Non-expansive Mapping in S-Metric Space

Manjusha P. Gandhi, Anushree A. Aserkar

Abstract—**The objective of this article is to create unique fixed point theorem for non-expansive functions within complete S-metric space by utilizing a ternary relation. Further, the fixed point theorems for partially ordered S-metric space have been developed. The authors have introduced a novel non-expansive condition and have defined the term f-invariant with this context. Thereby, the validation of the findings has been done through a concrete example. The results are the extension, modification, and enhancement of numerous theorems in existing literature.**

Index Terms—**Fixed Point, f-invariant, mapping, Non expansive, S-metric space.**

I. Introduction and Pre-requisite

The philosophy of fixed points hold a crucial role in the functional analysis, with the Banach principle [1] serving as a cornerstone of metric fixed point study. If $F : X \to X$ is a function of complete metric space and for all a, b in X, it fulfils the contraction constraint $d(Fa, Fb) \leq kd(a,b)$,for $0 < k \leq 1$, then there exists unique fixed point of F. This map F is termed as contraction map, where k is being referred to as the Lipschitz constant. However, for all a, b in X, when $d(Fa, Fb) \leq d(a,b)$, it is classified as non-expansive map. The idea of non-expansive mappings play a pivotal part in studies of fixed point theorems. Numerous investigators [2], [3], [10], [5] have established fixed point outcomes for non-expansive functions in metric spaces.

The concept of metric space is very important and has broader applications across varying fields in applied sciences. Several authors have contributed to the extensive view of metric spaces, for example Gahler [6] with the introduction of 2 metric spaces, the idea of D metric spaces given by Dhage [7]. Mustafa et al. [8] presented the more general metric space, recognized as G-metric spaces, paving a way for development of fixed point theories for different type of mappings within this framework. Recently Nallaselli et al [9] worked on orthogonal metric space and Ushabhavani et al [10] on Fuzzy soft metric space.

Sedghi et al. [11] pioneered the S-metric space concept besides providing the properties along with applications. Based upon this, Sedghi and Dung [12] extended the work by proving general fixed point outcomes in S-metric spaces. Further researchers [13], [14], [15], [16], [17] have put forth various features of the S metric spaces.

Despite extensive research on S-metric spaces, the study of non-expansive mappings with this background is notably missed out from existing literature. This gap in knowledge has motivated the authors to delve into this direction. The present paper aims to establish results in S metric space for a new kind of non-expansive mapping without relying on the condition of continuity. In addition to this, theorems are discussed in partially ordered metric space.

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Definition 1.1 [11]: Assume $S : X^3 \to [0, \infty)$ is a function, that satisfy the constraints, as here under for every $a_1, a_2, a_3, a_4 \in X$

1) $S(a_1, a_2, a_3) = 0$, iff $a_1 = a_2 = a_3$.
2) $S(a_1, a_2, a_3) \leq S(a_1, a_1, a_4) + S(a_2, a_2, a_4) + S(a_3, a_3, a_4)$

Couple (X, S) is named as S-metric space. Here X is non-empty set.

Example: Consider $X = R^n$ also $\|.\|$ be norm on X. Then below are S-metric spaces

$$S(a_1, a_2, a_3) = \|a_2 + a_3 - 2a_1\| + \|a_2 - a_3\|$$

$$S(a_1, a_2, a_3) = \|a_1 - a_3\| + \|a_2 - a_3\|$$

Remark: S-metric space is generalization of metric space where the distance function satisfies a relaxed set of axioms. In an S-metric space, the distance function may not necessarily be symmetric or satisfy the triangle inequality.

We have provided the subsequent new definition-

Definition 1.2: Assume $f : X \to X$ is mapping and M is subset of $X \times X \times X$, if $(fa, f^2a, f^3a) \in M$, when $(a, fa, f^2a) \in M$, then M is f-invariant.

We have proved the subsequent Lemmas:

Lemma 1: If $\{x_n\}$ is non-increasing sequence of non-negative real numbers, then

$$\left\{ \frac{2x_n + x_{n+1} + x_{n+2})}{x_n + x_{n+1} + x_{n+2})} \right\} \text{ is non-increasing too}$$

Proof:

$$\frac{2x_n + x_{n+1} + x_{n+2}}{x_n + x_{n+1} + x_{n+2}} \geq \frac{2x_{n+1} + x_{n+2} + x_{n+3}}{x_{n+1} + x_{n+2} + x_{n+3}} \text{ if and only if}$$

$$(2x_n + x_{n+1} + x_{n+2})(x_{n+1} + x_{n+2} + x_{n+3}) \geq (2x_{n+1} + x_{n+2} + x_{n+3})(x_n + x_{n+1} + x_{n+2})$$

This holds since $x_n \geq x_{n+1} \geq x_{n+2}$.

