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Related Theorems on Double Integral Transform Under Certain Conditions

A.K. Awasthi, *Member, IAENG,* Lukman Ahmed

***Abstract*— This article delves into the theoretical foundations of Double Integral Transforms (DIT), focusing on the pivotal role played by Fox's H-Function in their theoretical framework. The DIT denoted as $\phi(t)$, is formulated using Fox's H-Function and is expressed as a contour integral of Mellin-Barnes type, involving various parameters and conditions. The study provides a concise representation of DIT as $\phi(t) = DT[f(x, y)]$, and three theorems are presented utilizing the power series expansion of special functions including Laplace transforms, Hankel transforms, and additional transforms by Pathak and Narain. These theorems are proven analytically and the proofs of these theorems are validated through examples, demonstrating the manipulation of DIT under specific conditions.**

The Fox H-Function is a powerful mathematical tool explored for its significance in the analysis of DIT. As a hypergeometric series generalization, the H-Function finds widespread applications across mathematics, physics, and engineering. The application of these theorems extends to evaluating integrals that combine H-Functions with other functions. This comprehensive analysis highlights the crucial significance of Fox's H-Function in enhancing the theoretical landscape and applicability of DIT. Ultimately, this article highlights the transformative role of DIT in integral transform theory, offering a vigorous tool for researchers investigating structural stability and elasticity in various applied contexts.

***Index Terms*— Double Integral Transform, Laplace Transform, Hankel transform, H-Function.**

I. INTRODUCTION

INTEGRAL equations and transforms are crucial tools to solve many engineering problems related to the cracks of different shapes in the materials and these different types of cracks in the structure or materials can be solved by using integral equations and transforms along with some special function based on the type of problem. In dealing with such a complex problem in the mathematical world, we are using transforms like Laplace transforms and Hankel transforms along with certain generalized hypergeometric functions. The H-Function, combined with Laplace and Hankel transforms, enables effective solutions for crack problems in materials, enhancing elasticity analysis and structural integrity.

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A.K. Awasthi is a Professor in the Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara, Punjab, 144411, India (corresponding author to provide phone: +91 9315517628; e-mail: dramitawasthi@gmail.com).

Lukman Ahmed is a Ph.D. student in the Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara, Punjab, 144411, India (corresponding author to provide phone: +91 6005219243; e-mail: luqmanahmed513@ gmail.com).

Fig. 1 shows different types of cracks in materials before and after repair.

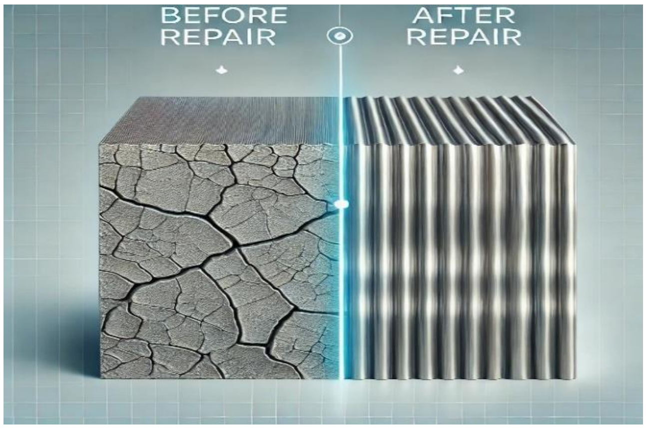

Fig. 1. A material with a visible crack before repair and after repair.

Mathematical transformations are a crucial component of the analytical tools used by scientists and engineers. In the realm of mathematical analysis and integral transforms, DIT plays a crucial role in the expression of functions with two variables using new variables. The DIT, denoted as $\phi(t)$ unfolds its mathematical richness, intricately connected to various renowned transforms such as the Laplace Transform, the Hankel Transform, and the specialized transforms introduced by Pathak and Narain. As we navigate through this terrain, the study unveils a concise representation of the DIT emphasizing its chain properties and the underlying theorems that shape its theoretical foundations. This transformative process involves the integration of a given function over a region on the plane. Erdogan [1] suggests that systems of simultaneous dual integral equations with trigonometric and Bessel kernels can be solved by reducing them to simultaneous singular integral equations. However, it points out that using a direct method to solve these dual integral equations may not accurately recover the oscillation character of the solution. Kalia [2] discussed the use of dual integral equations with Fox's H-function kernel to solve a class of mixed boundary value problems involving the potential of an electrified disc. Kashuri et al. [3] established a relationship between the double new integral transform and the double Laplace transform and presented various results in this regard. Al-Safi et al. [4] propose a new DIT called the Double Sumudu-Elzaki transform (DSET) combined with the variational iteration method for solving nonlinear partial differential equations (PDEs) of fractional order derivatives.

The Laplace transform technique is used to analyze linear time-invariant systems in mathematics, engineering, and physics. With the increasing complexity of engineering problems, Laplace transforms help in solving complex problems with a straightforward approach just like the applications of transfer functions to solve ODE. Davies & Martin [5] examined a multitude of diverse techniques for numerically reversing the Laplace transform and assessed them based on their suitability for real inversion problems, suitability for different function types, numerical precision, computational efficiency, and ease of implementation and programming. Guo [6] investigates the impact of the stereotype associated with the Laplace transform on students' perspective towards utilizing this approach for resolving ODE with distinct initial conditions. Cost [7] investigates various methods for Laplace transform inversion in viscoelastic stress analysis, their applications, and the effectiveness of these methods in solving specific engineering problems. Schiff [8] presents a comprehensive approach to the paper, focusing on the Laplace transform's applications, theoretical foundations, and accessibility for students, making it a valuable resource for learning.

