Positive Solutions for Fractional Switched System with Integral Boundary Conditions

Hui Xu*, Baiyan Xu

***Abstract*—In this article, We find positive functions which satisfy fractional order systems with integrals in the boundary conditions. We first equivalently transform the fractional-order switched system into an integral operator equation. Next we discuss the equivalent integral equation. We present the properties of Green's function for the above problem. Finally, as an application of Green's function, we discuss the above problem have positive solutions under certain conditions. Our results extend the existing results. The tool we use is the fixed point theorem about mixed monotone operator theory.**

***Index Terms*—Switched fractional differential equation; Positive solution; Mixed monotone operator theory; Integral boundary conditions.**

I. Introduction

FRACTIONAL calculus theory can effectively analyze many phenomena in engineering and natural sciences. Many physical problems, like earthquakes, electrodynamics of complex media, and measurements are related to fractional calculus theory [1-7,24].

Recently, problems with integral boundaries have become popular among scholars (see [8-10]). In [8], authors considered the following problems earliest

$$^{C}D_{0+}^{\beta}u(r)+f(r,u(r))=0,\quad 0<r<1,$$

$$u(1)=\lambda\int_0^1 u(e)de,\quad u(0)=u'(0)=0.$$

In [9], authors considered the

$$D_{0+}^{\beta}u(r)+\lambda f(r,u(r),u(r))=0,\quad 0<r<1,$$

$$u(1)=\int_0^1 q(r)u(r)dr,\quad u(0)=u'(0)=0.$$

As we all know, the switched systems consist of a switching signal and many subsystems. Many issues in the application fields may include switched systems. For example, the electric power system in the literature (Shtessel, Raznopolov, Ozerov, [11]), the Robotic system in the literature [12], the model of HIV infection in the literature [13] and multi-vehicle system in the literature (Zhang et al. [14]). The study of determining whether solutions exist is the most fundamental and important issue in switched systems. In [15], the authors verify that the solutions not only exist but are also unique of switched Hamiltonian systems. Positive functions which satisfy a class of fractional switched systems are considered by Lv et al. (2014)[16]. The conclusion that coupled implicit -Hilfer switched systems which is fractional order have solution has been proven by Ahmad, Zada, and Wang (2020) [17]. Under some time-varying switching law, in paper [18], it is shown that the solution of an on-off nonlinear switched system exists and is unique. For uncertain nonlinear switched systems, Luo et al. (2024) [19] presented event-triggered prescribed performance control.

Li and Liu [20] studied the following issues

$$x''(r)+f_{\sigma(r)}(r,x(r))=0,\qquad r\in J \text{ denotes interval } [0,1],$$

$$x(1)=\int_0^1 a(e)x(e)de,\qquad x(0)=0.$$

The switching signal $\sigma(r)$ maps J to M which is the interval $\{1,2,\cdots,N\}$.

In [21], Guo discussed the following issues

$$D_{0+}^{\beta}\varphi_p(D_{0+}^{\alpha}u(r))=f_{\sigma(r)}(r,u(r),D_{0+}^{\gamma}u(r)),$$

$$r\in J \text{ is the interval } [0,1],$$

$$kD_{0+}^{\alpha}u(\eta)=D_{0+}^{\alpha}u(0),$$

$$\mu\int_0^1 u(e)de+\lambda u(\xi)=u(0),\quad \xi,\eta\in[0,1].$$

Lv and Chen [16] gave positive functions which satisfied the following issues

$$^{C}D_{0+}^{\alpha}u(r)+f_{\sigma(r)}(r,u(r))+g_{\sigma(r)}(r,u(r))=0,$$

$$r\in J \text{ denotes interval } [0,1],$$

$$u(1)=\int_0^1 u(e)de,\quad u''(0)=0,\quad u(0)=0.$$

But one noticed eigenvalue of p-Laplacian fractional-order switched systems had not been studied. In light of the above research content, we analyze issues below in this paper

