Maximum Likelihood Estimation in Partially Observed Stochastic Fractional Differential Equations

Chao Wei, Jiahui Liu and Lijiao Wang

Abstract—**This study focuses on maximum likelihood estimation (MLE) within the context of partially observed stochastic fractional differential equations (POSFDEs). Initially, the equation for state estimation is presented, followed by the derivation of the maximum likelihood estimator (MLEr). Subsequently, the asymptotic properties of the estimator are established. Lastly, as an illustration to validate the findings, the Hyperbolic diffusion model is introduced.**

Index Terms—**MLE; POSFDEs; state estimation; consistency; asymptotic normality**

I. Introduction

Statistical inference has attracted considerable interest from many researchers ([5], [23]). For instance, Zhang et al. ([26]) presented a numerical methodology aimed at uncovering the structure and estimating line parameters without any pre-existing information regarding voltage angles. Similarly, Maldonado et al. ([16]) applied a sequential Bayesian method to deduce parameters within stochastic dynamic load models. In another study, Zhang et al. ([27]) examined the concurrent estimation in a particular subset of bilinear model. Ji and Kang ([14]) investigated innovative techniques for online parameter estimation in nonlinear systems. Additionally, Escobar et al. ([9]) suggested various approaches to address the challenges of parameter estimation in continuously operating stochastic systems. Ding ([7]) conducted an analysis of the properties of two distinct least squares methods that proficiently manage disturbances from both white and colored noise using conventional techniques commonly employed in the discipline. Shin and Park ([20]) employed a generator-regularized continuous conditional generative adversarial network to assess uncertain parameters. Amorino et al. ([1]) introduced a contrasting function designed to estimate parameters within a stochastic McKean-Vlasov equation. Mehmood and Raja ([17]) explored evolutionary heuristics of estimation in Hammerstein model. Brusa et al. ([4]) proposed an evolutionary optimization method aimed at refining approximate maximum likelihood estimation in discrete models. In practical applications, elements such as unpredictable communication environments, exemplified by population dynamics with time delays, require consideration of time lags. Consequently, the estimation of parameters for stochastic delay differential equations has gained considerable attention in recent times. Benke and Pap ([3]) investigated the convergence characteristics of the MLEr. Liu and Jia ([15]) employed the method of moments to derive parameter values from observations of discrete solutions. Zhu et al. ([29]) focused on determining parameters in a reaction-diffusion system related to rumor propagation that includes time delays. Jamilla et al. ([13]) applied a multi-parent crossover genetic algorithm to estimate parameters within three models of neutral delay differential equations featuring discrete delays. Long memory processes are extensively utilized across diverse domains. Fractional Brownian motion serves as an effective extension of standard Brownian motion and demonstrates long-range dependence. While employing a long-memory model to characterize certain physical process, accurately identifying the model's parameters becomes crucial. As a result, several researchers have explored the issue of parameter estimation for SFDE influenced by fractional Brownian motion. For example, Wu and Ding ([25]) presented a MLEr based on wavelet analysis, which demonstrates asymptotic normality. In a subsequent study, Dai et al. ([6]) derived a Girsanov-type formula and proceeded to estimate parameters utilizing the MLE approach. By using MLE, Dufitinema et al. ([8]) addressed the challenge of concurrently deriving estimators for all unknown parameters. Moreover, Feng et al. ([10]) developed a comprehensive neural network aimed at estimating both system and noise parameters based on a brief sample trajectory.

When managing a system, situations may arise where directly acquiring the system's state is not feasible, or the expense associated with obtaining this state is prohibitively high. In such cases, it becomes essential to apply specific algorithms to estimate the system's state. Over the past few decades, several researchers have explored the challenge of state estimation in relation to stochastic differential equations. For example, Basit and colleagues ([2]) introduced a distributed state estimator's design specifically tailored for nonlinear systems. Song and associates ([21]) examined state estimation pertaining to neural networks. Meanwhile, Wen et al. ([24]) explored fusion estimation challenges for a type of systems. Zhang et al. ([28]) employed physics-informed deep learning techniques to estimate traffic states utilizing the traffic flow model alongside the computational graph approach. When both parameters and states are simultaneously unknown, it is crucial to integrate theoretical approaches with algorithmic methods to estimate these elements. For instance, Stojanovic et al. ([22]) introduced two distinct strategies

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Chao Wei is a Professor of School of Mathematics and Statistics, An'yang Normal University, An'yang, 455000, China (email: aytongji@126.com).