The Hankel transform is another tool that plays a crucial role in solving problems with radial symmetry. This transform extends the concept of the Fourier transform to functions defined in polar coordinates. It is commonly used when dealing with problems that exhibit circular or cylindrical symmetry. The transform is defined as an integral involving a Bessel function, a special mathematical function that arises frequently in problems with circular symmetry. Garg et al. [9] introduced a finite integral transform that utilizes a combination of Bessel functions as a kernel, subject to certain conditions. Cinelli [10] introduced finite Hankel transforms that incorporate kernels along with their corresponding infinite series. These advancements now allow for the application of integral transform theory to solve Bessel's equation with asymmetric endpoint conditions. Ueda [11] delineates the primary aims and approaches of the study, emphasizing the utilization of the Hankel transform in tackling a particular fracture issue in piezoelectric substances subjected to thermal stress. It establishes a basis for the comprehensive examination and findings articulated in the manuscript. Yonglin et al. [12] proposed a new numerical. approach to the calculation of the Hankel transform, which is very important for solving multiple physics and engineering problems. This approach is most useful in high oscillation integrals using Bessel functions which are commonly encountered in various fields like physics, engineering, etc. For example, in areas such as wave propagation and electromagnetic physics.

Central to our investigation is Fox's H-Function, a powerful mathematical tool that transcends traditional boundaries. The three theorems established in this exploration leverage the power of series expansions, including those of Laplace Transforms, Hankel Transforms, and specialized transforms, providing a rigorous foundation for the analytical framework. Awasthi et. al. [13] considered the dual integral equations involving Fox's H-function and addressed the formal solution of specific dual integral equations simultaneously, which encompass Fox's H-function. Fox [14] defined the H-function as a Mellin-Barnes-type contour integral, which is symbolically denoted as

$$H^{m,n}_{p,q}\left[x\,\middle|\begin{matrix}\{(a_p,\alpha_p)\}\\\{(b_q,\beta_q)\}\end{matrix}\right] = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}\,x^s\,ds \qquad (1)$$

Where an empty product is interpreted as 1; m,n,p,q are integers such that $0\le m\le q$ and $0\le n\le p$; the $A_j, 1\le j\le p$ and $B_j, 1\le j\le q$, are all positive; the poles of the integrand in (1) are simple; L stands for a suitable contour of Mellin-Barnes type which runs from $-i\infty$ to $i\infty$ with indentations, if necessary in such a manner that all the poles of $\Gamma(b_j-\beta_j s), j=1,\dots,m$ are to the right and those of $\Gamma(1-a_j+\alpha_j s), j=1,\dots,n$ to the left of L.

In this paper, we discuss the chain properties connecting the double integral transform.

$$\phi(t)=\int_0^\infty\int_0^\infty x^{\alpha-1}\,y^{\beta-1}(x+y)^{\sigma}\,.\,H^{f,g}_{u,v}\left[\lambda(x+y)\,\middle|\begin{matrix}\{(A_u,\eta_u)\}\\\{(B_v,\xi_v)\}\end{matrix}\right].H^{m,n}_{p,q}\left[tx^{\alpha_1}y^{\beta_1}(x+y)^{\sigma_1}\,\middle|\begin{matrix}\{(c_p,\gamma_p)\}\\\{(d_q,\delta_q)\}\end{matrix}\right]f(x,y)\,dx\,dy \qquad (2)$$

Provided that $0\le m\le q,\ 0\le n\le p,\ 0\le f\le v,\ 0\le g\le u,\ R(\alpha)>0,\ R(\beta)>0, \alpha_1,\beta_1,\sigma_1\ge 0\,, -\min_{1\le j\le f} R(B_j|\xi_j) < R(\alpha+\beta+\sigma) < -\max_{1\le i\le g} R\{(A_i-1)/\eta_i\}, m,n,p,q,f,g,u,v$ are integers with various known transforms e.g. the Laplace transform, the Hankel transform, the $J^{\mu}_{\nu,\lambda}$ transfrom due to Pathak [16] and Ψ_{ν_1,k_1,m_1} transform due to Narain [17]. For the sake of brevity, we denote the double integral transform (2) as

$$\phi(t)=DT\,[f(x,y)]$$

The impetus for this scholarly article is derived from the finding (4) presented below, which was recently disclosed by Srivastava and Panda [18]. We prove three theorems frequently using the power series expansion of various special functions appearing in the integral. We give several examples based on these theorems thereby evaluating a few known or new integrals involving the product of H-functions and other functions. Fig. 2 shows the step-by-step process.

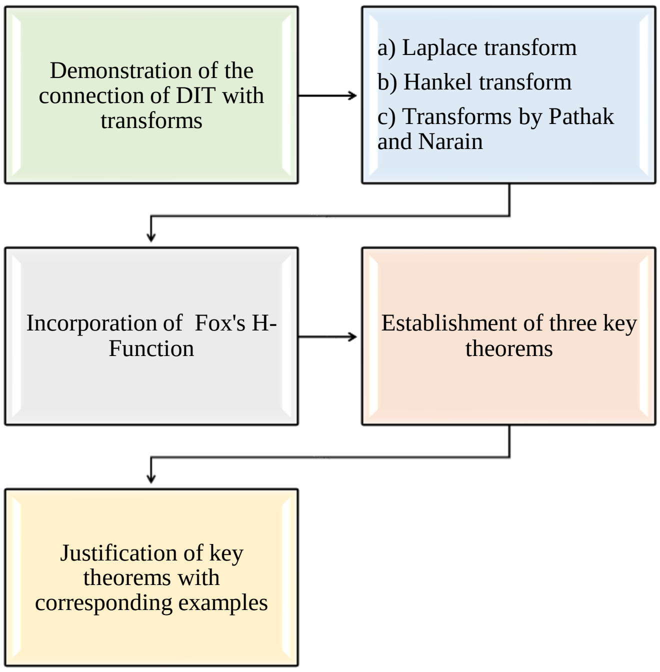

Fig. 2. The flowchart of the processes.

