Inventory Models with Algebraic Approach

Junling Zhang, Chiu-Tang Lin, Xiaolin Li

Abstract—**In this paper, we show that the benefit of using algebraic procedures is to derive the optimal value before knowing the optimal ordering quantity. It is a distinct feature that may have a significant influence on dealing with several dimensional functional problems such that we can save time and effort without knowing the optimal solutions for each variable.**

***Index Terms*—Economic production quantity, Algebraic approach, Inventory systems, Economic ordering quantity**

I. Introduction

After Grubbström and Erdem [1] invented a new approach to solving inventory models and systems by a non-analytic process to be of assistance to practitioners and researchers who are not used to the analytical method of differential equations and calculus, there are about one hundred papers following Grubbström and Erdem [1] to extend their approach in different directions to handle various inventory models. We provide a brief review of related papers in the following. Chang et al. [2] showed that the solution approach developed by Ronald et al. [3] is too difficult for ordinary practitioners and researchers such that they applied the traditional method to apply the square for the quadratic or the constant terms to handle the inventory systems of Cárdenas-Barrón [4] and Grubbström and Erdem [1]. Chang et al. [2] also raised an open problem for future study that is related to an alternative solution procedure. Ronald et al. [3] mentioned that the technical reasoning of Grubbström and Erdem [1] is not clear such that how to develop from $(x/(x+y))+(y/(x+y))=1$ to their desired decomposition is not well explained. Hence, Ronald et al. [3] constructed a two-step solution process to solve the inventory model examined by Grubbström and Erdem [1]. Wee et al. [5] extended algebraic approaches for inventory systems with a temporary on-sale price. Cárdenas-Barrón [4] mimicked the solution approach of Grubbström and Erdem [1] for the economic ordering quantity model to develop a similar process for the economic production quantity model. Sphicas [6] generalized the inventory models from one shortage cost to two kinds of shortages. However, his process is so sophisticated that several future papers still work on the same topic. Lan et al. [7] extended the inventory systems to a stochastic lead time environment. Wee et al. [8] constructed a new method, "cost-difference comparisons" to solve inventory problems without analytical tools. Teng [9] claimed he found a simple method to solve the optimal economic order quantity. However, his approach implicitly used previously derived findings that resulted in his simple method. Cárdenas-Barrón [10] also mentioned his simple method to locate the optimal economic order quantity. We must point out that he overlooked the sign before he took the square root such that his simple method is incomplete. Xiao et al. [11] apply algebraic methods for supply chains. Cárdenas-Barrón [12] elegantly applied the Schwarz inequality to create a new algebraic method to solve the optimal economic order quantity problem. However, for an economic production quantity model, Cárdenas-Barrón [12] should change the expressions of parameters and then his results for the economic order quantity model can be directly applied to the economic order production model. Leung [13] also declared his new simple method for solving inventory systems by algebraic approaches. However, his method is severely mistaken because he ignored the sign of the quantity in his square root. Omar et al. [14] committed similar errors to Leung [13] and Cárdenas-Barrón [10] which implies they only handle a small portion of the original problem. Cárdenas-Barrón [15] overlooked the sign of an item before taking the square root which resulted in his simplified version. It is an incomplete solution process because he pretended the difficult cases did not exist to create an illusion of his invalid approach that was challenged by a later paper to reveal Cárdenas-Barrón [15] containing several questionable derivations. Chuang and Cárdenas-Barrón [16] declared that they found a complete solution process for the economic ordering quantity and economic production quantity inventory models. However, we find that their solutions are only suitable for the interior optimal solution. For those degenerated cases, their solution is completely mistaken. Sphicas [17] tried to develop a family of new inventory models. However, we carefully examined to find that his extensions are redundant and contain several questionable findings. Tuan and Himalaya [18] used an algebraic procedure to solve the optimal solution of a supply chain under conditions of consumer return. Lau et al. [19] considered the open problem developed by Chang et al. [5] and then extended it to a general expression of a difference between (i) a linear function, and (ii) a square root of a cubic function. Their goal is to seek conditions to guarantee the existence of an interior minimum solution. Chiu et al. [20] pointed out that Lau et al. [19] committed several evaluation mistakes and their solution process contained differential calculus that is beyond the scope of algebraic methods. Lin and Deng [21] applied algebraic procedures for economic ordering quantity inventory systems with a temporary sale price condition. Lin et al. [22] showed that the solution

