Discussion for the Belton-Gear Normalization of Analytic Hierarchy Process

Du Peng, Jinyuan Liu, Shusheng Wu, Gino Yang

Abstract—**Belton and Gear claimed that their revised method could prevent the rank reversal phenomenon in analytic hierarchy process and kept the alternatives on the same ratio. In this paper, we will prove that their method was incorrect and the rank reversal problem remains. We will use the same example with two different approaches to illustrate and then point out that their method cannot keep the same ratio, even worse their method still implies a rank reversal phenomenon. We suggested that the researchers should back to the original theory of Saaty, and consider that rank reversal is not a mathematical or theoretical predicament but is a practical phenomenon in the decision-making process for various problems. Therefore, we should follow Saaty's four methods by the criteria characteristics to solve the rank reversal phenomenon.**

Index Terms—**Analytic method, Hierarchy lattice, Solution process, Normalization**

I. Introduction

SEVENTEEN papers have cited Belton and Gear [1] in their references. Owing to the high citation, it is worth providing a deep examination of their revised method. In these seventeen papers, the application in various fields included Ayag [2], Borenstein and Betencourt [3], Tam et al. [4], Leung et al. [5], Macharis et al. [6], Zahir [7], Leung and Cao [8], Sun [9], Al-Subhi and Al-Harbi [10], Al-Harbi [11], and Weck et al. [12].

The others explored the theory and methodology involved that Raharjo and Endah [13] such that they showed the simulation result of the rank reversal phenomenon with respect to the changing values of consistency ratio and the number of alternatives. Aguaron and Moreno-Jimenez [14] provided a sensitivity analysis for selecting the best alternativeand the ranking of all the alternatives. Zahir [15] extended the conventional analytic hierarchy process to a Euclidean vector space and develop formulations for aggregation of the alternative preferences with the criteria preferences. Mulye [16] provided an empirical comparison of two methods of attribute valuation: the analytic hierarchy process and conjoint analysis. Zanakis et al. [17] investigated the performance of eight methods including analytic hierarchy process. However, none of them pointed out the essential problem in the revised method proposed by Belton and Gear [1].

Manuscript received April 28, 2024; revised January 15, 2025.

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II. Review Results of Belton and Gear

In the discussion of criteria weights, Belton and Gear [18] firstly asserted that by their normalization could eliminate rank reversals such that the criteria weights are assumed to be equal importance. With the interpretation of weight for criteria of Belton and Gear [18], Saaty and Vargas [19] presented a counter example to illustrate the Belton and Gear's normalization [18] cannot prevent the rank reversal phenomenon. Later in Belton and Gear [1] they revised their normalization procedure such that the criteria weights are assumed to be proportional to the total contribution of the criteria.

Therefore, according to Belton and Gear [1], the multiplication of the criterion and the alternative are preserved to be equal. However, we will show that the revision of Belton and Gear [1] still cannot avoid the rank reversal phenomenon.

We will consider the same example as Belton and Gear [1]. It assumes that there are three alternatives A, B and C with three criteria C_1, C_2 and C_3. Before the alternative C is considered, we express the relative weights for alternatives A and B for criteria C_1, C_2 and C_3 in the next Table 1 with the original weights for criteria asfollows,

$$\left(C_1 : C_2 : C_3\right) = \left(\alpha_1 : \alpha_2 : \alpha_3\right). \tag{2.1}$$

After alternative C is considered, the relative weights for alternatives A, B and C corresponding to criteria C_1, C_2 and C_3 is expressed in Table 2 where the weights for criteria is denoted as

$$\left(C_1 : C_2 : C_3\right) = \left(\beta_1 : \beta_2 : \beta_3\right). \tag{2.2}$$

The revised normalization procedure of Belton and Gear [1] will yield the following rules:

(Property 1) The maximum value for alternatives on each criterion is unity.

(Property 2) In the beginning, the relative ratios among criterions are equal.

(Property 3) When a new alternative adds to discuss, the relative ratios among criterions need to adjust such that (a) the new ratio of criteria times the new weight of alternative, and (b) the old ratio of criteria times the old weight of alternative, are equal. It means that to adjust β_i with

$$a_{Ai}\alpha_i = b_{Ai}\beta_i. \tag{2.3}$$

In Belton and Gear [1], they demonstrated by an example to show that their method can preserve the relative ratio between A and B so there is no rank reversal phenomenon by their revised method. In this paper, we will consider the same example with two different approaches and then point out their method cannot keep the same ratio, and even worse their method still implies rank reversal phenomenon.

Since the final weights for alternatives will be adjusted to the total sum is unity, such that during the computation to normalize the total weights of criteria becoming one is an useless operation. Therefore, in this paper, we will not modify the total sum for criteria to unity.

III. Our counter-example for Belton and Gear

We will demonstrate that the normalization method proposed by Belton and Gear [1] contained questionable results that cannot handle the rank reversal phenomenon in analytic hierarchy process. We consider the same problem in Belton and Gear [1]. The relative weights for alternatives *A*, *B* and *C* with three criteria C_1, C_2 and C_3 are listed in Table 3.

Approach 1. For the first approach, we assume that it is a three-period problem such that (a) in the first period, there is only one alternative A; (b) in the second period, there are two alternatives A and B; (c) in the third period, there are three alternatives A, B and C.

For the first period, there is only one alternative A. From Property 2, the weights for criteria are assumed to be equal as

$$(C_1, C_2, C_3) = (1,1,1). \tag{3.1}$$

According the Property 1, the relative weight for alternative A is revised to one.

For the second period, we add the other alternative B then the relative ratios between alternatives and the relative ratios among criterions will be modified as follows. By Property 1, the maximum value of each criterion is adjusted to one. The relative weights of A corresponding to C_2 and C_3 are unchanged so the weights of C_2 and C_3 are kept to 1 without revision. Moreover, by Property 3, the weights for criteria C_1 is revised as

$$(C_1, C_2, C_3) = (2,1,1), \tag{3.2}$$

since the relative ratio of C_1 is changed from 1 to 2 such that Property 3 of Belton and Gear method, as $1 \times 1 = (1/2)2$, will hold.

For the third period, we add another alternative C, then the relative ratios between alternatives and the relative ratios among criterions will modify according to Property 1 and Property 3 with

$$(C_1, C_2, C_3) = (2,1,2), \tag{3.3}$$

since the relative ratio of C_3 is changed from 1 to 2 such that Property 3 of Belton and Gear's method, as $1 \times 1 = (1/2)2$ and $(1/3) \times 1 = (1/6)2$, will hold. On the other hand, the weights of C_1 and C_2 are unchanged, owing to the weights of the alternatives A and B corresponding to criterion C_1 and C_2 are the same.

We compute the relative weights for alternatives A, B and C, respectively, in the following:

$$(1/2)2 + 1 \times 1 + (1/2)2 + 1 \times 1 = 63/21, \tag{3.4}$$

$$1 \times 2 + (6/7)1 + (1/6)2 = 67/21, \tag{3.5}$$

and

$$1 \cdot 2 + 1 \cdot 1 + 1 \cdot 2 = 5 = 105/21. \tag{3.6}$$

Hence, the relative weights for alternatives is

$$(A:B:C) = (63/235, 67/235, 105/235). \tag{3.7}$$

The computation results are list in Table 4.