$$\begin{aligned}&D_{0+}^{\beta}\varphi_p(D_{0+}^{\alpha}\phi(t))+\lambda^{p-1}f_{\sigma(t)}(t,\phi(t),\phi(t))=0,\\&t\in J=[0,1],\end{aligned}\tag{1}$$

$$D_{0+}^{\alpha}\phi(0)=0,\ \phi(1)=\int_0^1 k(s)\phi(s)ds,\ \phi(0)=0,\tag{2}$$

here D_{0+}^{β}, D_{0+}^{α} represent the fractional derivatives of Riemann-Liouville style, $0<\beta\le 1,\ 1<\alpha\le 2$, $\varphi_p(s)$ is the p-Laplacian operator, $\sigma(t)$ is a piecewise function, $\sigma(t)$ maps J to $\{1,2,\cdots,N\}$, and each segment is a constant about t. Considering the discussed switching signal $\sigma(t)$, we construct

$$\{(i_0,t_0),\cdots,(i_j,t_j),\cdots,(i_k,t_k)|i_j \text{ belongs to the set } \{1,2,\cdots,N\}\},$$

here the value of j ranges from 0 to k. We first construct an iterative sequence, and then apply the theorem of monotonic operators with forcing terms to obtain the existence results to the studied problem.

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II. The preliminary lemmas

(H_1) k maps $[0,1]$ to $[0,\infty)$ and k is a L^1 function defined on the interval from 0 to 1. Let $\omega=\int_0^1 s^{\alpha-1}k(s)ds<1$.

Definition 2.1 [16]

$$I_{0^+}^{\alpha}h(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\rho)^{\alpha-1}h(\rho)d\rho,\quad \alpha>0.$$

We call the above expression R-L fractional integral of h.

Definition 2.2 [16]

$$D_{0^+}^{\alpha}h(t)=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\int_0^t (t-\rho)^{n-\alpha-1}h(\rho)d\rho.$$

We call the above expression R-L fractional derivative of h.

We denote P is a cone of Banach space E, order relation generated by the cone P likes $P\subset E, i.e., \nu-\mu\in P$ if and only if $\mu\le\nu$. we write $\nu>\mu$ or $\mu<\nu$, if $\mu\ne\nu$ and $\mu\le\nu$. We define P to be a normal cone, if there exists a positive number N satisfies $\|\mu\|\le N\|\nu\|$ for all μ,ν belong to the set E, $\theta\le\mu\le\nu$. We define the order interval between ν_1 and ν_2 to be the set $[\nu_1,\nu_2]=\{\nu\in E|\nu_1\le\nu\le\nu_2\}$ for $\nu_1,\nu_2\in E$. The operator T maps the set E to the set E, If $\mu\le\nu$, we say T is increasing (decreasing), if $T\mu\le T\nu$ $(T\mu\ge T\nu)$.

We define $\mu\sim\nu$ to be an equivalence relation, if two positive numbers λ and ϑ satisfy $\lambda\mu\le\nu\le\vartheta\mu$, for μ,ν belong to the set E. We define P_l to be $P_l=\{\mu\in E|\mu\sim l\}$, in which $l>\theta$. Clearly, $P_l\subset P$ and $\lambda P_l=P_l$ for all $\lambda>0$.

Definition 2.3 [23] If we choose $\mu_i,\nu_i(i=1,2)$ in P and $\mu_1\le\mu_2,\ \nu_1\ge\nu_2$, we say T is mixed monotone operator $T:\ P\times P\to P$ if $T(\mu_1,\ \nu_1)\le T(\mu_2,\ \nu_2)$. At this point, we also say T being monotonically increasing in μ and monotonically decreasing in ν. If μ belongs to P and satisfies $\mu=T(\mu,\mu)$, we call T has a fixed point μ.