II. THEOREM 1

If $\phi(\mathrm{t}) = \mathrm{DT}\,[f(x)]$ and $f(x)$ is the Laplace transform of $g(z)$ then

$$\phi(t) = \lambda^{-\alpha-\beta-\sigma} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \lambda^{-r}\; H_{p+v+2,\; q+u+1}^{m+g,\; 2+f+n} \left[t\lambda^{-\theta} \left| \begin{array}{c} (1-\alpha-r,\alpha_1),(1-\beta,\beta_1),\{(\epsilon_f,\theta\xi_f)\},\{(c_p,\gamma_p)\}, \\ (\epsilon_{f+1},\theta\xi_{f+1}),\ldots,(\epsilon_v,\theta\xi_v) \\ \{(\Psi_g\theta,\theta\eta_g)\},\{(d_q,\delta_q)\},(1-\alpha-\beta-r,\alpha_1+\beta_1), \\ (\Psi_{g+1},\theta\eta_{g+1}),\ \ldots,(\Psi_u,\theta\eta_u) \end{array} \right. \right] \int_0^{\infty} z^r g(z)dz \quad (3)$$

Where,

$$\begin{cases} \theta = \alpha_1 + \beta_1 + \sigma_1, \\ \epsilon_j = 1 - B_j - (\alpha+\beta+\sigma)\xi_j, \quad j = 1, \ldots, v, \\ \Psi_j = 1 - A_j - (\alpha+\beta+\sigma)\eta_j, \quad j = 1, \ldots, u, \end{cases}$$

Provided that the integrals $\int_0^{\infty} g(z)dz$ and $\int_0^{\infty} z^r g(z)dz,\ r \geq 1$, exist and all other conditions given with (2) are satisfied.

Proof: We have

$$f(x) = \int_0^{\infty} e^{-xz}\, g(z)dz$$

On substituting for $f(x)$ in (2) and changing the order of integration which is justifiable under the given conditions, we have

$$\phi(t) = \int_0^{\infty} g(z)dz \int_0^{\infty}\int_0^{\infty} e^{-xz} x^{\alpha-1} y^{\beta-1} (x+y)^{\sigma}$$
$$H_{u,v}^{f,g}\left[\lambda(x+y)\left|\begin{array}{c}\{(A_u,\eta_u)\}\\ \{(B_v,\xi_v)\}\end{array}\right.\right]$$
$$H_{p,q}^{m,n}\left[t x^{\alpha_1} y^{\beta_1} (x+y)^{\sigma_1}\left|\begin{array}{c}\{(c_p,\gamma_p)\}\\ \{(d_q,\delta_q)\}\end{array}\right.\right] dxdy$$

By the definition of Mellin transform $F(s)$ of a function $f(x)$ as

$$F(s) = M\{f(x):s\} = \int_0^{\infty} x^{s-1} f(x)dx$$

Where s is a complex number.

For convenience, we abbreviate the first member of equation (1) by

$$H_{p,q}^{f,g}[x]$$

Then by the Mellin inverse theorem [21], it follows from (1) that

$$M\{H_{p,q}^{f,g}[x]:s\} = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\ \prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}\ z^{-s}$$

Provided that $-\min_{1\leq j<m} Re\left(\frac{b_j}{\beta_j}\right) < Re(s) < \min_{1\leq j\leq n} Re\left(\frac{1-a_j}{\alpha_j}\right)$.

By using [18] the relation $\phi(x) = x^{\sigma} H_{p,q}^{f,g}[\lambda x]$ and expanding e^{-xz} in powers of xz and integrating term by term by using the following known integral due to Srivastava and Panda [18], we obtain

$$\int_0^{\infty}\int_0^{\infty} x^{\alpha-1} y^{\beta-1}\, (x+y)^{\sigma}\ H_{u,v}^{f,g}\left[\lambda(x+y)\left|\begin{array}{c}\{(A_u,\eta_u)\}\\ \{(B_v,\xi_v)\}\end{array}\right.\right]$$

$$H_{p,q}^{m,n}\left[t x^{\alpha_1} y^{\beta_1} (x+y)^{\sigma_1}\left|\begin{array}{c}\{(c_p,\gamma_p)\}\\ \{(d_q,\delta_q)\}\end{array}\right.\right] dxdy$$
$$= \lambda^{-\alpha-\beta-\sigma} H_{p+v+2,q+u+1}^{m+g,2+f+n}$$

$$\left[t\lambda^{-\theta} \left| \begin{array}{c} (1-\alpha,\alpha_1),(1-\beta,\beta_1),\{(\epsilon_f,\theta\xi_f)\},\{(c_p,\gamma_p)\}, \\ (\epsilon_{f+1},\theta\xi_{f+1}),\ldots,(\epsilon_v,\theta\xi_v) \\ \{(\Psi_g,\theta\eta_g)\},\{(d_q,\delta_q)\},(1-\alpha-\beta,\alpha_1+\beta_1), \\ (\Psi_{g+1},\theta\eta_{g+1}),\ldots,(\Psi_u,\theta\eta_u) \end{array} \right. \right] \quad (4)$$

Where all the conditions given in (3) are satisfied, we obtain the theorem.