Approach 2. We reconsider approach 1 but with different ordering to add criteria into consideration such that the alternatives in each period are (a) B, (b) B, A, and (c) B,A,C.

For the first period, there is only one alternative B and the by Property 2, it follows that

$$(C_1, C_2, C_3) = (1,1,1). \tag{3.8}$$

For the second period, we add the other alternative A then the relative ratios among criterions will modify by Property 3 as

$$(C_1 : C_2 : C_3) = (1 : 7/6 : 3). \tag{3.9}$$

For the third period, we add another alternative C then the relative weights among criterion will be modified as

$$(C_1 : C_2 : C_3) = (1 : 7/6 : 6). \tag{3.10}$$

We compute the relative weights for alternatives B, A and C, respectively, in the following:

$$1 \cdot 1 + (6/7) \cdot (7/6) + (1/6) \cdot 6 = 3 = 18/6, \tag{3.11}$$

$$(1/2) \cdot 1 + 1 \cdot (7/6) + (1/2) \cdot 6 = 28/6, \tag{3.12}$$

and

$$1 \cdot 1 + 1 \cdot (7/6) + 1 \cdot 6 = 49/6. \tag{3.13}$$

Hence, the relative weights for alternatives is

$$(B:A:C) = (18/95, 28/95, 49/95). \tag{3.14}$$

The computation results are list in Table 5.

Here, we compare the results in approaches 1 and 2. The relative ratio between A and B are $A/B = 63/67$ in approach 1 and $A/B = 28/18$ appeared in approach 2. The two ratios are in reverse order. Therefore, in Belton and Gear [1], Page 143, right column, Line 3 and 18, they claimed that their method will keep the same ratio is false, because we derive that in approach 1, $A < B$ and in approach 2, $A > B$. Their revised method still implies the rank reversal phenomenon with two different approaches 1 and 2.

Table 1. Before C is considered

	C_1	C_2	C_3
A	a_{A1}	a_{A2}	a_{A3}
B	a_{B1}	a_{B2}	a_{B3}

Table 2. After C is considered

	C_1	C_2	C_3
A	b_{A1}	b_{A2}	b_{A3}
B	b_{B1}	b_{B2}	b_{B3}
C	b_{C1}	b_{C2}	b_{C3}

Table 3. The relative weights

	C_1	C_2	C_3
A	1/5	7/20	3/10
B	2/5	6/20	1/10
C	2/5	7/20	6/10

Table 4. The relative weights by B-G method in different period for approach 1.

	C_1	C_2	C_3		C_1	C_2	C_3		C_1	C_2	C_3
A	1	1	1	A	1/2	1	1	A	1/2	1	1/2
				B	1	6/7	1/3	B	1	6/7	1/6
								C	1	1	1

Table 5. The relative weights by B-G method in different period for approach 2.

	C_1	C_2	C_3		C_1	C_2	C_3		C_1	C_2	C_3
B	1	1	1	B	1	6/7	1/3	B	1	6/7	1/6
				A	1/2	1	1	A	1/2	1	1/2
								C	1	1	1

IV. Rank Reversal Problem

A rank reversal problem had been raised by Dyer [20] to point out that adding a new alternative sometimes will shange the original ordering of alternatives. We consider the example proposed by Dyer [20] that the relative weight of alternative A corresponding to criteria C_2, C_3, and C_1 is expressed as $(9,8,1)^T$, where "T" denotes the transpose a row vector into a column vector. We will offer a detailed discussion to this relative weight to point out that it is not derived as the principal eigenvector corresponding the maximum eigenvalue for a comparison matrix. Consequently, we will demonstrate that the counterexample in Dyer [20] for the rank reversal phenomena will not happen.

We normalized $(9,8,1)^T$ to $(9/18, 8/18, 1/18)^T$, and then we show that $(9/18, 8/18, 1/18)^T$ is not a normalized principal eigenvector corresponding the maximum eigenvalue for a comparison matrix. Given a general three by three comparison matrix, say $A = [a_{ij}]_{3\times3}$, with entries a_{ij} that satisfies $a_{ij}a_{ji} = 1$, $a_{ii} = 1$, for $i, j \in \{1,2,3\}$, and $a_{ij} \in \{1,2,\dots,9, 1/2, 1/3, \dots, 1/9\}$.

For the matrix, $A = [a_{ij}]_{3\times3}$, the maximum eigenvalue, say λ_{max} and the normalized corresponding eigenvector, say $(b_1, b_2, b_3)^T$.

Moreover, we assume the distance between $(b_1, b_2, b_3)^T$ and $(9/18, 8/18, 1/18)^T$ as denoted by $d(A,(9,8,1))$. We begin to compute the minimum value for $d(A,(9,8,1))$ where $A = [a_{ij}]_{3\times3}$ is a comparison matrix with $a_{ij} \in \{1,2,\dots,9, 1/2, 1/3, \dots, 1/9\}$. Hence, with the help of computer program "Mathcad", we find the minimum value occurs at

$$A = \begin{bmatrix} 1 & 1 & 9 \\ 1 & 1 & 6 \\ 1/9 & 1/6 & 1 \end{bmatrix}, \tag{4.1}$$

with maximum eigenvalue

$$\lambda_{max} = 3.0193, \tag{4.2}$$

and the normalized corresponding eigenvector

$$(b_1, b_2, b_3) = (0.4998, 0.4366, 0.0636), \tag{4.3}$$

such that the distance between $(9/18, 8/18, 1/18)^T$ and $(b_1, b_2, b_3)^T$ is 0.0112.

It indicates that $(9/18, 8/18, 1/18)^T$ is not directly derived from any comparison matrix. Therefore, we must modify $(9/18, 8/18, 1/18)^T$ such that the new expression satisfies the requirement as the normalized corresponding eigenvector of a comparison matrix.

The first modification is based on the previous discussion. We may assume that $(9/18, 8/18, 1/18)^T$ is a simplified expression of $(b_1, b_2, b_3) = (0.4998, 0.4366, 0.0636)$ for the purpose to express as a fraction without the decimal expression. Our first modification is to use the original normalized corresponding eigenvector to avoid the possible error of the fraction expression.

The second modification is to consider what kind comparison matrix will derive the normalized corresponding eigenvector of $(9/18, 8/18, 1/18)^T$. In the beginning, we use the following matrix,

$$B = \begin{bmatrix} 1 & 9/8 & 9 \\ 8/9 & 1 & 8 \\ 1/9 & 1/8 & 1 \end{bmatrix}. \tag{4.4}$$

However, the entries a_{12} and a_{21} do not satisfy the condition of $a_{ij} \in \{1,2,\dots,9, 1/2, 1/3, \dots, 1/9\}$. Therefore, in the second modification, we revise the entries of the comparison matrix to improve a_{12} and a_{21}.