Jiahui Liu is a Postgraduate Student of School of Mathematics and Statistics, An'yang Normal University, An'yang, 455000, China (email: 1748342373@qq.com).

Lijiao Wang is a Postgraduate Student of School of Mathematics and Statistics, An'yang Normal University, An'yang, 455000, China (email: 2876323947@qq.com).

aimed at robust joint estimation of parameters and states in linear stochastic models, accommodating all conceivable faults and non-Gaussian noise. Hossain ([11]) provided a thorough overview of various techniques used for estimating the state of charge in batteries, which was complemented by a review of methods for estimating parameters in Li-ion battery models. Impraimakis and Smyth ([12]) analyzed the capabilities of a new unscented Kalman filter for estimating input parameters and states in both linear and nonlinear systems. Rodriguez et al. ([19]) introduced an innovative estimator aimed at accurately capturing vehicle dynamics.

While numerous authors have explored the parameter estimation problem associated with stochastic differential equations, there exists limited research specifically addressing parameter estimation for POSFDEs. In this research, the topic is explored. The state estimation equation is provided and the MLEr is derived. Furthermore, the asymptotic properties of the estimator are established.

The structure of the article is outlined below. Section 2 presents the state estimation equation and MLEr. Section 3 discusses the asymptotic properties of the estimator. The Hyperbolic diffusion model is provided as an illustration to validate the findings in Section 4. The conclusions are summarized in Section 5.

II. Problem formulation and preliminaries

Define a fundamental probability space denoted as $(\Omega, F, \mathbb{P})$, which comes with a right-continuous and non-decreasing collection of σ-algebras represented by $\{F_t\}_{t\geq 0}$.

The POSFDEs is considered as follows:

$$\begin{cases} dX_t = f(t, X_t, \theta)dt + g(t, X_t)dW_t^H \\ dY_t = u(t, X_t)dt + v(t, X_t)dW_t, \quad t \in [0, T], \\ X_0 = \xi, Y_0 = \eta. \end{cases} \tag{1}$$

In this context, W^H, for $H \in (\frac{1}{2}, 1)$ signifies the fractional Brownian motion, while W represents standard Brownian motion that is independent of W^H. The parameter θ remains unknown. We also assume that ξ and η have a fixed distribution, denoted as π_0.

It is assumed that the functions f, g, u and v are known and satisfy

Assumption 1: $|f(t, x, \theta)| + |g(t, x)| \leq K(1 + |x|)$ for all $t \in [0, T]$, $\quad K > 0$,

Assumption 2: $|f(t, x, \theta) - f(t, y, \theta)| + |g(t, x) - g(t, y)| \leq K(|x - y|)$ for all $t \in [0, T]$, $\quad K > 0$,

Assumption 3: $|u(t, x)| + |v(t, x)| \leq K'(1 + |x|)$ for all $t \in [0, T]$, $\quad K' > 0$,

Assumption 4: $|u(t, x) - u(t, y)| + |v(t, x) - v(t, y)| \leq K'(|x - y|)$ for all $t \in [0, T]$, $\quad K' > 0$.

Remark 1: Since stochastic differential equation (1) satisfy the linear growth condition and Lipschitz condition, there exists a unique solution.

Let $\Omega = \mathcal{C}([0, T]; \mathbb{R}^2)$ represent the collection of continuous functions mapping the interval $[0, T]$ into $\mathbb{R}^2$. We examine $(X, W^*) = (X_t, W_t^*, t \in [0, T])$ depend on Ω, where for each point $(x, y) \in \Omega$, we have $(X_t, W_t^*)(x, y) = (x_t, y_t)$. The probability measure $\widetilde{\mathbb{P}}$ is the distinguished probability measure on Ω such that, if we define the variable ξ as $\xi = W_0^*$ and let $\widetilde{W} = (\widetilde{W}_t)$ for $t \in [0, T]$ with $\widetilde{W}_t = W_t^* - W_0^*$, then the pair (X, ξ) is independent from the process $\widetilde{W}$, which behaves as fractional Brownian motion characterized by the Hurst parameter H. The canonical filtration on Ω is denoted as $(F_t, t \in [0, T])$, where $F_t = \sigma\{(X_s, W_s^*), 0 \leq s \leq t\} \vee \mathcal{N}$, with $\mathcal{N}$ representing the collection of null sets in the measure space $(\Omega, \widetilde{\mathbb{P}})$.

