Asymptotic Stability of Interactional Genetic Regulatory Networks with Reaction-Diffusion Terms

Zhiqiang Lv, Chunyun Xu, Chengye Zou, Zhigang Xie

Abstract—**This paper investigates the asymptotic stability of interactional genetic regulatory networks with reaction-diffusion terms, considering Dirichlet boundary conditions. For both stable and unstable genetic regulatory network interaction models, a new Lyapunov–Krasovskii functional is proposed. By applying Jensen's inequality, using Green's second identity, and employing a reciprocally convex combination, a delay-dependent stability criterion is derived, which does not require upper bounds on the derivative of the delays. Numerical simulations are conducted to validate the accuracy and effectiveness of the proposed theory.**

Index Terms—Interactional genetic regulatory networks; Asymptotic stability; Time-varying delays; Dirichlet boundary

I. Introduction

The evolution of gene regulatory networks (GRNs) can be traced back to the long-standing research in the fields of molecular biology and genetics. Since the birth of genetics, scientists have extensively explored genes, the basic units of hereditary information. In examining how genes influence the traits of organisms, scientists have found that the levels and regulation of gene expression are closely connected to cellular functions and characteristics. In recent years, the study of GRNs has garnered growing interest from both mathematicians and biologists. To further understand the deeper, underlying changes in biological systems, it has become essential to model GRNs using various techniques. Currently, research on systems or networks [17,23,27,28] is increasing.

In the characterization of GRNs, Boolean model [6-9], Bayesian model [10-12], and differential equation model [14-16] are among the most commonly used approaches and have therefore received increasing attention from scholars. Among these, differential equation models provide a better description of the dynamic behavior of GRNs, making them a key focus in the field of biology.

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Zhiqiang Lv is a postgraduate of Department of Mathematics and Statistics, Changchun University of Technology, Changchun, 130012, China. (e-mail: L13183302618@163.com).

Chunyun Xu is an associate senior doctor of Department of Dermatology, Maternity and Child Health Hospital of Qinhuangdao, 066000, China. (Corresponding Author to provide e-mail: 164013089@qq.com)

Chengye Zou is a lecturer of school of Information Science and Engineering, Yanshan University, Oinhuangdao, 066004. China. (Corresponding Author to provide e-mail: zouchengye@ysu.edu.cn)

Zhigang Xie is a lecturer of school of Information Science and Engineering, Yanshan University, Oinhuangdao, 066004. China. (e-mail: xiezhigang@ysu.edu.cn).

In real GRNs, there is a time delays issue [1-4] due to factors such as the slower rates of cellular transcription and translation. Accordingly, it is imperative to comprehend the role of time lags and incorporate them into GRN models.

In earlier models of GRN structure, it was generally assumed that the concentrations of mRNA and proteins were evenly distributed in space, which does not accurately reflect the true dynamics. Since spatial diffusion is a widely occurring phenomenon, it is crucial to incorporate it into GRN models [5,13,14,16,22,23,24]. Ma [26] developed asymptotic stability criteria for GRNs with reaction-diffusion terms under the constraints of Dirichlet, Neumann, and Robin boundary conditions, shedding light on the significant impact of these terms on the system. Zhang [27] studied the oscillatory behavior of GRNs incorporating terms that describe reactions and diffusion, Song [24] and colleagues have developed a state estimation approach for genetic regulatory networks that incorporates reaction-diffusion processes based on sampled-data and Han [5] proposed a method for addressing upper bounds on delays in GRNs that include reaction-diffusion terms.

Biological processes are the outcome of complex interactions, rather than the sole product of a single genetic regulatory networks (GRNs). GRNs are interconnected and influenced by interactions with neighboring networks. However, most existing studies primarily focus on analyzing GRNs in isolation. In reality, many biological characteristics result from the combined effects of multiple GRNs, such as the interactions between viruses and hosts [19,21], tumors and organisms [18,20]. Therefore, studying interacting genetic regulatory networks is more practically relevant than studying single GRNs in isolation. Therefore, it is necessary to study the model of interacting gene regulatory networks (GRNs) with reaction-diffusion terms, but it should be noted that the stability criteria for the existing models of interacting genetic regulatory networks that incorporate reaction-diffusion components are similar to those in some other articles, are valid only when the maximum value of the delay derivative is confined to be below 1. To address this, we have formulated criteria for the time-delay-related asymptotic stability of certain gene regulatory networks (GRNs), featuring time-varying delays and reaction-diffusion components, under the constraints of Dirichlet boundary conditions. This achievement was accomplished by introducing a novel Lyapunov–Krasovskii functional and leveraging the application of Green's second identity along with the lemma of reciprocally convex combination. This paper makes the following two key contributions:

(i) Unlike previous studies, we have eliminated the constraint that the maximum value of the delay derivative is confined to be below 1.

(ii) Employing Green's second identity and the lemma on reciprocally convex combinations to address the terms introduced.

Notation

We now introduce some standard notations for use throughout this paper. For real symmetric matrices X and Y, $X > Y (X \geq Y)$ denotes that $X - Y$ is positive definite (or positive semi-definite), $*$ represents the symmetric part of a symmetric matrix, A^T represents the transpose of matrix A, Ω represents a compact set with a smooth boundary $\partial\Omega$ in the vector space R^n, and $mes\Omega$ is the measure of Ω.

II. Model Description and Preliminaries

Two distinct nonlinear delayed GRNs, as characterized by Eq. (1) and Eq. (2):

$$\begin{cases} \dfrac{dm_{1i}(t)}{dt} = -a_{1i} m_{1i}(t,x) + \sum_{j=1}^{n} \omega_{ij} f_j(p_{1j}(t-\sigma(t),x)) \\ \dfrac{dp_{1i}(t)}{dt} = -c_{1i} p_{1i}(t,x) + b_{1i} m_{1i}(t-\tau(t),x), i = 1,2,...,n_1 \end{cases} \tag{1}$$

$$\begin{cases} \dfrac{dm_{2u}(t)}{dt} = -a_{2u} m_{2u}(t,x) + \sum_{v=1}^{n} \varpi_{uv} g_v(p_{1v}(t-\sigma'(t),x)) \\ \dfrac{dp_{2u}(t)}{dt} = -c_{2u} p_{2u}(t,x) + b_{2u} m_{2u}(t-\tau'(t),x), u = 1,2,...,n_2 \end{cases} \tag{2}$$

$m_{1i}(t), p_{1i}(t) \in R^{n1}, m_{2u}(t), p_{2u}(t) \in R^{n2}$ represent the concentrations of mRNA and protein for nodes i and u at time t. a_{1i} and a_{2u} are the rates at which mRNA degrades, while c_{1i} and c_{2u} represent the rates at which protein degrades, b_{1i} and b_{2u} indicate the rates of translation, while $f_j(x)$ and $g_v(y)$ are the Hill form regulatory functions describing the protein's feedback influence on transcription, as outlined in equation (3)

$$\begin{cases} f_j(x) = \dfrac{(\frac{x}{m_j})^{H_j}}{1+(\frac{x}{m_j})^{H_j}} \\ g_v(y) = \dfrac{(\frac{y}{n_v})^{H_v}}{1+(\frac{y}{n_v})^{H_v}} \end{cases} \tag{3}$$

where H_j and H_v serve as the Hill coefficients, m_j and n_v are designated as positive constants, and $\tau(t)$, $\tau'(t)$, $\sigma(t)$, $\sigma'(t)$, are time-varying delay term satisfying

$$\begin{cases} 0 \leq \tau(t) \leq \bar{\tau}, & \dot{\tau}(t) \leq \mu_1, \\ 0 \leq \sigma(t) \leq \bar{\sigma}, & \dot{\sigma}(t) \leq \mu_2, \\ 0 \leq \tau'(t) \leq \bar{\tau}', & \dot{\tau}'(t) \leq \mu_3, \\ 0 \leq \sigma'(t) \leq \bar{\sigma}', & \dot{\sigma}'(t) \leq \mu_4, \end{cases} \tag{4}$$

where $W_1 = (\omega_{ij}) \in R^{n_1 \times n_1}$ and $W_2 = (\varpi_{uv}) \in R^{n_2 \times n_2}$ are represented by Eq. (4) and Eq. (5), respectively, α_{ij} and β_{uv} represent the dimensionless transcription rates for gene i by transcription factor j and for gene u by transcription factor v.

$$\omega_{ij} = \begin{cases} \alpha_{ij} & \text{if the factor j acts as an activator for gene i,} \\ 0 & \text{if there exists no connection from gene j to i,} \\ -\alpha_{ij} & \text{if the factor j acts as a repressor for gene i,} \end{cases} \tag{5}$$

$$\varpi_{uv} = \begin{cases} \beta_{uv} & \text{if the factor v acts as an activator for gene u,} \\ 0 & \text{if there exists no connection from gene v to u,} \\ -\beta_{uv} & \text{if the factor v acts as a repressor for gene u.} \end{cases} \tag{6}$$

Taking into account the diffusion term, Eq. (1) and Eq. (2) can be rewritten as

$$\begin{cases} \dfrac{dm_{1i}(t)}{dt} = \sum_{k=1}^{l} \dfrac{\partial}{\partial x_k}\left(D_{ik} \dfrac{\partial m_{1i}(t,x)}{\partial x_k}\right) - a_{1i} m_{1i}(t,x) \\ \qquad + \sum_{j=1}^{n} \omega_{ij} f_j(p_{1j}(t-\sigma(t),x)) \\ \dfrac{dp_{1i}(t)}{dt} = \sum_{k=1}^{l} \dfrac{\partial}{\partial x_k}\left(D_{ik}^* \dfrac{\partial p_{1i}(t,x)}{\partial x_k}\right) - c_{1i} p_{1i}(t,x) \\ \qquad + b_{1i} m_{1i}(t-\tau(t),x), i = 1,2,...,n_1 \end{cases} \tag{7}$$

$$\begin{cases} \dfrac{dm_{2u}(t)}{dt} = \sum_{k=1}^{l} \dfrac{\partial}{\partial x_k}\left(d_{uk} \dfrac{\partial m_{2u}(t,x)}{\partial x_k}\right) - a_{2u} m_{2u}(t,x) \\ \qquad + \sum_{v=1}^{n} \varpi_{uv} g_v(p_{1v}(t-\sigma'(t),x)) \\ \dfrac{dp_{2u}(t)}{dt} = \sum_{k=1}^{l} \dfrac{\partial}{\partial x_k}\left(d_{uk}^* \dfrac{\partial p_{2u}(t,x)}{\partial x_k}\right) - c_{2u} p_{2u}(t,x) \\ \qquad + b_{2u} m_{2u}(t-\tau'(t),x), u = 1,2,...,n_2 \end{cases} \tag{8}$$

where $x = [x_1\ x_2 ... x_l]^T \in \Omega \subset R^l$, $\Omega = \{x \| x_k | \leq L_k, k = 1,2,...,l\}$, L_k is constant, $D_{ik} = D_{ik}(t,x) > 0$ and $D_{ik}^* = D_{ik}^*(t,x) > 0$ represent the diffusion operators for mRNA and protein along the i th gene, respectively, while $d_{uk} = d_{uk}(t,x) > 0$ and $d_{uk}^* = d_{uk}^*(t,x) > 0$ represent the diffusion operators for mRNA and protein along the u th gene, respectively.