Lemma 2: Let (X, S) be a S metric space, $f : X \to X$ be a non-expansive map and $a_0 \in X$ .If $\{a_n\}$ is a Picard sequence of initial point a_0, then the sequence

$$\left\{ \frac{2S(a_{n-1}, a_{n-1}, a_n) + S(a_{n+1}, a_{n+1}, a_n) + S(a_{n+1}, a_{n+1}, a_{n+2})}{S(a_{n-1}, a_{n-1}, a_n) + S(a_n, a_n, a_{n+1}) + S(a_{n+1}, a_{n+1}, a_{n+2}) + 1} \right\}$$ is non-increasing.

Proof: Since $f : X \to X$ is non-expansive, therefore $S(a_{n-1}, a_n, a_{n+1}) \geq S(a_n, a_{n+1}, a_{n+2}) \forall n \in N$ .Thus Lemma 2 is true by Lemma 1.

Lemma 3[11]: Let (X, S) be a S metric space, then $S(a_1, a_1, a_2) = S(a_2, a_2, a_1)$ for all $a_1, a_2 \in X$.

II. MAIN RESULT

We establish the following theorem on the fact of convergence of Picard sequences using ternary relation.

Theorem 2.1: Consider (X, S) is complete S metric space having ternary relation M on X and a non-expansive map $f : X \to X$ fulfills

$$S(fa, fb, fc) \leq \left[\frac{S(a, a, fb) + S(b, b, fa) + S(c, c, fc)}{S(a, a, fa) + S(b, b, fb) + S(c, c, fc) + 1} \right] S(a, b, c) \tag{2.1.1}$$

$\forall a, b, c \in M$.

Also suppose that

1) M is f -invariant
2) X has the sequence $\{a_n\}$,so $(a_{n-1}, a_{n-1}, a_n) \in M$, $\forall n \in N$ and $a_n \to \alpha$,when $n \to \infty$,implies $(a_{n-1}, a_{n-1}, \alpha) \in M$, $\forall n \in N$.

If $(a_0, a_0, fa_0) \in M$, where $a_0 \in X$ such that

$$\left[\frac{\left(2S(a_0, a_0, fa_0) + S(f^2a_0, f^2a_0, fa_0) + S(f^2a_0, f^2a_0, f^3a_0) \right)}{\left(S(a_0, a_0, fa_0) + S(fa_0, fa_0, f^2a_0) + S(f^2a_0, f^2a_0, f^3a_0) + 1 \right)} \right] < 1, \text{ then}$$

(i) An only fixed point $\alpha \in X$ exists for the function f .

(ii) Picard sequence, starting from the point $a_0 \in X$, converges towards fixed point of f .

Proof: Suppose a_0 is a point in X so $(a_0, a_0, fa_0) \in M$ and $\{a_n\}$ is Picards sequence of point $a_{n-1} = a_n$, for natural number n , then a_{n-1} is fixed point of f . Next, let's assume $a_{n-1} \neq a_n$ holds true $\forall n \in N$.

Due to the fact $(a_0, a_0, a_1) = (a_0, a_0, fa_0) \in M$, since M is f -invariant, one can infer that

$(a_1, a_1, a_2) \in (fa_0, fa_0, f^2a_0) \in M$, which indicates that

$(a_{n-1}, a_{n-1}, a_n) = (f^{n-1}a_0, f^{n-1}a_0, f^na_0) \in M$, $\forall n \in N$.