With respect to $9/8$ and $8/9$, we may say that the closest number in $\{1,2,\dots,9, 1/2, 1/3, \dots, 1/9\}$ is 1 such that we rewrite the matrix of B as follows,

$$C = \begin{bmatrix} 1 & 1 & 9 \\ 1 & 1 & 8 \\ 1/9 & 1/8 & 1 \end{bmatrix}, \tag{4.5}$$

and then the maximum eigenvalue,

$$\lambda_{max} = 3.0015, \tag{4.6}$$

and the normalized corresponding eigenvector,

$$(c_1, c_2, c_3) = (0.4815, 0.4629, 0.0556), \tag{4.7}$$

such that the distance to $(9/18, 8/18, 1/18)^T$ is 0.0262.

The third modification is to consider directly revise the relative weights form $(9,8,1)$ to a new setting and assume the comparison matrix is perfectly consistent as Dyer [20] and Finan and Hurley [21]. Since the ratio between 8 and 9 is not in the permissible set $\{1,2,\dots,9, 1/2, 1/3, \dots, 1/9\}$ and

$$1 = 9/9 = 8/8, \tag{4.8}$$

which is the colosest element in the permissible set with respect to $9/8$ or $8/9$. There are two possible ways to rewrite the relative weight as $(8,8,1)^T$ or $(9,9,1)^T$.

The distance between normalized new relative weight $(8/17, 8/17, 1/17)^T$ or $(9/19, 9/19, 1/19)^T$ to $(9/18, 8/18, 1/18)^T$ is computed as 0.0395 and 0.0394, respectively. Dyer [20] used a comparison matrix for alternatives A, B, C and D with $a_{24} = 9/8$, however, $9/8$ is not in the permissible comparison results in the set of $\{1,2,\dots,9, 1/2, 1/3, \dots, 1/9\}$.

V. Application of Our Revisions

Dyer [20] constructed the following table for the relative weights for three alternatives A, B and C, corresponding to three criteria C_1, C_2, and C_3,

$$\begin{array}{cccc} & C_1 & C_2 & C_3 \\ A & 1 & 9 & 8 \\ B & 9 & 1 & 9 \\ C & 1 & 1 & 1 \end{array}. \quad (5.1)$$

Dyer [20] claimed that these alternatives are evaluated by analytic hierarchy process using the principle of hierarchy composition and assuming equal weights on the criteria, and then the rankings in order are B, A and C. Belton and Gear [1] added a fourth alternative, say D, which is an exact copy of the alternative B to derive the new ranking as A, B and D (tie) and C such that the rank of B and A is reversal. Harker and Vargas [22] criticized the above example because alternative D is a copy of B. Dyer [20] provided a near copy of alternative B as new alternative D.

We have a plan to reduce the computation amount of comparison matrices from 17^3 to 9^3 such that we rearrange the order of alternatives from the largest to the smallest. We predict that the entries in the upper triangle will be selected from the set $\{1,2,\ldots,9\}$ such that $\{1/2, 1/3, \ldots, 1/9\}$ will not be appeared in the upper triangle. In the following, we present our theoretical result.

Lemma 1. We assume that $a_1 \geq a_2 \geq a_3 > 0$, $a_1 + a_2 + a_3 = 1$, $b_1 \geq b_2 \geq b_3 > 0$, $b_1 + b_2 + b_3 = 1$, and $a_3 = b_3$, and then we will show that (a_1, a_2, a_3) is more close to (b_1, b_2, b_3) than (a_2, a_1, a_3) to (b_1, b_2, b_3) that is we will try to show that

$$(a_2 - b_1)^2 + (a_1 - b_2)^2 \geq (a_1 - b_1)^2 + (a_2 - b_2)^2. \quad (5.2)$$

We will divide the proof procedure into three cases: Case I: $a_1 \geq a_2 \geq b_1 \geq b_2$, Case II: $a_1 \geq b_1 > a_2 \geq b_2$, and Case III: $a_1 \geq b_1 \geq b_2 > a_2$.

For Case I, under the restriction of $a_1 \geq a_2 \geq b_1 \geq b_2$, we assume three new parameters, $x = a_1 - a_2$, $y = a_2 - b_1$, and $z = b_1 - b_2$ to simplify the expressions. Our goal of Equation (5.2) is equivalent to the following,

$$(x + y + z)^2 + y^2 \geq (x + y)^2 + (y + z)^2. \quad (5.3)$$

We can simplify the inequality of Equation (5.3) as follows,

$$2xz \geq 0. \quad (5.4)$$

Based on our derivation of Equation (5.4), we imply that the inequality of Equation (5.3) is valid such that we finish the proof for Case I.

For Case II, under the restriction of $a_1 \geq b_1 > a_2 \geq b_2$, we define three new parameters, $\alpha = a_1 - b_1$, $\beta = b_1 - a_2$, and $\gamma = a_2 - b_2$ to simplify the expressions. Our goal of Equation (5.2) is equivalent to the following,

$$(-\beta)^2 + (\alpha + \beta + \gamma)^2 \geq \alpha^2 + \gamma^2. \quad (5.5)$$

We can simplify the inequality of Equation (5.5) as follows,

$$2\alpha\gamma + 2\beta(\alpha + \gamma) + 2\beta^2 \geq 0. \quad (5.6)$$

Referring to our finding of Equation (5.6), we claim that the inequality of Equation (5.5) is hold such that we terminate the proof for Case II.

For Case III, under the restriction of $a_1 \geq b_1 \geq b_2 > a_2$, we define three new parameters, $r = a_1 - b_1$, $s = b_1 - b_2$, and $t = b_2 - a_2$ to simplify the expressions. Our goal of Equation (5.2) is equivalent to the following,

$$(-s - t)^2 + (r + s)^2 \geq r^2 + s^2. \quad (5.7)$$

We can simplify the inequality of Equation (5.7) as follows,

$$2st + 2sr + s^2 + t^2 \geq 0. \quad (5.8)$$

According to our result of Equation (5.8), we derive that the inequality of Equation (5.7) is true such that we complete the verification of Case III.

With respect to Lemma 1, we conclude that the order of the closed eigenvector for $(9/18, 8/18, 1/18)^T$ may have the same ordering for the components, if $a_3 = b_3$.

VI. Revisions of Examples

We consider the numerical examples proposed by Dyer and Wendell [23], and Dyer [20]. Applying our second modification, we obtain the following results,

	C_1	C_2	C_3	C_4	Score	Rank
A_1	0.056	9/11	0.072	0.309	0.314	3
A_2	0.481	1/11	0.627	0.110	0.327	2
A_3	0.463	1/11	0.301	0.581	0.359	1

Based on our second modification, we derive the same rank as Dyer [20]. We consider the same numerical examples proposed by Dyer [20]. Applying our third modification, we derive the following findings,

	C_1	C_2	C_3	C_4	Score	Rank
A_1	1/19	9/11	1/13	3/10	0.312	3
A_2	9/19	1/11	8/13	1/10	0.320	2
A_3	9/19	1/11	4/13	6/10	0.368	1

Based on our third modification, we still derive the same rank as Dyer [20].