The initial condition is

$$\begin{cases} m_{1i}(s,x) = \psi_{1i}(s,x), & s \in (-\infty, 0], i = 1,2,...,n_1 \\ p_{1i}(s,x) = \psi_{1i}^*(s,x), & s \in (-\infty, 0], i = 1,2,...,n_1 \end{cases} \tag{9}$$

$$\begin{cases} m_{2u}(s,x) = \psi_{2u}(s,x), & s \in (-\infty, 0], u = 1,2,...,n_2 \\ p_{2u}(s,x) = \psi_{2u}^*(s,x), & s \in (-\infty, 0], u = 1,2,...,n_2 \end{cases} \tag{10}$$

Here, $\psi_{1i}(s,x)$, $\psi_{1i}^*(s,x), \psi_{2u}(s,x), \psi_{2u}^*(s,x)$ is bounded and continuous with respect to $(-\infty, 0] \times \Omega$.

Under Dirichlet boundary conditions

$$\begin{cases} m_{1i}(s,x) = 0, & x \in \partial\Omega, t \in [-K, +\infty), \\ p_{1i}(s,x) = 0, & x \in \partial\Omega, t \in [-K, +\infty), \end{cases} \tag{11}$$

$$\begin{cases} m_{2u}(s,x) = 0, & x \in \partial\Omega, t \in [-K, +\infty), \\ p_{2u}(s,x) = 0, & x \in \partial\Omega, t \in [-K, +\infty), \end{cases} \tag{12}$$

Systems (7) and (8) can be expressed in matrix notation as:

$$\begin{cases}\dfrac{dM_1(t,x)}{dt}=\sum_{k=1}^{l}\dfrac{\partial}{\partial x_k}\left(D_k\dfrac{\partial M_1(t,x)}{\partial x_k}\right)-A_1M_1(t,x)\\ \qquad +W_1F(P_1(t-\sigma(t),x))\\ \dfrac{dP_1(t,x)}{dt}=\sum_{k=1}^{l}\dfrac{\partial}{\partial x_k}\left(D_k^*\dfrac{\partial P_1(t,x)}{\partial x_k}\right)-C_1P_1(t,x)\\ \qquad +B_1M_1(t-\tau(t),x),\end{cases} \tag{13}$$

$$\begin{cases}\dfrac{dM_2(t,x)}{dt}=\sum_{k=1}^{l}\dfrac{\partial}{\partial x_k}\left(d_k\dfrac{\partial M_2(t,x)}{\partial x_k}\right)-A_2M_2(t,x)\\ \qquad +W_2G(P_2(t-\sigma'(t),x))\\ \dfrac{dP_2(t,x)}{dt}=\sum_{k=1}^{l}\dfrac{\partial}{\partial x_k}\left(d_k\dfrac{\partial M_2(t,x)}{\partial x_k}\right)-C_2P_2(t,x)\\ \qquad +B_2M_2(t-\tau'(t),x),\end{cases} \tag{14}$$

where

$A_1=\mathrm{diag}(a_{11},a_{12},...,a_{1n_1})$,

$B_1=\mathrm{diag}(b_{11},b_{12},...,b_{1n_1})$,

$C_1=\mathrm{diag}(c_{11},c_{12},...,c_{1n_1})$,

$A_2=\mathrm{diag}(a_{21},a_{22},...,a_{2n_2})$,

$B_2=\mathrm{diag}(b_{21},b_{22},...,b_{2n_2})$,

$C_2=\mathrm{diag}(c_{21},c_{22},...,c_{2n_2})$,

$D_k=\mathrm{diag}(D_{1k},D_{2k},...,D_{n_1k})$,

$D_k^*=\mathrm{diag}(D_{1k}^*,D_{2k}^*,...,D_{n_1k}^*)$,

$d_k=\mathrm{diag}(d_{1k},d_{2k},...,d_{n_2k})$,

$d_k^*=\mathrm{diag}(d_{1k}^*,d_{2k}^*,...,d_{n_2k}^*)$,

$M_1(t,x)=(m_{11}(t,x),m_{12}(t,x),...,m_{1n_1}(t,x))^T$,

$P_1(t,x)=(p_{11}(t,x),p_{12}(t,x),...,p_{1n_1}(t,x))^T$,

$M_2(t,x)=(m_{21}(t,x),m_{22}(t,x),...,m_{2n_2}(t,x))^T$,

$P_2(t,x)=(p_{21}(t,x),p_{22}(t,x),...,p_{2n_2}(t,x))^T$,

$$F(P_1(t-\sigma(t),x))=(f_1(p_{11}(t-\sigma(t),x)),f_2(p_{12}(t-\sigma(t),x)),\\ ...,f_n(p_{1n_1}(t-\sigma(t),x)))^T,$$

$$G(P_2(t-\sigma'(t),x))=(g_1(p_{21}(t-\sigma'(t),x)),g_2(p_{22}(t-\sigma'(t),x)),\\ ...,g_n(p_{2n_2}(t-\sigma'(t),x)))^T,$$

Since $f_i(\bullet)$ and $g_u(\bullet)$ are saturating monotonic increasing functions, $f_i(\bullet)$ and $g_u(\bullet)$ satisfy Ineq. (15) and Ineq. (16), respectively

$$0\le\frac{f_i(\kappa_i)}{\kappa_i}\le\xi_i,\quad \forall\kappa_i\ne0,i=1,2,...,n_1 \tag{15}$$

$$0\le\frac{g_u(\Upsilon_u)}{\Upsilon_u}\le\eta_u,\quad \forall\Upsilon_u\ne0,u=1,2,...,n_2 \tag{16}$$

i.e.

$$f^T(\kappa)(f(\kappa)-K_1\kappa)\le0 \tag{17}$$

$$g^T(\Upsilon)(g(\Upsilon)-K_2\Upsilon)\le0 \tag{18}$$

where

$K_1=diag(\xi_1,\xi_2,...,\xi_{n_1})>0$, $K_2=diag(\eta_1,\eta_2,...,\eta_{n_2})^T>0$, $\kappa=[\kappa_1,\kappa_2,...,\kappa_{n_1}]^T$ and $\Upsilon=[\Upsilon_1,\Upsilon_2,...,\Upsilon_{n_2}]^T$.

Lemma 1 Let $f(\mathbb{R})$ represent a function with real-valued on $[d,h]\subset R$, where $f(d)=f(h)=0$. If $f(\mathbb{R})\in C^1[d,h]$, then

$$\int_d^h f^2(\mathbb{R})dv\le\frac{(h-d)^2}{\pi^2}\int_d^h\left[f'(\mathbb{R})\right]^2dv \tag{19}$$

Lemma 2 Assume that Ω is an open set in R^n and $\varphi,\phi\in C^2(\overline{\Omega})$ with a C^1 boundary and is bounded , then

$$\int_\Omega\phi\Delta\varphi dx=\int_\Omega\phi\Delta\varphi dx+\int_{\partial\Omega}(\phi\frac{\partial\varphi}{\partial\overline{n}}-\varphi\frac{\partial\phi}{\partial\overline{n}})dS \tag{20}$$

where $\frac{\partial\varphi}{\partial\overline{n}}$ and $\frac{\partial\phi}{\partial\overline{n}}$ represent the directional derivatives of φ and ϕ, respectively, along the direction of the outward normal vector $\overline{n}$ to the surface element dS.