Theorem 2.4 [23] We use P to stand for a normal cone. Let T maps the set $P\times P$ to the set P, mixed monotone operator T satisfies the relations below:
(A_1) there is $l\in P,\ l>\theta$ satisfying $T(l,l)\in P_l$;
(A_2) for any μ,ν belong to the set P and γ belongs to the interval $(0,1)$, there is $\omega(\gamma)\in(t,1]$ satisfying $T(\gamma\mu,\gamma^{-1}\nu)\ge\omega(\gamma)T_\lambda(\mu,\nu)$.

Then there has a unique function $\mu^*\in P_l$ satisfying the expression $\mu=T(\mu,\mu)$. Moreover, for given $\mu_0,\nu_0\in P_l$, we construct a set of equations

$$\mu_m=T(\mu_{m-1},\nu_{m-1}),\ m \text{ can take values of } 1,2,\cdots,$$

$$\nu_m=T(\nu_{m-1},\mu_{m-1}),\ m \text{ can take values of } 1,2,\cdots,$$

then when $m\to\infty$, we have $\mu_m\to\mu^*$ and $\nu_m\to\mu^*$.

Lemma 2.5 [22] Given a continuous function h, the unique expression $\phi(t)$ satisfies the fractional sysytem

$$D_{0^+}^{\alpha}\phi(t)+h(t)=0,\ 0<t<1,\ 1<\alpha\le 2,\tag{3}$$

$$\phi(1)=\int_0^1 k(s)\phi(s)ds,\quad \phi(0)=0,\tag{4}$$

in which $\phi(t)$ looks like

$$\phi(t)=\int_0^1 H(t,s)h(s)ds,\tag{5}$$

here

$$H(t,s)=H_1(t,s)+H_2(t,s),\tag{6}$$

$$H_1(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}, & 0\le s\le t\le 1,\\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0\le t\le s\le 1,\end{cases}\tag{7}$$

$$H_2(t,s)=\frac{t^{\alpha-1}}{1-\omega}\int_0^1 H_1(\tau,s)k(\tau)d\tau.\tag{8}$$

Lemma 2.6 [22] The expression $H_1(t,s)$ given by (7) meets the relationship below

$$\frac{t^{\alpha-1}(1-t)s(1-s)^{\alpha-1}}{\Gamma(\alpha)}\le H_1(t,s)\le\frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha-1)},\tag{9}$$

$$\forall\ t,s\in[0,1].$$

Denote

$$k_A(s)=\int_0^1 H_1(\tau,s)k(\tau)d\tau.\tag{10}$$

Lemma 2.7 [22] Let $k_A(s)\ge 0,\ s\in[0,1]$, the expression $H(t,s)$ given by (6) meets:
(i) The function $H(t,s)$ is continuous on the interval t belongs to the inteval $(0,1)$ and s belongs to the inteval $(0,1)$;
(ii) $H(t,s)>0$ for each $s,t\in(0,1)\times(0,1)$;
(iii) $H(t,s)\le\frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-\omega)\Gamma(\alpha)}$ for t belongs to the inteval $(0,1)$ and s belongs to the inteval $(0,1)$;
(iv) $H(t,s)\ \ge\ \frac{t^{\alpha-1}}{1-\omega}\int_0^1 H_1(\tau,s)k(\tau)d\tau = \frac{k_A(s)}{1-\omega}t^{\alpha-1}$ for $s,t\in[0,1]$, here, ω is reflected in condition (H_1).