Convergence of Theorem 1: Theorem 1 provides a relation between the DIT and the Laplace transform. The convergence of this theorem is governed by the following factors:

a) Integral convergence

The Integral $\int g(z)dz$ and $\int r^r g(z)dz$ converge if the function $g(z)$ decays rapidly enough as $z \to \infty$. For example, $g(z) \sim e^{-az}$ (where $a > 0$) ensures the finiteness of these integrals.

b) Series Convergence

The series expansion in the theorem involves factorial growth in the denominator, ensuring convergence if $g(z)$ satisfies appropriate decay conditions.

The series also depends on the Fox's H-function, which must converge under the specified parameter constraints.

c) Fox's H-Function

The Fox's H-function converges as a Mellin-Barnes-type contour integral if the poles of $\Gamma(b_j - \beta_j s)$ and $\Gamma(1 - a_j + \alpha_j s)$ lie on opposite sides of the contour L. Parameter constraints such as $R(\alpha+\beta+\sigma) > 0$ ensure this condition is satisfied.

Example: Let $g(z) = z^{-\mu-\frac{1}{2}} K_{v+\frac{1}{2}}(bz)$

By using [18], we obtain

$$f(x) = \frac{\sqrt{\pi}\,\Gamma(-\mu+v+1)\Gamma(-\mu-v)}{(2b)^{\frac{1}{2}}} (x^2 - b^2)_v^{\mu/2}\ P_v^{\mu}(x/b) \quad (5)$$

Where $P_v^{\mu}(z)$ is the associated Legendre function with $R(\mu) - 1 < R(v) < -R(\mu)$

By putting $s = r - \mu + \frac{1}{2}$ in [20], we obtain

$$\int_0^{\infty} Z^r g(z)dz = b^{-r+\mu-\frac{1}{2}}\ 2^{r-\mu-\frac{3}{2}}\,\Gamma\left(\frac{1}{2}r - \frac{1}{2}\mu + \frac{1}{2} - \frac{1}{2}v\right)\Gamma\left(\frac{1}{2}r - \frac{1}{2}\mu + \frac{1}{2} + \frac{1}{2}v\right) \quad (6)$$

Finally, by putting the values from (5) and (6) in the theorem, we obtain

$$\int_0^{\infty}\int_0^{\infty} x^{\alpha-1} y^{\beta-1} (x^2 - b^2)^{\frac{1}{2}\mu}\ P_v^{\mu}\left(\frac{x}{b}\right)(x+y)^{\sigma}$$
$$H_{u,v}^{f,g}\left[\lambda(x+y)\left|\begin{array}{c}\{(A_u,\eta_u)\}\\ \{(B_v,\xi_v)\}\end{array}\right.\right] H_{p,q}^{m,n}\left[t x^{\alpha_1} y^{\beta_1} (x+y)^{\sigma_1}\left|\begin{array}{c}\{(c_p,\gamma_p)\}\\ \{(d_q,\delta_q)\}\end{array}\right.\right] dxdy$$
$$= \frac{\lambda^{-\alpha-\beta-\sigma}\ 2^{-\mu-1} b^{\mu}}{\sqrt{\pi}\,\Gamma(-\mu+v+1)\Gamma(-\mu-v)} \sum_{r=0}^{\infty} \frac{(-2/b\lambda)^r}{r!}.\Gamma\left(\frac{1}{2}r - \frac{1}{2}\mu + \frac{1}{2} - \frac{1}{2}v\right)\Gamma\left(\frac{1}{2}r - \frac{1}{2}\mu + \frac{1}{2} + \frac{1}{2}v\right) H_{p+v+2,\; q+u+1}^{m+n,\; 2+f+n}$$
$$\left[t\lambda^{-\theta} \left| \begin{array}{c} (1-\alpha-r,\alpha_1),(1-\beta,\beta_1),\{(\epsilon_f,\theta\xi_f)\},\ \{(c_p,\gamma_p)\}, \\ (\epsilon_{f+1},\theta\xi_{f+1}),\ldots,(\epsilon_v,\theta\xi_v) \\ \{(\Psi_g,\theta\eta_g)\},\{(d_g,\delta_g)\},(1-\alpha-\beta-r,\alpha_1+\beta_1), \\ (\Psi_{g+1},\theta\eta_{g+1}),\ldots,(\Psi_u,\theta\eta_u) \end{array} \right. \right] \quad (7)$$

Provided that $R(\mu) - 1 < R(v) < -R(\mu)$ and the conditions given in (3) are satisfied.

III. THEOREM 2

If

$$\phi(t) = DT[f(x)] \quad (8)$$

And $\quad f(x) = J_{v,\lambda_1}^{\mu}$ transform of $g(z) \quad (9)$

Where,

$$J_{v,\lambda_1}^{\mu}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\left(\frac{1}{2}r\right)^{v+2r+2\lambda_1}}{\Gamma(1+\lambda_1+r)\,\Gamma(1+\lambda_1+v+\mu_r)}, \mu > 0 \quad (10)$$