Lemma 3 According to Green's formula and under the constraints of Dirichlet boundary conditions, applying Lemmas 1 and Lemmas 2 yields the following Ineq.

$$2\int_\Omega m^T\sum_{k=1}^{l}\frac{\partial}{\partial x_k}(\frac{\partial m}{\partial x_k})dx\le-\frac{\pi^2}{2}\int_\Omega m^Tmdx \tag{21}$$

Lemma 4 $Z>0\in R^{n\times n}$, $\rho>0$ is a positive scalar, and x:$[0,\rho]\to R^n$ is a vector function such that the related integral is well-defined, and they exist:

$$\left(\int_0^\rho x(s)ds\right)^TZ\left(\int_0^\rho x(s)ds\right)\le\rho\left(\int_0^\rho x(s)Zx(s)ds\right) \tag{22}$$

Lemma 5 For every constant matrix $I^T=I>0$ of suitable size, any scalars h and l where $h<l$, and a vector function x:$[h,l]\to R^n$ such that the integrals below are well defined, the following Ineq. is satisfied:

$$\int_h^l x^T(s)dsI\int_h^l x(s)ds\le(h-l)\int_h^l x^T(s)Ix(s)ds \tag{23}$$

Lemma 6 For scalar ε, and vectors $\overline{X},\overline{Y}\in\mathrm{R}^n$ are any positive definite matrix, the following Ineq. is observed:

$$2\overline{X}^T\overline{Y}\le\varepsilon\overline{X}^T\overline{X}+\varepsilon^{-1}\overline{Y}^T\overline{Y} \tag{24}$$

Lemma 7 For any vector $\overline{X},\overline{Y}\in\mathrm{R}^n$ and scalar ε>0, an Ineq. of the following form is present:

$$2\overline{X}^TH\overline{Y}\le\varepsilon\overline{X}^TH\overline{X}+\varepsilon^{-1}\overline{Y}^TH\overline{Y} \tag{25}$$

Lemma 8 Let $Z_1>0$ and $Z_2>0$ are diagonal matrices; there exists:

$$\int_\Omega\frac{\partial m^T(s,x)}{\partial t}Z_1\sum_{k=1}^{l}\frac{\partial}{\partial x_k}(D_k\frac{\partial m(t,x)}{\partial x_k})dx\\ =\int_\Omega m^T(t,x)Z_1\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left[D_k\frac{\partial}{\partial x_k}(\frac{\partial m(t,x)}{\partial t})\right]dx \tag{26}$$

$$\int_\Omega\frac{\partial p^T(s,x)}{\partial t}Z_2\sum_{k=1}^{l}\frac{\partial}{\partial x_k}(D_k^*\frac{\partial p(t,x)}{\partial x_k})dx\\ =\int_\Omega p^T(t,x)Z_2\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left[D_k^*\frac{\partial}{\partial x_k}(\frac{\partial p(t,x)}{\partial t})\right]dx \tag{27}$$

Lemma 9 Let $h_1,h_2,...,h_N:R^m\to R$ all take on positive and finite values within an open set E of R^m. Consequently, the reciprocally convex combinations of h_i on E satisfy:

$$\min_{\{\beta_i:\beta_i>0,\sum_i\beta_i=1\}}\sum_i\frac{1}{\beta_i}h_i(t)=\sum_i h_i(t)+\max_{l_{i,j}(t)}\sum_{i\ne j}l_{i,j}(t) \tag{28}$$

subject to

$$l_{ij}: R^m \to R,\ l_{j,i}(t) = l_{i,j}(t),\ \begin{bmatrix} h_i(t) & l_{i,j}(t) \\ l_{i,j}(t) & h_j(t) \end{bmatrix} \geq 0 \ (29).$$

Definition 1 Let $C^1(X \times Y, R^n)$ be a Banach space that maps $X \times Y$ to R^n and is a function with continuous first-order derivatives. We define $\|\bullet\|$ and $\|\bullet\|_d$ as follows:

$$\| y(t,x) \| = (\int_\Omega y(t,x)^T y(t,x) dx)^{1/2}, \forall y(t,x) \in C^1((0,+\infty)\times\Omega, R^n)$$

$$\| z(t,x) \|_d = (\int_\Omega \sup_{-d\le t\le 0} z(t,x)^T z(t,x))^{1/2}, \forall z(t,x) \in C^1([-d,0]\times\Omega, R^n)$$

Definition 2 The trivial solution of Eq. (30) and Eq. (31) is asymptotically stable, if for any given $\varepsilon > 0$, there exist $\delta_1(\varepsilon) > 0$ and $\delta_2(\varepsilon) > 0$ such that:

(i) when the initial conditions are satisfied:

$$\|\psi_1(t,x)\|_d^2 \le \delta_1(\varepsilon), \quad \|\psi_1^*(t,x)\|_d^2 \le \delta_1(\varepsilon),$$

$$\|\psi_2(t,x)\|_d^2 \le \delta_2(\varepsilon), \quad \|\psi_2^*(t,x)\|_d^2 \le \delta_2(\varepsilon)$$

then

$$\| M_1(t,x) \|^2 \le \varepsilon, \quad \| P_1(t,x) \|^2 \le \varepsilon,$$

$$\| M_2(t,x) \|^2 \le \varepsilon, \quad \| P_2(t,x) \|^2 \le \varepsilon$$

(ii) $M_1(t,x) \to 0, P_1(t,x) \to 0, M_2(t,x) \to 0, P_2(t,x) \to 0$ when $t \to \infty$.

III. Model of Interactional GRNs

As stated by Eq. (13) and Eq. (14), W_1 or W_2 represents the interaction of expressed genes within a single GRN. Assuming the interactions between different GRNs are similar to those within a single GRN, consider the bidirectional coupling models (30) and (31) of time-delay GRNs incorporating reaction-diffusion terms, and investigate the stability criteria under Dirichlet boundary conditions.

$$\begin{cases} \dfrac{dM_1(t,x)}{dt} = \sum_{k=1}^{l} \dfrac{\partial}{\partial x_k}\left(D_k \dfrac{\partial M_1(t,x)}{\partial x_k}\right) - A_1 M_1(t,x) \\ \qquad + W_1 F(P_1(t-\sigma(t),x)) + W_1^* G(P_2(t-\sigma'(t),x) \\ \dfrac{dP_1(t,x)}{dt} = \sum_{k=1}^{l} \dfrac{\partial}{\partial x_k}\left(D_k^* \dfrac{\partial P_1(t,x)}{\partial x_k}\right) - C_1 P_1(t,x) \\ \qquad + B_1 M_1(t-\tau(t),x), \end{cases} \quad (30),$$

$$\begin{cases} \dfrac{dM_2(t,x)}{dt} = \sum_{k=1}^{l} \dfrac{\partial}{\partial x_k}(d_k \dfrac{\partial M_2(t,x)}{\partial x_k}) - A_2 M_2(t,x) \\ \qquad + W_2 G(P_2(t-\sigma'(t),x) + W_2^* F(P_1(t-\sigma(t),x) \\ \dfrac{dP_2(t,x)}{dt} = \sum_{k=1}^{l} \dfrac{\partial}{\partial x_k}(d_k^* \dfrac{\partial P_2(t,x)}{\partial x_k}) - C_2 P_2(t,x) \\ \qquad + B_2 M_2(t-\tau'(t),x), \end{cases} \quad (31).$$

here $W_1^* = (\omega_{iv}) \in R^{n_1 \times n_2}$, $W_2^* = (\varpi_{uj}^*) \in R^{n_2 \times n_1}$ is given by Eq. (32) and Eq. (33), α_{iv}^* and β_{uj}^* are the dimensionless transcription rates of transcription factor v for gene i in Eq. (30) and transcription factor u for gene j in Eq. (31), respectively:

$$\omega_{iv} = \begin{cases} \alpha_{iv}^* & \text{if the factor v acts as an activator for gene i,} \\ 0 & \text{if there exists no connection from gene v to i,} \\ -\alpha_{iv}^* & \text{if the factor v acts as a repressor for gene i,} \end{cases} \quad (32),$$

$$\varpi_{uj}^* = \begin{cases} \beta_{uj}^* & \text{if the factor j acts as an activator for gene u,} \\ 0 & \text{if there exists no connection from gene j to u,} \\ -\beta_{uj}^* & \text{if the factor j acts as a repressor for gene u.} \end{cases} \quad (33).$$

Theorem 1 For scalar $\tau(t)$, $\sigma(t)$, $\tau'(t)$, $\sigma'(t)$, $\bar{\tau}$, $\bar{\sigma}$, $\overline{\tau'}$, $\overline{\sigma'}$, μ_1, μ_2, μ_3, μ_4 that satisfies equation (4), if there exists a matrix $J_i^T = J_i > 0$, $\Lambda_i^T = \Lambda_i > 0$, $R_i^T = R_i > 0 (i = 1,...,4)$, $Q_j^T = Q_j > 0 (j = 1,...,10)$ a diagonal matrix $N_i^T = N_i > 0$, $(i = 1,...,4)$ and a matrix $G_i (i = 1,...,4)$ of suitable size so that the ensuing linear matrix inequality holds, then Eq. (30) and Eq. (31) are robustly stable under Dirichlet boundary conditions:

$$\begin{bmatrix} R_1 & G_1 \\ G_1^T & R_1 \end{bmatrix} \ge 0,\ \begin{bmatrix} R_2 & G_2 \\ G_2^T & R_2 \end{bmatrix} \ge 0,\ \begin{bmatrix} R_3 & G_3 \\ G_3^T & R_3 \end{bmatrix} \ge 0, \quad (34),$$

$$\begin{bmatrix} R_4 & G_4 \\ G_4^T & R_4 \end{bmatrix} \ge 0,\ \Xi_1 = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ * & \Xi_{13} \end{bmatrix} < 0,\ \Xi_2 = \begin{bmatrix} \Xi_{21} & \Xi_{22} \\ * & \Xi_{23} \end{bmatrix} < 0. \quad (35).$$

where

$$\Xi_{11} = \begin{bmatrix} \Pi_1 & G_1 & R_1 - G_1 & -N_1 A_1 & J_1 W_1 \\ * & \Pi_2 & R_1 - G_1 & 0 & 0 \\ * & * & \Pi_3 & 0 & 0 \\ * & * & * & \Pi_4 & N_1 W_1 \\ * & * & * & * & \Pi_5 \end{bmatrix}$$

$$\Xi_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & J_1 W_1^* \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_1 W_1^* \\ J_3 W_2^* & 0 & 0 & N_3 W_2^* & 0 \end{bmatrix}$$

$$\Xi_{13} = \begin{bmatrix} \Pi_6 & G_3 & R_3 - G_3 & -N_3 A_2 & 0 \\ * & \Pi_7 & R_3 - G_3 & 0 & 0 \\ * & * & \Pi_8 & 0 & 0 \\ * & * & * & \Pi_9 & N_3 W_2 \\ * & * & * & * & \Pi_{10} \end{bmatrix}$$

$$\Xi_{21} = \begin{bmatrix} \Omega_1 & G_2 & R_2 - G_2 & -N_2 C_1 & K_1 \Lambda_1 \\ * & \Omega_2 & R_2 - G_2 & 0 & 0 \\ * & * & \Omega_3 & 0 & 0 \\ * & * & * & \Omega_4 & 0 \\ * & * & * & * & \Omega_5 \end{bmatrix}$$