Now from (2.1.1)

$$S(a_n, a_{n+1}, a_{n+2}) = S(fa_{n-1}, fa_n, fa_{n+1})$$

$$\leq \left[\frac{\left(S\left(a_{n-1}, a_{n-1}, fa_n\right) + S\left(a_n, a_n, fa_{n-1}\right) + S\left(a_{n+1}, a_{n+1}, fa_{n+1}\right) \right)}{\left(S(a_{n-1}, a_{n-1}, fa_{n-1}) + S(a_n, a_n, fa_n) + S(a_{n+1}, a_{n+1}, fa_{n+1}) + 1 \right)} \right] S(a_{n-1}, a_n, a_{n+1})$$

$$= \left[\frac{\left(S(a_{n-1}, a_{n-1}, a_{n+1}) + S(a_n, a_n, a_n) + S(a_{n+1}, a_{n+1}, a_{n+2}) \right)}{\left(S(a_{n-1}, a_{n-1}, a_n) + S(a_n, a_n, a_{n+1}) + S(a_{n+1}, a_{n+1}, a_{n+2}) + 1 \right)} \right] S(a_{n-1}, a_n, a_{n+1})$$

$$\leq \left[\frac{\left(S(a_{n-1}, a_{n-1}, a_n) + S(a_{n-1}, a_{n-1}, a_n) + S(a_{n+1}, a_{n+1}, a_n) + S(a_{n+1}, a_{n+1}, a_{n+2}) \right)}{\left(S(a_{n-1}, a_{n-1}, a_n) + S(a_n, a_n, a_{n+1}) + S(a_{n+1}, a_{n+1}, a_{n+2}) + 1 \right)} \right] S(a_{n-1}, a_n, a_{n+1})$$

$$= \left[\frac{\left(2S(a_{n-1}, a_{n-1}, a_n) + S(a_{n+1}, a_{n+1}, a_n) + S(a_{n+1}, a_{n+1}, a_{n+2}) \right)}{\left(S(a_{n-1}, a_{n-1}, a_n) + S(a_n, a_n, a_{n+1}) + S(a_{n+1}, a_{n+1}, a_{n+2}) + 1 \right)} \right] S(a_{n-1}, a_n, a_{n+1})$$

$$\leq \left[\frac{\left(2S(a_0, a_0, a_1) + S(a_2, a_2, a_1) + S(a_2, a_2, a_3) \right)}{\left(S(a_0, a_0, a_1) + S(a_1, a_1, a_2) + S(a_2, a_2, a_3) + 1 \right)} \right] S(a_{n-1}, a_n, a_{n+1})$$

$= \lambda S(a_{n-1}, a_n, a_{n+1})$

Where

$$\lambda = \left[\frac{2S(a_0,a_0,a_1)+S(a_2,a_2,a_1)+S(a_2,a_2,a_3)}{S(a_0,a_0,a_1)+S(a_1,a_1,a_2)+S(a_2,a_2,a_3)+1}\right]$$

for all $n \in N$.

Therefore by Lemma 1 and Lemma 2

$$\left\{\frac{2S(a_{n-1},a_{n-1},a_n)+S(a_{n+1},a_{n+1},a_n)+S(a_{n+1},a_{n+1},a_{n+2})}{S(a_{n-1},a_{n-1},a_n)+S(a_n,a_n,a_{n+1})+S(a_{n+1},a_{n+1},a_{n+2})+1}\right\}$$

is a non-increasing sequence, so $\lambda < 1$.

$\therefore S(a_n, a_{n+1}, a_{n+2}) \le S(a_{n-1}, a_n, a_{n+1})$

Thus $\{a_n\}$ is Cauchy's sequence.

The sequence $\{a_n\}$ will converge to point $\alpha \in X$ due to the completeness of X . Our objective now, is to show that f has a fixed point α .Consider $S(a_n, a_n, \alpha) \in M$, as per assumption (2).

Now employing contractive condition

$$S(a_{n+1}, a_{n+1}, f\alpha) = S(fa_n, fa_n, f\alpha)$$

$$\le \left[\frac{\begin{pmatrix} S(a_n,a_n,fa_n)+S(a_n,a_n,fa_n) \\ +S(\alpha,\alpha,f\alpha)\end{pmatrix}}{\begin{pmatrix} S(a_n,a_n,fa_n)+S(a_n,a_n,fa_n) \\ +S(\alpha,\alpha,f\alpha)+1\end{pmatrix}}\right] S(a_n,a_n,\alpha)$$

$$\le \left[\frac{2S(a_n,a_n,a_{n+1})+S(\alpha,\alpha,f\alpha)}{2S(a_n,a_n,a_{n+1})+S(\alpha,\alpha,f\alpha)+1}\right] S(a_n,a_n,\alpha)$$

As $n \to \infty$, $S(\alpha,\alpha,f\alpha) \le S(\alpha,\alpha,\alpha) = 0 \Rightarrow f\alpha = \alpha$, hence α is fixed point of f .