Next, we examine our third modification with respect to the comparison matrix proposed by Dyer [20] to add another alternative, denoted as A_4, which is a copy of the alternative A_3, and then we list the new comparison matrix in the following,

	C_1	C_2	C_3	C_4	Score	Rank
A_1	1/21	9/11	1/21	3/16	0.259	1
A_2	8/21	1/11	8/21	1/16	0.227	4
A_3	8/21	1/11	4/21	6/16	0.257	2
A_4	4/21	1/11	8/21	6/16	0.257	2

Based on our third modification, we still derive the same rank reversal results as Dyer [20].

We reconsider the above examples with revised epressions of Dyer [20], and then apply our third modification to yield the next results,

	C_1	C_2	C_3	Score	Rank
A_1	1/11	9/11	8/17	0.460	1
A_2	9/11	1/11	8/17	0.460	1
A_3	1/11	1/11	1/17	0.080	3

Next, we examine our third modification with respect to the comparison matrix mentioned above to add another alternative, denoted as A_4, which is a copy of the alternative A_3, and then we list the new comparison matrix in the following,

	C_1	C_2	C_3	Score	Rank
A_1	1/20	9/11	8/25	0.374	1
A_2	9/20	1/11	8/25	0.284	2
A_3	1/20	1/11	1/25	0.058	4
A_4	9/20	1/11	8/25	0.284	2

We rerun the counter example of Dyer [20]. In Dyer [20], he claimed that before alternative A_4 is added, $A_2 \succ A_1$ and after alternative A_4 is added, $A_1 \succ A_2$ so Dyer [20] claimed that the rank reversal phenomenon happened. However, we reconsider the same problem by our third modification. Before the alternative A_4 is added, $A_2 = A_1$, and after alternative A_4 is added, $A_1 \succ A_2$ so the rank reversal phenomenon does not occur. It indicates that the counterexample of Dyer [20] contains questionable result.

We carefully examine our first modification is too close to Dyer's result such that we cannot prevent the rank reversal. Our second modification is a litter far from Dyer's result such that we still cannot prevent the rank reversal. Our third modification is easy to compute and change a lot such that the third method has the best possibility to prevent the rank reversal.

VII. A Further Examination

In the following, we will show that our second modification will imply the same rank reversal problem as Dyer [20]. With the fourth alternative, A_4, we consider the first column, $(1,9,1,8)^T$, and then we change the comparison matrix from

$$\begin{matrix} 1 & 1/9 & 1 & 1/8 \\ 9 & 1 & 9 & 9/8 \\ 1 & 1/9 & 1 & 1/8 \\ 8 & 8/9 & 8 & 1 \end{matrix}, \tag{7.1}$$

to

$$\begin{matrix} 1 & 1/9 & 1 & 1/8 \\ 9 & 1 & 9 & 1 \\ 1 & 1/9 & 1 & 1/8 \\ 8 & 1 & 8 & 1 \end{matrix}. \tag{7.2}$$

We obtain the maximum eigenvalue,

$$\lambda_{max} = 4.0017, \tag{7.3}$$

and the corresponding normalized eigenvector is

$$(0.0527, 0.4605, 0.0527, 0.4341)^T. \tag{7.4}$$

For the third column, $(8,9,1,8)^T$, we change the comparison matrix from

$$\begin{matrix} 1 & 8/9 & 8 & 1 \\ 9/8 & 1 & 9 & 9/8 \\ 1/8 & 1/9 & 1 & 1/8 \\ 1 & 8/9 & 8 & 1 \end{matrix}, \tag{7.5}$$

to

$$\begin{matrix} 1 & 1 & 8 & 1 \\ 1 & 1 & 9 & 1 \\ 1/8 & 1/9 & 1 & 1/8 \\ 1 & 1 & 8 & 1 \end{matrix}. \tag{7.6}$$

We obtain the maximum eigenvalue,

$$\lambda_{max} = 4.0017, \tag{7.7}$$

and the corresponding normalized eigenvector is

$$(0.3173, 0.3269, 0.0385, 0.3173)^T. \tag{7.8}$$

We derive the synthesized relative weight is

$$(0.3733, 0.2902, 0.0582, 0.2783)^T, \tag{7.9}$$

such that we claim that

$$A_1 \succ A_2. \tag{7.10}$$

Please refer to (9,8,1) and (8,9,1), in our previous computation, we use (9,8,1), but in Dyer's paper, he used (8,9,1). The synthesized relative weight is

$$(0.4573, 0.4635, 0.0791)^T, \tag{7.11}$$

such that we derive that

$$A_2 \succ A_1. \tag{7.12}$$

We conclude that for this counterexample our second method will imply the same rank reversal phenomenon as Dyer [20].

VIII. An In-death Analysis of Previous Results

Dyer [20] mentioned that the principal eigenvector is $(4,9,1)^T$, however, he did not offer the original comparison matrix. In the following, we will illustrate that his assertion of the principal eigenvector being $(4,9,1)^T$ which is not derived by the maximum eigenvalue and the corresponding eigenvector proposed by Belton and Gear [18], and Saaty and Vargas [19].

We normalized $(4,9,1)^T$ to $(4/14, 9/14, 1/14)^T$, and then we will show that $(4/14, 9/14, 1/14)^T$ is not a normalized principal eigenvector corresponding the maximum eigenvalue for a comparison matrix. Given a general three by three comparison matrix, say $A = [a_{ij}]_{3\times 3}$, with entries a_{ij} that satisfies $a_{ij}a_{ji} = 1$, $a_{ii} = 1$, for $i, j \in \{1,2,3\}$, and $a_{ij} \in \{1,2,\ldots,9,1/2,1/3,\ldots,1/9\}$, and then we find the maximum eigenvalue, say λ_{max} and the normalized corresponding eigenvector, say $V = (a_1, a_2, a_3)^T$.

Moreover, we assume the distance between $(a_1, a_2, a_3)^T$ and $(4/14, 9/14, 1/14)^T$ as denoted by $d(V,(4,9,1))$. We begin to compute the minimum value for $d(V,(4,9,1))$.

Hence, with the help of computer program "Mathcad", we find the minimum value occurs at

$$A = \begin{bmatrix} 1 & 1/4 & 7 \\ 4 & 1 & 5 \\ 1/7 & 1/5 & 1 \end{bmatrix}, \tag{8.1}$$

with maximum eigenvalue $\lambda_{max} = 3.0193$ and the normalized corresponding eigenvector,

$$V = (0.4366, 0.4998, 0.0636), \tag{8.2}$$

such that the distance between $(4/14, 9/14, 1/14)^T$ and $V = (a_1, a_2, a_3)^T$ is derived as

$$d(V,(4,9,1)) = 0.0011. \tag{8.3}$$

Our finding of Equation (8.3) points out that the assertion of $(4,9,1)^T$ proposed by Dyer [20] is not derived from any comparison matrix.

Therefore, we must modify $(4/14, 9/14, 1/14)^T$ such that the new expression satisfies the requirement as the normalized corresponding eigenvector of a comparison matrix.