$$\Xi_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 & J_1 W_1^* \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Xi_{23} = \begin{bmatrix} \Omega_6 & G_4 & R_4 - G_4 & -N_4 C_2 & K_2 \Lambda_3 \\ * & \Omega_7 & R_4 - G_4 & 0 & 0 \\ * & * & \Omega_8 & 0 & 0 \\ * & * & * & \Omega_9 & 0 \\ * & * & * & * & \Omega_{10} \end{bmatrix}$$

$$\Pi_1 = -\frac{\pi^2}{2} J_1 D_L - 2J_1 A_1 + Q_1 + Q_2 - R_1$$
$$\Pi_2 = -Q_2 - R_1$$
$$\Pi_3 = (\mu_1 - 1)Q_1 - 2R_1 + G_1 + G_1^T + B_1^T J_2 B_1 + B_1^T N_2 B_1$$
$$\Pi_4 = \bar{\tau}^2 R_1 - 2N_1$$
$$\Pi_5 = (\mu_2 - 1)Q_5 - \Lambda_2$$
$$\Pi_6 = -\frac{\pi^2}{2} J_3 d_L - 2J_3 A_2 + Q_6 + Q_7 - R_3 + \varepsilon J_3 W_2$$
$$\Pi_7 = -Q_7 - R_3$$
$$\Pi_8 = (\mu_3 - 1)Q_6 - 2R_3 + G_3 + G_3^T + B_2^T J_4 B_2 + B_2^T N_4 B_2$$
$$\Pi_9 = \bar{\tau}'^2 R_3 - 2N_3$$
$$\Pi_{10} = (\mu_4 - 1)Q_{10} - \Lambda_4 + \frac{J_3 W_2}{\varepsilon}$$
$$\Omega_1 = -\frac{\pi^2}{2} J_2 D_L^* - 2J_2 C_1 + Q_3 + Q_4 - R_2 + J_2$$
$$\Omega_2 = -Q_4 - R_2$$
$$\Omega_3 = (\mu_2 - 1)Q_3 - 2R_2 + G_2 + G_2^T + K_1 \Lambda_2 K_1$$
$$\Omega_4 = \bar{\sigma}^2 R_2 - N_2$$
$$\Omega_5 = Q_5 - 2\Lambda_1$$
$$\Omega_6 = -\frac{\pi^2}{2} J_4 d_L^* - 2J_4 C_2 + Q_8 + Q_9 - R_4 + J_4$$
$$\Omega_7 = -Q_9 - R_4$$
$$\Omega_8 = (\mu_4 - 1)Q_8 - 2R_4 + G_4 + G_4^T + K_2 \Lambda_4 K_2$$
$$\Omega_9 = \bar{\sigma}'^2 R_4 - N_4$$
$$\Omega_{10} = Q_{10} - 2\Lambda_3$$

Proof Construct the Lyapunov-Krasovskii functional for the system (13) and (14):

$$V(t,M,P) = \sum_{i=1}^{8} V_i(t,M,P) \tag{36}$$

At this point, the following terms are introduced into our Lyapunov–Krasovskii function

$$V_0(t,M) = \sum_{k=1}^{l} \int_\Omega \frac{\partial M^T(t,x)}{\partial x_k} N_1 D_k \frac{\partial M(t,x)}{\partial x_k} dx$$

Where

$$\begin{aligned} V_1(t,M,P) &= \int_\Omega M_1^T(t,x) J_1 M_1(t,x) dx + \int_\Omega P_1^T(t,x) J_2 P_1(t,x) dx \\ &+ \sum_{k=1}^{l} \int_\Omega \frac{\partial M_1^T(t,x)}{\partial x_k} N_1 D_k \frac{\partial M_1(t,x)}{\partial x_k} dx \\ &+ \sum_{k=1}^{l} \int_\Omega \frac{\partial P_1^T(t,x)}{\partial x_k} N_2 D_k^* \frac{\partial P_1(t,x)}{\partial x_k} dx \end{aligned} \tag{37}$$

$$\begin{aligned} V_2(t,M,P) &= \int_\Omega \int_{t-\tau(t)}^{t} M_1^T(s,x) Q_1 M_1(s,x) ds dx \\ &+ \int_\Omega \int_{t-\bar{\tau}}^{t} M_1^T(s,x) Q_2 M_1(s,x) ds dx \\ &+ \int_\Omega \int_{t-\sigma(t)}^{t} P_1^T(s,x) Q_3 P_1(s,x) ds dx \\ &+ \int_\Omega \int_{t-\bar{\sigma}}^{t} P_1^T(s,x) Q_4 P_1(s,x) ds dx \end{aligned} \tag{38}$$

$$V_3(t,M,P) = \int_\Omega \int_{t-\sigma(t)}^{t} F^T(P_1(s,x)) Q_5 F(P_1(s,x)) ds dx \tag{39}$$

$$\begin{aligned} V_4(t,M,P) &= \bar{\tau} \int_\Omega \int_{-\bar{\tau}}^{0} \int_{t+\theta}^{t} \frac{\partial M_1^T(s,x)}{\partial s} R_1 \frac{\partial M_1(s,x)}{\partial s} ds d\theta dx \\ &+ \bar{\sigma} \int_\Omega \int_{-\bar{\sigma}}^{0} \int_{t+\theta}^{t} \frac{\partial P_1^T(s,x)}{\partial s} R_2 \frac{\partial P_1(s,x)}{\partial s} ds d\theta dx \end{aligned} \tag{40}$$

$$\begin{aligned} V_5(t,M,P) &= \int_\Omega M_2^T(t,x) J_3 M_2(t,x) dx + \int_\Omega P_2^T(t,x) J_4 P_2(t,x) dx \\ &+ \sum_{k=1}^{l} \int_\Omega \frac{\partial M_2^T(t,x)}{\partial x_k} N_3 d_k \frac{\partial M_2(t,x)}{\partial x_k} dx \\ &+ \sum_{k=1}^{l} \int_\Omega \frac{\partial P_2^T(t,x)}{\partial x_k} N_4 d_k^* \frac{\partial P_2(t,x)}{\partial x_k} dx \end{aligned} \tag{41}$$

$$\begin{aligned} V_6(t,M,P) &= \int_\Omega \int_{t-\tau'(t)}^{t} M_2^T(s,x) Q_6 M_2(s,x) ds dx \\ &+ \int_\Omega \int_{t-\bar{\tau}'}^{t} M_2^T(s,x) Q_7 M_2(s,x) ds dx \\ &+ \int_\Omega \int_{t-\sigma'(t)}^{t} P_2^T(s,x) Q_8 P_2(s,x) ds dx \\ &+ \int_\Omega \int_{t-\bar{\sigma}'}^{t} P_2^T(s,x) Q_9 P_2(s,x) ds dx \end{aligned} \tag{42}$$

$$V_7(t,M,P) = \int_\Omega \int_{t-\sigma'(t)}^{t} G^T(P_2(s,x)) Q_{10} G(P_2(s,x)) ds dx \tag{43}$$

$$\begin{aligned} V_8(t,M,P) &= \bar{\tau}' \int_\Omega \int_{-\bar{\tau}'}^{0} \int_{t+\theta}^{t} \frac{\partial M_2^T(s,x)}{\partial s} R_3 \frac{\partial M_2(s,x)}{\partial s} ds d\theta dx \\ &+ \bar{\sigma}' \int_\Omega \int_{-\bar{\sigma}'}^{0} \int_{t+\theta}^{t} \frac{\partial P_2^T(s,x)}{\partial s} R_4 \frac{\partial P_2(s,x)}{\partial s} ds d\theta dx \end{aligned} \tag{44}$$

Upon performing the differentiation of $V_i(t,M,P)$ $(i = 1,\ldots,8)$, we obtain:

$$\begin{aligned} \frac{\partial V_1(t,M,P)}{\partial t} &= 2\int_\Omega M_1^T(t,x) J_1 \Bigg[\sum_{k=1}^{l} \frac{\partial}{\partial x_k} \left(D_k \frac{\partial M_1(t,x)}{\partial x_k} \right) - A_1 M_1(t,x) \\ &+ W_1 F(P_1(t-\sigma(t),x)) + W_1^* G(P_2(t-\sigma'(t),x) \Bigg] dx \\ &+ 2\int_\Omega P_1^T(t,x) J_2 \Bigg[\sum_{k=1}^{l} \frac{\partial}{\partial x_k} \left(D_k^* \frac{\partial P_1(t,x)}{\partial x_k} \right) - C_1 P_1(t,x) \\ &+ B_1 M_1(t-\tau(t),x) \Bigg] dx \\ &+ 2\sum_{k=1}^{l} \int_\Omega \frac{\partial M_1^T(t,x)}{\partial x_k} N_1 D_k \frac{\partial}{\partial x_k} \left(\frac{\partial M_1(t,x)}{\partial x_k} \right) dx \\ &+ 2\sum_{k=1}^{l} \int_\Omega \frac{\partial P_1^T(t,x)}{\partial x_k} N_2 D_k^* \frac{\partial}{\partial x_k} \left(\frac{\partial P_1(t,x)}{\partial x_k} \right) dx \end{aligned} \tag{45}$$

$$\begin{aligned} \frac{\partial V_2(t,M,P)}{\partial t} &= \int_\Omega M_1^T(t,x)(Q_1 + Q_2) M_1(t,x) dx \\ &- \int_\Omega M_1^T(t-\bar{\tau},x) Q_2 M_1(t-\bar{\tau},x) dx \\ &- (1-\dot{\tau}(t)) \int_\Omega M_1^T(t-\tau(t),x) Q_1 M_1(t-\tau(t),x) dx \\ &+ \int_\Omega P_1^T(t,x)(Q_3 + Q_4) P_1(t,x) dx \\ &- \int_\Omega P_1^T(t-\bar{\sigma},x) Q_4 P_1(t-\bar{\sigma},x) dx \\ &- (1-\dot{\sigma}(t)) \int_\Omega P_1^T(t-\sigma(t),x) Q_3 P_1(t-\sigma(t),x) dx \end{aligned} \tag{46}$$

