48) give

$$\begin{aligned}\sum_{i=1}^{2}\left\|\frac{\mathrm{d}\pi_i}{\mathrm{d}z}\right\|_{L^1[0,\infty)} &= \int_0^\infty |(\gamma+\lambda+\lambda_{12}+h(z))\pi_1(z)|\mathrm{d}z + \int_0^\infty |(\gamma+h(z))\pi_2(z)|\mathrm{d}z \\ &\leq (|\gamma|+\lambda+\lambda_{12}+\overline{h})\sum_{i=1}^{2}\|\pi_i\|_{L^1[0,\infty)} \\ &< \infty.\end{aligned} \tag{IV.49}$$

Moreover, (IV.46) and (IV.49) imply that $\pi \in \ker(\gamma I - A_m)$.

Note that the operator G is surjective. Hence,

$$G|_{\ker(\gamma I - A_m)} : \ker(\gamma I - A_m) \to \partial Z$$

is invertible for $\gamma \in \rho(A_0)$. Thus, for $\gamma \in \rho(A_0)$, we introduce the Dirichlet operator as

$$D_\gamma := \left(G|_{\ker(\gamma I - A_m)}\right)^{-1} : \partial Z \to \ker(\gamma I - A_m).$$

By Lemma IV.5, we get the explicit form of D_γ for $\gamma \in \rho(A_0)$ as follows:

$$D_\gamma \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \Gamma & 0 \\ \omega_1 & 0 \\ \omega_2 & \omega_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Here,

$$\Gamma = \frac{1}{(\gamma+2\lambda+\lambda_{12})}\int_0^\infty \omega_1 h(z)\mathrm{d}z,$$
$$\omega_1 = e^{-(\gamma+\lambda+\lambda_{12})z-\int_0^z h(\tau)]\mathrm{d}\tau},\ \omega_2 = e^{-\gamma z-\int_0^z h(\tau)\mathrm{d}\tau}.$$

Thus, using the definition of Φ and the expression of D_γ, it is easy to obtain the explicit form of ΦD_γ as follows:

$$\Phi D_\gamma \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2\lambda & 0 & H \\ \lambda_{12} & 0 & 0 \end{pmatrix} \begin{pmatrix} \Gamma & 0 \\ \omega_1 & 0 \\ \omega_2 & \omega_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2\lambda\Gamma + H\omega_2 & H\omega_2 \\ \lambda_{12}\Gamma & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{IV.50}$$

The following result can be found in Haji and Radl [21]:

Lemma IV.6 Suppose $\gamma \in \rho(A_0)$ and there exists $\gamma_0 \in \mathbb{C}$ such that $1 \notin \sigma(\Phi D_{\gamma_0})$, then

$$\gamma \in \sigma(A+B+C) \Longleftrightarrow 1 \in \sigma(\Phi D_\gamma).$$

By lemma IV.6 and Nagel [22], we derive the following lemma for the resolvent set of $A+B+C$.

Lemma IV.7 Let $h(z)$ be measurable and

$$0 < \underline{h} = \inf_{z\in[0,\infty)} h(z) \leq \sup_{z\in[0,\infty)} h(z) = \overline{h} < \infty.$$

Then, all points on the imaginary axis except zero belong to the resolvent set of $A+B+C$.

Proof Let $\vec{a} = (a_1, a_2)$, $|a_i| < \infty$ $(i = 1,2)$, and $\gamma = im$, $m \in \mathbb{R}\backslash\{0\}$. Then, by Riemann–Lebesgue lemma,

$$\lim_{m\to\infty}\int_0^\infty \kappa(z)\cos(mz)\mathrm{d}z = 0,$$
$$\lim_{m\to\infty}\int_0^\infty \kappa(z)\sin(mz)\mathrm{d}z = 0,\ \kappa \in L^1[0,\infty),\ \kappa(z) \geq 0.$$

Further, we know that there exists a finite positive constant $\mathcal{M} > 0$ and $\epsilon \in (0, \frac{1}{2})$ such that for $|m| > \mathcal{M}$

$$\begin{aligned}\left|\int_0^\infty \kappa(z)e^{-imz}\mathrm{d}z\right|^2 &= \left|\int_0^\infty \kappa(z)(\cos mz - i\sin mz)\mathrm{d}z\right|^2 \\ &= \left(\int_0^\infty \kappa(z)\sin(mz)\mathrm{d}z\right)^2 + \left(\int_0^\infty \kappa(z)\cos(mz)dz\right)^2 \\ &< \left(\epsilon\int_0^\infty \kappa(z)\mathrm{d}z\right)^2\end{aligned}$$

$$\Longrightarrow$$

$$\left|\int_0^\infty \kappa(z)e^{-imz}\mathrm{d}z\right| < \epsilon \int_0^\infty \kappa(z)\mathrm{d}z. \tag{IV.51}$$