The following modification is to consider what kind comparison matrix will derive the normalized corresponding eigenvector of $(4/14, 9/14, 1/14)^T$. In the beginning, we use the following matrix,

$$B = \begin{bmatrix} 1 & 4/9 & 4 \\ 9/4 & 1 & 9 \\ 1/4 & 1/9 & 1 \end{bmatrix}. \tag{8.4}$$

However, the entries of b_{12} and b_{21} do not satisfy the condition of $b_{ij} \in \{1,2,\ldots,9,1/2,1/3,\ldots,1/9\}$. Therefore, in the following modification, we revise the entries of the comparison matrix to improve b_{12} and b_{21}.

We may say that the closest number in the set proposed by Saaty and Vargas [19] as $\{1,2,\ldots,9,1/2,1/3,\ldots,1/9\}$ is denoted as $b_{12} = 1/2$ and $b_{21} = 2$ such that we rewrite the comparison matrix of Equation (8.4) in the following,

$$C = \begin{bmatrix} 1 & 1/2 & 4 \\ 2 & 1 & 9 \\ 1/4 & 1/9 & 1 \end{bmatrix}. \tag{8.5}$$

and then the maximum eigenvalue $\lambda_{max} = 3.0015$ and the normalized corresponding eigenvector,

$$W = (0.4815, 0.4629, 0.0556), \tag{8.6}$$

such that the distance between $(4/14, 9/14, 1/14)^T$ and W is derived as

$$d(W,(4,9,1)) = 0.0262. \tag{8.6}$$

The results of Equation (8.6) indicates that the finding by Mathcad is superior to the intuitive approach by modification of a consistent matrix of Equation (8.4).

Our further modification is to consider directly revise the relative weights form $(4,9,1)^T$ and assume the comparison matrix is perfectly consistent as Dyer [20] and Finan and Hurley [21]. Since the ratio between 4 and 9 is not in the permissible set $\{1,2,\ldots,9,1/2,1/3,\ldots,1/9\}$. We observe that

$$1/3 < 4/9 < 1/2, \tag{8.7}$$

and

$$2 < 9/4 < 3, \tag{8.8}$$

such that the closed way to express the ratio as

$$4/9 \sim 1/2, \tag{8.9}$$

and

$$9/4 \sim 2. \tag{8.10}$$

Therefore, we rewrite the relative weight as $(4,8,1)^{\mathrm{T}}$.

The distance between normalized new relative weight $\mathrm{U}=(4/13,8/13,1/13)^{\mathrm{T}}$ to $(4/14,9/14,1/14)^{\mathrm{T}}$ is computed as

$$\mathrm{d}\big(\mathrm{U},(4/14,9/14,1/14)\big)=0.0395, \tag{8.11}$$

which illustrate that the further modification through the resulting eigenvector will imply a large derivation.

With respect to the second example in Dyer [20], for the first column $(1,3,3/5,\ 3/5)^{\mathrm{T}}$, we change the comparison matrix from

$$\begin{matrix} 1 & 3 & 3/5 & 3/5 \\ 1/3 & 1 & 1/5 & 1/5 \\ 5/3 & 5 & 1 & 1 \\ 5/3 & 5 & 1 & 1 \end{matrix}, \tag{8.12}$$

to

$$\begin{matrix} 1 & 3 & 1/2 & 1/2 \\ 1/3 & 1 & 1/5 & 1/5 \\ 2 & 5 & 1 & 1 \\ 2 & 5 & 1 & 1 \end{matrix}, \tag{8.13}$$

such that the maximum eigenvalue $\lambda_{\max}=4.0042$ and the normalized eigenvector is $(0.1929,0.0704,0.3683,0.3683)^{\mathrm{T}}$, and then the relative weight is

$$\mathrm{P}=(0.2586,0.2327,0.2543,0.2543)^{\mathrm{T}}, \tag{8.14}$$

such that we still imply the same rank as Dyer [20].

IX. Another Discussion

We will provide another patch work for Dyer [20] to consider an improvement in an abstract setting. Dyer [20] used the relative ratio for three alternatives which was denoted as A_1, A_2 and A_3 for the criterion C_3 as $(8,9,1)$. Hence, we normalized the weight to $(1/18,9/18,8/18)$. We will point out that $(1/18,9/18,8/18)$ is not a normalized priority vector for any comparison matrix under the restriction of entries designed by the rule proposed by Saaty and Vargas [19]. Hence, it is not a suitable relative ratio for alternatives. Consequently, those examples are not obtained by any real comparison matrix in Dyer [20] which are based on debatable background so the rank reversal phenomenon in Dyer [20] required more investigation.

We consider the following comparison matrix

$$\begin{bmatrix} 1 & a_{12} & a_{13} \\ 1/a_{12} & 1 & a_{23} \\ 1/a_{13} & 1/a_{23} & 1 \end{bmatrix} \tag{9.1}$$

with $a_{ij}\in\{1,\ldots 9,1/2,\ldots,1/9\}$, to find the priority vector, say (v_1,v_2,v_3), corresponding to the maximum eigenvalue such that $v_1+v_2+v_3=1$. We try to show that the normalized vector of $(8,9,1)$, that is $(8/18,9/18,1/18)$, is not a priority vector for a comparison matrix.

We suppose that $\lfloor a_{ij}\rfloor$ is the comparison matrix for three alternative A_1, A_2 and A_3 with the priority vector (v_1,v_2,v_3) such that

$$\left[a_{ij}\right]\left[v_1,v_2,v_3\right]^T=\lambda_{\max}\left[v_1,v_2,v_3\right]^T. \tag{9.2}$$

If we interchange the position of A_1 and A_2 then the comparison matrix, say $\lfloor b_{ij}\rfloor$ that satisfies

$$\lfloor b_{ij}\rfloor=\begin{bmatrix} 1 & a_{21} & a_{23} \\ a_{12} & 1 & a_{13} \\ a_{32} & a_{31} & 1 \end{bmatrix}. \tag{9.3}$$

It means that $\lfloor b_{ij}\rfloor$ is made by (a) interchanging the first row and the second row of $\lfloor a_{ij}\rfloor$, where we assume the new matrix as $\lfloor c_{ij}\rfloor$, and then (b) interchanging the first column and the second column of $\lfloor c_{ij}\rfloor$.

Now we consider the characteristic functions of $\lfloor a_{ij}\rfloor$ as the determinant of $\lfloor a_{ij}-\lambda\delta_{ij}\rfloor$ where $\delta_{ij}=1$, when $i=j$ and $\delta_{ij}=0$, when $i\neq j$. On the other hand, the characteristic functions of $\lfloor b_{ij}\rfloor$ is the determinant of $\lfloor b_{ij}-\lambda\delta_{ij}\rfloor$. We know that interchange two rows (or two columns) will change the sign of the determinant. It implies that $\lfloor a_{ij}\rfloor$ and $\lfloor b_{ij}\rfloor$ have the same characteristic function so they have the same maximum eigenvalue. Moreover, from Equation (9.2), we obtain that

$$\left[b_{ij}\right]\left[v_2,v_1,v_3\right]^T=\lambda_{\max}\left[v_2,v_1,v_3\right]^T. \tag{9.4}$$

Equation (9.4) implies the priority vector corresponding to the maximum eigenvalue for interchanging two alternatives is the original priority vector to interchange two corresponding components.