$$\frac{\partial V_3(t,M,P)}{\partial t}=\int_\Omega F^T(P_1(t,x))Q_5F(P_1(t,x))dx$$
$$-(1-\dot{\sigma}(t))\int_\Omega F^T(P_1(t-\sigma(t),x)Q_5F(P_1(t-\sigma(t),x))dx \tag{47}$$

$$\frac{\partial V_4(t,M,P)}{\partial t}=\bar{\tau}^2\int_\Omega \frac{\partial M_1^T(t,x)}{\partial t}R_1\frac{\partial M_1(t,x)}{\partial t}dx$$
$$-\bar{\tau}\int_\Omega\int_{t-\bar{\tau}}^t \frac{\partial M_1^T(s,x)}{\partial s}R_1\frac{\partial M_1(s,x)}{\partial s}dsdx$$
$$+\bar{\sigma}^2\int_\Omega \frac{\partial P_1^T(t,x)}{\partial t}R_2\frac{\partial P_1(t,x)}{\partial t}dx$$
$$-\bar{\sigma}\int_\Omega\int_{t-\bar{\sigma}}^t \frac{\partial P_1^T(s,x)}{\partial s}R_2\frac{\partial P_1(s,x)}{\partial s}dsdx \tag{48}$$

$$\frac{\partial V_5(t,M,P)}{\partial t}=2\int_\Omega M_2^T(t,x)J_3\left[\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(d_k\frac{\partial M_2(t,x)}{\partial x_k}\right)-A_2M_2(t,x)\right.$$
$$\left.+W_2G(P_2(t-\sigma'(t),x)+W_2^*F(P_1(t-\sigma(t),x))\right]dx$$
$$+2\int_\Omega P_2^T(t,x)J_4\left[\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(d_k^*\frac{\partial P_2(t,x)}{\partial x_k}\right)-C_2P_2(t,x)\right.$$
$$\left.+B_2M_2(t-\tau'(t),x)\right]dx$$
$$+2\sum_{k=1}^{l}\int_\Omega \frac{\partial M_2^T(t,x)}{\partial x_k}N_3d_k\frac{\partial}{\partial x_k}\left(\frac{\partial M_2(t,x)}{\partial x_k}\right)dx$$
$$+2\sum_{k=1}^{l}\int_\Omega \frac{\partial P_2^T(t,x)}{\partial x_k}N_4d_k^*\frac{\partial}{\partial x_k}\left(\frac{\partial P_2(t,x)}{\partial x_k}\right)dx \tag{49}$$

$$\frac{\partial V_6(t,M,P)}{\partial t}=\int_\Omega M_2^T(t,x)(Q_6+Q_7)M_2(t,x)dx$$
$$-\int_\Omega M_2^T(t-\bar{\tau}',x)Q_7M_2(t-\bar{\tau}',x)dx$$
$$-(1-\dot{\tau}'(t))\int_\Omega M_2^T(t-\tau'(t),x)Q_6M_2(t-\tau'(t),x)dx$$
$$+\int_\Omega P_2^T(t,x)(Q_8+Q_9)P_2(t,x)dx$$
$$-\int_\Omega P_2^T(t-\bar{\sigma}',x)Q_9P_2(t-\bar{\sigma}',x)dx$$
$$-(1-\dot{\sigma}'(t))\int_\Omega P_2^T(t-\sigma'(t),x)Q_8P_2(t-\sigma'(t),x)dx \tag{50}$$

$$\frac{\partial V_7(t,M,P)}{\partial t}=\int_\Omega G^T(P_2(t,x))Q_{10}G(P_2(t,x))dx$$
$$-(1-\dot{\sigma}'(t))\int_\Omega G^T(P_2(t-\sigma'(t),x)Q_{10}G(P_2(t-\sigma'(t),x))dx \tag{51}$$

$$\frac{\partial V_8(t,M,P)}{\partial t}=\bar{\tau}'^2\int_\Omega \frac{\partial M_2^T(t,x)}{\partial t}R_3\frac{\partial M_2(t,x)}{\partial t}dx$$
$$-\bar{\tau}'\int_\Omega\int_{t-\bar{\tau}'}^t \frac{\partial M_2^T(s,x)}{\partial s}R_3\frac{\partial M_2(s,x)}{\partial s}dsdx$$
$$+\bar{\sigma}'^2\int_\Omega \frac{\partial P_2^T(t,x)}{\partial t}R_4\frac{\partial P_2(t,x)}{\partial t}dx$$
$$-\bar{\sigma}'\int_\Omega\int_{t-\bar{\sigma}'}^t \frac{\partial P_2^T(s,x)}{\partial s}R_4\frac{\partial P_2(s,x)}{\partial s}dsdx \tag{52}$$

According to Lemma 3, there are:

$$2\int_\Omega M_1^T(t,x)J_1\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_k\frac{\partial M_1(t,x)}{\partial x_k}\right)dx \le -\frac{\pi^2}{2}\int_\Omega M_1^T(t,x)J_1D_LM_1(t,x)dx \tag{53}$$

$$2\int_\Omega P_1^T(t,x)J_2\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_k^*\frac{\partial P_1(t,x)}{\partial x_k}\right)dx \le -\frac{\pi^2}{2}\int_\Omega P_1^T(t,x)J_2D_L^*P_1(t,x)dx \tag{54}$$

$$2\int_\Omega M_2^T(t,x)J_3\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(d_k\frac{\partial M_2(t,x)}{\partial x_k}\right)dx \le -\frac{\pi^2}{2}\int_\Omega M_2^T(t,x)J_3d_LM_2(t,x)dx \tag{55}$$

$$2\int_\Omega P_2^T(t,x)J_4\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(d_k^*\frac{\partial P_2(t,x)}{\partial x_k}\right)dx \le -\frac{\pi^2}{2}\int_\Omega P_2^T(t,x)J_4D_L^*P_2(t,x)dx$$

(56).

where

$$\left(M_1^T(t,x)J_1D_k\frac{\partial M_1(t,x)}{\partial x_k}\right)_{k=1}^{L}=\left(M_1^T(t,x)J_1D_k\frac{\partial M_1(t,x)}{\partial x_1},\ \dots, M_1^T(t,x)J_1D_k\frac{\partial M_1(t,x)}{\partial x_L}\right)$$

$$\left(P_1^T(t,x)J_2D_k^*\frac{\partial P_1(t,x)}{\partial x_k}\right)_{k=1}^{L}=\left(P_1^T(t,x)J_2D_k^*\frac{\partial P_1(t,x)}{\partial x_1},\ \dots, P_1^T(t,x)J_2D_k^*\frac{\partial P_1(t,x)}{\partial x_L}\right)$$

$$\left(M_2^T(t,x)J_3d_k\frac{\partial M_2(t,x)}{\partial x_k}\right)_{k=1}^{L}=\left(M_2^T(t,x)J_3d_k\frac{\partial M_2(t,x)}{\partial x_1},\ \dots, M_2^T(t,x)J_3d_k\frac{\partial M_2(t,x)}{\partial x_L}\right)$$

$$\left(P_2^T(t,x)J_4d_k^*\frac{\partial P_2(t,x)}{\partial x_k}\right)_{k=1}^{L}=\left(P_2^T(t,x)J_4d_k^*\frac{\partial P_2(t,x)}{\partial x_1},\ \dots, P_2^T(t,x)J_4d_k^*\frac{\partial P_2(t,x)}{\partial x_L}\right)$$

From Lemma 5 and Lemma 9, we have:

$$-\bar{\tau}\int_\Omega\int_{t-\bar{\tau}}^t \frac{\partial M_1^T(s,x)}{\partial s}R_1\frac{\partial M_1(s,x)}{\partial s}dsdx$$
$$=-\bar{\tau}\int_\Omega\int_{t-\bar{\tau}}^{t-\tau(t)} \frac{\partial M_1^T(s,x)}{\partial s}R_1\frac{\partial M_1(s,x)}{\partial s}dsdx$$
$$-\bar{\tau}\int_\Omega\int_{t-\tau(t)}^{t} \frac{\partial M_1^T(s,x)}{\partial s}R_1\frac{\partial M_1(s,x)}{\partial s}dsdx$$
$$\le\int_\Omega -\upsilon_1^T\begin{bmatrix}R_1 & G_1\\ G_1^T & R_1\end{bmatrix}\upsilon_1^T dx \tag{57}$$

where

$$\upsilon_1=\left[M_1^T(t-\tau(t),x)-M_1^T(t-\bar{\tau},x)\quad M_1^T(t,x)-M_1^T(t-\tau(t),x)\right]^T$$

$$-\bar{\sigma}\int_\Omega\int_{t-\bar{\sigma}}^t \frac{\partial P_1^T(s,x)}{\partial s}R_2\frac{\partial P_1(s,x)}{\partial s}dsdx$$
$$=-\bar{\sigma}\int_\Omega\int_{t-\bar{\sigma}}^{t-\sigma(t)} \frac{\partial P_1^T(s,x)}{\partial s}R_2\frac{\partial P_1(s,x)}{\partial s}dsdx$$
$$-\bar{\sigma}\int_\Omega\int_{t-\sigma(t)}^{t} \frac{\partial P_1^T(s,x)}{\partial s}R_2\frac{\partial P_1(s,x)}{\partial s}dsdx$$
$$\le\int_\Omega -\upsilon_2^T\begin{bmatrix}R_2 & G_2\\ G_2^T & R_2\end{bmatrix}\upsilon_2^T dx \tag{58}$$

where

$$\upsilon_2=\left[P_1^T(t-\sigma(t),x)-P_1^T(t-\overline{\sigma},x)\quad P_1^T(t,x)-P_1^T(t-\sigma(t),x)\right]^T$$