Now consider α and η are two distinct fixed point of f .Then

$$S(\alpha,\alpha,\eta) \le S(f\alpha, f\alpha, f\eta)$$

$$\le \left[\frac{S(\alpha,\alpha,f\alpha)+S(\alpha,\alpha,f\alpha)+S(\eta,\eta,f\eta)}{S(\alpha,\alpha,f\alpha)+S(\alpha,\alpha,f\alpha)+S(\eta,\eta,f\eta)+1}\right] S(\alpha,\alpha,\eta)$$

This implies $S(\alpha,\alpha,\eta) \le S(\alpha,\alpha,\eta)$

This contradicts our assumption, thus f possesses unique fixed point.

Theorem 2.2**:** Consider (X, S) as complete S metric space having ternary relation M on X and a non-expansive map $f : X \to X$ such that

$$S(fa, fb, fc) \le \left[\frac{S(a,a,fb)+S(b,b,fa)+S(c,c,fc)}{S(a,a,fa)+S(b,b,fb)+S(c,c,fc)+1}\right] S(a,b,c) + LS(b,b,fa) \tag{2.2.1}$$

$\forall a,b,c \in M$, L is non-negative real numbers. As well consider that

1) M is f -invariant
2) X have a sequence $\{a_n\}$ where $(a_{n-1}, a_{n-1}, a_n) \in M$ and $a_n \to \alpha$, when $n \to \infty$,implies $(a_{n-1}, a_{n-1}, \alpha) \in M$,for natural numbers n .

If there exists $a_0 \in X$ such that $(a_0, a_0, fa_0) \in M$ and (1.1.2) holds then

(i) An only fixed point $\alpha \in X$ exists for the function f .

(ii) Picard sequence, starting from the point a_0 in X , converges towards fixed point of f .

Proof: Assume a_0 is a point in X so $(a_0, a_0, fa_0) \in M$ and $\{a_n\}$ is Picards sequence of point $a_{n-1} = a_n$, for natural number n, then then a_{n-1} is fixed point of f . Next, let's assume $a_{n-1} \ne a_n$ holds true $\forall n \in N$ .Using contractive condition (2.1.1)

$$S(a_n, a_{n+1}, a_{n+2}) = S(fa_{n-1}, fa_n, fa_{n+1})$$

$$\le \left[\frac{\begin{pmatrix} S(a_{n-1},a_{n-1},fa_n) \\ +S(a_n,a_n,fa_{n-1}) \\ +S(a_{n+1},a_{n+1},fa_{n+1})\end{pmatrix}}{\begin{pmatrix} S(a_{n-1},a_{n-1},fa_{n-1}) \\ +S(a_n,a_n,fa_n) \\ +S(a_{n+1},a_{n+1},fa_{n+1})+1\end{pmatrix}}\right] S(a_{n-1},a_n,a_{n+1}) + LS(a_n, a_n, fa_{n-1})$$

$$= \left[\frac{\begin{pmatrix} 2S(a_{n-1},a_{n-1},a_n)+ \\ S(a_{n+1},a_{n+1},a_n)+ \\ S(a_{n+1},a_{n+1},a_{n+2})\end{pmatrix}}{\begin{pmatrix} S(a_{n-1},a_{n-1},a_n)+ \\ S(a_n,a_n,a_{n+1})+ \\ S(a_{n+1},a_{n+1},a_{n+2})+1\end{pmatrix}}\right] S(a_{n-1},a_n,a_{n+1}) + LS(a_n, a_n, a_n)$$

Continuing from the theorem 1's proof, one can say that $\{a_n\}$ forms Cauchy's sequence.

The sequence $\{a_n\}$ must converge to point $\alpha \in X$ because of completeness of X . By assumption (2), consider $S(a_n, a_n, \alpha) \in M$.