Proof: $(i),(ii)$ are easy to prove. Here we will not prove them. For (iii), it is evident by (7) that

$$H_1(t,s)\le\frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}\ \text{ for } s,t\in[0,1].\tag{11}$$

Thus, by (6),(8) and (11), we have

$$\begin{aligned} H(t,s)&=H_1(t,s)+H_2(t,s)\\ &\le H_1(t,s)+\frac{t^{\alpha-1}}{1-\omega}\int_0^1 H_1(\tau,s)k(\tau)d\tau\\ &\le\frac{(1-s)^{\alpha-1}t^{\alpha-1}}{\Gamma(\alpha)}\\ &+\frac{t^{\alpha-1}}{1-\omega}\int_0^1\frac{\tau^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}k(\tau)d\tau\\ &=\frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-\omega)\Gamma(\alpha)}.\end{aligned}\tag{12}$$

On the other hand, taking into account Lemma 2.5, expression of $H(t,s)$, one gets

$$\begin{aligned} H(t,s)&=H_1(t,s)+H_2(t,s)\\ &\ge H_2(t,s)=\frac{t^{\alpha-1}}{1-\omega}\int_0^1 H_1(\tau,s)k(\tau)d\tau\\ &=\frac{k_A(s)}{1-\omega}t^{\alpha-1}.\end{aligned}\tag{13}$$

■

Lemma 2.8 There is a unique function $\phi(t)$ satisfying the fractional switched system

$$D_{0^+}^{\beta}\varphi_p(D_{0^+}^{\alpha}\phi(t))+\lambda^{p-1}f_{\sigma(t)}(t,\phi(t),\phi(t))=0,\tag{14}$$

$$t\in J=[0,1],$$

$$D_{0+}^{\alpha}\phi(0)=0,\quad \phi(0)=0,\quad \phi(1)=\int_0^1 k(s)\phi(s)ds, \quad (15)$$

where ϕ is like this

$$\begin{aligned}\phi(t) &= \lambda\int_0^1 H(t,s)\varphi_q\big(\tfrac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1} \\ &\quad f_{\sigma(\tau)}(\tau,\phi(\tau),\phi(\tau))d\tau\big)ds,\end{aligned} \quad (16)$$

the function $H(t,s)$ is given by relationship (6).

Proof: From the system (14), (15), we get

$$\begin{aligned}&\varphi_p(D_{0+}^{\alpha}\phi(t)) \\ &= -\lambda^{p-1}\tfrac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}f_{\sigma(s)}(s,\phi(s),\phi(s))ds+C_0.\end{aligned}$$

Furthermore, the boundary condition $D_{0+}^{\alpha}\phi(0)=0$ implies that $C_0=0$. Thus, we have

$$\begin{aligned}&\varphi_p(D_{0+}^{\alpha}\phi(t)) \\ &= -\lambda^{p-1}\tfrac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}f_{\sigma(s)}(s,\phi(s),\phi(s))ds,\end{aligned}$$

this is equivalent to

$$\begin{aligned}&D_{0+}^{\alpha}\phi(t) \\ &= -\lambda\varphi_q\big(\tfrac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}f_{\sigma(s)}(s,\phi(s),\phi(s))ds\big).\end{aligned} \quad (17)$$

Considering the above equation (17), boundary condition (15) and Lemma 2.7, one can acquire the result of Lemma 2.8. ■

III. Main results

Under the standard norm

$$\|\phi\|=\max_{0\le t\le 1}|\phi(t)|,$$

continuous function on the interval $[0,1]$ defined as E is a Banach space. Mark P the normal cone by

$$P=\{\phi\in C[0,1]|\phi(t)\ge 0,\ t \text{ belongs to the interval } [0,1]\}.$$

Select any two quantities ϕ,ψ in space $C[0,1]$, the expression $\phi\le\psi$ is equivalent to $\phi(t)\le\psi(t)$, when t takes value in the range of 0 to 1.

Theorem 3.1 Let (H_1) is true and

(H_2) $f_i:\ [0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ are continuous, $f_i(t,\phi,\psi)$ is increasing in ϕ, when ϕ taking value in the range of 0 to $+\infty$, for fixed t taking value in the range of 0 to 1 and ψ taking value in the range of 0 to $+\infty$, decreasing in ψ taking value in the range of 0 to $+\infty$, for fixed t taking value in the range of 0 to 1 and ϕ taking value in the range of 0 to $+\infty$, for the value of i takes value in the range of 0 to N.