X. A Related Problem

The purpose of this section is fourfold. First, we use algebraic method to find the optimal solution for an economic ordering quantity model with imperfect quality such that this important extension of the traditional economic ordering quantity inventory model may introduce to those practitioners without the background of calculus. Second, we find the closed form solution for the maximum value of the economic ordering quantity model with imperfect quality. Third, using calculus, we proof that the optimal solution for economic ordering quantity model with imperfect quality is greater than the traditional optimal replenishment quantity. It reveals the real power of calculus. Fourth, the same numerical example is computed to demonstrate that our algebraic method is very easy to apply and derives the corrected solution.

To be compatible with Salameh and Jaber [24], we will use the same assumptions and notation as theirs. We review the economic ordering quantity model of Salameh and Jaber [24] with imperfect quality. Salameh and Jaber [24] considered the economic production quantity model where a lot of size y is delivered instantaneously with a purchasing price of c per unit and an ordering cost of *K*. It is assumed that each lot received contains percentage defectives, P, with a known probability density function, $f(p)$. The selling price of good-quality item is *s* per unit. A 100% percent screening process of the lot is conducted at a rate of *x* units per unit time.

To be comparable with previous studies, we used the same notation as Salameh and Jaber [24] and then add some extra notation to simplify the expression. The notation s are listed in the following.

a_{-1} denotes an abbreviation to stand for $a_{-1} = -DKM$.

a_0 denotes an abbreviation with $a_0 = D(s-c-d)M$.

a_1 denotes an abbreviation to stand for $a_1 = -\dfrac{1-E(P)}{2}h$.

y^*_{trad} denotes the traditional optimal solution, with the condition, $y^*_{trad} = \sqrt{\dfrac{2DK}{h}}$.

y^* denotes the optimal solution for this economic ordering quantity model with imperfect quality.

Ψ denotes the comparison between y^* and y^*_{trad} as the following condition, with $\Psi = \dfrac{y^*}{y^*_{trad}}$.

y denotes order size, the decision variable.

P denotes the percentage of defective items in *y* with probability density function $f(p)$.

$N(y,P)$ denotes the number of good items, with $N(y,P) = y - Py$.

M denotes the expected value of $\dfrac{1}{1-P}$, under the restriction of $M = E\left(\dfrac{1}{1-P}\right)$.

x denotes the screening rate per minute.

X denotes the total screening items per year, with $X = 8(60x)365$, under the restriction of $X > D$.

t denotes the total screening time of y units ordered per cycle, with the condition, $y = tx$.

v denotes the unit selling price of defective items.

s denotes the unit selling price of items of good quality.

d denotes the unit screening cost.

c denotes the purchasing price per unit.

K denotes the fixed cost of placing an order.

T denotes the cycle length.

D denotes the demand per year.

In Rosenblatt and Lee [25], the defective items can be reworked instantaneously at a cost. Salameh and Jaber [24] adopted alternative method to handle the defective items such that they are kept in stock and sold prior to receiving the next shipment as a single batch at a discounted price of *v* per unit. They tried to avoid shortages such that the number of good items, $N(y,P)$, is at least equal to the demand during screening time, *t*, that is

$$1 - \frac{D}{X} \geq P. \tag{10.1}$$

The expected total profit per unit time, $ETPU(y)$, is the total revenue less the total cost dividing by the cycle length. We directly quote the objective mapping of Salameh and Jaber [24] in the following,

$$ETPU(y) = D\left(s - v + \frac{h}{X}y\right) - \frac{1-E(P)}{2}hy,$$

$$+ D\left(v - \frac{h}{X}y - c - d - \frac{K}{y}\right)E\left(\frac{1}{1-P}\right). \tag{10.2}$$

In the objective mapping of Salameh and Jaber [24], they used the expression "*x*". However, *x* is represented the screening rate per minute. We modify the expression to replace *x* by *X*, the total screening items per year. Salameh and Jaber [24] used the calculus to find the solution for the first derivative of $ETPU(y)$ to derive the possible optimal order size and then used the second derivative of $ETPU(y)$ to verify that $ETPU(y)$ is a concave function. Hence, the solution for the first derivative becomes the optimal solution. We will use the algebraic method to solve the same problem such that this kind economic ordering quantity model with imperfect quality may introduce to those practitioners without the knowledge of calculus.

XI. Our Algebraic Approach

We rewrite Equation (10.2) abstractly as

$$ETPU(y) = -\left(a_1 y - a_0 + \frac{a_{-1}}{y}\right), \tag{11.1}$$

with three abbreviations,

$$a_1 = D\left(\frac{1-E(P)}{2D} + \frac{M-1}{X}\right)h, \tag{11.2}$$

$$a_0 = D(s-v) + D(v-c-d)M, \tag{11.3}$$

and

$$a_{-1} = DKM. \tag{11.4}$$

Our goal is to maximize $ETPU(y)$ under the constraint $1 - \dfrac{D}{X} \geq P$ in Equation (10.1). From Equation (11.1), it can be solved by algebraic method as

$$ETPU(y) = a_0 - 2\sqrt{a_1 a_{-1}},$$

$$-\left(\sqrt{a_1 y} - \sqrt{\frac{a_{-1}}{y}}\right)^2. \tag{11.5}$$

From Equation (11.5), we know that the maximum point, say

$$y^* = \sqrt{\frac{a_{-1}}{a_1}}, \tag{11.6}$$

and maximum value is derived as follows,

$$ETPU(y^*) = a_0 - 2\sqrt{a_1 a_{-1}}. \tag{11.7}$$

If we compare our optimal solution

$$y^* = \sqrt{\frac{a_{-1}}{a_1}},$$

$$= \sqrt{\frac{2DKM}{\left(1-E(P)+2D\frac{M-1}{X}\right)h}}, \quad (11.8)$$

to that of of Salameh and Jaber [24], their type error have been corrected by our result as Equation (11.8). Moreover, by our algebraic method, it shows that the closed form solution for the maximum value is expressed as

$$ETPU(y^*) = D(s-v) + D(v-c-d)M,$$

$$-2D\sqrt{\left(\frac{1-E(P)}{2D}+\frac{M-1}{X}\right)hKM}. \quad (11.9)$$

In the next section, we will compare the optimal solution for the inventory system with imperfect quality with the traditional inventory model.