Then using contractive condition (2.2.1)

$$S(a_{n+1}, a_{n+1}, f\alpha) = S(fa_n, fa_n, f\alpha)$$

$$\leq \left[\frac{\begin{pmatrix} S(a_n, a_n, fa_n) + \\ S(a_n, a_n, fa_n) + \\ S(\alpha, \alpha, f\alpha) \end{pmatrix}}{\begin{pmatrix} S(a_n, a_n, fa_n) + \\ S(a_n, a_n, fa_n) + \\ S(\alpha, \alpha, f\alpha) + 1 \end{pmatrix}} \right] S(a_n, a_n, \alpha) + L(a_n, a_n, fa_n)$$

$$\leq \left[\frac{2S(a_n, a_n, a_{n+1}) + S(\alpha, \alpha, f\alpha)}{2S(a_n, a_n, a_{n+1}) + S(\alpha, \alpha, f\alpha) + 1} \right] S(a_n, a_n, \alpha) + LS(a_n, a_n, a_{n+1})$$

As $n \to \infty$,

$S(\alpha, \alpha, f\alpha) \leq S(\alpha, \alpha, \alpha) = 0 \Rightarrow f\alpha = \alpha$. Hence α is fixed point of f.

Now consider α and η are two distinct fixed point of f. Then

$$S(\alpha, \alpha, \eta) \leq S(f\alpha, f\alpha, f\eta)$$

$$\leq \left[\frac{S(\alpha, \alpha, f\alpha) + S(\alpha, \alpha, f\alpha) + S(\eta, \eta, f\eta)}{S(\alpha, \alpha, f\alpha) + S(\alpha, \alpha, f\alpha) + S(\eta, \eta, f\eta) + 1} \right] S(\alpha, \alpha, \eta) + LS(\alpha, \alpha, f\alpha)$$

This implies $S(\alpha, \alpha, \eta) \leq S(\alpha, \alpha, \eta)$

This contradicts our assumption, consequently f possesses a unique fixed point.

III. Results for single valued functions

Following is the discussion regarding corollaries for single valued functions

Corollary 3.1: Consider (X, S) is complete S metric space and a non-expansive map $f : X \to X$ fulfills

$$S(fa, fb, fc) \leq \left[\frac{S(a, a, fb) + S(b, b, fa) + S(c, c, fc)}{S(a, a, fa) + S(b, b, fb) + S(c, c, fc) + 1} \right] S(a, b, c) \tag{3.1.1}$$

$\forall a, b, c \in X$.

Also suppose that for $a_0 \in X$

$$\left[\frac{\begin{pmatrix} 2S(a_0, a_0, fa_0) + S(f^2a_0, f^2a_0, fa_0) \\ +S(f^2a_0, f^2a_0, f^3a_0) \end{pmatrix}}{\begin{pmatrix} S(a_0, a_0, fa_0) + S(fa_0, fa_0, f^2a_0) + \\ S(f^2a_0, f^2a_0, f^3a_0) + 1 \end{pmatrix}} \right] < 1, \text{ then}$$

(i) An only fixed point $\alpha \in X$ exists for the function f.

(ii) Picard sequence, starting from the point $a_0 \in X$, converges towards fixed point f.

Proof: All conditions of theorem 2.1 are satisfied with $M = X \times X \times X$. Thus corollary 3.1 follows from theorem 2.1.

Corollary 3.2: Consider (X, S) as complete S metric space having ternary relation M on X and a non-expansive map $f : X \to X$ such that

$$S(fa, fb, fc) \leq \left[\frac{S(a, a, fb) + S(b, b, fa) + S(c, c, fc)}{S(a, a, fa) + S(b, b, fb) + S(c, c, fc) + 1} \right] S(a, b, c) + LS(b, b, fa) \tag{3.2.1}$$

$\forall a, b, c \in X$, L is non-negative real numbers. As well consider that

If there exists, then

(i)An only fixed point $\alpha \in X$ exists for the function f.

(ii)Picard sequence, starting from the point a_0 in X, converges towards fixed point of f.

Proof**:** All conditions of theorem 2.2 are satisfied with $M = X \times X \times X$. Thus corollary 3.2 follows from theorem 2.2.