Then

$$\phi(t) = \frac{\lambda^{-\alpha-\beta-\sigma-v-2\lambda_1+\frac{1}{2}}}{2^{v+2\lambda_1}} \sum_{r=0}^{\infty} \frac{(-1)^r (2\lambda)^{-2r}}{\Gamma(1+\lambda_1+r)\Gamma(1+\lambda_1+v+\mu_r)}.H_{p+v+2,\; q+u+1}^{g+m,\; 2+f+n}$$

$$\times\left[t\lambda^{-\theta}\left|\begin{matrix}\left(\frac{1}{2}-\alpha-\nu-2\lambda-2r,\alpha_1\right),(1-\beta,\beta_1),\{(\epsilon_f,\theta\xi_f)\},\{(c_p,\gamma_p)\},\\(\epsilon_{f+1},\theta\xi_{f+1}),\dots,(\epsilon_v,\theta\xi_v)\\\{(\Psi_g,\theta\eta_g)\},\{(d_g,\delta_g)\},\left(\frac{1}{2}-\alpha-\beta-\nu-2\lambda_1-2r,\alpha_1+\beta_1\right),\\(\Psi_{g+1},\theta\eta_{g+1}),\dots,(\Psi_u,\theta\eta_u)\end{matrix}\right.\right]$$

$$\times\int_0^\infty z^{\nu+2r+2\lambda_1+\frac{1}{2}}g(z)dz \quad (11)$$

Provided that the integrals

$\int_0^\infty z^{\frac{1}{2}}g(z)dz$ & $\int_0^\infty z^{\nu+2\lambda_1+2r+\frac{1}{2}}g(z)dz$ exist $\alpha_1.\beta_1,\sigma_1,\mu\geq 0, R\left(\alpha+\nu+2\lambda_1+\frac{1}{2}\right)>0, R(\beta)>0, -\min_{1\leq j\leq f}R(B_j/\xi_j)< R\left(\alpha+\beta+\sigma+\nu+2\lambda_1+\frac{1}{2}\right)<-\max_{1\leq i\leq g}R\{(A_i-1)/\eta_i\}$, and $\epsilon_j(j=1,2,\dots,v)$ and $\varphi_j(j=1,2,\dots,u)$ are the same as in (3).

Proof: On substituting for

$$f(x)=\int_0^\infty(xz)^{\frac{1}{2}}J^{\mu}_{\nu,\lambda_1}(xz)g(z)dz \quad (12)$$

The expression for $\phi(t)$ given by (2) and changing the order of integration which is suitable under the given conditions, we have

$$\phi(t)=\int_0^\infty z^{\frac{1}{2}}g(z)dz\int_0^\infty\int_0^\infty x^{\alpha-\frac{1}{2}}y^{\beta-1}J^{\mu}_{\nu,\lambda_1}(xz)\,(x+y)^{\sigma}H^{f,g}_{u,v}\left[\lambda(x+y)\left|\begin{matrix}\{(A_u,\eta_u)\}\\\{(B_v,\xi_v)\}\end{matrix}\right.\right]H^{m,n}_{p,q}\left[tx^{\alpha_1}y^{\beta_1}(x+y)^{\sigma_1}\left|\begin{matrix}\{(c_p,\gamma_p)\}\\\{(d_q,\delta_q)\}\end{matrix}\right.\right]dxdy \quad (13)$$

Now on substituting the series expansion for $J^{\mu}_{\nu,\lambda_1}(xz)$ from (10) and evaluating the inner integral with the help of (4) we obtain the theorem stated above.

Convergence of Theorem 2: Theorem 2 relates the DIT to the $J_{\nu,\mu}$ transform of a function $g(z)$. The convergence analysis for this theorem includes:

a) Integral Convergence

The integrals $\int z^r g(z)dz$ and $\int g(z)\,dz$ converge if $g(z)\sim e^{-az}$ or exhibits a similarly rapid decay. These conditions ensure the integral remains finite.

b) Series Convergence

The series expansion in Theorem 2 depends on the factorial term in the denominator, which dominates the growth of the numerator, ensuring convergence for well-behaved functions $g(z)$.

c) Fox's H-Function

The Fox's H-function must satisfy the Mellin-Barnes contour integral conditions for convergence. Constraints such as $-R(\frac{B_j}{\xi_j})<R(\alpha+\beta+\sigma)<-R(\frac{A_j-1}{\eta j})$ ensure the decay behavior necessary for series convergence.

Example: Let $g(z)=z^{-\frac{1}{2}}e^{-az}J_\nu(bz)$ on using [18], we get

$$f(x)=\int_0^\infty g(z)J_\nu(zx)(zx)^{1/2}dz$$

$$f(x)=\int_0^\infty z^{-1/2}e^{-az}J_\nu(bz)J_\nu(zx)(zx)^{1/2}dz$$

$$f(x)=\frac{1}{\pi}b^{-\frac{1}{2}}Q_{\nu-\frac{1}{2}}\left(\frac{a^2+b^2+x^2}{2bx}\right) \quad (14)$$

Provided that $R(a)>Im(b)>0$ and $R(\nu)>-\frac{1}{2}$.

By using [20], we obtain

$$\int_0^\infty z^{\nu+2r+\frac{1}{2}}g(z)\,dz=\frac{b^\nu\,\Gamma(2r+2\nu+1)}{2^\nu\,a^{2\nu+2r+1}\,\Gamma(\nu+1)}\,{}_2F_1\left(\nu+r+\frac{1}{2};\nu+r+1;\nu+1;\,-\frac{b^2}{a^2}\right)$$