Define the function $m(x, \theta)$ on the interval $[0, T]$ for every continuous function $x = (x_t, t \in [0, T])$ as follows:

$$m(x, \theta)(t) = \frac{f(t, X_t, \theta)}{g(t, x_t)}, t \in [0, T]). \tag{2}$$

Let $k_{m(x,\theta)}^t = (k_{m(x,\theta)}^t(s), 0 < s < t)$.

Let $A = (A_t, t \in [0, T])$, $< A >= (< A >_t, t \in [0, T])$ by

$$A_t := A_t^{m(X,\theta)}, \quad < A >_t := < A^{m(X,\theta)} >_t . \tag{3}$$

It is understood that A_t and $< A >_t$ are solely determined by $X^{(t)} = (X_s, 0 \leq s \leq t)$.

Define

$$< A, A^* >_t := < A^{m(X,\theta)}, A^* >_t = \int_0^t k_*^t(s) m(X, \theta)(s) ds, \tag{4}$$

and

$$n_t(X, \theta) := n_t^{m(X,\theta)} = \frac{d < A, A^* >_t}{d < A^* >_t}, \tag{5}$$

where $\widetilde{n}_t(X) := \frac{n_t(X,\theta)}{\theta}, t \in [0, T]$.

Define the processes

$$\widetilde{A}_t(x, \theta) = \int_0^t k_{m(x,\theta)}^t d\widetilde{W}_s^H, \tag{6}$$

$$< \widetilde{A} >_t (x, \theta) := \int_0^t m(x, \theta)(s) k_m^t(s) ds, \quad t \in [0, T], \tag{7}$$

where $\widetilde{A}_t(\theta, x)$ is a Gaussian martingale.

Let

$$\Upsilon_t(x, \theta) = e^{\widetilde{A}_t(x,\theta) - \frac{1}{2} < \widetilde{A} >_t (x,\theta)}, \quad t \in [0, T], \tag{8}$$

and

$$\Upsilon_t(\theta) = \Upsilon_t(X, \theta). \tag{9}$$

Let $\mathbb{P} = \Upsilon_T(\theta)\widetilde{\mathbb{P}}$, $\mathcal{Y}_t = \sigma(\{Y_s, 0 \leq s \leq t\})$, $\pi_t(\phi) = \mathbb{E}[\phi(X_t)|\mathcal{Y}_t]$, $\sigma_t(\phi) = \widetilde{\mathbb{E}}[\phi(X_t)\Upsilon_t|\mathcal{Y}_t]$.

Then, we have

$$\pi_t(\phi) = \frac{\sigma_t(\phi)}{\sigma_t(1)}, \quad t \in [0, T]. \tag{10}$$

Define

$$Z_t = \int_0^t k_{m(X,\theta)}^t(s) g^{-1}(s, X_s) dY_s, \quad t \in [0, T], \tag{11}$$

$$Z_t^* = \int_0^t k_*^t(s) g^{-1}(s, X_s) dY_s, \quad t \in [0, T]. \tag{12}$$

Then, Z and Z^* are semimartingales as follows:

$$Z_t = < A >_t + A_t, \quad t \in [0, T], \tag{13}$$

$$Z_t^* = < A, A^* >_t + A_t^*, \quad t \in [0, T]. \tag{14}$$

Thus,

$$Z_t = \int_0^t n_s^2(X, \theta) d < N^* >_s + \int_0^t n_s(X, \theta) dN_s^*, \tag{15}$$

$$Z_t^* = \int_0^t n_s(X, \theta) d < N^* >_s + N_t^*, \tag{16}$$

where $t \in [0, T]$.

Then, we have

$$Z_t = \int_0^t n_s(X, \theta) dZ_t^*, \quad t \in [0, T]. \tag{17}$$

Let

$$\alpha_t = Z_t^* - \int_0^t \pi_s(n) d < N^* >_s, \quad t \in [0, T]. \tag{18}$$

The process $\pi_s(n) = \mathbb{E}[n_s(X, \theta|\mathcal{Y}_s], \quad 0 \le s \le t.$

For $t \in [0, T]$, the unnormalized filter is

$$\widetilde{\Upsilon}_t(\theta) = \sigma_t(1) = \widetilde{\mathbb{E}}[\Upsilon_t|\mathcal{Y}_t]. \tag{19}$$

Then, we have

$$\widetilde{\Upsilon}_T(\mathcal{Y}_t, \theta) = e^{\theta \int_0^T \pi_s(\widetilde{n}) dZ_s^* - \frac{\theta^2}{2} \int_0^T \pi_s^2(\widetilde{n}) d<N^*>_s}. \tag{20}$$

Then, the MLEr is

$$\widehat{\theta}_T = \frac{\int_0^T \pi_s(\widetilde{n}) dZ_s^*}{\int_0^T \pi_s^2(\widetilde{n}) d < N^* >_s}. \tag{21}$$

In the following section, we will demonstrate the asymptotic properties of the MLEr.