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process of Cárdenas-Barrón [12] severely violated the rule of academic writing because Cárdenas-Barrón [12] repeatedly applied similar algebraic skills for (a) the economic ordering quantity, and (b) economic production quantity inventory models. Luo and Chou [23] reconsidered the open dilemma constructed by Chang et al. [5] and then extended and studied by Lau et al. [19] and Chiu et al. [20] to provide an improved algebraic process. Lin [24] showed that Cárdenas-Barrón [15] not only overlooked the algebraic rules but also violated the principle mentioned in Cárdenas-Barrón [10] for applying the Schwarz inequality. Lin [25] showed that Teng [9] implicitly used the findings of Wee et al. [8], but Teng [9] did not offer the credit of Wee et al. [8] in his paper. Moreover, Lin [25] pointed out that the challenge of Leung [13] to Teng [9] is invalid. Feng [26] discussed the two motivations mentioned in Sphicas [17] to show these two motivations are not reasonable. Lin [27] examined invalid algebraic improvements of Teng [9] and Leung [13]. Lin [28] first showed that the solution process of Chung and Cárdenas-Barrón [16] is false when the optimal solution degenerated to two boundaries and then he pointed out that Omar et al. [14] also had false derivations for the optimal boundary solution and then provided revisions. Yen [29] developed a new approach to solving inventory systems by intuitive algebraic methods. Yen [30] presented an amendment of Çalışkan [31] for inventory models with compound interest rates and then he offered an intuitive approach for the economic ordering quantity inventory model. Based on our previous review, so many papers reveal that applying algebraic approaches to handle inventory systems is still a hot research topic. The purpose of this article is to motivate researchers to use algebraic methods.

II. Assumptions and Notation

We follow the notation used in Yen [29] because Yen [29] is the latest paper in a long research trend of Grubbström and Erdem [1], and Cárdenas-Barrón [4]. Grubbström and Erdem [1] studied the economic ordering quantity model, on the other hand, Cárdenas-Barrón [4] examined the economic production quantity system. However, Cárdenas-Barrón [4] applied diverse notation apart from that of Grubbström and Erdem [1]. Following this currency, various papers adopted a variety of expressions in their papers, for example, Chang et al. [2], Ronald et al. [3], Wee et al. [5], Sphicas [6], Lan et al. [7], Wee et al. [8], Teng [9], Cárdenas-Barrón [10], Xiao et al. [11], Cárdenas-Barrón [12], Leung [13], Omar et al. [14], Cárdenas-Barrón [15], Chuang and Cárdenas-Barrón [16], Sphicas [17], Tuan and Himalaya [18], Lau et al. [19], Chiu et al. [20], Lin and Deng [21], Lin et al. [22], Luo and Chou [23], Lin [24], Lin [25], Feng [26], Lin [27], and Lin [28]. The expressions in the above list are different such that the same item has various names on different papers. Consequently, we must select one paper as the base to begin our discussion. We predict that Yen [29] is the proper selection.

The notation used in this paper is denoted as follows.

$AC(Q,B)$ is the average cost per unit of time;

$\rho = (P-D)/P$ is an abbreviation;

Q is the maximum inventory level;

P is the production rate, with $P > D$, for EPQ models;

K is the setup cost;

h is the holding cost per unit per unit of time;

$f(k) = (h+bk^2)/(1+k)^2$, is an auxiliary function proposed by Ronald et al. [4];

D is the demand rate;

c is the cost of production per unit;

b is the backorder cost per unit per unit of time;

B is the maximum backorder level.

III. The Classical Inventory Model

Harris [32] is the first paper to develop the Economic Ordering Quantity (EOQ) inventory model to introduce a mathematical system into control and management academic society. In the following, we provide an outline of his original work.

Harris [32] used h, the holding cost per unit per unit of time; D, the demand for a unit of time; K, the setup cost per ordering; Q, the ordering quantity (the decision variable) to construct his EOQ inventory system. The goal is to select the optimal ordering quantity, Q^*, to balance (i) the average setup cost, and (ii) the average holding cost.

Harris [32] assumed that the ordering quantity is Q, then the planning horizon is Q/D, that is the duration period for one replenishment cycle. On the other hand, the total inventory holding cost is computer as follows,

$$h\int_0^{Q/D}(Q-Dt)dt = hQ^2/2D. \tag{3.1}$$

The total cost for one replenishment cycle is derived as

$$K + (hQ^2/2D). \tag{3.2}$$

The average cost per unit of time is denoted as $AC(Q)$, and then Harris [32] obtained

$$AC(Q) = [K + (hQ^2/2D)]/(Q/D) = (KD/Q) + (hQ/2). \tag{3.3}$$

Based on Equation (3.3), Harris [32] derived that

$$dAC(Q)/dQ = (-KD/Q^2) + (h/2), \tag{3.4}$$

and

$$d^2AC(Q)/dQ^2 = 2KD/Q^3. \tag{3.5}$$

Owing to Equation (3.5), Harris [32] obtained that $d^2AC(Q)/dQ^2 > 0$, for $Q > 0$ such that $AC(Q)$ is a convex function and the solution of $dAC(Q)/dQ = 0$ is the minimum (optimal) solution.