XII. Comparison with Traditional Inventory Model

In Salameh and Jaber [24], they tried to compare

$$y^* = \sqrt{\frac{2DKM}{\left(1-E(P)+2D\frac{M-1}{X}\right)h}}, \quad (12.1)$$

of Equation (11.8) with the traditional optimal solution, say y^*_{trad}, with

$$y^*_{trad} = \sqrt{\frac{2DK}{h}}. \quad (12.2)$$

For uniform distribution, they only used the numerical method to compare the ratio, say Ψ, with

$$\Psi = \frac{y^*}{y^*_{trad}}, \quad (12.3)$$

to conclude that the value of Ψ is decreased from 1.65 to 1.32 when the value of $E(P)$ is changed from 25% to 2 %. They mentioned that the optimal solution, y^*, is greater than the traditional optimal solution, y^*_{trad} that contradicted the findings of Rosenblatt and Lee [25]. In Rosenblatt and Lee [25], their results showed that the optimal solution is less than the y^*_{trad} that is consistent with decreasing of ordering quantity.

In the following, we will analytically prove that for the economic ordering quantity model with imperfect quality, the optimal solution, y^* is indeed greater than the traditional optimal solution, y^*_{trad}. To verify that $\Psi > 1$ is equivalent to show that

$$E(P) + (M-1)\left(1-\frac{2D}{X}\right) > 0. \quad (12.4)$$

We assume that P is a uniform distribution with probability density function $f(p) = \frac{1}{\alpha}$ for $0 \le p \le \alpha$ such that $\alpha \le 1 - \frac{D}{x}$. Therefore, the constraint in Equation (10.1) is satisfied. We know that

$$E(P) = \int_0^\alpha pf(p)dp = \frac{\alpha}{2}, \quad (12.5)$$

and

$$M = E\left(\frac{1}{1-P}\right) =,$$

$$\int_0^\alpha \frac{1}{1-p} f(p)dp = \frac{-1}{\alpha}\ln(1-\alpha). \quad (12.6)$$

Hence, we rewrite Equation (12.4) as

$$\frac{\alpha}{2} + \left(\frac{-1}{\alpha}\ln(1-\alpha)-1\right)\left(1-\frac{2D}{x}\right) > 0. \quad (12.7)$$

Motivated by Equation (12.7), we assume an auxiliary mapping in the following, which is denoted as $G(\alpha)$, such that

$$G(\alpha) = \frac{\alpha}{2} + \left(\frac{-1}{\alpha}\ln(1-\alpha)-1\right)\left(1-\frac{2D}{X}\right), \quad (12.8)$$

for $0 < \alpha \le 1 - \frac{D}{X}$, and then our goal is to prove that

$$G(\alpha) > 0, \quad (12.9)$$

for $\alpha > 0$. By the Hospital's rule, it yields that

$$\lim_{\alpha \to 0^+} \frac{-\ln(1-\alpha)}{\alpha} = 1, \quad (12.10)$$

so we obtain that

$$\lim_{\alpha \to 0^+} G(\alpha) = 0. \quad (12.11)$$

On the other hand, we derive that

$$G'(\alpha) = \frac{1}{2} +,$$

$$\frac{(1-\alpha)\ln(1-\alpha)+\alpha}{\alpha^2(1-\alpha)}\left(1-\frac{2D}{X}\right). \quad (12.12)$$

From the Taylor's series expansion of logarithm mapping, we know that

$$\ln(1-\alpha) = -\sum_{k=1}^{\infty}\frac{\alpha^k}{k}, \quad (12.13)$$

and then we imply that

$$(1-\alpha)\ln(1-\alpha)+\alpha = \sum_{k=2}^{\infty}\frac{\alpha^k}{(k-1)k} > \frac{\alpha^2}{2}. \quad (12.14)$$

If we combine Equations (12.12) and (12.14), and then we obtain that

$$G'(\alpha) > \frac{1}{2} + \frac{1}{2}\left(1-\frac{2D}{X}\right),$$

$$= \frac{X-D}{X} > 0, \quad (12.15)$$

since the annual screening items is greater than the annual demand, that is

$$X > D. \quad (12.16)$$

From Equation (12.15), it yields that $G(\alpha)$ is an increasing function with $\lim_{\alpha \to 0^+} G(\alpha) = 0$. Therefore, we conclude that Equation (12.9) is valid.

XIII. Numerical Example

For completeness, we rerun the numerical example in Salameh and Jaber [24] with the following data: the demand rate $D = 50000$ units/year, the ordering cost $K = 100$ /cycle, the holding cost $h = 5$ /unit/year, the screening rate $x = 1$ unit/min, the screening cost $d = 0.5$ /unit, the purchase cost $c = 25$ /unit, the selling price of good quality items $s = 50$ /unit, the selling price of imperfect quality items $v = 20$ /unit, the annual screening rate $X = 1 \times 60 \times 8 \times 365 = 175200$, the percentage defective random variable, P, is uniformly distributed with its probability density function as $f(p) = 25$ for $0 \le p \le 0.04$.

Therefore, we derive that

$$E(P) = 0.02, \tag{13.1}$$

and

$$M = E\left(\frac{1}{1-P}\right) = 1.02055. \tag{13.2}$$

The optimal replenishment quantity in Salameh and Jaber [24] is calculated as

$$y^* = 1439, \tag{13.3}$$

units and the maximum profit per year is obtained as

$$ETPU(y^*) = 1212235, \tag{13.4}$$

per year.

XIV. Another Related Issue

We try to examine the inventory model constructed by Lin and Hou [26] to revise their findings for a pair of an upper bound and a lower bound with respect to the bisection procedure. For research propose, we will adopt the same assumptions and notation as and Lin and Hou [26] and Rosenblatt and Lee [27].

(a) T is the planning period.

(b) t is the production period.

(c) d is the constant demand of the item per unit of time.

(d) p is the production rate of the produces.

(e) r is the reinstallation or alteration cost.

(f) Under the production period of t, and the planning period of T, the model implies that $dT = pt$.

(g) $(p-d)t$ is the utmost inventory level.

(h) The system is constructed under the condition of production rate of p and the constant demand of d, with the restriction, $p > d$.

(i) h denoted the holding cost per unit of time per item.

(j) After working for a period of time, the system will run an inspection. A new set up cost, k, will be charged.

(k) The inspection period will be defined as out of managed status.

(l) Those deteriorated product will be amended under a cost denoted as s.

(m) When the model is under out of managed status, with a reinstallation or alteration cost, r, back to the normal production status.

(n) Because of manufacture is interrupted, a product has the probability density distribution of θ_1, when the model is under normal status.

(o) A product has the probability density distribution of θ_2, when the model is under out of managed status, with the restriction, $\theta_1 < \theta_2$.

(p) A product can be categorized into two kinds: normal product, or deterioration item, after the inspection check.

