Generalized Essential Interior Ideals Their Fuzzifications of Semigroups

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***Abstract*—In this paper, we give the concepts of essential interior ideals in semigroups. We proved properties and relationships between essential fuzzy interior ideals and essential interior ideals in semigroups. Finally, we extend essential fuzzy n-interior ideals and essential n-interior ideals.**

***Index Terms*—Essential interior ideal, Essential n-interior ideals, Essential fuzzy interior ideals, Essential fuzzy n-interior ideals.**

I. Introduction

THE THEORY of fuzzy sets was published by L. A. Zadeh in 1965 [1]. In 1979 N. Kuroki [2] investigated fuzzy left (right) ideals and fuzzy bi-ideals in semigroups. Tiprachot et al. discussed the notion of n-interior ideals as a generalization of interior ideals and characterized many classes of ordered semigroups in terms of (m, n)-ideals and n-interior ideals [3]. In 2023, Tiprachot et al. [4] extend n-interior ideals and (m, n)-ideals to hybrid in ordered semigroups.

In 1971 U. Medhi et al. [5] discussed the essential fuzzy ideals of ring. In 2013, U. Medhi and H.K. Saikia [6] studied the concept of T-fuzzy essential ideals and the properties of T-fuzzy essential ideals. In 2017 S. Wani and K. Pawar [7] extended the concept of essential ideals in semigroups to ternary semiring and studied essential ideals in ternary semiring. In 2020, S. Baupradist et al. [8] studied essential ideals and essential fuzzy ideals in semigroups. Together with 0-essential ideals and 0-essential fuzzy ideals in semigroups. Later, in 2021, R. Chinram and T. Gaketem [9] extend essential (m, n)-ideals and fuzzy essential fuzzy (m, n)-ideals in semigroups. In 2022 T. Gaketem et al. [10] studied essential bi-ideals and fuzzy essential bi-ideals in semigroups. Moreover, T. Gaketem and A. Iampan [11], [12] used knowledge of essential ideals in semigroups go to study essential ideals in UP-algebra. In the same year P. Khamrot and T. Gaketem, [13], [14] studied essential ideals in an interval valued fuzzy set and bipolar fuzzy set. In 2023, R. Rittichuai et al. [15] studied essential ideals and essential fuzzy ideals in ternary semigroups. Recently, N. Kaewmanee and T. Gaketem [16] studied essential hyperideals and essential fuzzy hyperideals in hypersemigroups.

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This paper studies essential interior ideals and n-interior ideals semigroups. We proved properties and relationships between essential fuzzy essential interior ideals and essential fuzzy interior ideals in semigroups. Finally, we study properties relationships between essential fuzzy n-interior ideals and essential n-interior ideals.

II. Preliminaries

In this section, we review the concept's basic definitions and the theorem used to prove all results in the next section.

In this topic, we review basic definitions and theorems used in the next section.

A non-empty subset $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ is called a *subsemigroup* (SSG) of $\mathfrak{S}$ if $\mathfrak{I}^2 \subseteq \mathfrak{I}$. A non-empty subset $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ is called a *left ideal* (LID) of $\mathfrak{S}$ if $\mathfrak{S}\mathfrak{I} \subseteq \mathfrak{I}$. A non-empty subset $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ is called a *right ideal* (RID) of $\mathfrak{S}$ if $\mathfrak{I}\mathfrak{S} \subseteq \mathfrak{I}$. An *ideal* (ID) $\mathfrak{I}$ of $\mathfrak{S}$ is a non-empty subset which is both a LID and a RID of S. A subsemigroup $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ is called an *interior ideal* (IID) of $\mathfrak{S}$ if $\mathfrak{S}\mathfrak{I}\mathfrak{S} \subseteq \mathfrak{I}$. A non-empty subset $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ is called a *weakly interior ideal* (WIID) of $\mathfrak{S}$ if $\mathfrak{S}\mathfrak{I}\mathfrak{S} \subseteq \mathfrak{I}$, [17].

It is well-known, that every ID of a semigroup $\mathfrak{S}$ is an IID of $\mathfrak{S}$ and every IID of a semigroup $\mathfrak{S}$ is a WIID of $\mathfrak{S}$.

A SSG $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ is called an *n-interior ideal* (n-IID) of $\mathfrak{S}$ if $\mathfrak{S}\mathfrak{I}^n\mathfrak{S} \subseteq \mathfrak{I}$ where n is an integer number.

For any $t, r \in [0, 1]$, we have

$$t \vee r = \max\{t, r\} \quad \text{and} \quad t \wedge r = \min\{t, r\}.$$

A fuzzy set (FS) ω of a non-empty set $\mathfrak{T}$ is function from $\mathfrak{T}$ into unit closed interval $[0, 1]$ of real numbers, i.e., $\omega : \mathfrak{T} \to [0, 1]$.

For any two FSs of ω and τ of a non-empty set of $\mathfrak{T}$, we defined the *support* of ω instead of $\text{supp}(\omega) = \{t \in \mathfrak{T} \mid \omega(t) \neq 0\}$, $\omega \leq \tau$ if $\omega(t) \leq \tau(t)$, $(\omega \vee \tau)(t) = \max\{\omega(t), \tau(t)\} = \omega(t) \vee \tau(t)$ and $(\omega \wedge \tau)(t) = \min\{\omega(t), \tau(t)\} = \omega(t) \wedge \tau(t)$ for all $t \in \mathfrak{T}$.

For two FSs ω and τ in a semigroup $\mathfrak{S}$, define the product $\omega \circ \tau$ as follows : for all $t \in \mathfrak{S}$,

$$(\omega \circ \tau)(t) = \begin{cases} \bigvee\limits_{(w,d) \in F_t} \{\{\omega(w) \wedge \tau(d)\}\} & \text{if } F_t \neq \emptyset, \\ 0 & \text{if } F_t = \emptyset, \end{cases}$$

where $F_t := \{(w, d) \in \mathfrak{S} \times \mathfrak{S} \mid t = wd\}$.

Let ω be a FS of a semigroup. Then

(1) A *fuzzy subsemigroup* (FSSG) ω of $\mathfrak{S}$ if $\omega(tr) \geq \omega(t) \wedge \omega(r)$ for all $t, r \in \mathfrak{S}$.

(2) A *fuzzy left ideal* (FLID) ω of $\mathfrak{S}$ if $\omega(tr) \geq \omega(r)$ for all $t, r \in \mathfrak{S}$.
(3) A *fuzzy right ideal* (FRID) ω of $\mathfrak{S}$ if $\omega(tr) \geq \omega(t)$ for all $t, r \in \mathfrak{S}$.
(4) A *fuzzy ideal* (FID) ω of $\mathfrak{S}$ if it both FLID and FRID of $\mathfrak{S}$.
(5) A FSSG ω of $\mathfrak{S}$ is called a *fuzzy interior ideal* (FIID) of $\mathfrak{S}$ if $\omega(trm) \geq \omega(r)$ for all $t, r, m \in \mathfrak{S}$.
(6) A *fuzzy weakly interior ideal* (FWIID) ω of $\mathfrak{S}$ if $\omega(trm) \geq \omega(r)$ for all $t, r, m \in \mathfrak{S}$.
(7) A FFSG ω of a semigroups $\mathfrak{S}$ is said to be a *fuzzy n-interior ideal* (F n-IID) of $\mathfrak{S}$ if

$$\omega(td_1^n w) \geq \omega(d_1) \wedge \omega(d_2) \wedge ... \wedge \omega(d_n)$$

for all $t, d_i, w \in \mathfrak{S}$ and $i \in \{1, 2, ..., n\}$.

It is well-known, that every FID of a semigroup $\mathfrak{S}$ is a FIID of $\mathfrak{S}$ and every FIID of a semigroup $\mathfrak{S}$ is a FWIID of $\mathfrak{S}$.

The characteristic function $\chi_{\mathfrak{I}}$ of a subset $\mathfrak{I}$ of a non-empty set $\mathfrak{T}$ is a FS of $\mathfrak{S}$

$$\chi_{\mathfrak{I}}(t) = \begin{cases} 1 & \text{if} \quad t \in \mathfrak{I} \\ 0 & \text{if} \quad t \notin \mathfrak{I}. \end{cases}$$

for all $t \in \mathfrak{T}$.

The following theorems are true.

Theorem 2.1. *[17] Let $\mathfrak{S}$ be a semigroup. Then $\mathfrak{I}$ is a SSG (LID, RID, IID, IID, WIID, n-IID) of $\mathfrak{S}$ if and only if characteristic function $\chi_{\mathfrak{I}}$ is a FSSG (FLID, FRID, FIID, FWIID, F n-IID) of $\mathfrak{S}$.*

Lemma 2.2. *Let $\chi_{\mathfrak{L}}$ and $\chi_{\mathfrak{L}}$ be any two FSs of a semigroup $\mathfrak{S}$. Then, the following properties hold:*

(1) $\chi_{\mathfrak{K} \cap \mathfrak{L}} = \chi_{\mathfrak{K}} \wedge \chi_{\mathfrak{L}}$.
(2) $\chi_{\mathfrak{K}\mathfrak{L}} = \chi_{\mathfrak{K}} \circ \chi_{\mathfrak{L}}$.

Theorem 2.3. *[17] Let ω be a nonzero FS of a semigroup $\mathfrak{S}$. Then ω is a FSSG (FLID, FRID, FWIID, FIID, F n-IID) of $\mathfrak{S}$ if and only if* $\text{supp}(\omega)$ *is a SSG (LID, RID, IID, IID, WIID, n-IID) of $\mathfrak{S}$.*

Next, we will review essential ideals and fuzzy essential ideals in a semigroup and their properties.

Definition 2.4. *[8] An* essential ideal *(EID) $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ if $\mathfrak{I}$ is an ID of $\mathfrak{S}$ and $\mathfrak{I} \cap \mathfrak{J} \neq \emptyset$ for every ID $\mathfrak{J}$ of $\mathfrak{S}$.*

Theorem 2.5. *[8] Let $\mathfrak{I}$ be an EID of a semigroup $\mathfrak{S}$. If $\mathfrak{I}_1$ is an ID of $\mathfrak{S}$ containing $\mathfrak{I}$, then $\mathfrak{I}_1$ is also an EID of $\mathfrak{S}$.*

Theorem 2.6. *[8] Let $\mathfrak{I}$ and $\mathfrak{J}$ be EIDs of a semigroup $\mathfrak{S}$. Then $\mathfrak{I} \cup \mathfrak{J}$ and $\mathfrak{I} \cap \mathfrak{J}$ are EIDs of $\mathfrak{S}$.*

Definition 2.7. *[8] An* essential fuzzy ideal *(EFID) ω of a semigroup $\mathfrak{S}$ if ω is a nonzero FID of $\mathfrak{S}$ and $\omega \wedge \tau \neq 0$ for every nonzero FID τ of $\mathfrak{S}$.*

Theorem 2.8. *Let ω be an EFID of a semigroup $\mathfrak{S}$. If ω_1 is a FID of $\mathfrak{S}$ such that $\omega \leq \omega_1$, then ω_1 is also an EFID of $\mathfrak{S}$.*

Theorem 2.9. *Let ω_1 and ω_2 be EFIDs of a semigroup $\mathfrak{S}$. Then $\omega_1 \vee \omega_2$ and $\omega_1 \wedge \omega_2$ are EFIDs of $\mathfrak{S}$.*

Theorem 2.10. *[8] Let $\mathfrak{I}$ be an ID of a semigroup $\mathfrak{S}$. Then $\mathfrak{I}$ is an EID of $\mathfrak{S}$ if and only if $\chi_{\mathfrak{I}}$ is an EFID of $\mathfrak{S}$.*

Theorem 2.11. *[8] Let ω be a nonzero FID of a semigroup $\mathfrak{S}$. Then ω is an EFID of $\mathfrak{S}$ if and only if* $\text{supp}(\omega)$ *is an EID of $\mathfrak{S}$.*

III. Essential subsemigroups and Essential fuzzy subsemigroups

In this section, we will study concepts of essential subsemigroups in a semigroup and fuzzy essential subsemigroups in a semigroup and their properties.