Referring to Equation (3.4), Harris [32] solved $dAC(Q)/dQ = 0$ to find that

$$Q^* = \sqrt{2KD/h}. \tag{3.6}$$

Harris [32] plugged the finding of Equation (3.6) into Equation (3.3) to derive the minimum value as

$$AC(Q^*) = (KD/Q^*) + (hQ^*/2) = \sqrt{KDh/2} + \sqrt{KDh/2} = \sqrt{2KDh}. \tag{3.7}$$

The above reviewing is based on analytic procedure, which is the major approach adopted by most researchers. In the next section, we will reconsider this inventory system from a different point of view for those practitioners who are not familiar with calculus.

IV. An Alternative Approach

In this section, we try to recall Harris' EOQ model from an algebraic point of view.

Without the integration of Equation (3.1), we can consider that the demand is a constant such that the inventory level is a linear function (that is a straight line) to imply the accumulated inventory level is a triangle with height Q and weight Q/D, and then the area of this triangle can be directly computed as

$$Q(Q/D)(1/2) = Q^2/2D, \quad (4.1)$$

to imply the total holding cost is $hQ^2/2D$ which is consistent with Equation (3.1).

In the following, we will use an algebraic method to solve the minimum problem of Equation (3.3), without referring to the analytic process of calculus.

We recall the complete square operation in algebra. Usually, we complete the square for the linear term such that we have to adjust the expression of constant term. For example, we try to find the minimum of the following function,

$$f(t) = 3t^2 - 12t + 147. \quad (4.2)$$

for $t > 0$.

We rewrite Equation (4.1) as

$$f(t) = 3(t^2 - 2 \times 2t) + 147. \quad (4.3)$$

Owing $6^2 = 36$ with the coefficient 3, we decompose the constant as follows,

$$147 = 3 \times 2^2 + 135, \quad (4.4)$$

referring to Equation (4.4), we rewrite Equation (4.3) as

$$f(t) = 3(t^2 - 2 \times 2t + 2^2) + 135. \quad (4.5)$$

Owing to $t^2 - 2 \times 2t + 2^2$ is a perfect square, then we know that

$$t^2 - 2 \times 2t + 2^2 = (t - 2)^2. \quad (4.6)$$

Using the result of Equation (4.5), we rewrite Equation (4.4) as

$$f(t) = 3(t - 2)^2 + 135. \quad (4.7)$$

We observe Equation (4.7) to find that the coefficient of the square term, $(t - 2)^2$ is a positive number, 3. If we try to minimize $f(t)$, then the value of $(t - 2)^2$ should be zero. Consequently, we derive

$$f(t^*) = 135. \quad (4.8)$$

and

$$t^* = 2. \quad (4.9)$$

However, with respect to Equation (3.3), we need another skill to complete the square. We use the following example to illustrate the procedure.

We assume the objective, denoted as

$$g(t) = f(t)/t, \quad (4.10)$$

for $t > 0$.

We rewrite Equation (4.10) as

$$g(t) = 3\big(t - 12 + (49/t)\big). \quad (4.11)$$

We know that

$$\left[\sqrt{t} - \left(7/\sqrt{t}\right)\right]^2 = t - 14 + (49/t). \quad (4.12)$$

Owing to Equation (4.12), we imply that

$$t + (49/t) = 14 + \left[\sqrt{t} - \left(7/\sqrt{t}\right)\right]^2. \quad (4.13)$$

Referring to Equation (4.13), we rewrite Equation (4.11) as follows,

$$g(t) = 3\left(2 + \left[\sqrt{t} - \left(7/\sqrt{t}\right)\right]^2\right). \quad (4.14)$$

We observe Equation (4.14) to find that the coefficient of the square term, $\left[\sqrt{t} - \left(7/\sqrt{t}\right)\right]^2$, is a positive number 3, such that to attain the minimum, the square term should be equal to zero, then we obtain that

$$g(t^*) = 6. \quad (4.15)$$

and

$$t^* = 7. \quad (4.16)$$

Similarly, we treat Equation (3.3) as

$$AC(Q) = \left(\sqrt{KD/Q} - \sqrt{hQ/2}\right)^2 + \left(2\sqrt{KD/Q}\sqrt{hQ/2}\right). \quad (4.17)$$

We observe Equation (4.17) to imply that

$$AC(Q^*) = \sqrt{2KDh}, \quad (4.18)$$

and

$$Q^* = \sqrt{2KD/h}. \quad (4.19)$$

Our results of Equations (4.18) and (4.19) are the same as those findings derived by the calculus of Equations (3.6) and (3.7) to indicate that our algebraic method is capable of solving this kind of inventory system.