III. Main Results and Proofs

Theorem 1: Under conditions 1–4, as $T \to \infty$, the MLEr $\widehat{\theta}_T$ is consistent, that is to say

$$\widehat{\theta}_T \overset{a.s.}{\to} \theta.$$

Proof: Since

$$dZ_t^* = \pi_t(n) d < N^* >_t + d\alpha_t. \tag{22}$$

We know that α is a continuous martingale on $(\mathcal{Y}_t, \mathbb{P})$ and $< \alpha > = < N^* >$.

Then, we obtain

$$\widehat{\theta}_T = \frac{\int_0^T \pi_s(\widetilde{n}) dZ_s^*}{\int_0^T \pi_s^2(\widetilde{n}) d < N^* >_s} = \theta + \frac{\int_0^T \pi_s(\widetilde{n}) d\alpha_s}{\int_0^T \pi_s^2(\widetilde{n}) d < N^* >_s}, \tag{23}$$

in other words

$$\widehat{\theta}_T - \theta = \frac{\int_0^T \pi_s(\widetilde{n}) d\alpha_s}{\int_0^T \pi_s^2(\widetilde{n}) d < N^* >_s}. \tag{24}$$

Thus, we get

$$\frac{\int_0^T \pi_s(\widetilde{n}) d\alpha_s}{\int_0^T \pi_s^2(\widetilde{n}) d < N^* >_s} \overset{a.s.}{\to} 0. \tag{25}$$

Hence, we have

$$\widehat{\theta}_T \overset{a.s.}{\to} \theta.$$

The proof is complete. ∎

Remark 2: From conditions 1–4, we could have

$$\limsup_T \frac{Q_T^{\frac{1}{2}} |\widehat{\theta}_T - \theta|}{(2 \log \log Q_T)^{\frac{1}{2}}} = 1, a.s.$$

where $Q_T = \int_0^T \pi_s^2(\widetilde{n}) d < N^* >_s$.

Theorem 2: Under conditions 1–4,

$$\sqrt{\int_0^T \pi_s^2(\widetilde{n}) d < N^* >_s} (\widehat{\theta}_T - \theta) \overset{d}{\to} \mathcal{N}(0, 1).$$

as $T \to \infty$.

Proof:

$$\sqrt{\int_0^T \pi_s^2(\widetilde{n}) d < N^* >_s} (\widehat{\theta}_T - \theta) = \frac{\int_0^T \pi_s(\widetilde{n}) d\nu_s}{\sqrt{\int_0^T \pi_s^2(\widetilde{n}) d < N^* >_s}}.$$

Then, we have

$$\frac{\int_0^T \pi_s(\widetilde{n}) d\nu_s}{\sqrt{\int_0^T \pi_s^2(\widetilde{n}) d < N^* >_s}} \overset{d}{\to} \mathcal{N}(0, 1).$$

Thus,

$$\sqrt{\int_0^T \pi_s^2(\widetilde{n}) d < N^* >_s} (\widehat{\theta}_T - \theta) \overset{d}{\to} \mathcal{N}(0, 1). \tag{26}$$

The proof is complete. ∎

IV. Example

Consider the partially observed fractional Hyperbolic diffusion model as follows:

$$\begin{cases} dX_t = \theta \dfrac{X_t}{\sqrt{1 + X_t^2}} dt + dW_t^H \\ dY_t = X_t dt + dW_t, \quad t \in [0, T], \\ Y_0 = 0, X_0 = 0, \end{cases} \tag{27}$$

in which W^H, for $H \in (\frac{1}{2}, 1)$ signifies the fractional Brownian motion, while W represents standard Brownian motion that is independent of W^H. The parameter θ remains unknown.