$$\begin{aligned}&-\overline{\tau'}\int_\Omega\int_{t-\overline{\tau'}}^{t}\frac{\partial M_2^T(s,x)}{\partial s}R_3\frac{\partial M_2(s,x)}{\partial s}dsdx\\&=-\overline{\tau'}\int_\Omega\int_{t-\overline{\tau'}}^{t-\tau'(t)}\frac{\partial M_2^T(s,x)}{\partial s}R_3\frac{\partial M_2(s,x)}{\partial s}dsdx\\&\quad-\overline{\tau'}\int_\Omega\int_{t-\tau'(t)}^{t}\frac{\partial M_2^T(s,x)}{\partial s}R_3\frac{\partial M_2(s,x)}{\partial s}dsdx\\&\le\int_\Omega-\upsilon_3^T\begin{bmatrix}R_3&G_3\\G_3^T&R_3\end{bmatrix}\upsilon_3^Tdx\end{aligned}\tag{59}$$

where

$\upsilon_3=$

$$\left[M_2^T(t-\tau'(t),x)-M_2^T(t-\overline{\tau'},x)\quad M_2^T(t,x)-M_2^T(t-\tau'(t),x)\right]^T$$

$$\begin{aligned}&-\overline{\sigma'}\int_\Omega\int_{t-\overline{\sigma'}}^{t}\frac{\partial P_2^T(s,x)}{\partial s}R_4\frac{\partial P_2(s,x)}{\partial s}dsdx\\&=-\overline{\sigma'}\int_\Omega\int_{t-\overline{\sigma'}}^{t-\sigma'(t)}\frac{\partial P_2^T(s,x)}{\partial s}R_4\frac{\partial P_2(s,x)}{\partial s}dsdx\\&\quad-\overline{\sigma'}\int_\Omega\int_{t-\sigma'(t)}^{t}\frac{\partial P_2^T(s,x)}{\partial s}R_4\frac{\partial P_2(s,x)}{\partial s}dsdx\\&\le\int_\Omega-\upsilon_4^T\begin{bmatrix}R_4&G_4\\G_4^T&R_4\end{bmatrix}\upsilon_4^Tdx\end{aligned}\tag{60}$$

where

$\upsilon_4=$

$$\left[P_2^T(t-\sigma'(t),x)-P_2^T(t-\overline{\sigma'},x)\quad P_2^T(t,x)-P_2^T(t-\sigma'(t),x)\right]^T$$

For diagonal matrices N_1, N_2, N_3, N_4 are all positive definite, it can be seen that:

$$2\int_\Omega\frac{\partial M_1^T(s,x)}{\partial t}N_1\left[\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_k\frac{\partial M_1(t,x)}{\partial x_k}\right)-A_1M_1(t,x)+W_1F(P_1(t-\sigma(t),x)-\frac{\partial M_1(t,x)}{\partial t}\right]dx=0\tag{61}$$

$$2\int_\Omega\frac{\partial P_1^T(s,x)}{\partial t}N_2\left[\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_k^*\frac{\partial P_1(t,x)}{\partial x_k}\right)-C_1P_1(t,x)+B_1M_1(t-\tau(t),x)-\frac{\partial P_1(t,x)}{\partial t}\right]dx=0\tag{62}$$

$$2\int_\Omega\frac{\partial M_2^T(s,x)}{\partial t}N_3\left[\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(d_k\frac{\partial M_2(t,x)}{\partial x_k}\right)-A_2M_2(t,x)+W_2G(P_2(t-\sigma'(t),x)-\frac{\partial M_2(t,x)}{\partial t}\right]dx=0\tag{63}$$

$$2\int_\Omega\frac{\partial P_2^T(s,x)}{\partial t}N_4\left[\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(d_k^*\frac{\partial P_2(t,x)}{\partial x_k}\right)-C_2P_2(t,x)+B_2M_2(t-\tau'(t),x)-\frac{\partial P_2(t,x)}{\partial t}\right]dx=0\tag{64}$$

From Lemma 8, the imposition of Dirichlet boundary conditions and Green formula, we deduce that:

$$\begin{aligned}&2\int_\Omega\frac{\partial M_1^T(s,x)}{\partial t}N_1\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_k\frac{\partial M_1(t,x)}{\partial x_k}\right)dx\\&=2\int_\Omega M_1^T(t,x)N_1\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left[D_k\frac{\partial}{\partial x_k}\left(\frac{\partial M_1(t,x)}{\partial t}\right)\right]dx\\&=-2\sum_{k=1}^{l}\int_\Omega\frac{\partial M_1^T(t,x)}{\partial x_k}N_1D_k\frac{\partial}{\partial x_k}\left(\frac{\partial M_1(t,x)}{\partial t}\right)dx\end{aligned}\tag{65}$$

Similarly

$$\begin{aligned}&2\int_\Omega\frac{\partial P_1^T(s,x)}{\partial t}N_2\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_k^*\frac{\partial P_1(t,x)}{\partial x_k}\right)dx\\&=2\int_\Omega P_1^T(t,x)N_2\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left[D_k^*\frac{\partial}{\partial x_k}\left(\frac{\partial P_1(t,x)}{\partial t}\right)\right]dx\\&=-2\sum_{k=1}^{l}\int_\Omega\frac{\partial P_1^T(t,x)}{\partial x_k}N_2D_k^*\frac{\partial}{\partial x_k}\left(\frac{\partial P_1(t,x)}{\partial t}\right)dx\end{aligned}\tag{66}$$

$$\begin{aligned}&2\int_\Omega\frac{\partial M_2^T(s,x)}{\partial t}N_3\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(d_k\frac{\partial M_2(t,x)}{\partial x_k}\right)dx\\&=2\int_\Omega M_2^T(t,x)N_3\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left[d_k\frac{\partial}{\partial x_k}\left(\frac{\partial M_2(t,x)}{\partial t}\right)\right]dx\\&=-2\sum_{k=1}^{l}\int_\Omega\frac{\partial M_2^T(t,x)}{\partial x_k}N_3d_k\frac{\partial}{\partial x_k}\left(\frac{\partial M_2(t,x)}{\partial t}\right)dx\end{aligned}\tag{67}$$

$$\begin{aligned}&2\int_\Omega\frac{\partial P_2^T(s,x)}{\partial t}N_4\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(d_k^*\frac{\partial P_2(t,x)}{\partial x_k}\right))dx\\&=2\int_\Omega P_2^T(t,x)N_4\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left[d_k^*\frac{\partial}{\partial x_k}\left(\frac{\partial P_2(t,x)}{\partial t}\right)\right]dx\\&=-2\sum_{k=1}^{l}\int_\Omega\frac{\partial P_2^T(t,x)}{\partial x_k}N_4d_k^*\frac{\partial}{\partial x_k}\left(\frac{\partial P_2(t,x)}{\partial t}\right)dx\end{aligned}\tag{68}$$

From Ineq. (17) and Ineq. (18), for diagonal matrices $\Lambda_1>0$, $\Lambda_2>0$, $\Lambda_3>0$, $\Lambda_4>0$,the following Ineqs. holds:

$$2F^T(P_1(t,x))\Lambda_1F(P_1(t,x))-2P_1(t,x)K_1\Lambda_1F(P_1(t,x))\le0\tag{69}$$

$$\begin{aligned}&2F^T(P_1(t-\sigma(t),x)\Lambda_2F(P_1(t-\sigma(t),x))\\&-2P_1^T(t-\sigma(t),x)K_1\Lambda_2F(P_1(t-\sigma(t),x)\le0\end{aligned}\tag{70}$$

$$2G^T(P_2(t,x))\Lambda_3G(P_2(t,x))-2P_2(t,x)K_2\Lambda_3G(P_2(t,x))\le0\tag{71}$$

$$\begin{aligned}&2G^T(P_2(t-\sigma'(t),x)\Lambda_4G(P_2(t-\sigma'(t),x)\\&-2P_2^T(t-\sigma'(t),x)K_2\Lambda_4G(P_2(t-\sigma'(t),x)\le0\end{aligned}\tag{72}$$

From Lemma 6, the subsequent Ineqs. can be derived:

$$\begin{aligned}&2P_1^T(t-\sigma(t),x)K_1\Lambda_2F(P_1(t-\sigma(t),x))\\&\le P_1^T(t-\sigma(t),x)K_1\Lambda_2K_1P_1(t-\sigma(t),x)\\&\quad+F^T(P_1(t-\sigma(t),x))\Lambda_2F(P_1(t-\sigma(t),x))\end{aligned}\tag{73}$$

$$\begin{aligned}&2P_2^T(t-\sigma'(t),x)K_2\Lambda_4G(P_2(t-\sigma'(t),x))\\&\le P_2^T(t-\sigma'(t),x)K_2\Lambda_4K_2P_2(t-\sigma'(t),x)\\&\quad+G^T(P_2(t-\sigma'(t),x))\Lambda_4G(P_2(t-\sigma'(t),x))\end{aligned}\tag{74}$$

$$\begin{aligned}&2P_1^T(t,x)J_2B_1M_1(t-\tau(t),x)\\&\le P_1^T(t,x)J_2P_1(t,x)+M_1(t-\tau(t),x)B_1^TJ_2B_1M_1(t-\tau(t),x)\end{aligned}\tag{75}$$

$$2P_2^T(t,x)J_4B_2M_2(t-\tau'(t),x)$$
$$\le P_2^T(t,x)J_4P_2(t,x)+M_2(t-\tau'(t),x)B_2^TJ_4B_2M_2(t-\tau'(t),x) \quad (76),$$

$$2\frac{\partial P_1^T(t,x)}{\partial t}N_2B_1M_1(t-\tau(t),x)$$
$$\le \frac{\partial P_1^T(t,x)}{\partial t}N_2\frac{\partial P_1(t,x)}{\partial t}+M_1(t-\tau(t),x)B_1^TN_2B_1M_1(t-\tau(t),x) \quad (77),$$

$$2\frac{\partial P_2^T(t,x)}{\partial t}N_4B_2M_2(t-\tau'(t),x)$$
$$\le \frac{\partial P_2^T(t,x)}{\partial t}N_4\frac{\partial P_2(t,x)}{\partial t}+M_2(t-\tau'(t),x)B_2^TN_4B_2M_2(t-\tau'(t),x) \quad (78).$$