V. Comparison between two Methods

We recall the solution procedure of Sections 3 and 4 to find that through an analytic approach, researchers must find the minimum solution in the first place, and then plug the minimum point into the objective function to derive the minimum value.

On the other hand, using an algebraic process, the minimum value and minimum point are derived simultaneously. Hence, the tedious computation of the minimum value can be avoided to save precious computation time.

The same observation can be traced in Equations (4.8) and (4.9), and Equations (4.15) and (4.16).

It is a distinct feature of our algebraic method to find the minimum value without deriving the minimum point in advance.

VI. Inventory Models with Backlogging

Grubbström and Erdem [1] considered solving inventory models with backlogging by algebraic methods that aroused the attention of practitioners such that about a hundred papers followed their approach to examine various inventory models through algebraic methods. Hence, the original paper of Grubbström and Erdem [1] deserved a detailed review. For the next inventory system with two variables: the ordering quantity, Q, and back ordered quantity, B, researchers considered the balance among three costs: the holding cost, the back ordered cost, and the setup cost. The duration period for a stock is Q/D, and then based on Equations (3.1) or (4.1), the average holding cost is expressed as $hQ^2/2D$, where h is the holding cost per unit per unit of time.

Similarly, the average back-ordered cost is expressed as $bB^2/2D$, where b is the back-ordered cost per unit per unit of time.

Therefore, the average cost of inventory system with the backlogging cost is derived as

$$AC(Q,B) = [K + (bB^2/2D) + (hQ^2/2D)]/((Q+B)/D) = (KD/(Q+B)) + \big((bB^2 + hQ^2)/2\,(Q+B)\big). \quad (6.1)$$

Grubbström and Erdem [1] assumed that

$$KD = \frac{h}{2}\left(\sqrt{\frac{2bKD}{h(h+b)}}\right)^2 + \frac{b}{2}\left(\sqrt{\frac{2hKD}{b(h+b)}}\right)^2. \quad (6.2)$$

However, Grubbström and Erdem [1] did not provide any motivation for their assumption of Equation (6.2).

In the following, we will provide a possible method to derive the result of Equation (6.2).

From the addition of real number is commutative, we imply that

$$h + b = b + h. \quad (6.3)$$

We divide the two sides of Equation (6.3) by $(h + b)$ to show that

$$1 = \frac{b}{h+b} + \frac{h}{h+b}. \quad (6.4)$$

We multiply two sides of Equation (6.4) by KD to show that

$$KD=\left(\sqrt{\frac{bKD}{h+b}}\right)^2+\left(\sqrt{\frac{hKD}{h+b}}\right)^2. \quad (6.5)$$

Owing to

$$\frac{h}{2}\left(\sqrt{\frac{2}{h}}\right)^2=1, \quad (6.6)$$

and

$$\frac{b}{2}\left(\sqrt{\frac{2}{b}}\right)^2=1, \quad (6.7)$$

we plug Equation (6.6) into the first term of the right-hand side of Equation (6.5) to find that

$$KD=\frac{h}{2}\left(\sqrt{\frac{2bKD}{h(h+b)}}\right)^2+\left(\sqrt{\frac{hKD}{h+b}}\right)^2. \quad (6.8)$$

Similarly, we plug Equation (6.7) into the second term of the right hand side of Equation (6.8) to show that

$$KD=\frac{h}{2}\left(\sqrt{\frac{2bKD}{h(h+b)}}\right)^2+\frac{b}{2}\left(\sqrt{\frac{2hKD}{b(h+b)}}\right)^2. \quad (6.9)$$

Our result of Equation (6.9) is identical to the assumption of Grubbström and Erdem [1] of Equation (6.2) to indicate the validity of Equation (6.2). In the next section, we will present a possible direction for the motivation implicitly used by Grubbström and Erdem [1] for their assumption of Equation (6.2).