So, for any $|m| > \mathcal{M}$, using (IV.50), (IV.51), and $\int_0^\infty \mu(z)e^{-\int_0^z \mu(\tau)\mathrm{d}\tau}\mathrm{d}z = 1$, we obtain

$$\begin{aligned}
\|\Phi D_\gamma \overrightarrow{a}\| &= |2\lambda\Gamma a_1 + H\omega_2 a_1 + H\omega_2 a_2| + |\lambda_{12}\Gamma a_1| \\
&\le \frac{(2\lambda+\lambda_{12})|a_1|}{\sqrt{m^2+(2\lambda+\lambda_{12})^2}} \\
&\quad \times \int_0^\infty e^{-imz-(\lambda+\lambda_{12})z-\int_0^z h(\tau)\mathrm{d}\tau} h(z)\mathrm{d}z \\
&\quad + (|a_1|+|a_2|)\int_0^\infty e^{-imz-\int_0^z h(\tau)\mathrm{d}\tau} h(z)\mathrm{d}z \\
&< |a_1|\epsilon \int_0^\infty e^{-\int_0^z h(\tau)\mathrm{d}\tau} h(z)\mathrm{d}z \\
&\quad + (|a_1|+|a_2|)\epsilon \int_0^\infty e^{\int_0^z h(\tau)\mathrm{d}\tau} h(z)\mathrm{d}z \\
&< 2(|a_1|+|a_2|)\epsilon < (|a_1|+|a_2|) = \|\overrightarrow{a}\| \\
&\Longrightarrow
\end{aligned}$$

$$\|\Phi D_\gamma\| < 1. \tag{IV.52}$$

Here, (IV.52) implies that spectral radius $r(\Phi D_\gamma) \le \|\Phi D_\gamma\| < 1$ when $|m| > \mathcal{M}$. This means $1 \notin \sigma(\Phi D_\gamma)$ when $|m| > \mathcal{M}$. This result together with Lemma IV.6 indicates that $\gamma \notin \sigma(A+B+C)$ for $|m| > \mathcal{M}$. That is,

$$\begin{aligned}
&\{im \mid |m| > \mathcal{M}\} \subset \rho(A+B+C), \\
&\{im \mid |m| \le \mathcal{M}\} \subset \sigma(A+B+C) \cap i\mathbb{R}.
\end{aligned} \tag{IV.53}$$

Moreover, since $T(t)$ is a positive contraction C_0-semigroup with spectral bound zero as per Theorem III.1 and Lemma IV.1, from Nagel [22], we know that $\sigma(A+B+C)\cap i\mathbb{R}$ is imaginary addictively cyclic, which indicates that

$$\begin{aligned}
&im \in \sigma(A+B+C)\cap i\mathbb{R} \\
&\Rightarrow imk \in \sigma(A+B+C)\cap i\mathbb{R}, \quad k \in \mathbb{N}.
\end{aligned} \tag{IV.54}$$

From this result, (IV.53), and Lemma IV.1, we conclude that $\sigma(A+B+C)\cap i\mathbb{R} = \{0\}$.

Since Theorem III.1, Lemma IV.1, Lemma IV.3, and Lemma IV.7 satisfy the conditions of Theorem 14 of Gupur et al. [19], we conclude the desired result in this paper by combining them.

Theorem IV.1 Let $h(z)$ be measurable and

$$0 < \underline{h} = \inf_{z\in[0,\infty)} h(z) \le \sup_{z\in[0,\infty)} h(z) = \overline{h} < \infty.$$

Then, the time-dependent solution of the system (2.1) strongly converges to its steady-state solution, i.e.,

$$\lim_{t\to\infty}\|\pi(t,\cdot) - \beta\pi(\cdot)\| = 0,$$

where $\pi(z)$ is the eigenvector used in Lemma IV.1.

V. Conclusion

This paper investigated the existence and asymptotic behavior of a time-dependent solution of a system with two identical components whose lifetimes follow a bivariate exponential distribution. We discussed the well-posedness and asymptotic properties of the two-unit system using the C_0-semigroup theory via functional analysis. By introducing a state space and operators and their domains, the corresponding model of the two-unit system was transformed into an abstract Cauchy problem in an appropriate Banach space. Then, we discussed the existence of a unique nonnegative time-dependent solution of the system model and verified that this time-dependent solution converges strongly to the system steady-state solution.

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