ε is a positive scalar and from Lemma 7, it is known that there exists:

$$2M_2(t,x)J_3W_2G(P_2(t-\sigma'(t),x)$$
$$\le \varepsilon M_2^T(t,x)J_3W_2M_2(t,x)+\frac{1}{\varepsilon}G^T(P_2(t-\sigma'(t),x)J_3W_2G(P_2(t-\sigma'(t),x) \quad (79).$$

By combining Eq. (45) to Eq. (79), we can obtain:

$$\frac{\partial V(t,M,P)}{\partial t}=\sum_{i=1}^{8}\frac{\partial V_i(t,M,P)}{\partial t}$$
$$\le \int_\Omega\left[\chi_1^T(t,x)\Xi_1\chi_1(t,x)+\chi_2^T(t,x)\Xi_2\chi_2(t,x)\right]dx$$
$$\le -\lambda_{1\min}(-\Xi_1)(\|M_1(t,x)\|^2+\|P_1(t,x)\|^2) \quad (80),$$
$$-\lambda_{2\min}(-\Xi_2)(\|M_2(t,x)\|^2+\|P_2(t,x)\|^2)$$
$$\le 0$$

where

$$\chi_1=\Big[M_1^T(t,x),M_1^T(t-\bar{\tau},x),M_1^T(t-\tau(t),x),\frac{\partial M_1^T(t,x)}{\partial t},F^T(P_1(t-\sigma(t),x)),$$
$$M_2^T(t,x),M_2^T(t-\bar{\tau}',x),M_2^T(t-\tau'(t),x),\frac{\partial M_2^T(t,x)}{\partial t},G^T(P_2(t-\sigma'(t),x))\Big]^T$$

$$\chi_2=\Big[P_1^T(t,x),P_1^T(t-\bar{\sigma},x),P_1^T(t-\sigma(t),x),\frac{\partial P_1^T(t,x)}{\partial t},F^T(P_1(t,x)),$$
$$P_2^T(t,x),P_2^T(t-\bar{\sigma}',x),P_2^T(t-\sigma'(t),x),\frac{\partial P_2^T(t,x)}{\partial t},G^T(P_2(t,x))\Big]^T$$

From Eq. (80), it can be seen

$$\frac{\partial V(t,M,P)}{\partial t}\le -\lambda_{1\min}(-\Xi_1)\|M_1(t,x)\|^2$$
$$\frac{\partial V(t,M,P)}{\partial t}\le -\lambda_{1\min}(-\Xi_1)\|P_1(t,x)\|^2$$
$$\frac{\partial V(t,M,P)}{\partial t}\le -\lambda_{2\min}(-\Xi_2)\|M_2(t,x)\|^2$$
$$\frac{\partial V(t,M,P)}{\partial t}\le -\lambda_{2\min}(-\Xi_2)\|P_2(t,x)\|^2$$

By integrating the aforementioned Ineqs., we derive:

$$\int_0^t\frac{\partial V(s,M,P)}{\partial s}ds\le\int_0^t-\lambda_{1\min}(-\Xi_1)\|M_1(s,x)\|^2ds$$
$$\int_0^t\frac{\partial V(s,M,P)}{\partial s}ds\le\int_0^t-\lambda_{1\min}(-\Xi_1)\|P_1(s,x)\|^2ds$$
$$\int_0^t\frac{\partial V(s,M,P)}{\partial s}ds\le\int_0^t-\lambda_{2\min}(-\Xi_2)\|M_2(s,x)\|^2ds$$
$$\int_0^t\frac{\partial V(s,M,P)}{\partial s}ds\le\int_0^t-\lambda_{2\min}(-\Xi_2)\|P_2(s,x)\|^2ds$$

Then

$$V(t,M,P)\le\int_0^t-\lambda_{1\min}(-\Xi_1)\|M_1(s,x)\|^2ds+V(0,M(0,x),P(0,x)) \quad (81),$$
$$V(t,M,P)\le\int_0^t-\lambda_{1\min}(-\Xi_1)\|P_1(s,x)\|^2ds+V(0,M(0,x),P(0,x)) \quad (82),$$
$$V(t,M,P)\le\int_0^t-\lambda_{2\min}(-\Xi_2)\|M_2(s,x)\|^2ds+V(0,M(0,x),P(0,x)) \quad (83),$$
$$V(t,M,P)\le\int_0^t-\lambda_{2\min}(-\Xi_2)\|P_2(s,x)\|^2ds+V(0,M(0,x),P(0,x)) \quad (84).$$

Therefore

$\|M_1(t,x)\|^2\to 0$, $\|P_1(t,x)\|^2\to 0$,

$\|M_2(t,x)\|^2\to 0$, $\|P_2(t,x)\|^2\to 0(t\to\infty)$

From Ineqs. (81)-(84), it can be observed that

$$V(t,M,P)\le V(0,M(0,x),P(0,x)) \quad (85).$$

For ψ_{1i} , ψ_{1i}^* , ψ_{2u} , ψ_{2u}^* in Eq. (9) and Eq. (10), there are non-negative real numbers that exist ϑ , ϑ^* , ζ , ζ^* , δ , δ^* , γ , γ^* , such that

$$\left|\frac{\partial\psi_{1i}(s,x)}{\partial t}\right|\le\vartheta,\quad\left|\frac{\partial\psi_{1i}(s,x)}{\partial x_k}\right|\le\vartheta^*$$
$$\left|\frac{\partial\psi_{1i}^*(s,x)}{\partial t}\right|\le\zeta,\quad\left|\frac{\partial\psi_{1i}^*(s,x)}{\partial x_k}\right|\le\zeta^*$$
$$\left|\frac{\partial\psi_{2u}(s,x)}{\partial t}\right|\le\delta,\quad\left|\frac{\partial\psi_{2u}(s,x)}{\partial x_k}\right|\le\delta^*$$
$$\left|\frac{\partial\psi_{2u}^*(s,x)}{\partial t}\right|\le\gamma,\quad\left|\frac{\partial\psi_{2u}^*(s,x)}{\partial x_k}\right|\le\gamma^*$$

Hence

$$\sum_{i=1}^{n_1}\int_\Omega N_{1i}\sum_{k=1}^{l}D_{ik}(\frac{\partial m_{1i}(0,x)}{\partial x_k})^2dx\le mes(\Omega)\sum_{i=1}^{n_1}\sum_{k=1}^{l}N_{1i}D_{ik}(\vartheta^*)^2$$

and

$$\bar{\tau}\int_\Omega\int_{-\bar{\tau}}^0\int_\theta^0\frac{\partial M_1^T(s,x)}{\partial s}R_1\frac{\partial M_1(s,x)}{\partial s}dsd\theta dx\le\frac{1}{2}\lambda_{1\max}(R_1)\vartheta^2\bar{\tau}^3mes(\Omega)$$

Similarly

$$\sum_{i=1}^{n_1}\int_\Omega N_{2i}\sum_{k=1}^{l}D_{ik}^*(\frac{\partial p_{1i}(0,x)}{\partial x_k})^2dx\le mes(\Omega)\sum_{i=1}^{n_1}\sum_{k=1}^{l}N_{2i}D_{ik}^*(\zeta^*)^2$$

and

$$\bar{\sigma}\int_\Omega\int_{-\bar{\sigma}}^0\int_\theta^0\frac{\partial P_1^T(s,x)}{\partial s}R_2\frac{\partial P_1(s,x)}{\partial s}dsd\theta dx\le\frac{1}{2}\lambda_{1\max}(R_2)\zeta^2\bar{\sigma}^3mes(\Omega)$$

$$\sum_{u=1}^{n_2}\int_\Omega N_{3u}\sum_{k=1}^{l}d_{uk}(\frac{\partial m_{2u}(0,x)}{\partial x_k})^2dx\le mes(\Omega)\sum_{u=1}^{n_2}\sum_{k=1}^{l}N_{3u}d_{uk}(\delta^*)^2$$

and

$$\bar{\tau}'\int_\Omega\int_{-\bar{\tau}'}^0\int_\theta^0\frac{\partial M_2^T(s,x)}{\partial s}R_3\frac{\partial M_2(s,x)}{\partial s}dsd\theta dx\le\frac{1}{2}\lambda_{2\max}(R_3)\delta^2\bar{\tau}'^3mes(\Omega)$$

$$\sum_{u=1}^{n_2}\int_\Omega N_{4u}\sum_{k=1}^{l}d_{uk}^*(\frac{\partial p_{2u}(0,x)}{\partial x_k})^2dx\le mes(\Omega)\sum_{u=1}^{n_2}\sum_{k=1}^{l}N_{4u}d_{uk}^*(\gamma^*)^2$$

and

$$\bar{\sigma}'\int_\Omega\int_{-\bar{\sigma}'}^0\int_\theta^0\frac{\partial P_2^T(s,x)}{\partial s}R_4\frac{\partial P_2(s,x)}{\partial s}dsd\theta dx\le\frac{1}{2}\lambda_{2\max}(R_4)\gamma^2\bar{\sigma}'^3mes(\Omega)$$

It is easy to see that there exist non-negative non-complex numbers M_1, M_2, M_3, M_4, such that the following Ineqs. holds

$$mes(\Omega)(\sum_{i=1}^{n_1}\sum_{k=1}^{l}N_{1i}D_{ik}(\vartheta^*)^2+\frac{1}{2}\lambda_{1\max}(R_1)\vartheta^2\bar{\tau}^3 mes(\Omega))=M_1\left\|\psi_1(t,x)\right\|_d^2$$

$$mes(\Omega)(\sum_{i=1}^{n_1}\sum_{k=1}^{l}N_{2i}D_{ik}^*(\zeta^*)^2+\frac{1}{2}\lambda_{1\max}(R_2)\zeta^2\bar{\sigma}^3 mes(\Omega))=M_2\left\|\psi_1^*(t,x)\right\|_d^2$$

$$mes(\Omega)(\sum_{u=1}^{n_2}\sum_{k=1}^{l}N_{3u}d_{uk}(\delta^*)^2+\frac{1}{2}\lambda_{2\max}(R_3)\delta^2\bar{\tau}'^3 mes(\Omega))=M_3\left\|\psi_2(t,x)\right\|_d^2$$

$$mes(\Omega)(\sum_{u=1}^{n_2}\sum_{k=1}^{l}N_{4u}d_{uk}^*(\gamma^*)^2+\frac{1}{2}\lambda_{2\max}(R_4)\gamma^2\bar{\sigma}'^3 mes(\Omega))=M_4\left\|\psi_2^*(t,x)\right\|_d^2$$