(H_3) $f_i(t,0,1)\ne 0,\ f_i(t,1,0)\ne 0,$ if t belongs to the interval $[0,1]$, for i taking value in the range of 1 to N;

(H_4) choose any γ from 0 to 1, there exists a constant $\omega(\gamma)$ in the interval from γ to 1 meets the relationship

$$f_i(t,\gamma\phi,\gamma^{-1}\psi)\ge(\omega(\gamma))^{p-1}f_i(t,\phi,\psi),$$

for all t in the interval $[0,1]$, ϕ,ψ in the interval $[0,+\infty)$. Thus for any positive parameter λ, there exists a unique positive function ϕ_λ^* belonging to P_l,satisfies the fractional switched system (1), (2), here $l(t)=t^{\alpha-1},\ t$ belongs to the interval $[0,1]$. Moreover, choose any ϕ_0,ψ_0 from the interval P_l, denote

$$\begin{aligned}\phi_{n+1}(t) &= \lambda\int_0^1 H(t,s)\varphi_q\big(\tfrac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1} \\ &\quad f_{\sigma(\tau)}(\tau,\phi_n(\tau),\psi_n(\tau))d\tau\big)ds,\quad n=0,1,2,\cdots,\end{aligned}$$

$$\begin{aligned}\psi_{n+1}(t) &= \lambda\int_0^1 H(t,s)\varphi_q\big(\tfrac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1} \\ &\quad f_{\sigma(\tau)}(\tau,\psi_n(\tau),\phi_n(\tau))d\tau\big)ds,\quad n=0,1,2,\cdots,\end{aligned}$$

there are

$$\phi_n(t) \text{ converge to } \phi_\lambda^*(t),\quad \psi_n(t) \text{ converge to } \phi_\lambda^*(t),\ (n\to\infty),$$

we can find $H(t,s)$ from Lemma 2.5.

Proof: As we know, the expression $\phi(t)$ satisfies the fractional switched system (1),(2) is equivalent to the following

$$\begin{aligned}\phi(t) &= \lambda\int_0^1 H(t,s)\varphi_q\big(\tfrac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1} \\ &\quad f_{\sigma(\tau)}(\tau,\phi(\tau),\psi(\tau))d\tau\big)ds,\end{aligned}$$

we can find $H(t,s)$ from Lemma 2.5. For any ϕ,ψ belongs to set P, we write

$$\begin{aligned}&T_\lambda(\phi,\psi)(t)=\lambda\int_0^1 H(t,s) \\ &\varphi_q\big(\tfrac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}f_{\sigma(\tau)}(\tau,\phi(\tau),\psi(\tau))d\tau\big)ds.\end{aligned} \quad (18)$$

From the condition (H_2) and the equation (18), for any $i=1,2,\ \phi_i,\psi_i$ belongs to set P, and $\phi_1\ge\phi_2,\ \psi_1\le\psi_2$, there are

$$\begin{aligned}&\lambda\int_0^1 H(t,s) \\ &\varphi_q\big(\tfrac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}f_i(\tau,\phi_1(\tau),\psi_1(\tau))d\tau\big)ds \\ &\ge\lambda\int_0^1 H(t,s) \\ &\varphi_q\big(\tfrac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}f_i(\tau,\phi_2(\tau),\psi_2(\tau))d\tau\big)ds, \\ &i=1,2,\cdots,N,\end{aligned}$$

so we have

$$T_\lambda(\phi_1,\psi_1)(t)\ge T_\lambda(\phi_2,\psi_2)(t),$$

this equals

$$T_\lambda(\phi_1,\psi_1)\ge T_\lambda(\phi_2,\psi_2).$$

From this, we can conclude that T_λ is a mixed monotone operator, here T_λ maps the set $P\times P$ to the set P..