Definition 3.1. *An* essential subsemigroup *(ESSG) $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ if $\mathfrak{I}$ is a SSG of $\mathfrak{S}$ and $\mathfrak{I} \cap \mathfrak{J} \neq \emptyset$ for every SSG $\mathfrak{J}$ of $\mathfrak{S}$.*

Example 3.2. (1) *Let E be set of all even integers. Then $(E, +)$ and $(\mathbb{N}, +)$ are SSGs of $(\mathbb{Z}, +)$. Thus $(E, +) \cap (\mathbb{N}, +) \neq \emptyset$. Hence, $(E, +)$ is an ESSG of $(\mathbb{Z}, +)$.*
(2) *Let $A = \{2n \mid n \in \mathbb{Z}\}$ and $B = \{3n \mid n \in \mathbb{Z}\}$. Then $(A, \cdot)$ and $(B, \cdot)$ are SSGs of $(\mathbb{Z}, \cdot)$. Thus $(A, \cdot) \cap (B, \cdot) \neq \emptyset$. Hence $(A, \cdot)$ is an ESSG.*

Theorem 3.3. *Let $\mathfrak{I}$ be an ESSG of a semigroup $\mathfrak{S}$. If $\mathfrak{I}_1$ is an SSG of $\mathfrak{S}$ with $\mathfrak{I} \subseteq \mathfrak{I}_1$, then $\mathfrak{I}_1$ is also an ESSG of $\mathfrak{S}$.*

Proof: Suppose that $\mathfrak{I}_1$ is a SSG of $\mathfrak{S}$ such that $\mathfrak{I} \subseteq \mathfrak{I}_1$ and let $\mathfrak{J}$ be any SSG of $\mathfrak{S}$. Thus, $\mathfrak{I} \cap \mathfrak{J} \neq \emptyset$. Hence, $\mathfrak{I}_1 \cap \mathfrak{J} \neq \emptyset$. Therefore $\mathfrak{I}_1$ is an ESSG of $\mathfrak{S}$. ■

Theorem 3.4. *Let $\mathfrak{I}$ and $\mathfrak{J}$ be ESSGs of a semigroup $\mathfrak{S}$. Then $\mathfrak{I} \cup \mathfrak{J}$ and $\mathfrak{I} \cap \mathfrak{J}$ are ESSG of $\mathfrak{S}$.*

Proof: Since $\mathfrak{I} \subseteq \mathfrak{I} \cup \mathfrak{J}$ and $\mathfrak{I}$ is an ESSG we have $\mathfrak{I} \cup \mathfrak{J}$ is an ESSG of $\mathfrak{S}$, by Theorem 3.3.

Since $\mathfrak{I}$ and $\mathfrak{J}$ are ESSGs of $\mathfrak{S}$ we have $\mathfrak{I}$ and $\mathfrak{J}$ are ESSGs of $\mathfrak{S}$. Thus, $\mathfrak{I} \cap \mathfrak{J}$ is a SSG of $\mathfrak{S}$. Let $\mathfrak{K}$ be a SSG of $\mathfrak{S}$. Then $\mathfrak{I} \cap \mathfrak{K} \neq \emptyset$. Thus there exists $t, d \in \mathfrak{I} \cap \mathfrak{K}$. Let $t, d \in \mathfrak{J}$. Then $td \in (\mathfrak{I} \cap \mathfrak{J}) \cap \mathfrak{K}$. Thus, $(\mathfrak{I} \cap \mathfrak{J}) \cap \mathfrak{K} \neq \emptyset$. Hence, $\mathfrak{I} \cap \mathfrak{J}$ is an ESSG of $\mathfrak{S}$. ■

Definition 3.5. *An* essential fuzzy subsemigroup *(EFSSG) ω of a semigroup $\mathfrak{S}$ if ω is a nonzero FSSG of $\mathfrak{S}$ and $\omega \wedge \tau \neq 0$ for every nonzero FSSG τ of $\mathfrak{S}$.*

Theorem 3.6. *Let ω be an EFSSG of a semigroup $\mathfrak{S}$. If ω_1 is a FSSG of $\mathfrak{S}$ such that $\omega \leq \omega_1$, then ω_1 is also an EFSSG of $\mathfrak{S}$.*

Proof: Let ω_1 be a FSSG of $\mathfrak{S}$ such that $\omega \leq \omega_1$ and let τ be any FSSG of $\mathfrak{S}$. Thus, $\omega \wedge \tau \neq 0$. So $\omega_1 \wedge \tau \neq 0$. Hence ω_1 is an EFSSG of $\mathfrak{S}$. ■

Theorem 3.7. *Let ω_1 and ω_2 be EFSSGs of a semigroup $\mathfrak{S}$. Then $\omega_1 \vee \omega_2$ and $\omega_1 \wedge \omega_2$ are EFSSGs of $\mathfrak{S}$.*

Proof: Let ω_1 and ω_2 be EFSSGs of $\mathfrak{S}$. Then by Theorem 3.6, $\omega_1 \vee \omega_2$ is an EFSSG of $\mathfrak{S}$. Since ω_1 and ω_2 are EFSSGs of $\mathfrak{S}$ we have $\omega_1 \wedge \omega_2$ is a FSG of $\mathfrak{S}$. Let τ be a nonzero FSG of $\mathfrak{S}$. Then $\omega_1 \wedge \tau \neq 0$. Thus, there exists $t \in \mathfrak{S}$ such that $\omega_1(t) \neq 0$ and $\tau(t) \neq 0$. Since $\omega_2 \neq 0$ and let $d \in \mathfrak{S}$ such that $\omega_2(d) \neq 0$. Since ω_1 and ω_2 are FSGs of $\mathfrak{S}$ we have $\omega_1(td) \geq \omega_1(t) \wedge \omega_1(d) > 0$ and $\omega_2(td) \geq \omega_2(t) \wedge \omega_2(d) > 0$. Thus, $(\omega_1 \cap \omega_2)(td) = \omega_1(td) \wedge \omega_2(td) \neq 0$. Since τ is a FSG of $\mathfrak{S}$ and $\tau(t) \neq 0$ we have $\tau(td) \neq 0$ for all $t, d \in \mathfrak{S}$. Thus, $[(\omega_1 \wedge \omega_2) \cap \tau](td) \neq 0$. Hence, $[(\omega_1 \cap \omega_2) \wedge \tau] \neq 0$. Therefore, $\omega_1 \cap \omega_2$ is an EFSG of $\mathfrak{S}$. ■

Theorem 3.8. *Let $\mathfrak{I}$ be a SSG of a semigroup $\mathfrak{S}$. Then $\mathfrak{I}$ is an ESSG of $\mathfrak{S}$ if and only if $\chi_{\mathfrak{I}}$ is an EFSSG of $\mathfrak{S}$.*

Proof: Suppose that $\mathfrak{I}$ is an ESSG of $\mathfrak{S}$ and let ω be a nonzero FSSG of $\mathfrak{S}$. Then by Theorem 2.3, $\text{supp}(\omega)$ is a SSG of $\mathfrak{S}$. Since $\mathfrak{I}$ is an ESSG of $\mathfrak{S}$ we have $\mathfrak{I}\cap\text{supp}(\omega) \neq \emptyset$. Thus, there exists $\mathfrak{t}, \mathfrak{d} \in \mathfrak{I} \cap \text{supp}(\omega)$ such that $\omega(\mathfrak{td}) \neq 0$ and $\chi_{\mathfrak{I}}(\mathfrak{td}) \neq 0$. So $(\omega\wedge\chi_{\mathfrak{I}})(\mathfrak{td}) \neq 0$ implies that $\chi_{\mathfrak{I}}\wedge\omega \neq 0$. Therefore, $\chi_{\mathfrak{I}}$ is an essential FWIID of $\mathfrak{S}$.

Conversely, assume that $\chi_{\mathfrak{I}}$ is an EWFIID of $\mathfrak{S}$ and let $\mathfrak{J}$ be a WIID of $\mathfrak{S}$. Then by Theorem 2.1, $\chi_{\mathfrak{J}}$ is a nonzero FWIID of $\mathfrak{S}$. Since $\chi_{\mathfrak{I}}$ is an EWFIID of $\mathfrak{S}$ we have $\chi_{\mathfrak{I}} \wedge \chi_{\mathfrak{J}} \neq 0$. Thus, by Lemma 2.2, $\chi_{\mathfrak{I}\cap\mathfrak{J}} \neq 0$. Hence, $\mathfrak{I} \cap \mathfrak{J} \neq \emptyset$. Therefore, $\mathfrak{I}$ is an EWIID of $\mathfrak{S}$. ■

Theorem 3.9. *Let ω be a nonzero FSSG of a semigroup $\mathfrak{S}$. Then ω is an EFSSG of $\mathfrak{S}$ if and only if $\text{supp}(\omega)$ is an ESSG of $\mathfrak{S}$.*

Proof: Assume that ω is an EFSSG of $\mathfrak{S}$. Then ω is a FSSG of $\mathfrak{S}$. Thus, by Theorem 2.3, $\text{supp}(\omega)$ is a SSG of $\mathfrak{S}$. Let $\mathfrak{I}$ be a SSG of $\mathfrak{S}$. Then by Theorem 3.8, $\chi_{\mathfrak{I}}$ is a FSSG of $\mathfrak{S}$. By assumption, $\omega \wedge \chi_{\mathfrak{I}} \neq 0$. So, there exists $t \in \mathfrak{S}$ such that $(\omega \wedge \chi_{\mathfrak{I}})(t) \neq 0$. It implies that $\omega(t) \neq 0$ and $\chi_{\mathfrak{I}}(t) \neq 0$. Hence, $t \in \text{supp}(\omega)\cap\mathfrak{I}$ so $\text{supp}(\omega)\cap\mathfrak{I} \neq \emptyset$. Therefore, $\text{supp}(\omega)$ is an ESSG of $\mathfrak{S}$.

Conversely, assume that $\text{supp}(\omega)$ is an ESSG of $\mathfrak{S}$. Then $\text{supp}(\omega)$ is a SSG of $\mathfrak{S}$. Let τ be a FSSG of $\mathfrak{S}$. Then by Theorem 2.3, $\text{supp}(\tau)$ is a SSG of $\mathfrak{S}$. By assumption, $\text{supp}(\omega) \cap \text{supp}(\tau) \neq \emptyset$. So, there exists $t \in \text{supp}(\omega) \cap \text{supp}(\tau)$, this implies that $\omega(t) \neq 0$ and $\tau(t) \neq 0$. Hence, $(\omega \wedge \tau)(t) \neq 0$. Therefore, $\omega \wedge \tau \neq 0$. We conclude that ω is an EFSSG of $\mathfrak{S}$. ■

IV. Essential interior ideals Their Fuzzification

In this section, we defined essential interior ideals and essential fuzzy interior ideals in semigroup and integrated properties of its.

Definition 4.1. *An* essential interior ideal *(EIID) $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ if $\mathfrak{I}$ is an IID of $\mathfrak{S}$ and $\mathfrak{I} \cap \mathfrak{J} \neq \emptyset$ for every IID $\mathfrak{J}$ of $\mathfrak{S}$.*

Theorem 4.2. *Let $\mathfrak{I}$ be an EIID of a semigroup $\mathfrak{S}$. If $\mathfrak{I}_1$ is an IID of $\mathfrak{S}$ with $\mathfrak{I} \subseteq \mathfrak{I}_1$, then $\mathfrak{I}_1$ is also an EIID of $\mathfrak{S}$.*

Proof: Suppose that $\mathfrak{I}_1$ is an IID of $\mathfrak{S}$ such that $\mathfrak{I} \subseteq \mathfrak{I}_1$ and let $\mathfrak{J}$ be any IID of $\mathfrak{S}$. Thus, $\mathfrak{I}\cap\mathfrak{J} \neq \emptyset$. Hence, $\mathfrak{I}_1\cap\mathfrak{J} \neq \emptyset$. Therefore, $\mathfrak{I}_1$ is an EIID of $\mathfrak{S}$. ■

Theorem 4.3. *Let $\mathfrak{I}$ and $\mathfrak{J}$ be EIIDs of a semigroup $\mathfrak{S}$. Then $\mathfrak{I} \cup \mathfrak{J}$ and $\mathfrak{I} \cap \mathfrak{J}$ are EIIDs of $\mathfrak{S}$.*

Proof: Since $\mathfrak{I}$ and $\mathfrak{J}$ are EIIDs of $\mathfrak{S}$ we have $\mathfrak{I}$ and $\mathfrak{J}$ are essential subsemigroups of a $\mathfrak{S}$. Thus by Theorem 3.8, $\mathfrak{I} \cup \mathfrak{J}$ and $\mathfrak{I} \cap \mathfrak{J}$ are ESSGs of $\mathfrak{S}$. Since $\mathfrak{I} \subseteq \mathfrak{I} \cup \mathfrak{J}$ and $\mathfrak{I}$ is an EIID we have $\mathfrak{I} \cup \mathfrak{J}$ is an EIID of $\mathfrak{S}$.

Let $\mathfrak{K}$ be an IID of $\mathfrak{S}$. Then $\mathfrak{I} \cap \mathfrak{K} \neq \emptyset$. Thus, there exists t, d and $w \in \mathfrak{I} \cap \mathfrak{K}$. Let t, d and $w \in \mathfrak{J}$. Then $tdw \in (\mathfrak{I} \cap \mathfrak{J}) \cap \mathfrak{K}$. Thus, $(\mathfrak{I} \cap \mathfrak{J}) \cap \mathfrak{K} \neq \emptyset$. Hence $\mathfrak{I} \cap \mathfrak{J}$ is an EIID of $\mathfrak{S}$. ■

The following theorem we will use the basic knowledge of ID and IID in semigroups to prove EIID in semigroup.