Based on Equation (6.2), Grubbström and Erdem [1] rewrote Equation (6.1) as follows,

$$AC(Q,B)=$$
$$=\left[\frac{h}{2}\left(\sqrt{\frac{2bKD}{h(h+b)}}\right)^2+\frac{b}{2}\left(\sqrt{\frac{2hKD}{b(h+b)}}\right)^2\right]\Big/\left((Q+B)\right),$$
$$+\left((bB^2+hQ^2)/2\,(Q+B)\right),$$
$$=[h/2(Q+B)]\left[Q^2+\left(\sqrt{\frac{2bKD}{h(h+b)}}\right)^2\right],$$
$$+[b/2(Q+B)]\left[B^2+\left(\sqrt{\frac{2hKD}{b(h+b)}}\right)^2\right]. \quad (6.10)$$

Applying the similar algebraic method of Equations (4.10-4.13), Grubbström and Erdem [1] derived that

$$AC(Q,B)=$$
$$=[h/2(Q+B)]\left[Q^2-2Q\sqrt{\frac{2bKD}{h(h+b)}}+\left(\sqrt{\frac{2bKD}{h(h+b)}}\right)^2\right],$$
$$+[b/2(Q+B)]\left[B^2-2B\sqrt{\frac{2hKD}{b(h+b)}}+\left(\sqrt{\frac{2hKD}{b(h+b)}}\right)^2\right],$$
$$+[h/2(Q+B)]2Q\sqrt{\frac{2bKD}{h(h+b)}},$$
$$+[b/2(Q+B)]2B\sqrt{\frac{2hKD}{b(h+b)}}. \quad (6.11)$$

The third and the fourth terms of Equation (6.11) can be simplified as follows,

$$[h/2(Q+B)]2Q\sqrt{\frac{2bKD}{h(h+b)}},$$
$$+[b/2(Q+B)]2B\sqrt{\frac{2hKD}{b(h+b)}},$$
$$=\left[\left(Q\sqrt{\frac{2hbKD}{h+b}}+B\sqrt{\frac{2hbKD}{h+b}}\right)/(Q+B)\right],$$
$$=\sqrt{\frac{2hbKD}{h+b}}. \quad (6.12)$$

The first and the second terms of (6.11) are perfect square to imply that

$$AC(Q,B)=\sqrt{\frac{2hbKD}{h+b}},$$
$$+[h/2(Q+B)]\left(Q-\sqrt{\frac{2bKD}{h(h+b)}}\right)^2,$$
$$+[b/2(Q+B)]\left[B-\sqrt{\frac{2hKD}{b(h+b)}}\right]^2. \quad (6.13)$$

Based on Equation (6.13), Grubbström and Erdem [1] located that the minimum value,

$$AC(Q^*,B^*)=\sqrt{\frac{2hbKD}{h+b}}, \quad (6.14)$$

with the optimal ordering quantity,

$$Q^*=\sqrt{\frac{2bKD}{h(h+b)}}, \quad (6.15)$$

and the optimal back ordered quantity,

$$B^*=\sqrt{\frac{2hKD}{b(h+b)}}. \quad (6.16)$$

The above derivations proposed by Grubbström and Erdem [1] are quite amazing and deserve to be published.

However, the motivation for the assumption of Equation (6.2) is missing in Grubbström and Erdem [1]. Consequently, several papers tried to solve this kind of inventory model through different points of view with algebraic approaches.

VII. Our Explanation for the Assumption

If we observe the final results of Equations (6.14-6.16), then we can predict that the objective function of Equation (6.1) will be transformed into the following expression,

$$AC(Q,B)=F(Q,B)\left[Q-\sqrt{\frac{2bKD}{h(h+b)}}\right]^2,$$
$$+G(Q,B)\left[B-\sqrt{\frac{2hKD}{b(h+b)}}\right]^2+\sqrt{\frac{2hbKD}{h+b}}, \quad (7.1)$$

where $F(Q,B)$ and $G(Q,B)$ are two positive functions.

We expand the expressions of Equation (7.1) to yield that

$$AC(Q,B)=\sqrt{\frac{2hbKD}{h+b}},$$
$$+F(Q,B)\left[Q^2-2Q\sqrt{\frac{2bKD}{h(h+b)}}+\left(\sqrt{\frac{2bKD}{h(h+b)}}\right)^2\right],$$
$$+G(Q,B)\left[B^2-2B\sqrt{\frac{2hKD}{b(h+b)}}+\left(\sqrt{\frac{2hKD}{b(h+b)}}\right)^2\right]. \quad (7.2)$$

We compare Equations (7.2) and (6.1) for the terms containing Q^2 to imply that

$$F(Q,B)=h/[2(Q+B)], \quad (7.3)$$

which is a positive function.