Provided that $R(a)>Im(b)>0$ and $R(\nu)>-\frac{1}{2}$. Hence using the result (11) with $\lambda_1=0$ and $\mu=1$, we have

$$\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}(x+y)^{\sigma}\,Q_{\nu-\frac{1}{2}}\left(\frac{a^2+b^2+x^2}{2bx}\right)H^{f,g}_{u,v}\left[\lambda(x+y)\left|\begin{matrix}\{(A_u,\eta_u)\}\\\{(B_v,\xi_v)\}\end{matrix}\right.\right]H^{m,n}_{p,q}\left[tx^{\alpha_1}y^{\beta_1}(x+y)^{\sigma_1}\left|\begin{matrix}\{(c_p,\gamma_p)\}\\\{(d_q,\delta_q)\}\end{matrix}\right.\right]dxdy=$$
$$\frac{\lambda^{-\alpha-\beta-\sigma-\nu+\frac{1}{2}}b^{\nu+\frac{1}{2}\pi}}{2^{2\nu}\Gamma(1+\nu)a^{2\nu+1}}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\frac{(2\lambda a)^{-2r}\Gamma(2\nu+2r+1)}{\Gamma(\nu+r+1)}\cdot{}_2F_1\left(\nu+r+\frac{1}{2},\nu+r+1;\nu+1;\frac{-b^2}{a^2}\right)H^{m+g,\;2+f+n}_{p+v+2,\;q+u+1}$$

$$\left[t\lambda^{-\theta}\left|\begin{matrix}\left(\frac{1}{2}-\alpha-\nu-2r,\alpha_1\right),(1-\beta,\beta_1),\{(\epsilon_f,\theta\xi_f)\},\\\{(c_p,\gamma_p)\},(\epsilon_{f+1},\theta\xi_{f+1}),\dots,(\epsilon_v,\theta\xi_v)\\\{(\Psi_g,\theta\eta_g)\},\{(d_q,\delta_q)\},\left(\frac{1}{2}-\alpha-\beta-\nu-2r,\alpha_1+\beta_1\right),\\(\Psi_{g+1},\theta\eta_{g+1}),\dots,(\Psi_u,\theta\eta_u)\end{matrix}\right.\right] \quad (15)$$

Provided that $R(a)>Im(b)>0, R(\nu)>-\frac{1}{2},\alpha_1,\beta_1,\sigma_1\geq 0$ and the other conditions given in (3) are satisfied.

IV. Theorem 3

If

$$\phi(t)=DT[f(x)] \quad (16)$$

And $f(x)$ is self-reciprocal in the ψ_{ν_1},k_1,m_1 transform then

$$\phi(t)=\frac{\lambda^{-\alpha-\beta-\sigma-\nu_1-\frac{1}{2}}}{2^{\nu_1}}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\frac{\Gamma(2m_1-r)\Gamma\left(\frac{1}{2}-k_1+m_1+\nu_1+r\right)}{\Gamma(1+2m_1+\nu_1+r)}$$
$$2^{-2r}\,\frac{1}{\Gamma(1+\nu_1+r)\Gamma\left(-k_1+m_1+\frac{1}{2}-r\right)}H^{g+m,\;2+f+n}_{p+v+2,q+u+1}$$
$$\left[t\lambda^{-\theta}\left|\begin{matrix}\left(\frac{1}{2}-\alpha-\nu_1-2r,\alpha_1\right),(1-\beta,\beta_1),\{(\epsilon_f,\theta\xi_f)\},\;\{(c_p,\gamma_p)\},\\(\epsilon_{f+1},\theta\xi_{f+1}),\dots,(\epsilon_v,\theta\xi_v)\\\{(\Psi_g,\theta\eta_g)\},\{(d_q,\delta_q)\},\left(\frac{1}{2}-\alpha-\beta-\nu_1-2r,\alpha_1+\beta_1\right),\\(\Psi_{g+1},\theta\eta_{g+1}),\dots,(\Psi_u,\theta\eta_u)\end{matrix}\right.\right]$$
$$\int_0^\infty z^{\nu_1+2r+\frac{1}{2}}f(z)dz \quad (17)$$

Provided that the integrals

$\int_0^\infty z^{\nu_1+2r+\frac{1}{2}}f(z)dz$ *and* $\int_0^\infty z^{-\nu_1+\frac{1}{2}}f(z)dz$ exist $\alpha_1,\beta_1,\sigma_1\geq 0, R\left(\nu_1-k_1+m_1+\frac{1}{2}\right)>0, R(m_1)>0,\nu_1<0,\nu_1<-2m_1, 2m_1$ is not an integer; $-\delta'<R\left(\alpha+\beta+\sigma+\nu_1+\frac{1}{2}\right)<-\beta'$, *and* $\theta,\epsilon_j(j=1,\dots,v),\psi_j(j=1,\dots,u)$ are the same as in (3).

Proof: We have

$$f(x)=2^{\nu_1}\int_0^\infty(xz)^{-\nu_1+\frac{1}{2}}.$$
$$H^{2,1}_{2,4}\left[\frac{x^2z^2}{4}\left|\begin{matrix}\left(k_1-m_1-\frac{1}{2},1\right),\left(\nu_1-k_1+2m_1+\frac{1}{2},1\right)\\(\nu_1,1),(\nu_1+2m_1,1),(-2m_1,1),(0,1)\end{matrix}\right.\right]f(z)dz \quad (18)$$

On substituting for $f(x)$ from (18) in (16) and changing the order of integration which is justifiable under the above conditions, we obtain

$$\phi(t)=2^{\nu_1}\int_0^\infty z^{-\nu_1+\frac{1}{2}}f(z)dz\int_0^\infty\int_0^\infty x^{\alpha-\nu_1+\frac{1}{2}-1}y^{\beta-1}(x+y)^{\sigma}\;H^{f,g}_{u,v}\left[\lambda(x+y)\left|\begin{matrix}\{(A_u,\eta_u)\}\\\{(B_v,\xi_v)\}\end{matrix}\right.\right]H^{m,n}_{p,q}\left[tx^{\alpha_1}y^{\beta_1}(x+y)^{\sigma_1}\left|\begin{matrix}\{(c_p,\gamma_p)\}\\\{(d_q,\delta_q)\}\end{matrix}\right.\right]H^{2,1}_{2,4}\left[\frac{x^2z^2}{4}\left|\begin{matrix}\left(k_1-m_1-\frac{1}{2},1\right),\left(\nu_1-k_1+m_1+\frac{1}{2},1\right)\\(\nu_1,1),(\nu_1+2m_1,1),(-2m_1,1),(0,1)\end{matrix}\right.\right]dxdy \quad (19)$$