Theorem 4.4. *EID of semigroup $\mathfrak{S}$ is an EIID of $\mathfrak{S}$.*

Proof: The proof is obvious. ■

Definition 4.5. *An* essential fuzzy interior ideal *(EFIID) ω of a semigroup $\mathfrak{S}$ if ω is a nonzero FIID of $\mathfrak{S}$ and $\omega \wedge \tau \neq 0$ for every nonzero FIID τ of $\mathfrak{S}$.*

Theorem 4.6. *Let ω be an EFIID of a semigroup $\mathfrak{S}$. If ω_1 is a FIID of $\mathfrak{S}$ such that $\omega \leq \omega_1$, then ω_1 is also an EFIID of $\mathfrak{S}$.*

Proof: Let ω_1 be a FIID of $\mathfrak{S}$ such that $\omega \leq \omega_1$ and let τ be any FIID of $\mathfrak{S}$. Thus, $\omega \wedge \tau \neq 0$. So $\omega_1 \wedge \tau \neq 0$. Hence, ω_1 is an EFIID of $\mathfrak{S}$. ■

Theorem 4.7. *Let ω_1 and ω_2 be EFIIDs of a semigroup $\mathfrak{S}$. Then $\omega_1 \vee \omega_2$ and $\omega_1 \wedge \omega_2$ are EFIIDs of $\mathfrak{S}$.*

Proof: Let ω_1 and ω_2 be EFIIDs of $\mathfrak{S}$. Then by Theorem 4.6, $\omega_1 \vee \omega_2$ is an EFIID of $\mathfrak{S}$. Since ω_1 and ω_2 are EFIIDs of $\mathfrak{S}$ we have ω_1 and ω_2 is an EFSSG of $\mathfrak{S}$. Thus $\omega_1 \wedge \omega_2$ is an EFSSG of $\mathfrak{S}$. Let τ be a nonzero FIID of $\mathfrak{S}$. Then $\omega_1 \wedge \tau \neq 0$. Thus there exists $t, d \in \mathfrak{S}$ such that $\omega_1(td) \neq 0$ and $(\tau)(td) \neq 0$. Since $\omega_2 \neq 0$ and let $w \in \mathfrak{S}$ such that $\omega_2(w) \neq 0$. Since ω_1 and ω_2 are FSSGs of $\mathfrak{S}$ we have $\omega_1(tdw) \geq \omega_1(d) > 0$ and $\omega_2(tdw) \geq \omega_2(d) > 0$. Thus, $(\omega_1\wedge\omega_2)(tdw) = \omega_1(tdw)\wedge\omega_2(tdw) \neq 0$. Since τ is a FSSG of $\mathfrak{S}$ and $\tau(d) \neq 0$ we have $\tau(tdw) \neq 0$ for all $t, d, w \in \mathfrak{S}$. Thus $[(\omega_1 \wedge \omega_2) \wedge \tau](tdw) \neq 0$. Hence $[(\omega_1 \wedge \omega_2) \wedge \tau] \neq 0$. Therefore, $\omega_1 \wedge \omega_2$ is an EFIID of $\mathfrak{S}$. ■

The following theorem we will use the basic knowledge of ID and IID in semigroups to prove EIID in semigroup.

Theorem 4.8. *Every EFID of semigroup $\mathfrak{S}$ is an EFIID of $\mathfrak{S}$.*

Proof: The proof is obvious. ■

Theorem 4.9. *Let $\mathfrak{I}$ be an IID of a semigroup $\mathfrak{S}$. Then $\mathfrak{I}$ is an EIID of $\mathfrak{S}$ if and only if $\chi_{\mathfrak{I}}$ is an EFIID of $\mathfrak{S}$.*

Proof: Suppose that $\mathfrak{I}$ is an EIID of $\mathfrak{S}$ and let ω be a nonzero FIID of $\mathfrak{S}$. Then by Theorem 2.3, $\text{supp}(\omega)$ is a IID of $\mathfrak{S}$. Since $\mathfrak{I}$ is an EIID of $\mathfrak{S}$ we have $\mathfrak{I} \cap \text{supp}(\omega) \neq \emptyset$. Thus, there exists $t, d, w \in \mathfrak{I}\cap\text{supp}(\omega)$ such that $\omega(tdw) \neq 0$ and $\chi_{\mathfrak{I}}(tdw) \neq 0$. So $(\omega \wedge \chi_{\mathfrak{I}})(tdw) \neq 0$ implies that $\chi_{\mathfrak{I}} \wedge \omega \neq 0$. Therefore, $\chi_{\mathfrak{I}}$ is an essential FIID of $\mathfrak{S}$.

Conversely, assume that $\chi_{\mathfrak{I}}$ is an EFIID of $\mathfrak{S}$ and let $\mathfrak{J}$ be an IID of $\mathfrak{S}$. Then by Theorem 2.1, $\chi_{\mathfrak{J}}$ is a nonzero FIID of $\mathfrak{S}$. Since $\chi_{\mathfrak{I}}$ is an EFIID of $\mathfrak{S}$ we have $\chi_{\mathfrak{I}} \wedge \chi_{\mathfrak{J}} \neq 0$. Thus, by Lemma 2.2, $\chi_{\mathfrak{I}\cap\mathfrak{J}} \neq 0$. Hence, $\mathfrak{I} \cap \mathfrak{J} \neq \emptyset$. Therefore, $\mathfrak{I}$ is an EIID of $\mathfrak{S}$. ■

Theorem 4.10. *Let ω be a nonzero FIID of a semigroup $\mathfrak{S}$. Then ω is an EFIID of $\mathfrak{S}$ if and only if $\text{supp}(\omega)$ is an EIID of $\mathfrak{S}$.*

Proof: Assume that ω is an EFIID of $\mathfrak{S}$. Then ω is a FIID of $\mathfrak{S}$. Thus, by Theorem 2.3, $\text{supp}(\omega)$ is an IID of $\mathfrak{S}$. Let $\mathfrak{I}$ be an IID of $\mathfrak{S}$. Then by Theorem 2.1, $\chi_{\mathfrak{I}}$ is a FIID of $\mathfrak{S}$. By assumption, $\omega \wedge \chi_{\mathfrak{I}} \neq 0$. So, there exists $t \in \mathfrak{S}$ such that $(\omega \wedge \chi_{\mathfrak{I}})(t) \neq 0$. It implies that $\omega(t) \neq 0$ and $\chi_{\mathfrak{I}}(t) \neq 0$. Hence, $t \in \text{supp}(\omega) \cap \mathfrak{I}$ so $\text{supp}(\omega) \cap \mathfrak{I} \neq \emptyset$. Therefore, $\text{supp}(\omega)$ is an EIID of $\mathfrak{S}$.

Conversely, assume that $\text{supp}(\omega)$ is an EIID of $\mathfrak{S}$. Then $\text{supp}(\omega)$ is a IID of $\mathfrak{S}$. Let τ be a FIID of $\mathfrak{S}$. Then by Theorem 2.3, $\text{supp}(\tau)$ is an IID of $\mathfrak{S}$. By assumption,

$\mathrm{supp}(\omega) \cap \mathrm{supp}(\tau) \neq \emptyset$. So, there exists $t \in \mathrm{supp}(\omega) \cap \mathrm{supp}(\tau)$, this implies that $\omega(t) \neq 0$ and $\tau(t) \neq 0$. Hence, $(\omega \wedge \tau)(t) \neq 0$. Therefore, $\omega \wedge \tau \neq 0$. We conclude that ω is an EFIID of $\mathfrak{S}$. ■

Definition 4.11. *An EIID $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ is called*

(1) *a* minimal *(MiEIID) if for every EIID of $\mathfrak{J}$ of $\mathfrak{S}$ such that $\mathfrak{J} \subseteq \mathfrak{I}$, we have $\mathfrak{J} = \mathfrak{I}$,*
(2) *a* maximal *(MaEIID) if for every EIID of $\mathfrak{J}$ of $\mathfrak{S}$ such that $\mathfrak{I} \subseteq \mathfrak{J}$, we have $\mathfrak{J} = \mathfrak{I}$,*

Definition 4.12. *An EFIID ω of a semigroup $\mathfrak{S}$ is called*

(1) *a* minimal *(MiEFIID) if for every EFIID of τ of $\mathfrak{S}$ such that $\tau \leq \omega$, we have $\mathrm{supp}(\tau) = \mathrm{supp}(\omega)$,*
(2) *a* maximal *(MaEFIID) if for every EFIID of τ of $\mathfrak{S}$ such that $\omega \leq \tau$, we have $\mathrm{supp}(\omega) = \mathrm{supp}(\tau)$.*

Theorem 4.13. *Let $\mathfrak{I}$ be a non-empty subset of a semigroup $\mathfrak{S}$. Then the following statement holds.*

(1) *$\mathfrak{I}$ is a MiEIID of $\mathfrak{S}$ if and only if $\chi_{\mathfrak{I}}$ is a MiEFIID of $\mathfrak{S}$,*
(2) *$\mathfrak{I}$ is a MaEIID of $\mathfrak{S}$ if and only if $\chi_{\mathfrak{I}}$ is a MaEFIID of $\mathfrak{S}$.*

Proof:

(1) Suppose that $\mathfrak{I}$ is a MiEIID of $\mathfrak{S}$. Then $\mathfrak{I}$ is an EIID of $\mathfrak{S}$. By Theorem 4.9, $\chi_{\mathfrak{I}}$ is an EFIID of $\mathfrak{S}$. Let τ be an EFIID of $\mathfrak{S}$ such that $\tau \leq \chi_{\mathfrak{I}}$. Then $\mathrm{supp}(\tau) \subseteq \mathrm{supp}(\chi_{\mathfrak{I}})$. Thus, $\mathrm{supp}(\tau) \subseteq \mathrm{supp}(\chi_{\mathfrak{I}}) = \mathfrak{I}$. Hence, $\mathrm{supp}(\tau) \subseteq \mathfrak{I}$. Since τ is an EFIID of $\mathfrak{S}$ we have $\mathrm{supp}(\tau)$ is an EIID of $\mathfrak{S}$. By assumption, $\mathrm{supp}(\tau) = \mathfrak{I} = \mathrm{supp}(\chi_{\mathfrak{I}})$. Hence, $\chi_{\mathfrak{I}}$ is a MiEFIID of $\mathfrak{S}$.
Conversely, $\chi_{\mathfrak{I}}$ is a MiEFIID of $\mathfrak{S}$. Then $\chi_{\mathfrak{I}}$ is an EFIID of $\mathfrak{S}$. By Theorem 4.9, $\mathfrak{I}$ is an EIID of $\mathfrak{S}$. Let $\mathfrak{J}$ be an EIID of $\mathfrak{S}$ such that $\mathfrak{J} \subseteq \mathfrak{I}$. Then $\mathfrak{J}$ is an IID of $\mathfrak{S}$. Thus by Theorem 4.9, $\chi_{\mathfrak{J}}$ is an EFIID of $\mathfrak{S}$ such that $\chi_{\mathfrak{J}} \leq \chi_{\mathfrak{I}}$. Hence, $\mathfrak{J} = \mathrm{supp}(\chi_{\mathfrak{J}}) \subseteq \mathrm{supp}(\chi_{\mathfrak{I}}) = \mathfrak{I}$. By assumption, $\mathfrak{J} = \mathrm{supp}(\chi_{\mathfrak{J}}) = \mathfrak{J} = \mathrm{supp}(\chi_{\mathfrak{I}}) = \mathfrak{I}$. So, $\mathfrak{J} = \mathfrak{I}$. Hence, $\mathfrak{I}$ is a MiEIID of $\mathfrak{S}$.
(2) Suppose that $\mathfrak{I}$ is a MaEIID of $\mathfrak{S}$. Then $\mathfrak{I}$ is an EIID of $\mathfrak{S}$. By Theorem 4.9, $\chi_{\mathfrak{I}}$ is an EFIID of $\mathfrak{S}$. Let τ be an EFIID of $\mathfrak{S}$ such that $\chi_{\mathfrak{I}} \leq \tau$. Then $\mathrm{supp}(\chi_{\mathfrak{I}}) \subseteq \mathrm{supp}(\tau)$. Thus, $\mathfrak{I} = \mathrm{supp}(\chi_{\mathfrak{I}}) \subseteq \mathrm{supp}(\tau)$. Hence, $\mathfrak{I} \subseteq \mathrm{supp}(\tau)$. Since τ is an EFIID of $\mathfrak{S}$ we have $\mathrm{supp}(\tau)$ is an EIID of $\mathfrak{S}$. By assumption, $\mathrm{supp}(\tau) = \mathfrak{I} = \mathrm{supp}(\chi_{\mathfrak{I}})$. Hence, $\chi_{\mathfrak{I}}$ is a MaEFIID of $\mathfrak{S}$.
Conversely, $\chi_{\mathfrak{I}}$ is a MaEFIID of $\mathfrak{S}$. Then $\chi_{\mathfrak{I}}$ is an EFIID of $\mathfrak{S}$. By Theorem 4.9, $\mathfrak{I}$ is an EIID of $\mathfrak{S}$. Let $\mathfrak{J}$ be an EIID of $\mathfrak{S}$ such that $\mathfrak{I} \subseteq \mathfrak{J}$. Then $\mathfrak{J}$ is an IID of $\mathfrak{S}$. Thus by Theorem 4.9, $\chi_{\mathfrak{J}}$ is an EFIID of $\mathfrak{S}$ such that $\chi_{\mathfrak{I}} \leq \chi_{\mathfrak{J}}$. Hence, $\mathfrak{I} = \mathrm{supp}(\chi_{\mathfrak{I}}) \subseteq \mathrm{supp}(\chi_{\mathfrak{J}}) = \mathfrak{J}$. By assumption, $\mathfrak{J} = \mathrm{supp}(\chi_{\mathfrak{J}}) = \mathfrak{J} = \mathrm{supp}(\chi_{\mathfrak{I}}) = \mathfrak{I}$. So, $\mathfrak{J} = \mathfrak{I}$. Hence, $\mathfrak{I}$ is a MaEIID of $\mathfrak{S}$.