Similarly, we compare Equations (7.2) and (6.1) for the terms containing B^2 to imply that

$$G(Q,B)=b/[2(Q+B)], \quad (7.4)$$

which also is a positive function.

The remaining computations are similar to Equations (6.10-6.13) but in a reverse ordering.

For completeness, we list them in the following,

$$\sqrt{\frac{2hbKD}{h+b}},$$
$$+(h/[2(Q+B)])\left[Q^2-2Q\sqrt{\frac{2bKD}{h(h+b)}}+\left(\sqrt{\frac{2bKD}{h(h+b)}}\right)^2\right],$$
$$+(b/[2(Q+B)])\left[B^2-2B\sqrt{\frac{2hKD}{b(h+b)}}+\left(\sqrt{\frac{2hKD}{b(h+b)}}\right)^2\right],$$
$$=(bB^2+hQ^2)/[2\,(Q+B)]+\sqrt{\frac{2hbKD}{h+b}},$$

$$-(\mathrm{h}Q/(Q+B))\sqrt{\frac{2bKD}{h(h+b)}}-(bB/(Q+B))\sqrt{\frac{2hKD}{b(h+b)}},$$
$$(h/[2(Q+B)])\frac{2bKD}{h(h+b)}+(b/[2(Q+B)])\frac{2hKD}{b(h+b)}. \quad (7.5)$$

We can simply the results of the third and the fourth terms of the right-hand side of Equation (7.5) as follows,

$$-(hQ/(Q+B))\sqrt{\frac{2bKD}{h(h+b)}}-(bB/(Q+B))\sqrt{\frac{2hKD}{b(h+b)}},$$
$$=-\left(\frac{Q}{Q+B}+\frac{B}{Q+B}\right)\sqrt{\frac{2hbKD}{h+b}},$$
$$=-\sqrt{\frac{2hbKD}{h+b}}. \quad (7.6)$$

We can simply the results of the fifth and the sixth terms of the right-hand side of Equation (7.5) as follows,

$$(h/2(Q+B))\frac{2bKD}{h(h+b)}+(b/2(Q+B))\frac{2hKD}{b(h+b)},$$
$$=\frac{1}{Q+B}\left(\frac{b}{h+b}+\frac{h}{h+b}\right)KD,$$
$$=KD/(Q+B). \quad (7.7)$$

We plug the findings of Equations (7.6) and (7.7) into Equation (7.5) to yield that

$$\sqrt{\frac{2hbKD}{h+b}},$$
$$+(h/2(Q+B))\left[Q^2-2Q\sqrt{\frac{2bKD}{h(h+b)}}+\left(\sqrt{\frac{2bKD}{h(h+b)}}\right)^2\right],$$
$$+(b/2(Q+B))\left[B^2-2B\sqrt{\frac{2hKD}{b(h+b)}}+\left(\sqrt{\frac{2hKD}{b(h+b)}}\right)^2\right],$$
$$=(KD/(Q+B))+\left((bB^2+hQ^2)/2\,(Q+B)\right). \quad (7.8)$$

We recall Equation (6.1) to derive that

$$[h/2(Q+B)]\left(Q-\sqrt{\frac{2bKD}{h(h+b)}}\right)^2,$$
$$+[b/2(Q+B)]\left[B-\sqrt{\frac{2hKD}{b(h+b)}}\right]^2+\sqrt{\frac{2hbKD}{h+b}},$$
$$=AC(Q,B). \quad (7.9)$$

We observe Equation (7.7) and overlook the quantity $(Q+B)$ in the denominator, then the amazing assumption of Equation (6.2) appears.

According to our above discussion, a reasonable motivation for the assumption of Equation (6.2) proposed by Grubbström and Erdem [1] was presented by us. However, the above explanation has a vital shorting that researchers must know that final results of Equations (6.14-6.16) in advance.