In fact, T_λ meets all the requirements of the Theorem 2.4. Next we will prove it, for any ϕ,ψ belong to the set P and γ belongs to the interval $(0,1)$, one can get

$$\begin{aligned}&\lambda\int_0^1 H(t,s) \\ &\varphi_q\big(\tfrac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}f_i(\tau,\gamma\phi(\tau),\gamma^{-1}\psi(\tau))d\tau\big)ds \\ &\ge\lambda\int_0^1 H(t,s) \\ &\varphi_q\big(\tfrac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}(\omega(\gamma))^{p-1}f_i(\tau,\phi(\tau),\psi(\tau))d\tau\big)ds, \\ &=\lambda\omega(\gamma)\int_0^1 H(t,s) \\ &\varphi_q\big(\tfrac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}f_i(\tau,\phi(\tau),\psi(\tau))d\tau\big)ds,\end{aligned}$$

here i takes value in the range of 1 to N. so one has

$$T_\lambda(\gamma\phi,\gamma^{-1}\psi)(t)\ge\omega(\gamma)T_\lambda(\phi,\psi)(t),$$

it means that

$$T_\lambda(\gamma\phi,\gamma^{-1}\psi)\ge\omega(\gamma)T_\lambda(\phi,\psi)$$

for any ϕ,ψ belong to the set P and γ belongs to the interval $(0,1)$. So, the requirement (A_2) of Theorem 2.4 can be satisfied. Then, according to the conditions $(H_1),(H_2)$ and the conclusion of the Lemma 2.7, we can derive

$$\begin{aligned}&\int_0^1 H(t,s) \\ &\varphi_q\big(\tfrac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}f_i(\tau,l(\tau),l(\tau))d\tau\big)ds \\ &\ge\tfrac{1}{1-\omega}t^{\alpha-1}\int_0^1 k_A(s) \\ &\varphi_q\big(\tfrac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}f_i(\tau,0,1)d\tau\big)ds,\end{aligned}$$

here i takes value in the range of 1 to N. On the contrary, according to the condition (H_2) and the conclusion of the Lemma 2.7, for any t takes value in the range of 0 to 1 and $i = 1, 2, \cdots, N$, we can derive

$$\begin{aligned}
&\int_0^1 H(t,s) \\
&\varphi_q\big(\tfrac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f_i(\tau, l(\tau), l(\tau))d\tau\big)ds \\
&\le \tfrac{1}{(1-\omega)\Gamma(\alpha)} t^{\alpha-1}\int_0^1 (1-s)^{\alpha-1} \\
&\varphi_q\big(\tfrac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f_i(\tau, 1, 0)d\tau\big)ds,
\end{aligned}$$

here i takes value in the range of 1 to N. For i taking value from 1 to N, let

$$\begin{aligned}
m_i = &\frac{1}{(1-\omega)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} \\
&\varphi_q\big(\tfrac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f_i(\tau, 1, 0)d\tau\big)ds,
\end{aligned} \tag{19}$$

$$\begin{aligned}
n_i = &\frac{1}{1-\omega}\int_0^1 k_A(s) \\
&\varphi_q\big(\tfrac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f_i(\tau, 0, 1)d\tau\big)ds.
\end{aligned} \tag{20}$$

Conditions $f_i(t, 0, 1) \neq 0,\ f_i(t, 1, 0) \neq 0$ imply that

$$\begin{aligned}
&\int_0^1 (1-s)^{\alpha-1}\varphi_q\big(\tfrac{1}{\Gamma(\beta)} \\
&\int_0^s (s-\tau)^{\beta-1} f_i(\tau, 1, 0)d\tau\big)ds \\
&> 0,
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 k_A(s)\varphi_q\big(\tfrac{1}{\Gamma(\beta)} \\
&\int_0^s (s-\tau)^{\beta-1} f_i(\tau, 0, 1)d\tau\big)ds \\
&> 0.
\end{aligned} \tag{21}$$