■

Definition 4.14. *An EIID $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ is called a* 0-minimal *(0-MiEIID) if for every EIID of $\mathfrak{J}$ of $\mathfrak{S}$ such that $\mathfrak{J} \subseteq \mathfrak{I}$, we have $\mathfrak{J} = \mathfrak{I}$,*

Definition 4.15. *An EFIID ω of a semigroup $\mathfrak{S}$ is called a* 0-minimal *(0-MiEFIID) if for every EFIID of τ of $\mathfrak{S}$ such that $\tau \leq \omega$, we have $\mathrm{supp}(\tau) = \mathrm{supp}(\omega)$,*

Theorem 4.16. *Let $\mathfrak{I}$ be a non-empty subset of a semigroup $\mathfrak{S}$. Then the following statement holds. $\mathfrak{I}$ is a 0-MiEIID of $\mathfrak{S}$ if and only if $\chi_{\mathfrak{I}}$ is a 0-MiEFIID of $\mathfrak{S}$.*

Proof: Suppose that $\mathfrak{I}$ is a 0-MiEIID of $\mathfrak{S}$. Then $\mathfrak{I}$ is an EIID of $\mathfrak{S}$. By Theorem 4.9, $\chi_{\mathfrak{I}}$ is an EFIID of $\mathfrak{S}$. Let τ be an EFIID of $\mathfrak{S}$ such that $\tau \leq \chi_{\mathfrak{I}}$. Then $\mathrm{supp}(\tau) \subseteq \mathrm{supp}(\chi_{\mathfrak{I}})$. Thus, $\mathrm{supp}(\tau) \subseteq \mathrm{supp}(\chi_{\mathfrak{I}}) = \mathfrak{I}$. Hence, $\mathrm{supp}(\tau) \subseteq \mathfrak{I}$. Since τ is an EFIID of $\mathfrak{S}$ we have $\mathrm{supp}(\tau)$ is an EIID of $\mathfrak{S}$. By assumption, $\mathrm{supp}(\tau) = \mathfrak{I} = \mathrm{supp}(\chi_{\mathfrak{I}})$. Hence, $\chi_{\mathfrak{I}}$ is a 0-MiEFIID of $\mathfrak{S}$.

Conversely, $\chi_{\mathfrak{I}}$ is a 0-MiEFIID of $\mathfrak{S}$. Then $\chi_{\mathfrak{I}}$ is an EFIID of $\mathfrak{S}$. By Theorem 4.9, $\mathfrak{I}$ is an EIID of $\mathfrak{S}$. Let $\mathfrak{J}$ be an EIID of $\mathfrak{S}$ such that $\mathfrak{J} \subseteq \mathfrak{I}$. Then $\mathfrak{J}$ is an IID of $\mathfrak{S}$. Thus by Theorem 4.9, $\chi_{\mathfrak{J}}$ is an EFIID of $\mathfrak{S}$ such that $\chi_{\mathfrak{J}} \leq \chi_{\mathfrak{I}}$. Hence, $\mathfrak{J} = \mathrm{supp}(\chi_{\mathfrak{J}}) \subseteq \mathrm{supp}(\chi_{\mathfrak{I}}) = \mathfrak{I}$. By assumption, $\mathfrak{J} = \mathrm{supp}(\chi_{\mathfrak{J}}) = \mathfrak{J} = \mathrm{supp}(\chi_{\mathfrak{I}}) = \mathfrak{I}$. So, $\mathfrak{J} = \mathfrak{I}$. Hence, $\mathfrak{I}$ is a 0-MiEIID of $\mathfrak{S}$. ■

Definition 4.17. *An EIID $\mathfrak{I}$ of a semigroup $\mathfrak{S}$. Then $\mathfrak{I}$ is said to be:*

(1) prime *(PEIID) if $td \in \mathfrak{I}$ implies $t \in \mathfrak{I}$ or $d \in \mathfrak{I}$, for all $t, d \in \mathfrak{S}$.*
(2) semiprime *(SPEIID) if $t^2 \in \mathfrak{I}$ implies $t \in \mathfrak{I}$, for all $t \in \mathfrak{S}$.*

Definition 4.18. *Let ω be an EFIID of a semigroup $\mathfrak{S}$. Then ω is said to be:*

(1) prime *(PEFIID) if $\omega(td) \leq \omega(t) \vee \omega(d)$ for all $t, d \in \mathfrak{S}$.*
(2) semiprime *(SPEFIID) if $\omega(t^2) \leq \omega(t)$ for all $t, d \in \mathfrak{S}$.*

It is clear, every PEFIIDs of is SPEFIIDs in semigroup.

Theorem 4.19. *Let $\mathfrak{I}$ be a non-empty subset of a semigroup $\mathfrak{S}$. Then the following statement holds.*

(1) *$\mathfrak{I}$ is a PEIID of $\mathfrak{S}$ if and only if $\chi_{\mathfrak{I}}$ is a PEFIID of $\mathfrak{S}$,*
(2) *$\mathfrak{I}$ is a SPEIID of $\mathfrak{S}$ if and only if $\chi_{\mathfrak{I}}$ is a SPEFIID of $\mathfrak{S}$,*

Proof:

(1) Suppose that $\mathfrak{I}$ is a PEIID of $\mathfrak{S}$ and let $t, d \in \mathfrak{S}$. Then $\mathfrak{I}$ is an EIID of $\mathfrak{S}$. By Theorem 4.9, $\chi_{\mathfrak{I}}$ is an EFIID of $\mathfrak{S}$.
Case 1: $td \in \mathfrak{I}$, then $t \in \mathfrak{I}$ or $d \in \mathfrak{I}$. Thus, $\chi_{\mathfrak{I}}(td) = 1 = \chi_{\mathfrak{I}}(t) = \chi_{\mathfrak{I}}(d)$. Hence, $\chi_{\mathfrak{I}}(td) \leq \chi_{\mathfrak{I}}(t) \vee \chi_{\mathfrak{I}}(d)$.
Case 2: $td \notin \mathfrak{I}$, then $\chi_{\mathfrak{I}}(td) \leq \chi_{\mathfrak{I}}(t) \vee \chi_{\mathfrak{I}}(d)$.
From two cases, we have $\chi_{\mathfrak{I}}$ is a PEFIID of $\mathfrak{S}$.
Conversely, $\chi_{\mathfrak{I}}$ is a PEFIID of $\mathfrak{S}$. Then $\chi_{\mathfrak{I}}$ is an EFIID of $\mathfrak{S}$. By Theorem 4.9, $\mathfrak{I}$ is an EIID of $\mathfrak{S}$. Let $t, d \in \mathfrak{S}$ and $td \in \mathfrak{I}$. Then $\chi_{\mathfrak{I}}(td) = 1$. If $t \notin \mathfrak{I}$ and $d \notin \mathfrak{I}$, then $\chi_{\mathfrak{I}}(t) = 0 = \chi_{\mathfrak{I}}(d)$. Thus, $0 = \chi_{\mathfrak{I}}(t) \vee \chi_{\mathfrak{I}}(d) \leq \chi_{\mathfrak{I}}(td) = 1$. By assumption, $\chi_{\mathfrak{I}}(td) \leq \chi_{\mathfrak{I}}(t) \vee \chi_{\mathfrak{I}}(d)$. It is a contradiction, so $t \in \mathfrak{I}$ or $d \in \mathfrak{I}$. Thus, $\mathfrak{I}$ is a PEIID of $\mathfrak{S}$.
(2) Suppose that $\mathfrak{I}$ is a SEIID of $\mathfrak{S}$ and let $t \in \mathfrak{S}$. Then $\mathfrak{I}$ is an EIID of $\mathfrak{S}$. By Theorem 4.9, $\chi_{\mathfrak{I}}$ is an EFIID of $\mathfrak{S}$.
Case 1: $t^2 \in \mathfrak{I}$, then $t \in \mathfrak{I}$. Thus, $\chi_{\mathfrak{I}}(t^2) = 1 = \chi_{\mathfrak{I}}(t)$. Hence, $\chi_{\mathfrak{I}}(t^2) \leq \chi_{\mathfrak{I}}(t)$.
Case 2: $t^2 \notin \mathfrak{I}$, then $\chi_{\mathfrak{I}}(t^2) \leq \chi_{\mathfrak{I}}(t)$.
From two cases, we have $\chi_{\mathfrak{I}}$ is a SEFIID of $\mathfrak{S}$.
Conversely, $\chi_{\mathfrak{I}}$ is a SEFIID of $\mathfrak{S}$. Then $\chi_{\mathfrak{I}}$ is an EFIID of $\mathfrak{S}$. By Theorem 4.9, $\mathfrak{I}$ is an EIID of $\mathfrak{S}$. Let $t \in \mathfrak{S}$

and $t^2 \in \mathfrak{I}$. Then $\chi_{\mathfrak{I}}(t^2) = 1$. If $t \notin \mathfrak{I}$, then $\chi_{\mathfrak{I}}(t) = 0$. Thus, $0 = \chi_{\mathfrak{I}}(t) \leq \chi_{\mathfrak{I}}(t^2) = 1$. By assumption, $\chi_{\mathfrak{I}}(t^2) \leq \chi_{\mathfrak{I}}(t)$. It is a contradiction, so $t \in \mathfrak{I}$. Thus, $\mathfrak{I}$ is a SEIID of $\mathfrak{S}$. ■

V. Essential weakly interior ideals Their Fuzzification

In this section, we defined essential weakly interior ideals and essential fuzzy weakly interior ideals in semigroup and integrated properties of its.

Definition 5.1. *An* essential weakly interior ideal *(EWIID) $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ if $\mathfrak{I}$ is a WIID of $\mathfrak{S}$ and $\mathfrak{I} \cap \mathfrak{J} \neq \emptyset$ for every WIID $\mathfrak{J}$ of $\mathfrak{S}$.*

Theorem 5.2. *Let $\mathfrak{I}$ be an EWIID of a semigroup $\mathfrak{S}$. If $\mathfrak{I}_1$ is a WIID of $\mathfrak{S}$ with $\mathfrak{I} \subseteq \mathfrak{I}_1$, then $\mathfrak{I}_1$ is also an EWIID of $\mathfrak{S}$.*

Proof: Suppose that $\mathfrak{I}_1$ is a WIID of $\mathfrak{S}$ such that $\mathfrak{I} \subseteq \mathfrak{I}_1$ and let $\mathfrak{J}$ be any WIID of $\mathfrak{S}$. Since $\mathfrak{I}$ is an EWIID of $\mathfrak{S}$ we have $\mathfrak{I} \cap \mathfrak{J} \neq \emptyset$. By assumption, we have $\mathfrak{I}_1 \cap \mathfrak{J} \neq \emptyset$. Therefore, $\mathfrak{I}_1$ is an EWIID of $\mathfrak{S}$. ■

Theorem 5.3. *Let $\mathfrak{I}$ and $\mathfrak{J}$ be EWIIDs of a semigroup $\mathfrak{S}$. Then $\mathfrak{I} \cup \mathfrak{J}$ and $\mathfrak{I} \cap \mathfrak{J}$ are EWIIDs of $\mathfrak{S}$.*

Proof: Since $\mathfrak{I}$ and $\mathfrak{J}$ are EWIIDs of $\mathfrak{S}$ such that $\mathfrak{I} \subseteq \mathfrak{I} \cup \mathfrak{J}$ and $\mathfrak{I}$ is an EWIID we have $\mathfrak{I} \cup \mathfrak{J}$ is an EWIID of $\mathfrak{S}$.