But from the power series expansion due to Mukherji and Prasad [15]

$$H_{p,q+1}^{m+1,n}\left[ax^{\sigma}\left|\begin{matrix}\{(a_p,\alpha_p)\}\\(b_0,\beta_0),\{(b_q,\beta_q)\}\end{matrix}\right.\right]=\frac{1}{\beta_0}.$$
$$\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\,\rho_r)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\,\rho_r)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\,\rho_r)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\,\rho_r)}a^{\rho r}x^{\sigma\rho r}\quad(20)$$

Where,

$\rho_r=\frac{b_0+r}{\beta_0},\beta<R\left(\frac{b_0}{\beta_0}\right)<\delta,|\arg a|<\frac{1}{2}\lambda\pi,\lambda>0,A>0$, We have

$$H_{2,4}^{2,1}\left[\frac{x^2z^2}{4}\left|\begin{matrix}\left(k_1-m_1-\frac{1}{2},1\right),\left(v_1-k_1+m_1+\frac{1}{2},1\right)\\(v_1,1),(v_1+2m_1,1),(-2m_1,1),(0,1)\end{matrix}\right.\right]=$$
$$\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\frac{\Gamma(2m-r)\Gamma\left(\frac{1}{2}-k_1+m_1+v_1+r\right)}{\Gamma(1+2m_1+v_1+r)\,\Gamma(1+v_1+r)\Gamma\left(-k_1+m_1+\frac{1}{2}-r\right)}.\left(\frac{xz}{2}\right)^{2v_1+2r}\quad(21)$$

Provided that $R\left(v_1-k_1+m_1+\frac{1}{2}\right)>0,R(m_1)>0,v_1<0,-v_1<-2m_1,$ and $2m_1$ is not a positive integer. On substituting the results (21) in (19) interchanging the order of integration and summation and finally using the result (4), we obtain the theorem.

Convergence of Theorem 3: Theorem 3 relates the DIT to a function $f(x)$ that is self-reciprocal in the $\psi_{k,m}$ transform. The convergence considerations include:

a) Integral Convergence

The Integral $\int z^r g(z)dz$ converges if $f(z)\sim e^{-az^2}$ or exhibits sufficient decay. For Gaussian-like functions, this condition is automatically satisfied.

b) Series Convergence

The series expansion converges due to the rapid growth of $r!$ in the denominator, ensuring the terms diminish as $r\to\infty$. This is further supported by the asymptotic behavior of the Fox's H-function.

c) Fox's H-Function

The Fox's H-function converges if the Mellin-Barnes integral conditions are met. Constraints such as $-\delta<R(\alpha+\beta+\sigma)<-\beta$ ensure that the integral is well-behaved.

Example: Let us consider $f(x)=e^{-x^2}$

Where the function $f(x)$ is a Gaussian function that often satisfies self-reciprocal properties in integral transforms.

Since $f(x)$ is self-reciprocal in the ψ_{v_1},k_1,m_1 transform. On expressing $f(x)$ in an integral form involving Fox's H-function, we obtain equation (18) as

$$f(x)=2^{v_1}\int_0^{\infty}(xz)^{-v_1+\frac{1}{2}}\,H_{2,4}^{2,1}\left[\frac{x^2z^2}{4}\left|\begin{matrix}\left(k_1-m_1-\frac{1}{2},1\right),\left(v_1-k_1+2m_1+\frac{1}{2},1\right)\\(v_1,1),(v_1+2m_1,1),(-2m_1,1),(0,1)\end{matrix}\right.\right]\times f(z)dz$$

Using $\alpha=1,\beta=1,\sigma=0;\ v_1=-1,k_1=0,m_1=1$ and $\lambda=1$, we get

$$f(x)=2^{-1}\int_0^{\infty}(xz)^1H_{2,4}^{2,1}\left[\frac{x^2z^2}{4}\left|\begin{matrix}\left(-\frac{1}{2},1\right),\left(\frac{1}{2},1\right)\\(-1,1),(1,1),(-2,1),(0,1)\end{matrix}\right.\right]e^{-z^2}dz\quad(22)$$

Further on substituting equation (22) in (16), we get

$$\phi(t)=2^{-1}\int_0^{\infty}ze^{-z^2}dz\int_0^{\infty}\int_0^{\infty}x^{\frac{1}{2}}y^0(x+y)^0\,H_{u,v}^{f,g}\left(\lambda(x+y)\left|\begin{matrix}\{(A_u,\eta_u)\}\\\{(B_v,\xi_v)\}\end{matrix}\right.\right)H_{p,q}^{m,n}\left(tx^1y^1\left|\begin{matrix}\{(c_p,\gamma_p)\}\\\{(d_q,\delta_q)\}\end{matrix}\right.\right).$$
$$H_{2,4}^{2,1}\left(\frac{x^2z^2}{4}\left|\begin{matrix}\left(-\frac{1}{2},1\right),\left(\frac{1}{2},1\right)\\(-1,1),(1,1),(-2,1),(0,1)\end{matrix}\right.\right)dxdy\quad(23)$$

Using the series expansion for Fox's H-function as provided by Mukherji and Prasad [15], we get

$$H_{2,4}^{2,1}\left(\frac{x^2z^2}{4}\left|\begin{matrix}\left(-\frac{1}{2},1\right),\left(\frac{1}{2},1\right)\\(-1,1),(1,1),(-2,1),(0,1)\end{matrix}\right.\right)=$$
$$\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\frac{\Gamma(2m-r)\Gamma\left(\frac{1}{2}-k_1+m_1+v_1+r\right)}{\Gamma(1+2m_1+v_1+r)\,\Gamma(1+v_1+r)\Gamma\left(-k_1+m_1+\frac{1}{2}-r\right)}.\left(\frac{xz}{2}\right)^{2v_1+2r}$$

On substituting the above expression into equation (23), on interchanging the summation, and on simplifying each term by performing the integrations involving x and y separately we shall obtain the final expression in the form of equation (17). This result demonstrates the use of the series representation and convergence properties of Fox's H-function in the context of integral transforms.