Thus,

$$m_i > 0, \qquad n_i > 0, \tag{22}$$

here the value of i ranges from 1 to N. Set

$n = \min\{n_i,$ here the value of i ranges from 1 to $N\}$ and

$m = \max\{m_i,$ here the value of i ranges from 1 to $N\}$, we have $n > 0$ and $m > 0$. Therefore,

$$\lambda n l(t) \le T_\lambda(l, l) \le \lambda m l(t), \tag{23}$$

this means that

$$T_\lambda(l, l) \in P_l. \tag{24}$$

Then the requirement (A_1) of Theorem 2.4 can be satisfied. Then, from the conclusion of Theorem 2.4, we can find a unique ϕ_λ^* belongs to the set P_l, here ϕ_λ^* meets the relationship $T_\lambda(\phi_\lambda^*, \phi_\lambda^*) = \phi_\lambda^*$, we can conclude that the unique positive function ϕ_λ^* satisfies the switched system (1), (2). For the value ϕ_0, ψ_0 belongs to the set $P_{t^{\alpha-1}}$, we establish the sequence

$$\phi_{n+1} = T_\lambda(\phi_n, \psi_n),\ n = 0, 1, 2, \cdots,$$

$$\psi_{n+1} = T_\lambda(\psi_n, \phi_n),\ n = 0, 1, 2, \cdots,$$

one can derive

$$\phi_n \text{ converge to } \phi_\lambda^*,$$

$$\psi_n \text{ converge to } \phi_\lambda^*,\ (n \text{ converge to } \infty),$$

i.e.,

$$\begin{aligned}
&\phi_{n+1}(t) \\
&= \lambda\int_0^1 H(t,s)\varphi_q\big(\tfrac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} \\
&f_{\sigma(\tau)}(\tau, \phi_n(\tau), \psi_n(\tau))d\tau\big)ds, \\
&\to \phi_\lambda^*(t), \quad n \to \infty,
\end{aligned}$$

$$\begin{aligned}
&\psi_{n+1}(t) \\
&= \lambda\int_0^1 H(t,s)\varphi_q\big(\tfrac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} \\
&f_{\sigma(\tau)}(\tau, \psi_n(\tau), \phi_n(\tau))d\tau\big)ds, \\
&\to \phi_\lambda^*(t), \quad n \to \infty.
\end{aligned}$$

This result is proved. ■

REFERENCES

[1] Z. Guo, H. Yuan, "Pricing European option under the time-changed mixed Brownian-fractional Brownian model," *Physica A*, vol. 406, pp. 73-79, 2014.

[2] V. Tarasov, "Lattice model of fractional gradient and integral elasticity:long-range interaction of Grnwald-Letnikov-Riesz type," *Mech. Mater*, vol. 70, pp. 106-114, 2014.

[3] H. El-Saka, "The fractional-order SIS epidemic model with variable population size," *J. Egyptian Math. Soc.*, vol. 22, pp. 50-54, 2014.

[4] D. Yin, X. Duan, X. Zhou, "Fractional time-dependent deformation component models for characterizing viscoelastic Poisson's ratio," *Eur. J. Mech. A Solids*, vol. 42, pp. 422-429, 2013.

[5] M. Paola, M. Zingales, "The multiscale stochastic model of Fractional Hereditary Materials (FHM)," *Procedia IUTAM*, vol. 6, pp. 50-59, 2013.

[6] M. Paola, F. Pinnola, M. Zingales, "Fractional differential equations and related exact mechanical models," *Comput. Math. Appl.*, vol. 66, pp. 608-620, 2013.

[7] S. Das, K. Maharatna, "Fractional dynamical model for the generation of ECG like signals from filtered coupled Van-der," *Comput. Methods Programs Biomed.*, vol. 122, pp. 490-507, 2013.