Let $\mathfrak{K}$ be a WIID of $\mathfrak{S}$. Since $\mathfrak{I}$ is an EWIID of $\mathfrak{S}$ we have $\mathfrak{I} \cap \mathfrak{K} \neq \emptyset$. Thus, there exists t, d and $w \in \mathfrak{I} \cap \mathfrak{K}$. Since $\mathfrak{J}$ is an EWIID of $\mathfrak{S}$, there exists t, d and $w \in \mathfrak{J}$. Then $tdw \in (\mathfrak{I} \cap \mathfrak{J}) \cap \mathfrak{K}$. Thus, $(\mathfrak{I} \cap \mathfrak{J}) \cap \mathfrak{K} \neq \emptyset$. Hence, $\mathfrak{I} \cap \mathfrak{J}$ is an EWIID of $\mathfrak{S}$. ■

The following theorem we will use the basic knowledge of ID and WIID in semigroups to prove EWIID in semigroup.

Theorem 5.4. *Every EID of semigroup $\mathfrak{S}$ is an EWIID of $\mathfrak{S}$.*

Proof: The proof is obvious. ■

Definition 5.5. *An* essential fuzzy weakly interior ideal *(EFWIID) ω of a semigroup $\mathfrak{S}$ if ω is a nonzero FWIID of $\mathfrak{S}$ and $\omega \wedge \tau \neq 0$ for every nonzero FWIID τ of $\mathfrak{S}$.*

Theorem 5.6. *Let $\mathfrak{I}$ be a WIID of a semigroup $\mathfrak{S}$. Then $\mathfrak{I}$ is an EWIID of $\mathfrak{S}$ if and only if $\chi_{\mathfrak{I}}$ is an EFWIID of $\mathfrak{S}$.*

Proof: Suppose that $\mathfrak{I}$ is an EWIID of $\mathfrak{S}$ and let ω be a nonzero FWIID of $\mathfrak{S}$. Then by Theorem 2.3, $\mathrm{supp}(\omega)$ is a WIID of $\mathfrak{S}$. Since $\mathfrak{I}$ is an EWIID of $\mathfrak{S}$ we have $\mathfrak{I} \cap \mathrm{supp}(\omega) \neq \emptyset$. Thus, there exists $t, d, w \in \mathfrak{I} \cap \mathrm{supp}(\omega)$ such that $\omega(tdw) \neq 0$ and $\chi_{\mathfrak{I}}(tdw) \neq 0$. So $(\omega \wedge \chi_{\mathfrak{I}})(tdw) \neq 0$ implies that $\chi_{\mathfrak{I}} \wedge \omega \neq 0$. Therefore, $\chi_{\mathfrak{I}}$ is an essential FWIID of $\mathfrak{S}$.

Conversely, assume that $\chi_{\mathfrak{I}}$ is an EWFIID of $\mathfrak{S}$ and let $\mathfrak{J}$ be a WIID of $\mathfrak{S}$. Then by Theorem 2.1, $\chi_{\mathfrak{J}}$ is a nonzero FWIID of $\mathfrak{S}$. Since $\chi_{\mathfrak{I}}$ is an EWFIID of $\mathfrak{S}$ we have $\chi_{\mathfrak{I}} \wedge \chi_{\mathfrak{J}} \neq 0$. Thus, by Lemma 2.2, $\chi_{\mathfrak{I} \cap \mathfrak{J}} \neq 0$. Hence, $\mathfrak{I} \cap \mathfrak{J} \neq \emptyset$. Therefore, $\mathfrak{I}$ is an EWIID of $\mathfrak{S}$. ■

Theorem 5.7. *Let ω be a nonzero FWIID of a semigroup $\mathfrak{S}$. Then ω is an EFWIID of $\mathfrak{S}$ if and only if $\mathrm{supp}(\omega)$ is an EWIID of $\mathfrak{S}$.*

Proof: Assume that ω is an EFWIID of $\mathfrak{S}$. Then ω is a FWIID of $\mathfrak{S}$. Thus, by Theorem 2.3, $\mathrm{supp}(\omega)$ is a WIID of $\mathfrak{S}$. Let $\mathfrak{I}$ be a WIID of $\mathfrak{S}$. Then by Theorem 5.6, $\chi_{\mathfrak{I}}$ is a FWIID of $\mathfrak{S}$. By assumption, $\omega \wedge \chi_{\mathfrak{I}} \neq 0$. So, there exists $t \in \mathfrak{S}$ such that $(\omega \wedge \chi_{\mathfrak{I}})(t) \neq 0$. It implies that $\omega(t) \neq 0$ and $\chi_{\mathfrak{I}}(t) \neq 0$. Hence, $t \in \mathrm{supp}(\omega) \cap \mathfrak{I}$ so $\mathrm{supp}(\omega) \cap \mathfrak{I} \neq \emptyset$. Therefore, $\mathrm{supp}(\omega)$ is an EWIID of $\mathfrak{S}$.

Conversely, assume that $\mathrm{supp}(\omega)$ is an EWIID of $\mathfrak{S}$. Then $\mathrm{supp}(\omega)$ is a WIID of $\mathfrak{S}$. Let τ be a FWIID of $\mathfrak{S}$. Then by Theorem 2.3, $\mathrm{supp}(\tau)$ is a WIID of $\mathfrak{S}$. By assumption, $\mathrm{supp}(\omega) \cap \mathrm{supp}(\tau) \neq \emptyset$. So, there exists $t \in \mathrm{supp}(\omega) \cap \mathrm{supp}(\tau)$, this implies that $\omega(t) \neq 0$ and $\tau(t) \neq 0$. Hence, $(\omega \wedge \tau)(t) \neq 0$. Therefore, $\omega \wedge \tau \neq 0$. We conclude that ω is an EFWIID of $\mathfrak{S}$. ■

Theorem 5.8. *Let ω be an EFWIID of a semigroup $\mathfrak{S}$. If ω_1 is a FWIID of $\mathfrak{S}$ such that $\omega \leq \omega_1$, then ω_1 is also an EFWIID of $\mathfrak{S}$.*

Proof: Let ω_1 be a FWIID of $\mathfrak{S}$ such that $\omega \leq \omega_1$ and let τ be any FWIID of $\mathfrak{S}$. Thus $\omega \wedge \tau \neq 0$. So $\omega_1 \wedge \tau \neq 0$. Hence, ω_1 is an EFWIID of $\mathfrak{S}$. ■

Theorem 5.9. *Let ω_1 and ω_2 be EFWIIDs of a semigroup $\mathfrak{S}$. Then $\omega_1 \vee \omega_2$ and $\omega_1 \wedge \omega_2$ are EFWIIDs of $\mathfrak{S}$.*

Proof: Let ω_1 and ω_2 be EFWIIDs of $\mathfrak{S}$. Then by Theorem 5.8, $\omega_1 \vee \omega_2$ is an EFWIID of $\mathfrak{S}$. Let τ be a nonzero FWIID of $\mathfrak{S}$. Then $\omega_1 \wedge \tau \neq 0$. Thus, there exists $t, d \in \mathfrak{S}$ such that $\omega_1(td) \neq 0$ and $\tau(td) \neq 0$. Since $\omega_2 \neq 0$ and let $w \in \mathfrak{S}$ such that $\omega_2(w) \neq 0$. Since ω_1 and ω_2 are FWIID s of $\mathfrak{S}$ we have $\omega_1(tdw) \geq \omega_1(d) > 0$ and $\omega_2(tdw) \geq \omega_2(d) > 0$. Thus $(\omega_1 \wedge \omega_2)(tdw) = \omega_1(tdw) \wedge \omega_2(tdw) \neq 0$. Since τ is a FWIID of $\mathfrak{S}$ and $\tau(d) \neq 0$ we have $\tau(tdw) \neq 0$ for all $t, d, w \in \mathfrak{S}$. Thus, $[(\omega_1 \wedge \omega_2) \wedge \tau](tdw) \neq 0$. Hence $[(\omega_1 \wedge \omega_2) \wedge \tau] \neq 0$. Therefore, $\omega_1 \wedge \omega_2$ is an EFWIID of $\mathfrak{S}$. ■

The following theorem we will use the basic knowledge of FIID and FWIID in semigroups to prove EFWIID in semigroup.

Theorem 5.10. *Every EFIID of semigroup $\mathfrak{S}$ is an EFWIID of $\mathfrak{S}$.*

Proof: The proof is obvious. ■

Definition 5.11. *An EWIID $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ is called*

(1) *a* minimal *(MiEWIID) if for every EWIID of $\mathfrak{J}$ of $\mathfrak{S}$ such that $\mathfrak{J} \subseteq \mathfrak{I}$, we have $\mathfrak{J} = \mathfrak{I}$,*
(2) *a* maximal *(MaEWIID) if for every EWIID of $\mathfrak{J}$ of $\mathfrak{S}$ such that $\mathfrak{I} \subseteq \mathfrak{J}$, we have $\mathfrak{J} = \mathfrak{I}$,*

Definition 5.12. *An EFWIID ω of a semigroup $\mathfrak{S}$ is called*

(1) *a* minimal *(MiEFWIID) if for every EFWIID of τ of $\mathfrak{S}$ such that $\tau \leq \omega$, we have* $\mathrm{supp}(\tau) = \mathrm{supp}(\omega)$,
(2) *a* maximal *(MaEFWIID) if for every EFWIID of τ of $\mathfrak{S}$ such that $\omega \leq \tau$, we have* $\mathrm{supp}(\omega) = \mathrm{supp}(\tau)$.

Theorem 5.13. *Let $\mathfrak{I}$ be a non-empty subset of a semigroup $\mathfrak{S}$. Then the following statement holds.*

(1) *$\mathfrak{I}$ is a MiEWIID of $\mathfrak{S}$ if and only if $\chi_{\mathfrak{I}}$ is a MiEFWIID of $\mathfrak{S}$,*
(2) *$\mathfrak{I}$ is a MaEWIID of $\mathfrak{S}$ if and only if $\chi_{\mathfrak{I}}$ is a MaEFWIID of $\mathfrak{S}$.*

Proof:

(1) Suppose that $\mathfrak{I}$ is a MiEWIID of $\mathfrak{S}$. Then $\mathfrak{I}$ is an EWIID of $\mathfrak{S}$. By Theorem 5.6, $\chi_{\mathfrak{I}}$ is an EFWIID of $\mathfrak{S}$. Let τ be an EFWIID of $\mathfrak{S}$ such that $\tau \leq \chi_{\mathfrak{I}}$. Then $\mathrm{supp}(\tau) \subseteq \mathrm{supp}(\chi_{\mathfrak{I}})$. Thus, $\mathrm{supp}(\tau) \subseteq \mathrm{supp}(\chi_{\mathfrak{I}}) = \mathfrak{I}$. Hence, $\mathrm{supp}(\tau) \subseteq \mathfrak{I}$. Since τ is an EFWIID of $\mathfrak{S}$ we have $\mathrm{supp}(\tau)$ is an EWIID of $\mathfrak{S}$. By assumption, $\mathrm{supp}(\tau) = \mathfrak{I} = \mathrm{supp}(\chi_{\mathfrak{I}})$. Hence, $\chi_{\mathfrak{I}}$ is a MiEFWIID of $\mathfrak{S}$.
Conversely, $\chi_{\mathfrak{I}}$ is a MiEFWIID of $\mathfrak{S}$. Then $\chi_{\mathfrak{I}}$ is an EFWIID of $\mathfrak{S}$. By Theorem 5.6, $\mathfrak{I}$ is an EWIID of $\mathfrak{S}$. Let $\mathfrak{J}$ be an EWIID of $\mathfrak{S}$ such that $\mathfrak{J} \subseteq \mathfrak{I}$. Then $\mathfrak{J}$ is an WIID of $\mathfrak{S}$. Thus by Theorem 4.9, $\chi_{\mathfrak{J}}$ is an EFWIID of $\mathfrak{S}$ such that $\chi_{\mathfrak{J}} \leq \chi_{\mathfrak{I}}$. Hence, $\mathfrak{J} = \mathrm{supp}(\chi_{\mathfrak{J}}) \subseteq \mathrm{supp}(\chi_{\mathfrak{I}}) = \mathfrak{I}$. By assumption, $\mathfrak{J} = \mathrm{supp}(\chi_{\mathfrak{J}}) = \mathfrak{J} = \mathrm{supp}(\chi_{\mathfrak{I}}) = \mathfrak{I}$. So, $\mathfrak{J} = \mathfrak{I}$. Hence, $\mathfrak{I}$ is a MiEWIID of $\mathfrak{S}$.