V. Conclusion

This article contributes to the theoretical framework of Double Integral Transforms (DITs), through the lens of Fox's H-Function, highlighting its crucial role in shaping and understanding these mathematical tools. Firstly, we have demonstrated the DIT, denoted as $\phi(t)=DT[f(x,y)]$ is intricately connected to established transforms such as the Laplace and Hankel transforms. The established theorems are based on certain conditions which are solved by making use of Fox's H-function, defined as a Mellin-Barnes-type contour integral which is symbolically denoted as in equation (1). Next, we discussed the chain properties connecting the DIT as given in equation (2). Further, we established three key theorems that demonstrate the correlation between the DIT, the Laplace transform, the Hankel transform, and the specialized transforms given by Pathak and Narain. The established theorems one, two, and three are then proven analytically with corresponding examples. The three presented theorems provide valuable insights for researchers dealing with crack problems and working in integral transformation with special functions.

The findings in this study lay a foundation for future research to expand DIT applications. Future work could explore advanced transforms like Fourier-Bessel and Mellin, broadening the theorems' reach in mathematical physics and engineering. Developing numerical methods for these transforms would also enable practical use, especially for complex applied sciences.

References

[1] Erdogan, F.F., 1968. Simultaneous dual integral equations with trigonometric and Bessel kernels. ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik, 48(4), pp.217-225.

[2] Kalia, R.N., 1984. Dual integral equations with Fox's H-function kernel. International Journal of Mathematics and Mathematical Sciences, 7, pp.257-262.

[3] Kashuri, A., Fundo, A. and Liko, R., 2013. On double new integral transform and double Laplace transform. European Scientific Journal, 9(33).

[4] AL-Safi, M.G., Fawzi, R.M. and Abd AL-Hussein, W.R., 2023. Numerical Solutions for the Nonlinear PDEs of Fractional Order by Using a New Double Integral Transform with Variational Iteration Method. Baghdad Science Journal, 20(3 (Suppl.)), pp.1087-1087.

[5] Davies, B. and Martin, B., 1979. Numerical inversion of the Laplace transform: a survey and comparison of methods. Journal of computational physics, 33(1), pp.1-32.

[6] Guo, W., 2021. The Laplace transform as an alternative general method for solving linear ordinary differential equations. STEM Education, 1(4), pp.309-329.

[7] Cost, T.L., 1964. Approximate Laplace transform inversions in visco-elastic stress analysis. AIAA journal, 2(12), pp.2157-2166.

[8] Schiff, J.L., 2013. The Laplace transform: theory and applications. Springer Science & Business Media.

[9] Garg, M., Rao, A. and Kalla, S.L., 2007. On a generalized finite Hankel transform. Applied Mathematics and Computation, 190(1), pp.705-711.

[10] Cinelli, G., 1965. An extension of the finite Hankel transform and applications. International Journal of Engineering Science, 3(5), pp.539-559.

[11] Ueda, S., 2012. Application of Hankel transform for solving a fracture problem of a cracked piezoelectric strip under thermal loading (Vol. 10). chapter.

[12] Yonglin, Y.A.N.G., Xing, L.I., Shenghu, D.I.N.G. and Wenshuai, W.A.N.G., 2022. A numerical algorithm for arbitrary real-order Hankel transform. *Wuhan University Journal of Natural Sciences*, *27*(1), pp.26-34.

[13] Awasthi, A.K., Rachna, R., Tripathi, R.K. and Emadifar, H., 2023, June. The solution of simultaneous dual integral equations involving fox's H-function. In AIP Conference Proceedings (Vol. 2768, No. 1). AIP Publishing

[14] C. Fox, The G and H functions as symmetrical Fourier kernels, Trans. Amer. Math. Soc. 98 (l 961), 395-429.

[15] Mukherji, S.N. and Prasad, Y.N., 1971. Some infinite integrals involving the products of H. function. The Mathematics Education, 5, pp.6-12.

[16] Pathak, R.S., 1966.: Certain convergence theorems and asymptotic properties of a generalization of Lommel and Maitland transformations. Proc. Natl. Acad. Sci. India A-36 (1), 81-86.

[17] Narain, R., 1956. On a generalization of Hankel transform and self-reciprocal functions. Rendi. Sem. Mat. Torino, 16, pp.269-300.

[18] Srivastava, H.M. and Panda, R., 1973, January. Some operational techniques in the theory of special functions. In Indagationes Mathematicae (Proceedings) (Vol. 76, No. 4, pp. 308-319). North-Holland.

[19] Carr, A.R., 1977. Table errata: Tables of integral transforms, Vol. II (McGraw-Hill, New York, 1954) by A. Erdélyi, W. Magnus, F. Oberhettinger and FG Tricomi. Mathematics of Computation, 31(138), p.614.

[20] Erdelyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G., 1954. Tables of Integral Transforms Vol. I McGraw–Hill Book Company. Inc., New York.

[21] WIDDER, D. V., (1952). The Laplace transform, Third printing, Uni versity Press, Princeton.