Example: Let $X = R^+$. Assume Q as S metric space stated by

$$S(a_1, a_2, a_3) = \left| \frac{1}{a_1} - \frac{1}{a_3} \right| + \left| \frac{1}{a_1} + \frac{1}{a_3} - \frac{2}{a_2} \right|, \text{ for}$$

$a_1, a_2, a_3 \in Q$. If $a_1 = 1, a_2 = 2, a_3 = 5$. Then

L.H.S. $= S(1,2,5) = \left|1 - \frac{1}{5}\right| + \left|1 + \frac{1}{5} - \frac{2}{2}\right| = 1.0$

R.H.S. $= S(1,1,20) + S(2,2,20) + S(5,5,20)$

$$= \left|1 - \frac{1}{20}\right| + \left|1 + \frac{1}{20} - \frac{2}{1}\right| + \left|\frac{1}{2} - \frac{1}{20}\right| + \left|\frac{1}{2} + \frac{1}{20} - \frac{2}{2}\right| + \left|\frac{1}{5} - \frac{1}{20}\right| + \left|\frac{1}{5} + \frac{1}{20} - \frac{2}{5}\right| = 2.2$$

L.H.S. < R.H.S., so conditions of S metric space are satisfied.

Let $f(\alpha) = \begin{cases} \alpha + 15, & \text{for } \alpha \in \{1, 3\} \\ 20, & \text{otherwise} \end{cases}$

Then we verify the contraction condition

L.H.S.=

$$S(f1, f2, f5) = S(16, 17, 20) = \left|\frac{1}{16} - \frac{1}{20}\right| + \left|\frac{1}{16} + \frac{1}{20} - \frac{2}{17}\right| = 0.007353$$

R.H.S. =

$$\left[\frac{S(1,1,f2) + S(2,2,f1) + S(5,5,f5)}{S(1,1,f1) + S(2,2,f2) + S(5,5,f5) + 1} \right] S(1,2,5)$$

$$= \left[\frac{S(1,1,17) + S(2,2,16) + S(5,5,20)}{S(1,1,16) + S(2,2,17) + S(5,5,20) + 1} \right] S(1,2,5)$$

$= 0.75498$

$\therefore$ L.H.S. < R.H.S.

Thus (2.1.1) is verified. Hence $f(20) = 20$ is single fixed point, as shown in Fig 1 below.

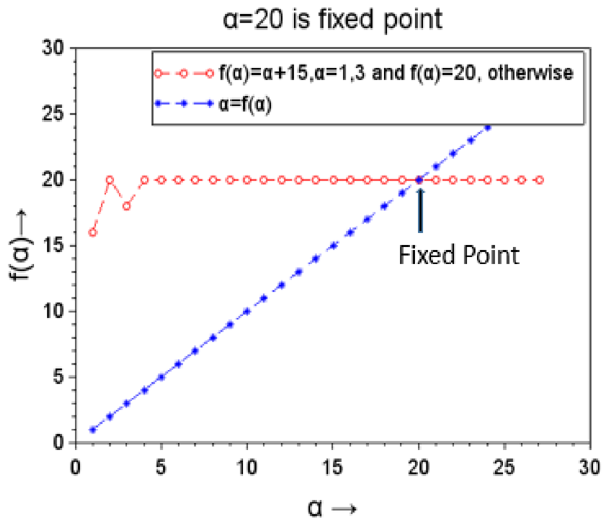

Fig.1. Unique Fixed Point

IV. . RESULTS IN PARTIALLY ORDERED S-METRIC SPACE

We have extended the study of non-expansive mappings in partially ordered S metric space.

Let (X, S) be a S metric space and $(X, \leq)$ be a partially ordered set, then $(X, S, \leq)$ is called partially ordered S metric space. Also $a_1, a_2 \in X$ are called comparable if $a_1 \leq a_2$ or $a_2 \leq a_1$ holds. A self- mapping $f : X \to X$ is called non-decreasing if $f(a_1) \leq f(a_2)$ whenever $a_1 \leq a_2$ for all $a_1, a_2 \in X$.

The following 2 theorems can be proved for non-expansive mappings in partially ordered S metric space.

Theorem 4.1: Let $(X, S, \leq)$ be a partially ordered S metric space. Let mapping $f : X \to X$ be a non-decreasing mapping such that

$$S(fa, fb, fc) \leq \left[\frac{S(a,a,fb)+S(b,b,fa)+S(c,c,fc)}{S(a,a,fa)+S(b,b,fb)+S(c,c,fc)+1}\right] S(a,b,c)$$

For all comparable elements $a,b,c \in X$ .Assume that $\{a_n\}$ is non-decreasing sequence in X such that $a_n \to \alpha$, then $a_n \leq \alpha$, for all $n \in N$. If there exists $a_0 \in X$ such that $a_0 \leq f(a_0)$ then f has a fixed point.