(2) Suppose that $\mathfrak{I}$ is a MaEWIID of $\mathfrak{S}$. Then $\mathfrak{I}$ is an EWIID of $\mathfrak{S}$. By Theorem 5.6, $\chi_{\mathfrak{I}}$ is an EFWIID of $\mathfrak{S}$. Let τ be an EFWIID of $\mathfrak{S}$ such that $\chi_{\mathfrak{I}} \leq \tau$. Then $\mathrm{supp}(\chi_{\mathfrak{I}}) \subseteq \mathrm{supp}(\tau)$. Thus, $\mathfrak{I} = \mathrm{supp}(\chi_{\mathfrak{I}}) \subseteq \mathrm{supp}(\tau)$. Hence, $\mathfrak{I} \subseteq \mathrm{supp}(\tau)$. Since τ is an EFWIID of $\mathfrak{S}$ we have $\mathrm{supp}(\tau)$ is an EWIID of $\mathfrak{S}$. By assumption, $\mathrm{supp}(\tau) = \mathfrak{I} = \mathrm{supp}(\chi_{\mathfrak{I}})$. Hence, $\chi_{\mathfrak{I}}$ is a MaEFWIID of $\mathfrak{S}$.
Conversely, $\chi_{\mathfrak{I}}$ is a MaEFWIID of $\mathfrak{S}$. Then $\chi_{\mathfrak{I}}$ is an EFWIID of $\mathfrak{S}$. By Theorem 5.6, $\mathfrak{I}$ is an EWIID of $\mathfrak{S}$. Let $\mathfrak{J}$ be an EWIID of $\mathfrak{S}$ such that $\mathfrak{I} \subseteq \mathfrak{J}$. Then $\mathfrak{J}$ is a WIID of $\mathfrak{S}$. Thus by Theorem 4.9, $\chi_{\mathfrak{J}}$ is an EFWIID of $\mathfrak{S}$ such that $\chi_{\mathfrak{I}} \leq \chi_{\mathfrak{J}}$. Hence, $\mathfrak{I} = \mathrm{supp}(\chi_{\mathfrak{I}}) \subseteq \mathrm{supp}(\chi_{\mathfrak{J}}) = \mathfrak{J}$. By assumption, $\mathfrak{J} = \mathrm{supp}(\chi_{\mathfrak{J}}) = \mathfrak{J} = \mathrm{supp}(\chi_{\mathfrak{I}}) = \mathfrak{I}$. So, $\mathfrak{J} = \mathfrak{I}$. Hence, $\mathfrak{I}$ is a MaEWIID of $\mathfrak{S}$.

■

Definition 5.14. *An EIID $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ is called a* 0-minimal *(0-MiEWIID) if for every EIID of $\mathfrak{J}$ of $\mathfrak{S}$ such that $\mathfrak{J} \subseteq \mathfrak{I}$, we have $\mathfrak{J} = \mathfrak{I}$,*

Definition 5.15. *An EFIID ω of a semigroup $\mathfrak{S}$ is called a* 0-minimal *(0-MiEFWIID) if for every EFIID of τ of $\mathfrak{S}$ such that $\tau \leq \omega$, we have $\mathrm{supp}(\tau) = \mathrm{supp}(\omega)$,*

Theorem 5.16. *Let $\mathfrak{I}$ be a non-empty subset of a semigroup $\mathfrak{S}$. Then the following statement holds. $\mathfrak{I}$ is a 0-MiEWIID of $\mathfrak{S}$ if and only if $\chi_{\mathfrak{I}}$ is a 0-MiEFWIID of $\mathfrak{S}$.*

Proof: Suppose that $\mathfrak{I}$ is a 0-MiEWIID of $\mathfrak{S}$. Then $\mathfrak{I}$ is an EWIID of $\mathfrak{S}$. By Theorem 5.6, $\chi_{\mathfrak{I}}$ is an EFWIID of $\mathfrak{S}$. Let τ be an EFWIID of $\mathfrak{S}$ such that $\tau \leq \chi_{\mathfrak{I}}$. Then $\mathrm{supp}(\tau) \subseteq \mathrm{supp}(\chi_{\mathfrak{I}})$. Thus, $\mathrm{supp}(\tau) \subseteq \mathrm{supp}(\chi_{\mathfrak{I}}) = \mathfrak{I}$. Hence, $\mathrm{supp}(\tau) \subseteq \mathfrak{I}$. Since τ is an EFWIID of $\mathfrak{S}$ we have $\mathrm{supp}(\tau)$ is an EWIID of $\mathfrak{S}$. By assumption, $\mathrm{supp}(\tau) = \mathfrak{I} = \mathrm{supp}(\chi_{\mathfrak{I}})$. Hence, $\chi_{\mathfrak{I}}$ is a 0-MiEFWIID of $\mathfrak{S}$.

Conversely, $\chi_{\mathfrak{I}}$ is a 0-MiEFWIID of $\mathfrak{S}$. Then $\chi_{\mathfrak{I}}$ is an EFIID of $\mathfrak{S}$. By Theorem 4.9, $\mathfrak{I}$ is an EWIID of $\mathfrak{S}$. Let $\mathfrak{J}$ be an EWIID of $\mathfrak{S}$ such that $\mathfrak{J} \subseteq \mathfrak{I}$. Then $\mathfrak{J}$ is a WIID of $\mathfrak{S}$. Thus by Theorem 5.6, $\chi_{\mathfrak{J}}$ is an EFWIID of $\mathfrak{S}$ such that $\chi_{\mathfrak{J}} \leq \chi_{\mathfrak{I}}$. Hence, $\mathfrak{J} = \mathrm{supp}(\chi_{\mathfrak{J}}) \subseteq \mathrm{supp}(\chi_{\mathfrak{I}}) = \mathfrak{I}$. By assumption, $\mathfrak{J} = \mathrm{supp}(\chi_{\mathfrak{J}}) = \mathfrak{J} = \mathrm{supp}(\chi_{\mathfrak{I}}) = \mathfrak{I}$. So, $\mathfrak{J} = \mathfrak{I}$. Hence, $\mathfrak{I}$ is a 0-MiEWIID of $\mathfrak{S}$. ■

Definition 5.17. *An EWIID $\mathfrak{I}$ of a semigroup $\mathfrak{S}$. Then $\mathfrak{I}$ is said to be:*

(1) prime *(PEWIID) if $td \in \mathfrak{I}$ implies $t \in \mathfrak{I}$ or $d \in \mathfrak{I}$, for all $t, d \in \mathfrak{S}$.*

(2) semiprime *(SPEWIID) if $t^2 \in \mathfrak{I}$ implies $t \in \mathfrak{I}$, for all $t \in \mathfrak{S}$.*

Definition 5.18. *Let ω be a EFWIID of a semigroup $\mathfrak{S}$. Then ω is said to be:*

(1) prime *(PEFWIID) if $\omega(td) \leq \omega(t) \vee \omega(d)$ for all $t, d \in \mathfrak{S}$.*

(2) semiprime *(SPEFWIID) if $\omega(t^2) \leq \omega(t)$ for all $t \in \mathfrak{S}$.*

It is clear, every PEFWIIDs is SPEFWIIDs in semigroups.

Theorem 5.19. *Let $\mathfrak{I}$ be a non-empty subset of a semigroup $\mathfrak{S}$. Then the following statement holds.*

(1) *$\mathfrak{I}$ is a PEWIID of $\mathfrak{S}$ if and only if $\chi_{\mathfrak{I}}$ is a PEFWIID of $\mathfrak{S}$,*

(2) *$\mathfrak{I}$ is a SEWIID of $\mathfrak{S}$ if and only if $\chi_{\mathfrak{I}}$ is a SPEFWIID of $\mathfrak{S}$,*

Proof:

(1) Suppose that $\mathfrak{I}$ is a PEWIID of $\mathfrak{S}$ and let $t, d \in \mathfrak{S}$. Then $\mathfrak{I}$ is an EWIID of $\mathfrak{S}$. By Theorem 5.6, $\chi_{\mathfrak{I}}$ is an EFWIID of $\mathfrak{S}$.
Case 1: If $td \in \mathfrak{I}$, then $t \in \mathfrak{I}$ or $d \in \mathfrak{I}$. Thus, $\chi_{\mathfrak{I}}(td) = 1 = \chi_{\mathfrak{I}}(t) = \chi_{\mathfrak{I}}(d)$. Hence, $\chi_{\mathfrak{I}}(td) \leq \chi_{\mathfrak{I}}(t) \vee \chi_{\mathfrak{I}}(d)$.
Case 2: If $td \notin \mathfrak{I}$, then $\chi_{\mathfrak{I}}(td) \leq \chi_{\mathfrak{I}}(t) \vee \chi_{\mathfrak{I}}(d)$.
From two cases, we have $\chi_{\mathfrak{I}}$ is a PEFWIID of $\mathfrak{S}$.
Conversely, $\chi_{\mathfrak{I}}$ is a PEFWIID of $\mathfrak{S}$. Then $\chi_{\mathfrak{I}}$ is an EFWIID of $\mathfrak{S}$. By Theorem 5.6, $\mathfrak{I}$ is an EWIID of $\mathfrak{S}$. Let $t, d \in \mathfrak{S}$ and $td \in \mathfrak{I}$. Then $\chi_{\mathfrak{I}}(td) = 1$. If $t \notin \mathfrak{I}$ and $d \notin \mathfrak{I}$, then $\chi_{\mathfrak{I}}(t) = 0 = \chi_{\mathfrak{I}}(d)$. Thus, $0 = \chi_{\mathfrak{I}}(t) \vee \chi_{\mathfrak{I}}(d) \leq \chi_{\mathfrak{I}}(td) = 1$. By assumption, $\chi_{\mathfrak{I}}(td) \leq \chi_{\mathfrak{I}}(t) \vee \chi_{\mathfrak{I}}(d)$. It is a contradiction, so $t \in \mathfrak{I}$ or $d \in \mathfrak{I}$. Thus, $\mathfrak{I}$ is a PEWIID of $\mathfrak{S}$.

(2) Suppose that $\mathfrak{I}$ is a SPEWIID of $\mathfrak{S}$ and let $t \in \mathfrak{S}$. Then $\mathfrak{I}$ is an EWIID of $\mathfrak{S}$. By Theorem 5.6, $\chi_{\mathfrak{I}}$ is an EFWIID of $\mathfrak{S}$.
Case 1: If $t^2 \in \mathfrak{I}$, then $t \in \mathfrak{I}$. Thus, $\chi_{\mathfrak{I}}(t^2) = 1 = \chi_{\mathfrak{I}}(t)$. Hence, $\chi_{\mathfrak{I}}(t^2) \leq \chi_{\mathfrak{I}}(t)$.
Case 2: If $t^2 \notin \mathfrak{I}$, then $\chi_{\mathfrak{I}}(t^2) \leq \chi_{\mathfrak{I}}(t)$..
From two cases, we have $\chi_{\mathfrak{I}}$ is a SPEFWID of $\mathfrak{S}$.
Conversely, $\chi_{\mathfrak{I}}$ is a SPEFWIID of $\mathfrak{S}$. Then $\chi_{\mathfrak{I}}$ is an EFWIID of $\mathfrak{S}$. By Theorem 5.6, $\mathfrak{I}$ is an EWIID of $\mathfrak{S}$. Let $t \in \mathfrak{S}$ and $t^2 \in \mathfrak{I}$. Then $\chi_{\mathfrak{I}}(t^2) = 1$. If $t \notin \mathfrak{I}$, then $\chi_{\mathfrak{I}}(t) = 0$. Thus, $0 = \chi_{\mathfrak{I}}(t) \leq \chi_{\mathfrak{I}}(t^2) = 1$. By assumption, $\chi_{\mathfrak{I}}(t^2) \leq \chi_{\mathfrak{I}}(t)$. It is a contradiction, so $t \in \mathfrak{I}$. Thus, $\mathfrak{I}$ is a SPEWIID of $\mathfrak{S}$.

■

VI. Essential n-interior ideals and essential fuzzy n-interior ideals

In this section, we will study concepts of essential fuzzy n-interior ideals in a semigroup and properties of those.