Since $|\theta \frac{x}{\sqrt{1+x^2}} - \theta \frac{y}{\sqrt{1+y^2}}| \le 2\theta |x - y|$ and $|\theta \frac{x}{\sqrt{1+x^2}}| \le \theta(1 + |x|)$, it can be checked that the equation satisfies Assumptions 1–4.

Define $\widehat{X}_t = \mathbb{E}(X_t|\mathcal{Y}_t)$, $K_t = \mathbb{E}([X_t - \widehat{X}_t]^2|\mathcal{Y}_t)$.

Then,

$$\begin{cases} d\widehat{X}_t = \theta \dfrac{\widehat{X}_t}{\sqrt{1 + \widehat{X}_t^2}} dt + K_t d\nu_t, t \in [0, T] \\ \widehat{X}_0 = 0, \end{cases} \tag{28}$$

$$\begin{cases} dK_t = dt + 2\theta K_t dt - K_t^2 d < N^* >_t \\ K_0 = 0. \end{cases} \tag{29}$$

Thus,

$$K_t \to \theta + \sqrt{\theta^2 + 1}, \tag{30}$$

as $t \to \infty$.

Let $K_\theta = \theta + \sqrt{\theta^2 + 1}$.

Then, we have

$$\begin{cases} d\widehat{X}_t = \theta \dfrac{\widehat{X}_t}{\sqrt{1 + \widehat{X}_t^2}} dt + K_\theta d\nu_t, t \in [0, T] \\ \widehat{X}_0 = 0. \end{cases} \tag{31}$$

The verification of the MLEr's conformity with the asymptotic properties is straightforward.

Next, we make the simulations of MLEr. We utilized Paxson's method ([18]) for simulation. In Table 1, $H = 0.65$, the size of n ranges from 1000 to 5000. In Table 2, $H = 0.75$, the size of n ranges from 1000 to 5000. In Table 3, $H = 0.65$, the size of n ranges from 10000 to 50000. In Table 4, $H = 0.75$, the size of n ranges from 10000 to 50000. In Table 5, $H = 0.8$, the size of n ranges from 10000 to 50000.

TABLE I
MLEr simulation results.

True Value	Average Value		Absolute Error
θ	Size n	$\widehat{\theta}_n$	$\|\theta - \widehat{\theta}_n\|$
2	1000	2.0517	0.0517
	2000	2.0368	0.0368
	5000	2.0149	0.0149
3	1000	3.0621	0.0621
	2000	3.0408	0.0408
	5000	3.0193	0.0193

TABLE II
MLEr simulation results.

True Value	Average Value		Absolute Error
θ	Size n	$\widehat{\theta}_n$	$\|\theta - \widehat{\theta}_n\|$
1	1000	1.0625	0.0625
	2000	1.0481	0.0481
	5000	1.0173	0.0173
2	1000	2.0592	0.0592
	2000	2.0313	0.0313
	5000	2.0169	0.0169

TABLE III
MLEr simulation results.

True Value	Average Value		Absolute Error
θ	Size n	$\widehat{\theta}_n$	$\|\theta - \widehat{\theta}_n\|$
2	10000	2.0358	0.0358
	20000	2.0071	0.0071
	50000	2.0005	0.0005
3	10000	3.0296	0.0296
	20000	3.0052	0.0052
	50000	3.0007	0.0007

TABLE IV
MLEr simulation results.

True Value	Average Value		Absolute Error
θ	Size n	$\widehat{\theta}_n$	$\|\theta - \widehat{\theta}_n\|$
1	10000	1.0263	0.0263
	20000	1.0105	0.0105
	50000	1.0011	0.0011
2	10000	2.0217	0.0217
	20000	2.0091	0.0091
	50000	2.0008	0.0008

V. Conclusion

The purpose of this research is to explore the issue of MLE for POSFDEs. We presented the state estimation equation. We derived the asymptotic properties such as consistency and asymptotic normality for the MLEr by utilizing the probability theory. Future investigations will focus on addressing estimation challenges for POSFDE driven by fractional Lévy process.

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TABLE V
MLER SIMULATION RESULTS.

True Value		Average Value	Absolute Error
θ	Size n	$\widehat{\theta}_n$	$\lvert\theta - \widehat{\theta}_n\rvert$
1	10000	1.0286	0.0286
	20000	1.0127	0.0127
	50000	1.0016	0.0016
2	10000	2.0235	0.0235
	20000	2.0103	0.0103
	50000	2.0012	0.00012

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