VIII. An Analytic Method

Most researchers applied calculus for the inventory model, $AC(Q,B)$, of Equation (6.1) to derive

$$\frac{\partial}{\partial Q}AC(Q,B)=\frac{-bB^2}{2(B+Q)^2}$$
$$+\frac{h\left[2Q(B+Q)-Q^2\right]}{(B+Q)^2}-\frac{DK}{(B+Q)^2}, \quad (8.1)$$

and

$$\frac{\partial}{\partial B}AC(Q,B)=\frac{b\left[2B(B+Q)-B^2\right]}{(B+Q)^2}$$
$$-\frac{hQ^2}{2(B+Q)^2}-\frac{DK}{(B+Q)^2}, \quad (8.2)$$

then they solved the system of $\frac{\partial}{\partial Q}AC(Q,B)=0$ and $\frac{\partial}{\partial B}AC(Q,B)=0$ to yield that

$$bB^2+2DK=hQ^2+2hBQ, \quad (8.3)$$

and

$$hQ^2+2DK=bB^2+2bBQ. \quad (8.4)$$

Usually, they added Equations (8.3) and (8.4) to cancel out the common factors bB^2 and hQ^2 to obtain that

$$2(b+h)BQ=4DK, \quad (8.5)$$

to find a relation for the optimal solution of B and Q as

$$B=\frac{2DK}{(b+h)Q}. \quad (8.6)$$

We plug the finding of Equation (8.6) into Equation (8.3) to show that

$$b\left(\frac{2DK}{(b+h)Q}\right)^2+2DK$$
$$=hQ^2+2h\left(\frac{2DK}{(b+h)Q}\right)Q, \quad (8.7)$$

and then simplify Equation (8.7) as

$$hQ^2-\left(\frac{4bD^2K^2}{(b+h)^2}\right)\frac{1}{Q^2}=\frac{2(b-h)DK}{b+h}. \quad (8.8)$$

We express Equation (8.8) as a quadratic polynomial of Q^2, as

$$h\left(Q^2\right)^2-\frac{2(b-h)DK}{b+h}Q^2-\left(\frac{4bD^2K^2}{(b+h)^2}\right)=0. \quad (8.9)$$

From the quadratic formula, we know that

$$Q^2=\frac{1}{2h}\left[\frac{2(b-h)DK}{b+h}\pm\sqrt{\Omega}\right], \quad (8.10)$$

where

$$\Omega=\left(\frac{2(b-h)DK}{b+h}\right)^2+4h\frac{4bD^2K^2}{(b+h)^2},$$
$$=\frac{4D^2K^2}{(b+h)^2}\left[(b-h)^2+4bh\right],$$
$$=\frac{4D^2K^2}{(b+h)^2}(b+h)^2,$$
$$=4D^2K^2. \quad (8.11)$$

By Equation (8.11), we derive that

$$Q^2=\frac{2DK}{2h}\left[\frac{b-h}{b+h}+1\right]$$
$$=\frac{2bDK}{h(b+h)}, \quad (8.12)$$

or

$$Q^2=\frac{2DK}{2h}\left[\frac{b-h}{b+h}-1\right]$$

$$= \frac{-2DK}{b+h}. \quad (8.13)$$

The result of Equation (8.13) violates the restriction of $Q \geq 0$, then

$$Q = \sqrt{\frac{2bDK}{h(b+h)}}. \quad (8.14)$$

We plug the finding of Equation (8.14) into Equation (8.6) to yield that

$$B = \frac{2DK}{(b+h)}\sqrt{\frac{h(b+h)}{2bDK}}$$

$$= \sqrt{\frac{2hDK}{b(b+h)}}. \quad (8.15)$$

We recall the numerical example proposed by Grubbström and Erdem [1] with $b=3$, $D=100$, $h=1$ and $K=75$, and then we execute numerical examination of Equations (8.14) and (8.15), to derive that

$$Q = 75\sqrt{2}, \quad (8.16)$$

and

$$B = 25\sqrt{2}, \quad (8.17)$$

which is consistent with our derivations by algebraic methods.

After we find the optimal solution of Q^* and B^* by Equations (8.14) and (8.15), we still need a lengthy computation for $AC(B^*, Q^*)$ as follows.

First, we derive

$$B^* + Q^* = \sqrt{\frac{2DK}{b+h}}\left(\sqrt{\frac{h}{b}} + \sqrt{\frac{b}{h}}\right),$$

$$= \sqrt{\frac{2DK}{b+h}}\frac{b+h}{\sqrt{bh}},$$

$$= \sqrt{\frac{2(b+h)DK}{bh}}. \quad (8.18)$$

Next, we compute

$$AC(B^*, Q^*) = \frac{b(B^*)^2}{2(B^*+Q^*)} + \frac{h(Q^*)^2}{2(B^*+Q^*)} + \frac{DK}{B^*+Q^*},$$

$$= \sqrt{\frac{bh}{2(b+h)DK}}\left[\left(\frac{b}{2}\right)\frac{2hDK}{b(b+h)} + \left(\frac{h}{2}\right)\frac{2bDK}{h(b+h)} + DK\right],$$