Definition 6.1. *An* essential n-interior ideal *(E n-IID) $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ if $\mathfrak{I}$ is an n-IID of $\mathfrak{S}$ and $\mathfrak{I} \cap \mathfrak{J} \neq \emptyset$ for every n-IID $\mathfrak{J}$ of $\mathfrak{S}$.*

Theorem 6.2. *Let $\mathfrak{I}$ be an E n-IID of a semigroup $\mathfrak{S}$. If $\mathfrak{I}_1$ is an n-IID of $\mathfrak{S}$ with $\mathfrak{I} \subseteq \mathfrak{I}_1$, then $\mathfrak{I}_1$ is also an E n-IID of $\mathfrak{S}$.*

Proof: Suppose that $\mathfrak{I}_1$ is an n-IID of $\mathfrak{S}$ such that $\mathfrak{I} \subseteq \mathfrak{I}_1$ and let $\mathfrak{J}$ be any n-IID of $\mathfrak{S}$. Thus, $\mathfrak{I} \cap \mathfrak{J} \neq \emptyset$. Hence, $\mathfrak{I}_1 \cap \mathfrak{J} \neq \emptyset$. Therefore, $\mathfrak{I}_1$ is an E n-IID of $\mathfrak{S}$. ■

Theorem 6.3. *Let $\mathfrak{I}$ and $\mathfrak{J}$ be E n-IIDs of a semigroup $\mathfrak{S}$. Then $\mathfrak{I} \cup \mathfrak{J}$ and $\mathfrak{I} \cap \mathfrak{J}$ are E n-IIDs of $\mathfrak{S}$.*

Proof: Since $\mathfrak{I}$ and $\mathfrak{J}$ are E n-IIDs of $\mathfrak{S}$ we have $\mathfrak{I}$ and $\mathfrak{J}$ are ESSGs of a $\mathfrak{S}$. Thus by Theorem 3.8, $\mathfrak{I} \cup \mathfrak{J}$ and $\mathfrak{I} \cap \mathfrak{J}$ are ESSGs of $\mathfrak{S}$. Since $\mathfrak{I} \subseteq \mathfrak{I} \cup \mathfrak{J}$ and $\mathfrak{I}$ is an E n-IID we have $\mathfrak{I} \cup \mathfrak{J}$ is an E n-IID of $\mathfrak{S}$.

Let $\mathfrak{K}$ be an n-IID of $\mathfrak{S}$. Then $\mathfrak{I} \cap \mathfrak{K} \neq \emptyset$. Thus, there exists t, d and $w \in \mathfrak{I} \cap \mathfrak{K}$. Let t, d and $w \in \mathfrak{J}$. Then $tdw \in (\mathfrak{I} \cap \mathfrak{J}) \cap \mathfrak{K}$. Thus, $(\mathfrak{I} \cap \mathfrak{J}) \cap \mathfrak{K} \neq \emptyset$. Hence $\mathfrak{I} \cap \mathfrak{J}$ is an E n-IID of $\mathfrak{S}$. ■

Theorem 6.4. *EID of semigroup $\mathfrak{S}$ is an E n-IID of $\mathfrak{S}$.*

Proof: The proof is obvious. ■

Definition 6.5. *An* essential fuzzy n-interior ideal *(EF n-IID) ω of a semigroup $\mathfrak{S}$ if ω is a nonzero F n-IID of $\mathfrak{S}$ and $\omega \wedge \tau \neq 0$ for every nonzero F n-IID τ of $\mathfrak{S}$.*

Theorem 6.6. *Let ω be an EF n-IID of a semigroup $\mathfrak{S}$. If ω_1 is a F n-IID of $\mathfrak{S}$ such that $\omega \leq \omega_1$, then ω_1 is also an EF n-IID of $\mathfrak{S}$.*

Proof: Let ω_1 be a F n-IID of $\mathfrak{S}$ such that $\omega \leq \omega_1$ and let τ be any F n-IID of $\mathfrak{S}$. Thus, $\omega \wedge \tau \neq 0$. So $\omega_1 \wedge \tau \neq 0$. Hence, ω_1 is an EF n-IID of $\mathfrak{S}$. ■

Theorem 6.7. *Let ω_1 and ω_2 be EF n-IIDs of a semigroup $\mathfrak{S}$. Then $\omega_1 \vee \omega_2$ and $\omega_1 \wedge \omega_2$ are EF n-IIDs of $\mathfrak{S}$.*

Proof: Let ω_1 and ω_2 be EF n-IIDs of $\mathfrak{S}$. Then by Theorem 4.6, $\omega_1 \vee \omega_2$ is an EF n-IID of $\mathfrak{S}$. Since ω_1 and ω_2 are EF n-IIDs of $\mathfrak{S}$ we have ω_1 and ω_2 is an EFSSG of $\mathfrak{S}$. Thus $\omega_1 \wedge \omega_2$ is an EFSSG of $\mathfrak{S}$. Let τ be a nonzero F n-IID of $\mathfrak{S}$. Then $\omega_1 \wedge \tau \neq 0$. Thus there exists $t, d \in \mathfrak{S}$ such that $\omega_1(td) \neq 0$ and $\tau(td) \neq 0$. Since $\omega_2 \neq 0$ and let $w \in \mathfrak{S}$ such that $\omega_2(w) \neq 0$. Since ω_1 and ω_2 are FSSGs of $\mathfrak{S}$ we have $\omega_1(td^n w) \geq \omega_1(d) > 0$ and $\omega_2(tdw) \geq \omega_2(d) > 0$. Thus, $(\omega_1 \wedge \omega_2)(td_i^n w) = \omega_1(td_i^n w) \wedge \omega_2(td_i^n w) \neq 0$ where $i \in \{1, 2, ..., n\}$. Since τ is a FSSG of $\mathfrak{S}$ and $\tau(d) \neq 0$ we have $\tau(td_i^n w) \neq 0$ for all $t, d_i^n, w \in \mathfrak{S}$ where $i \in \{1, 2, ..., n\}$. Thus $[(\omega_1 \wedge \omega_2) \wedge \tau](td_1^n w) \neq 0$. Hence $[(\omega_1 \wedge \omega_2) \wedge \tau] \neq 0$. Therefore, $\omega_1 \wedge \omega_2$ is an EF n-IID of $\mathfrak{S}$. ■

Theorem 6.8. *Let $\mathfrak{I}$ be an n-IID of a semigroup $\mathfrak{S}$. Then $\mathfrak{I}$ is an E n-IID of $\mathfrak{S}$ if and only if $\chi_{\mathfrak{I}}$ is an EF n-IID of $\mathfrak{S}$.*

Proof: Suppose that $\mathfrak{I}$ is an E n-IID of $\mathfrak{S}$ and let ω be a nonzero F n-IID of $\mathfrak{S}$. Then by Theorem 2.3, $\operatorname{supp}(\omega)$ is a n-IID of $\mathfrak{S}$. Since $\mathfrak{I}$ is an E n-IID of $\mathfrak{S}$ we have $\mathfrak{I} \cap \operatorname{supp}(\omega) \neq \emptyset$. Thus, there exists $d_1, d_2, ..., d_n, t, w \in \mathfrak{I} \cap \operatorname{supp}(\omega)$ such that $\omega(td_1^n w) \neq 0$ and $\chi_{\mathfrak{I}}(td_1^n w) \neq 0$. So $(\omega \wedge \chi_{\mathfrak{I}})(td_1^n w) \neq 0$ implies that $\chi_{\mathfrak{I}} \wedge \omega \neq 0$. Therefore, $\chi_{\mathfrak{I}}$ is an essential F n-IID of $\mathfrak{S}$.

Conversely, assume that $\chi_{\mathfrak{I}}$ is an E n-FIID of $\mathfrak{S}$ and let $\mathfrak{J}$ be a n-IID of $\mathfrak{S}$. Then by Theorem 2.1, $\chi_{\mathfrak{J}}$ is a nonzero F n-IID of $\mathfrak{S}$. Since $\chi_{\mathfrak{I}}$ is an E n-FIID of $\mathfrak{S}$ we have $\chi_{\mathfrak{I}} \wedge \chi_{\mathfrak{J}} \neq 0$. Thus, by Lemma 2.2, $\chi_{\mathfrak{I} \cap \mathfrak{J}} \neq 0$. Hence, $\mathfrak{I} \cap \mathfrak{J} \neq \emptyset$. Therefore, $\mathfrak{I}$ is an E n-IID of $\mathfrak{S}$. ■

Theorem 6.9. *Let ω be a nonzero F n-IID of a semigroup $\mathfrak{S}$. Then ω is an EF n-IID of $\mathfrak{S}$ if and only if $\sup(\omega)$ is an E n-IID of $\mathfrak{S}$.*

Proof: Assume that ω is an EF n-IID of $\mathfrak{S}$. Then ω is a F n-IID of $\mathfrak{S}$. Thus, by Theorem 2.3, $\operatorname{supp}(\omega)$ is a n-IID of $\mathfrak{S}$. Let $\mathfrak{I}$ be an n-IID of $\mathfrak{S}$. Then by Theorem 6.8, $\chi_{\mathfrak{I}}$ is a F n-IID of $\mathfrak{S}$. By assumption, $\omega \wedge \chi_{\mathfrak{I}} \neq 0$. So, there exists $d_1, d_2, ..., d_n, t, w \in \mathfrak{S}$ such that $(\omega \wedge \chi_{\mathfrak{I}})(td_1^n w) \neq 0$. It implies that $\omega(td_1^n w) \neq 0$ and $\chi_{\mathfrak{I}}(td_1^n w) \neq 0$. Hence, $td_1^n w \in \operatorname{supp}(\omega) \cap \mathfrak{I}$ so $\operatorname{supp}(\omega) \cap \mathfrak{I} \neq \emptyset$. Therefore, $\operatorname{supp}(\omega)$ is an E n-IID of $\mathfrak{S}$.

Conversely, assume that $\operatorname{supp}(\omega)$ is an E n-IID of $\mathfrak{S}$. Then $\operatorname{supp}(\omega)$ is an n-IID of $\mathfrak{S}$. Let τ be a F n-IID of $\mathfrak{S}$. Then by Theorem 2.3, $\operatorname{supp}(\tau)$ is an n-IID of $\mathfrak{S}$. By assumption, $\operatorname{supp}(\omega) \cap \operatorname{supp}(\tau) \neq \emptyset$. So, there exists $d_1, d_2, ..., d_n, t, w \in \operatorname{supp}(\omega) \cap \operatorname{supp}(\tau)$, this implies that $\omega(td_1^n w) \neq 0$ and $\tau(td_1^n w) \neq 0$. Hence, $(\omega \wedge \tau)(td_1^n w) \neq 0$. Therefore, $\omega \wedge \tau \neq 0$. We conclude that ω is an EFn-IID of $\mathfrak{S}$. ■

Definition 6.10. *An E n-IID $\mathfrak{I}$ of a semigroup $\mathfrak{S}$ is called*
(1) *a* minimal *(MiE n-IID) if for every E n-IID of $\mathfrak{J}$ of $\mathfrak{S}$ such that $\mathfrak{J} \subseteq \mathfrak{I}$, we have $\mathfrak{J} = \mathfrak{I}$,*
(2) *a* maximal *(MaE n-IID) if for every E n-IID of $\mathfrak{J}$ of $\mathfrak{S}$ such that $\mathfrak{I} \subseteq \mathfrak{J}$, we have $\mathfrak{J} = \mathfrak{I}$,*

Definition 6.11. *An EF n-IID ω of a semigroup $\mathfrak{S}$ is called*
(1) *a* minimal *(MiEF n-IID) if for every EF n-IID of τ of $\mathfrak{S}$ such that $\tau \leq \omega$, we have* $\operatorname{supp}(\omega) = \operatorname{supp}(\tau)$,
(2) *a* maximal *(MaEF n-IID) if for every EF n-IID of τ of $\mathfrak{S}$ such that $\tau \geq \omega$, we have* $\operatorname{supp}(\tau) = \operatorname{supp}(\omega)$,

Theorem 6.12. *Let $\mathfrak{I}$ be a non-empty subset of a semigroup $\mathfrak{S}$. Then the following statement holds.*
(1) *$\mathfrak{I}$ is a MiE n-IID of $\mathfrak{S}$ if and only if $\chi_{\mathfrak{I}}$ is a MiEF n-IID of $\mathfrak{S}$,*
(2) *$\mathfrak{I}$ is a MaE n-IID of $\mathfrak{S}$ if and only if $\chi_{\mathfrak{I}}$ is a MaEF n-IID of $\mathfrak{S}$,*