[8] A. Cabada, G. Wang, "Positive solutions of nonlinear fractional differential equations with integral boundary value conditions," *Abstr. Appl. Anal.*, vol. 2012, Article ID 403, 2012.

[9] H. Feng, C. Zhai, "Properties of positive solutions to a class of fractional differential equations with parameters and integral boundary value conditions," *Appl. Math. Mech.*, vol. 38(7), pp. 818-826, 2017 (in Chinese).

[10] H. Zhang, "Iterative solutions for fractional nonlocal boundary value problems involving integral conditions," *Bound. Value Probl.*, vol. 2016, Article ID 3, 2016.

[11] Y. B. Shtessel, O. A. Raznopolov, L. A. Ozerov, "Control of multiple modular dc-to-dc power converts in conventional and dynamic sliding surfaces," *IEEE Transactions on Circuits and Systems*, vol. 45(10), pp. 1091-1100, 1998.

[12] T. Bak, J. Bendtsen, A. P. Ravn, "Hybrid control design for a wheeled mobile robot," *Hybrid Systems: Computation and Control*, vol. 2623, pp. 50-65, 2003.

[13] E. Hemandez-Vargas, P. Colaneri, R. Middleton, F. Blanchili, "Discretetime control for switched positive systems with application to mitigating viral escape," *International Journal of Robust and Nonlinear Control*, vol. 21(10), pp. 1093-1111, 2011.

[14] D. Zhang, Z. Xu, G. Feng, H. Li, "Asynchronous resilient output consensus of switched heterogeneous linear multivehicle systems with communication delay," *IEEE/ASME Transactions on Mechatronics*, vol. 24(6), pp. 2627-2640, 2019.

[15] L. Zhu, G. Feng, "On the existence and uniqueness of solutions of switched Hamiltonian systems,"*In Proceedings of the 31st Chinese control conference*, (pp. 2106-2111, 2012.

[16] Z. Lv, B. Chen, "Existence and uniqueness of positive solutions for a fractional switched system," *Abstract and Applied Analysis*, vol. 70, pp. 1-7, 2014.

[17] M. Ahmad, A. Zada, X. Wang, "Existence, uniqueness and stability of implicit switched coupled fractional differential equations of -Hilfer type," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 21, no. 3-4, pp. 327-337, 2020.

[18] H. Li, Y. Peng, K. Wu, "The existence and uniqueness of the solutions of the nonlinear on-off switched systems with switching at variable times," *Nonlinear Dynamics*, vol. 103(3), pp. 2287-2298, 2021.

[19] Y. Luo, N. Zhao, X. Ouyang, D. Liu, "Event-Triggered Prescribed Performance Control for Switched Uncertain Nonlinear Systems," *Engineering Letters*, vol. 32, no. 5, pp. 897-903, 2024.

[20] H. Li, Y.Liu, "On the uniqueness of the positive solution for a second-order integral boundary value problem with switched nonlinearity,"*Appl. Math. Lett.*, vol. 24, pp. 2201-2205, 2011.

[21] X. Guo, "Existence of unique solution to switched fractional differential equations with p-Laplacian operator," *Turk. J. Math.*, vol. 39, pp. 864-871, 2015.

[22] Y. Sun, M. Zhao, "Positive solutions for a class of fractional differential equations with integral boundary conditions," *Applied Mathematics Letters*, vol. 34, pp. 17-21, 2014.

[23] Q. Song, Z. Bai, “Positive solutions of fractional differential equations involving the Riemann-Stieltjes integral boundary condition,” *Advances in Difference Equations*, vol. 183, pp. 1-7, 2018.

[24] L. Wang, H. Dong, G. Chen, “New Lyapunov-type Inequalities for Fractional Differential Equations with Bi-ordinal Psi-Hilfer Fractional Derivative Involving Multi-point Boundary Conditions,” *Engineering Letters*, vol. 31, no.2, pp. 648-655, 2023.