Lin and Hou [26] constructed a novel economic manufacture quantity system under different a probability density distribution for normal and deterioration product, and reworked cost. The objective mapping, which is denoted as $TC(t)$, proposed by Lin and Hou [26] is directly cited in the following. The interested readers please referred to the original article of Lin and Hou [26] for their detailed development.

$$TC(t) = \beta \frac{1-e^{-\lambda t}}{t} + sd\theta_2 + \frac{h(p-d)t}{2} + \frac{dk}{pt}, \tag{14.1}$$

under an abbreviation,

$$\beta = \frac{(\theta_1 - \theta_2)ds}{\lambda} + \frac{dr}{p}. \tag{14.2}$$

Lin and Hou [26] obtained that

$$\frac{dTC(t)}{dt} = \left(\left(e^{-\lambda t}(1+\lambda t) - 1\right)\beta + \frac{h(p-d)}{2}t^2 - \frac{dk}{p}\right)\frac{1}{t^2}. \tag{14.3}$$

For further discussion, Lin and Hou [26] defined a new auxiliary mapping, which is expressed to be f(t), in the following,

$$f(t)=\left(e^{-\lambda t}(1+\lambda t)-1\right)\beta$$

$$+\frac{h(p-d)}{2}t^2-\frac{dk}{p}, \tag{14.4}$$

where the solution of $f(t)=0$ will be a candidate of the optimal solution. Before running the bisection algorithm to seek the solution of $f(t)=0$, Lin and Hou [26] tried to find a pair of an upper bound and a lower bound to shrink the search domain. Therefore, for the exponential term,

$$e^{-x}(1+x)-1, \tag{14.5}$$

in the Equation (14.4), Lin and Hou [26] tried to approximate it with an algebraic expression, and then Lin and Hou [26] can apply the quadratic formula for polynomials to locate a pair of an upper bound and a lower bound.

Lin and Hou [26] obtained a lowed bound,

$$t_1=\sqrt{\frac{2kd}{(p-d)ph}}, \tag{14.6}$$

and derived an upper bound,

$$t_2=\sqrt{\frac{2kd}{(p-d)ph-\lambda^2\beta p}}, \tag{14.7}$$

The optimal solution is expressed as t^* by Lin and Hou [26]. Their Proposition 2 is cited as follows. If $\beta\le 0$ then

$$0<t^*\le t_1, \tag{14.8}$$

and if $\beta>0$ then

$$t_1<t^*<t_2. \tag{14.9}$$

We provide our amendment in the next section to improve the pair of an upper bound and a lower bound with respect to a bisection procedure studied by Lin and Hou [26].

XV. Our Improvement

We consider the approximation for the negative exponential function. Chung and Lin [28], evaluated e^{-x} by a pair of an upper bound and a lower bound,

$$\frac{2-x}{x+2}<e^{-x}<\frac{x^2}{2}-x+1. \tag{15.1}$$

and Chung [29] and Lan et al. [30] independently derived the following estimation,

$$\frac{2-x}{x+2}<e^{-x}<\frac{x^2-4x+6}{6+2x+6}. \tag{15.2}$$

Yang et al. [31] obtained that

$$\frac{2+x}{2-x}\ge\frac{6+4x+x^2}{6-2x}\ge e^x. \tag{15.3}$$

We observe the above approximation for the negative exponential mapping to show relationship for the lower bounds,

$$\frac{2-x}{x+2}\le\frac{-2x+6}{x^2+4x+6}\le e^{-x}. \tag{15.4}$$

On the other hand, we begin to evaluate the next exponential term of Equation (14.5),

$$e^{-x}(x+1)-1, \tag{15.5}$$

In Lin and Hou [26], they claimed that

$$0>e^{-x}(x+1)-1, \tag{15.6}$$

The estimation from Chung and Lin [32] is expressed as

$$\frac{x^3-x^2}{2}>(1+x)e^{-x}-1>\frac{-x^2}{2+x}. \tag{15.7}$$

Chung [29] and Lan et al. [30] independently obtained that

$$\frac{x^3-3x^2}{6+2x}>(1+x)e^{-x}-1>\frac{-x^2}{2+x}. \tag{15.8}$$

Yang et al. [31] showed the following findings,

$$\frac{x^3-3x^2}{6+2x}>(1+x)e^{-x}-1>\frac{-3x^2}{6+4x+x^2}. \tag{15.9}$$

According to our above examination, researchers can found several pairs of an upper bound and a lower bound to execute the bisection procedure whish have a smaller searching domain than that of Lin and Hou [26].

XVI. Possible Directions for Further Development

There are several recently published articles that are valuable for researchers to consider and then we list them in this section. Considering wandering salesman issues, Xu and Zhang [33] applied Wolf optimization method under revised gray procedure. According to occurrence logic chart, Li et al. [34] acquired evolutionary judgment with respect to municipal urgent situation. Based on heuristic combination of big data, Kusuma and Prasasti [35] gained an optimal approach with entities and members. Referred to internet of things, Sangeetha, and Ravi [36] derived woodland categorization with accidental regressive judgment by satisfied healthcare information. To learn YOLOv7 Algorithm, Zhuang and Liu [37] obtained recognition procedure for submarine genetic goal. For brand new goods allocation, Wang et al. [38] solved multiple partition automobile direction-finding issue by mixture inherent methods. With respect to associated plug-in hybrid electric vehicles, Sun et al. [39] adopted deep learning power managing policy by corresponding corroboration from cloud computation. Under a interior structure, Polo et al. [40] employed a disorganized path for hardware accomplishment. Based on the energetic performance of the panic outcome, Chong et al. [41] examined the Volterra and Lotka system. In the enhanced YOLOv5s, Shan et al. [42] found out a pneumonia discovery procedure. Owing to Partial Backlogging, Inflation, and demand depending to price and time, with respect to decay product, Pathak et al. [43] developed inventory systems under own and rented warehouses conditions. Related to petroleum compartment with covering switch over, Li et al. [44] considered preheating problems with short temperature to optimize the control. To realize the group structure, Khamrot et al. [45] examined vague internal ideals with complex and bipolar restrictions. Referring to neglected covariates, Ji et al. [46] studied parameter systems with single index system by quintile regression. Through aligned magnetic field, chemical reaction and temperature generation, Buzuzi et al. [47] applied a widen and disposed area to construct stable current. According to our above citation and discussion, researchers can find interesting study topics for their further development.

XVII. Conclusion

As preferences do not exist in isolation from the decision making in analytic hierarchy process, there are different interpretations on scaling ratio by researchers who advocate multiattribute value models. However, Belton and Gear [18] and Belton and Gear [1] elicited specific normalization procedure to claim that their method can prevent rank reversal phenomenon. We demonstrate that their revision is useless. Barzilai and Golany [48] have indicated that normalization itself can not avoid rank reversals, because there will always exist a set of vectors exhibiting rank reversal for any normalization. Nevertheless, normalization in analytic hierarchy process provides information on the total dominance and is not only a mathematical operation, Saaty and Vargas [19]. Although the analytic hierarchy process of Saaty and Vargas [19] has become a popular and practical technique for dealing with complex decision problems and solving broad range ofmulti-criteria decision problems. It provides a ranking for the decision alternatives. Successful applications can be found in business, industry, government, education, military and various research fields. In this paper we wish our clarification of the questionable results in Belton and Gear [1] instead of criticizing or mimicking will thereafter polish analytic hierarchy process to a better state. At last, the other two papers that have discussed the rank reversal problem in analytic hierarchy process: Brugha [49], and Salo and Homoloinen [50], that are worthy to mention to provide a bird view of this research problem.

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