Proof:
(1) Suppose that $\mathfrak{I}$ is a MiE n-IID of $\mathfrak{S}$. Then $\mathfrak{I}$ is an essential ideal of $\mathfrak{S}$. By Theorem 6.8, $\chi_{\mathfrak{I}}$ is an EF n-IID of $\mathfrak{S}$. Let τ be an EF n-IID of $\mathfrak{S}$ such that $\tau \leq \chi_{\mathfrak{I}}$. Then $\operatorname{supp}(\tau)$ is an E n-IID of $\mathfrak{S}$ such that $\operatorname{supp}(\tau) \subseteq \operatorname{supp}(\chi_{\mathfrak{I}})$. Thus, $\operatorname{supp}(\tau) \subseteq \operatorname{supp}(\chi_{\mathfrak{I}}) = \mathfrak{I}$. Hence, $\operatorname{supp}(\tau) \subseteq \mathfrak{I}$. By assumption, $\operatorname{supp}(\tau) = \mathfrak{I} = \operatorname{supp}(\chi_{\mathfrak{I}})$. Hence, $\chi_{\mathfrak{I}}$ is a MiEF n-IID of $\mathfrak{S}$.
Conversely, $\chi_{\mathfrak{I}}$ is a MiEF n-IID of $\mathfrak{S}$. Then $\chi_{\mathfrak{I}}$ is an EF n-IID of $\mathfrak{S}$. By Theorem 6.8, $\mathfrak{I}$ is an E n-IID of $\mathfrak{S}$. Let $\mathfrak{J}$ be an E n-IID of $\mathfrak{S}$ such that $\mathfrak{J} \subseteq \mathfrak{I}$. Then $\mathfrak{J}$ is an n-IID of $\mathfrak{S}$. Thus by Theorem 6.8, $\chi_{\mathfrak{J}}$ is an EF n-IID of $\mathfrak{S}$ such that $\chi_{\mathfrak{J}} \leq \chi_{\mathfrak{I}}$. Hence, $\mathfrak{J} = \operatorname{supp}(\chi_{\mathfrak{J}}) \subseteq \operatorname{supp}(\chi_{\mathfrak{I}}) = \mathfrak{I}$. By assumption, $\mathfrak{J} = \operatorname{supp}(\chi_{\mathfrak{J}}) = \mathfrak{I} = \operatorname{supp}(\chi_{\mathfrak{I}}) = \mathfrak{I}$. So, $\mathfrak{J} = \mathfrak{I}$. Hence, $\mathfrak{I}$ is a MiE n-IID of $\mathfrak{S}$.
(2) Suppose that $\mathfrak{I}$ is a MaE n-IID of $\mathfrak{S}$. Then $\mathfrak{I}$ is an E n-IID of $\mathfrak{S}$. By Theorem 6.8, $\chi_{\mathfrak{I}}$ is an EF n-IID of $\mathfrak{S}$. Let ω be an EF n-IID of $\mathfrak{S}$ such that $\chi_{\mathfrak{I}} \leq \omega$. Then $\operatorname{supp}(\omega)$ is an E n-ID of $\mathfrak{S}$ such that $\operatorname{supp}(\omega) \subseteq \operatorname{supp}(\chi_{\mathfrak{I}})$. Thus, $\operatorname{supp}(\omega) \subseteq \operatorname{supp}(\chi_{\mathfrak{I}}) = \mathfrak{I}$. Hence, $\operatorname{supp}(\omega) \subseteq \mathfrak{I}$. By assumption, $\operatorname{supp}(\mathfrak{I}) = \mathfrak{I} = \operatorname{supp}(\chi_{\mathfrak{I}})$. Hence, $\chi_{\mathfrak{I}}$ is a MaEF n-IID of $\mathfrak{S}$.
Conversely, $\chi_{\mathfrak{I}}$ is a MaEF n-IID of $\mathfrak{S}$. Then $\chi_{\mathfrak{I}}$ is an EF n-ID of $\mathfrak{S}$. By Theorem 6.8, $\mathfrak{I}$ is an E n-IID of $\mathfrak{S}$. Let $\mathfrak{J}$ be an E n-IID of $\mathfrak{S}$ such that $\mathfrak{I} \subseteq \mathfrak{J}$. Then $\mathfrak{J}$ is an n-IID of $\mathfrak{S}$. Thus by Theorem 6.8, $\chi_{\mathfrak{J}}$ is an EF n-IID of $\mathfrak{S}$ such that $\chi_{\mathfrak{I}} \leq \chi_{\mathfrak{J}}$. Hence, $\mathfrak{I} = \operatorname{supp}(\chi_{\mathfrak{I}}) \subseteq \operatorname{supp}(\chi_{\mathfrak{J}}) = \mathfrak{J}\mathfrak{J}$. By assumption, $\mathfrak{J} = \operatorname{supp}(\chi_{\mathfrak{J}}) = \mathfrak{J} = \operatorname{supp}(\chi_{\mathfrak{I}}) = \mathfrak{I}$. So, $\mathfrak{J} = \mathfrak{I}$. Hence, $\mathfrak{I}$ is a MaE n-IID of $\mathfrak{S}$.

■

Definition 6.13. *An E n-IID $\Im$ of a semigroup $\mathfrak{S}$ is called a* 0-minimal *(0-MiE n-IID) if for every EIID of $\mathfrak{J}$ of $\mathfrak{S}$ such that $\mathfrak{J} \subseteq \Im$, we have $\mathfrak{J} = \Im$,*

Definition 6.14. *An EF n-IID ω of a semigroup $\mathfrak{S}$ is called a* 0-minimal *(0-MiEF n-IID) if for every EFIID of τ of $\mathfrak{S}$ such that $\tau \leq \omega$, we have* $\operatorname{supp}(\tau) = \operatorname{supp}(\omega)$,

Theorem 6.15. *Let $\Im$ be a non-empty subset of a semigroup $\mathfrak{S}$. Then the following statement holds. $\Im$ is a 0-MiE n-IID of $\mathfrak{S}$ if and only if $\chi_\Im$ is a 0-MiEFIID of $\mathfrak{S}$.*

Proof: Suppose that $\Im$ is a 0-MiE n-IID of $\mathfrak{S}$. Then $\Im$ is an EIID of $\mathfrak{S}$. By Theorem 6.8, $\chi_\Im$ is an EF n-IID of $\mathfrak{S}$. Let τ be an EFIID of $\mathfrak{S}$ such that $\tau \leq \chi_\Im$. Then $\operatorname{supp}(\tau) \subseteq \operatorname{supp}(\chi_\Im)$. Thus, $\operatorname{supp}(\tau) \subseteq \operatorname{supp}(\chi_\Im) = \Im$. Hence, $\operatorname{supp}(\tau) \subseteq \Im$. Since τ is an EF n-IID of $\mathfrak{S}$ we have $\operatorname{supp}(\tau)$ is an E n-IID of $\mathfrak{S}$. By assumption, $\operatorname{supp}(\tau) = \Im = \operatorname{supp}(\chi_\Im)$. Hence, $\chi_\Im$ is a 0-MiEF n-IID of $\mathfrak{S}$.

Conversely, $\chi_\Im$ is a 0-MiEF n-IID of $\mathfrak{S}$. Then $\chi_\Im$ is an EF n-IID of $\mathfrak{S}$. By Theorem 4.9, $\Im$ is an E n-IID of $\mathfrak{S}$. Let $\mathfrak{J}$ be an E n-IID of $\mathfrak{S}$ such that $\mathfrak{J} \subseteq \Im$. Then $\mathfrak{J}$ is an n-IID of $\mathfrak{S}$. Thus by Theorem 6.8, $\chi_\mathfrak{J}$ is an EF n-IID of $\mathfrak{S}$ such that $\chi_\mathfrak{J} \leq \chi_\Im$. Hence, $\mathfrak{J} = \operatorname{supp}(\chi_\mathfrak{J}) \subseteq \operatorname{supp}(\chi_\Im) = \Im$. By assumption, $\mathfrak{J} = \operatorname{supp}(\chi_\mathfrak{J}) = \mathfrak{J} = \operatorname{supp}(\chi_\Im) = \Im$. So, $\mathfrak{J} = \Im$. Hence, $\Im$ is a 0-MiE n-IID of $\mathfrak{S}$. ■

Definition 6.16. *An E n-IID $\Im$ of a semigroup $\mathfrak{S}$. Then $\Im$ is said to be:*

(1) prime *(PE n-IID) if $td \in \Im$ implies $t \in \Im$ or $d \in \Im$, for all $t, d \in \mathfrak{S}$.*
(2) semiprime *(SPE n-IID) if $t^2 \in \Im$ implies $t \in \Im$, for all $t \in \mathfrak{S}$.*

Definition 6.17. *Let ω be an EF n-IID of a semigroup $\mathfrak{S}$. Then ω is said to be:*

(1) prime *(PEF n-IID) if $\omega(td) \leq \omega(t) \vee \omega(d)$ for all $t, d \in \mathfrak{S}$.*
(2) semiprime *(SPEF n-IID) if $\omega(t^2) \leq \omega(t)$ for all $t \in \mathfrak{S}$.*

It is clear, every PE n-IIDs is PEF n-IIDs in semigroups

Theorem 6.18. *Let $\Im$ be a non-empty subset of a semigroup $\mathfrak{S}$. Then the following statement holds.*

(1) *$\Im$ is a PE n-IID of $\mathfrak{S}$ if and only if $\chi_\Im$ is a PEF n-IID of $\mathfrak{S}$,*
(2) *$\Im$ is a SPE n-IID of $\mathfrak{S}$ if and only if $\chi_\Im$ is a SPEF n-IID of $\mathfrak{S}$,*

Proof:

(1) Suppose that $\Im$ is a PE n-IID of $\mathfrak{S}$ and Let $t, d \in \mathfrak{S}$. Then $\Im$ is an E n-IID of $\mathfrak{S}$. By Theorem 6.8, $\chi_\Im$ is an EF n-IID of $\mathfrak{S}$.
Case 1: If $td \in \Im$, then $t \in \Im$ or $d \in \Im$. Thus, $\chi_\Im(td) = 1 = \chi_\Im(t) = \chi_\Im(d)$. Hence, $\chi_\Im(td) \leq \chi_\Im(t) \vee \chi_\Im(d)$.
Case 2: If $td \notin \Im$, then $\chi_\Im(td) \leq \chi_\Im(t) \vee \chi_\Im(d)$.
From two cases, we have $\chi_\Im$ is a PEF n-IID of $\mathfrak{S}$.
Conversely, $\chi_\Im$ is a PEF n-IID of $\mathfrak{S}$. Then $\chi_\Im$ is an EF n-IID of $\mathfrak{S}$. By Theorem 6.8, $\Im$ is an E n-IID of $\mathfrak{S}$. Let $t, d \in \mathfrak{S}$ and $td \in \Im$. Then $\chi_\Im(td) = 1$.
If $t \notin \Im$ and $d \notin \Im$, then $\chi_\Im(t) = 0 = \chi_\Im(d)$. Thus, $0 = \chi_\Im(t) \vee \chi_\Im(d) \leq \chi_\Im(td) = 1$. By assumption, $\chi_\Im(td) \leq \chi_\Im(t) \vee \chi_\Im(d)$. It is a contradiction, so $t \in \Im$ or $d \in \Im$. Thus, $\Im$ is a PE n-IID of $\mathfrak{S}$.
(2) Suppose that $\Im$ is a SPE n-IID of $\mathfrak{S}$ and $t \in \mathfrak{S}$. Then $\Im$ is an E n-IID of $\mathfrak{S}$. By Theorem 6.8, $\chi_\Im$ is an EFn-IID of $\mathfrak{S}$.
Case 1: If $t^2 \in \Im$, then $t \in \Im$. Thus, $\chi_\Im(t^2) = 1 = \chi_\Im(t)$. Hence, $\chi_\Im(t^2) \leq \chi_\Im(t)$.
Case 2: If $t^2 \notin \Im$, then $\chi_\Im(t^2) \leq \chi_\Im(t)$.
From two cases, we have $\chi_\Im$ is a SPEF n-IID of $\mathfrak{S}$.
Conversely, $\chi_\Im$ is a SPEF n-IID of $\mathfrak{S}$. Then $\chi_\Im$ is an EF n-IID of $\mathfrak{S}$. By Theorem 6.8, $\Im$ is an E n-IID of $\mathfrak{S}$. Let $t \in \mathfrak{S}$ and $t^2 \in \Im$. Then $\chi_\Im(t^2) = 1$. If $t \notin \Im$, then $\chi_\Im(t) = 0$. Thus, $0 = \chi_\Im(t) \leq \chi_\Im(t^2) = 1$. By assumption, $\chi_\Im(t^2) \leq \chi_\Im(t)$. It is a contradiction, so $t \in \Im$. Thus, $\Im$ is a SPE n-IID of $\mathfrak{S}$.

■

VII. Conclusion

In Section IV, we define essential interior ideals and essential fuzzy interior ideals. We present that the union and intersection of essential interior ideals are also essential interior ideals of semigroups. Moreover, we prove some relationship between essential interior ideals and essential fuzzy interior ideals. In Section V, we define essential weakly interior ideals and essential fuzzy weakly interior ideals. We present that the union and intersection of essential interior ideals are also essential interior ideals of semigroups. Moreover, we prove some relationship between essential weakly interior ideals and essential weakly fuzzy interior ideals. In Section VI, we define essential n-interior ideals and essential fuzzy n-interior ideals. We present that the union and intersection of essential fuzzy n-interior ideals ideals are also essential n-interior ideals of semigroups. Moreover, we prove some relationship between essential n-interior ideals and essential fuzzy n-interior ideals. In the future work, we can discuss essential i-ideals and essential fuzzy i-ideals in n-ary semigroups and algebraic systems.

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