$$= \sqrt{\frac{bhDK}{2(b+h)}}\left[\frac{h}{(b+h)} + \frac{b}{(b+h)} + 1\right],$$

$$= \sqrt{\frac{2bhDK}{(b+h)}}. \quad (8.19)$$

For completeness, we compute the minimum cost with the following parameters, $b=3$, $D=100$, $h=1$ and $K=75$, such that we obtain that

$$AC(B^*, Q^*) = 75\sqrt{2}. \quad (8.20)$$

From the above computation, through an analytical procedure, researchers must find the optimal solutions of (i) the ordering quantity, and (ii) the backlogging quantity, and then they need a lengthy computation to find the optimal minimum value.

On the other hand, by algebraic methods, the optimal minimum point and optimal minimum value can be derived simultaneously.

IX. An Alternative through Analytic Process

We provide the second method to solve the system of partial derivatives. We subtract Equation (8.3) from Equation (8.4) to obtain

$$bB^2 - hQ^2 = hQ^2 + 2hBQ - bB^2 - 2bBQ. \quad (9.1)$$

We rewrite Equation (9.1) as

$$2bB^2 + 2bBQ = 2hQ^2 + 2hBQ, \quad (9.2)$$

that is

$$2bB(B+Q) = 2hQ(B+Q), \quad (9.3)$$

and then we cancel out the positive common factor $2(B+Q)$ from Equation (9.3) to yield that

$$bB = hQ. \quad (9.4)$$

We plug the finding of Equation (9.4) into Equation (8.3) to imply

$$b\left(\frac{hQ}{b}\right)^2 + 2DK = hQ^2 + 2h\left(\frac{hQ}{b}\right)Q, \quad (9.5)$$

and then we derive that

$$h(b+h)Q^2 = 2bDK, \quad (9.6)$$

to find the maximum inventory level,

$$Q = \sqrt{\frac{2bDK}{h(b+h)}}. \quad (9.7)$$

Consequently, we derive the maximum back order quantity as follows,

$$B = Q\frac{h}{b} = \sqrt{\frac{2DKh}{(b+h)b}}. \quad (9.8)$$

We compare our results of Equations (9.7) and (9.8) which are identical to Equations (8.14) and (8.15) to indicate that our second solution procedure is valid.

After Equations (8.14) and (8.15), we obtain the optimal ordering quantity and the optimal backlogged quantity. However, to derive the optimal minimum cost, we have to first find the value of the sum of (i) the optimal ordering quantity and (ii) the optimal backlogged quantity, and then through a lengthy computation expressed in Equation (8.19), we finally find the optimal minimum cost.

If we recall the algebraic approach, several times, the optimal minimum value is obtained before the derivation of (i) the optimal ordering quantity and (ii) the optimal backlogged quantity.

Consequently, the significant advantage of algebraic methods is to derive the optimal minimum value before knowing (i) the optimal ordering quantity and (ii) the optimal backlogged quantity.

X. Direction for Future Study

Chen and Yang [33] constructed a family of exponentially increasing ratios that is between the constant ratio and linear ratio proposed by Montgomery et al. [34] such that they can make a good approximation to the real-world partially backlogging ratio. Using the system of first partial derivatives, Chu [35] directly solved the partially backlogging inventory model proposed by Montgomery et al. [34] to find the criteria to derive an interior optimal solution against the local minimum on the boundary. Under the stationary condition, Fei et al. [36] changed two variables problem into one variable problem, to solve the partially backlogging inventory model proposed by Montgomery et al. [34] to find the criteria to derive interior optimal solution. Several recently published papers are valuable to discuss to designate the research tendency. We list them in the following: Jiang et al. [37], Timpitak and Pochai [38], Ahmad et al. [39], Nunez-Agurto et al. [40], Long et al. [41], and Wang et al. [42]. Moreover, we refer to Wei et al. [43] that was studied filter voltage with quality factor and electric control. Yu et al. [44] developed the Archimedes optimization algorithm and solubility optimization algorithm for vector machines. Gudekote et al. [45] examined peristaltic transport fluid by the transfer effect of mass and heat.

XI. Conclusion

We demonstrate the benefit of algebraic methods to directly locate the optimal value. Moreover, we provide a second process to locate the optimal solutions for each variable that is faster than the traditional approach used by ordinary practitioners. Our paper can help researchers to realize the benefit of algebraic procedures. For completeness, we also discuss analytic approaches to illustrate that our algebraic methods is superior to analytic approaches.

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