Proof: If $a_0 = f(a_0)$, then nothing to prove. Suppose $a_0 < f(a_0)$ .Since f is non-decreasing function, we have

$$a_0 < f(a_0) \leq f^2(a_0) \leq f^3(a_0) \leq ...... \leq f^n(a_0) \leq ......$$

(2.3.1)

Put $a_{n+1} = f(a_n) \forall n \geq 0$.

(2.3.2)

By Lemma 2 $S(a_n, a_n, a_{n+1}) = S(a_n, a_{n+1}, a_{n+1}) \forall n \in N$

If there exists $n \geq 1$ such that $a_n = a_{n+1}$, then from (2.3.2) $a_n = a_{n+1} = f(a_n)$ .i.e. a_n is fixed point of f and so proof is completed.

Now assume $a_n \neq a_{n+1}$, then for all $n \geq 1$, using non-expansive condition, we have

$$S(a_n,a_n,a_{n+1}) = S(fa_{n-1}, fa_{n-1}, fa_n)$$

$$\leq \left[\frac{S(a_{n-1},a_{n-1},fa_{n-1})+S(a_{n-1},a_{n-1},fa_{n-1})+S(a_n,a_n,fa_n)}{S(a_{n-1},a_{n-1},fa_{n-1})+S(a_{n-1},a_{n-1},fa_{n-1})+S(a_n,a_n,fa_n)+1}\right] S(a_{n-1},a_{n-1},a_n)$$

$$\leq \left[\frac{2S(a_{n-1},a_{n-1},a_n)+S(a_n,a_n,a_{n+1})}{2S(a_{n-1},a_{n-1},a_n)+S(a_n,a_n,a_{n+1})+1}\right] S(a_{n-1},a_{n-1},a_n)$$

$$\leq \frac{2S(a_0,a_0,a_1)+S(a_1,a_1,a_2)}{2S(a_0,a_0,a_1)+S(a_1,a_1,a_2)+1} S(a_{n-1},a_{n-1},a_n)$$

Thus $S(a_n, a_n, a_{n+1}) S(a_{n-1}, a_{n-1}, a_n)$ .Therefore $\{a_n\}$ is a Cauchy sequence. Now as in theorem 2.1, we can prove the presence of unique fixed point in f .

Theorem 4.2: Let $(X, S, \leq)$ be a partially ordered S metric space. Let mapping $f : X \to X$ be a non-decreasing mapping such that

$$S(fa, fb, fc) \leq \left[\frac{S(a,a,fb)+S(b,b,fa)+S(c,c,fc)}{S(a,a,fa)+S(b,b,fb)+S(c,c,fc)+1}\right] S(a,b,c) + LS(b,b,fa)$$

For all comparable elements $a,b,c \in X$ .Assume that $\{a_n\}$ is non-decreasing sequence in X such that $a_n \to \alpha$, then $a_n \leq \alpha$, for all $n \in N$. If there exists $a_0 \in X$ such that $a_0 \leq f(a_0)$ then f has a fixed point.

Proof: Proceeding similar to proofs of theorems 2.2 and 2.3, we can prove the existence of unique common fixed point of f .

V. APPLICATIONS

The fixed point theorems on non-expansive mappings in S-metric space are applicable in following fields:

- The application of S-metric spaces extend to various fields where the notion of distance or similarity doesn't strictly adhere to the axioms of a metric space but still requires a formal framework for analysis and computation.
- Non-expansive mappings play a significant role in the study of S-metric spaces, as they help to use the notion of contraction without relying on the traditional metric space structure.
- Non-expansive mappings are significant to fixed-point theory in S-metric spaces. They help to establish the existence and uniqueness of fixed points, which have

applications in various areas such as optimization, economics, and mathematical modelling.

- Non-expansive mappings are widely used in iterative methods for solving equations and optimization problems in S-metric spaces. These methods include fixed-point iteration, gradient descent, and proximal algorithms among others.
- Non-expansive mappings provide a useful framework for analyzing the behavior of dynamical systems defined on S-metric spaces. They help to characterize stability properties, attractors, and long-term behavior of such systems.

VI. Conclusion

In the present article, two fixed-point theorems have been formulated for a novel category of non-expansive mappings within S-metric space, though the function is not continuous. We have established two more theorems on partially ordered S-metric space for non-expansive mappings. The results in this paper are improvement, modification and extension of results available in existing literature.

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