From these, it follows that

$$\begin{aligned}
V(0,M(0,x),P(0,x)) &= \int_\Omega M_1^T(0,x)J_1M_1(0,x)dx \\
&+\int_\Omega P_1^T(0,x)J_2P_1(0,x)dx \\
&+\sum_{i=1}^{n_1}\int_\Omega N_{1i}\sum_{k=1}^{l}D_{ik}\left(\frac{\partial m_{1i}(0,x)}{\partial x_k}\right)^2dx \\
&+\sum_{i=1}^{n_1}\int_\Omega N_{2i}\sum_{k=1}^{l}D_{ik}^*\left(\frac{\partial p_{1i}(0,x)}{\partial x_k}\right)^2dx \\
&+\int_\Omega\int_{-\tau(0)}^{0}M_1^T(s,x)Q_1M_1(s,x)dsdx \\
&+\int_\Omega\int_{-\bar{\tau}}^{0}M_1^T(s,x)Q_2M_1(s,x)dsdx \\
&+\int_\Omega\int_{-\sigma(0)}^{0}P_1^T(s,x)Q_3P_1(s,x)dsdx \\
&+\int_\Omega\int_{-\bar{\sigma}}^{0}P_1^T(s,x)Q_4P_1(s,x)dsdx \\
&+\int_\Omega\int_{-\sigma(0)}^{0}F^T(P_1(s,x))Q_5F(P_1(s,x))dsdx \\
&+\bar{\tau}\int_\Omega\int_{-\bar{\tau}}^{0}\int_\theta^0\frac{\partial M_1^T(s,x)}{\partial s}R_1\frac{\partial M_1(s,x)}{\partial s}dsd\theta dx \\
&+\bar{\sigma}\int_\Omega\int_{-\bar{\sigma}}^{0}\int_\theta^0\frac{\partial P_1^T(s,x)}{\partial s}R_2\frac{\partial P_1(s,x)}{\partial s}dsd\theta dx \\
&+\int_\Omega M_2^T(0,x)J_3M_2(0,x)dx \\
&+\int_\Omega P_2^T(0,x)J_4P_2(0,x)dx \\
&+\sum_{u=1}^{n_2}\int_\Omega N_{3u}\sum_{k=1}^{l}d_{uk}\left(\frac{\partial m_{2u}(0,x)}{\partial x_k}\right)^2dx \\
&+\sum_{u=1}^{n_2}\int_\Omega N_{4u}\sum_{k=1}^{l}d_{uk}^*\left(\frac{\partial p_{2u}(0,x)}{\partial x_k}\right)^2dx \\
&+\int_\Omega\int_{-\tau'(0)}^{0}M_2^T(s,x)Q_6M_2(s,x)dsdx \\
&+\int_\Omega\int_{-\bar{\tau}'}^{0}M_2^T(s,x)Q_7M_2(s,x)dsdx \\
&+\int_\Omega\int_{-\sigma'(0)}^{0}P_2^T(s,x)Q_8P_2(s,x)dsdx \\
&+\int_\Omega\int_{-\bar{\sigma}'}^{0}P_2^T(s,x)Q_9P_2(s,x)dsdx \\
&+\int_\Omega\int_{-\sigma'(0)}^{0}G^T(p_2(s,x))Q_{10}G(p_2(s,x))dsdx \\
&+\bar{\tau}'\int_\Omega\int_{-\bar{\tau}'}^{0}\int_\theta^0\frac{\partial M_2^T(s,x)}{\partial s}R_3\frac{\partial M_2(s,x)}{\partial s}dsd\theta dx \\
&+\bar{\sigma}'\int_\Omega\int_{-\bar{\sigma}'}^{0}\int_\theta^0\frac{\partial P_2^T(s,x)}{\partial s}R_4\frac{\partial P_2(s,x)}{\partial s}dsd\theta dx \\
&\le \lambda_{11}\left\|\psi_1(t,x)\right\|_d^2+\lambda_{12}\left\|\psi_1^*(t,x)\right\|_d^2 \\
&+\lambda_{21}\left\|\psi_2(t,x)\right\|_d^2+\lambda_{22}\left\|\psi_2^*(t,x)\right\|_d^2 \qquad (86).
\end{aligned}$$

where

$$\lambda_{11}=\lambda_{1\max}(J_1)+\bar{\tau}\lambda_{1\max}(Q_1)+\bar{\tau}\lambda_{1\max}(Q_2)+M_1$$

$$\begin{aligned}\lambda_{12}&=\lambda_{1\max}(J_2)+\bar{\sigma}\lambda_{1\max}(Q_3)+\bar{\sigma}\lambda_{1\max}(Q_4)\\&+\bar{\sigma}\lambda_{1\max}(Q_5)\lambda_{1\max}(K_1^TK_1)+M_2\end{aligned}$$

$$\lambda_{21}=\lambda_{2\max}(J_3)+\bar{\tau}'\lambda_{2\max}(Q_6)+\bar{\tau}'\lambda_{2\max}(Q_7)+M_3$$

$$\begin{aligned}\lambda_{22}&=\lambda_{2\max}(J_4)+\bar{\sigma}'\lambda_{2\max}(Q_8)+\bar{\sigma}'\lambda_{2\max}(Q_9)\\&+\bar{\sigma}'\lambda_{2\max}(Q_{10})\lambda_{2\max}(K_2^TK_2)+M_4\end{aligned}$$

In other words

$$V(t,M,P)\ge\lambda_{1\min}(J_{1,2})\,||\,M_1(t,x)\,||^2 \qquad (87),$$

$$V(t,M,P)\ge\lambda_{1\min}(J_{1,2})\,||\,P_1(t,x)\,||^2 \qquad (88),$$

$$V(t,M,P)\ge\lambda_{2\min}(J_{3,4})\,||\,M_2(t,x)\,||^2 \qquad (89),$$

$$V(t,M,P)\ge\lambda_{2\min}(J_{3,4})\,||\,P_2(t,x)\,||^2 \qquad (90).$$

Here, $\lambda_{1\min}(J_{1,2})$ represents the smallest eigenvalue of $diag(J_1,J_2)$, and $\lambda_{2\min}(J_{3,4})$ represents the smallest eigenvalue of $diag(J_3,J_4)$.

From Eqs. (85)-(90), it can be derived that

$$||\,M_1(t,x)\,||^2\le\frac{\lambda_{11}\left\|\psi_1(t,x)\right\|_d^2+\lambda_{12}\left\|\psi_1^*(t,x)\right\|_d^2}{\lambda_{1\min}(J_{1,2})}$$

$$||\,P_1(t,x)\,||^2\le\frac{\lambda_{11}\left\|\psi_1(t,x)\right\|_d^2+\lambda_{12}\left\|\psi_1^*(t,x)\right\|_d^2}{\lambda_{1\min}(J_{1,2})}$$

$$||\,M_2(t,x)\,||^2\le\frac{\lambda_{21}\left\|\psi_2(t,x)\right\|_d^2+\lambda_{22}\left\|\psi_2^*(t,x)\right\|_d^2}{\lambda_{2\min}(J_{3,4})}$$

$$||\,P_2(t,x)\,||^2\le\frac{\lambda_{21}\left\|\psi_2(t,x)\right\|_d^2+\lambda_{22}\left\|\psi_2^*(t,x)\right\|_d^2}{\lambda_{2\min}(J_{3,4})}$$

For any $\varepsilon>0$, it is found that

$$\delta_1:=\min\left\{\frac{\varepsilon\lambda_{1\min}(J_{1,2})}{2\lambda_{11}},\frac{\varepsilon\lambda_{1\min}(J_{1,2})}{2\lambda_{12}}\right\},$$

$$\delta_2:=\min\left\{\frac{\varepsilon\lambda_{2\min}(J_{3,4})}{2\lambda_{21}},\frac{\varepsilon\lambda_{2\min}(J_{3,4})}{2\lambda_{22}}\right\}.$$

such that

$$||\,M_1(t,x)\,||^2\le\varepsilon,\quad ||\,P_1(t,x)\,||^2\le\varepsilon,$$

$$||\,M_2(t,x)\,||^2\le\varepsilon,\quad ||\,P_2(t,x)\,||^2\le\varepsilon$$

The proof is completed. therefore, according to Definition 2, it can be known that the trivial solutions of Eq. (30) and Eq. (31) exhibit asymptotic stability under the constraints of Dirichlet boundary conditions.

IV. NUMERICAL SIMULATION

The following numerical simulations are presented. In these simulations, Eq. (1) is stable, while Eq. (2) is unstable.

For the time-delayed Eq. (30) and Eq. (31), which incorporate reaction-diffusion dynamics, are considered under the constraints of Dirichlet boundary conditions, here are the specified parameters:

$A_1=\mathrm{diag}(3.0,3.0,3.0,3.0,3.0)$,

$B_1=\mathrm{diag}(0.8,0.8,0.8,0.8,0.8)$,

$C_1 = \mathrm{diag}(2.5, 2.5, 2.5, 2.5, 2.5)$,
$D_1 = \mathrm{diag}(0.1, 0.1, 0.1, 0.1, 0.1)$,
$D_1^* = \mathrm{diag}(0.2, 0.2, 0.2, 0.2, 0.2)$,
$d_1 = \mathrm{diag}(0.1, 0.1, 0.1)$,
$d_1^* = \mathrm{diag}(0.2, 0.2, 0.2)$,
$A_2 = \mathrm{diag}(0.1, 0.1, 0.1)$,
$B_2 = \mathrm{diag}(1.8, 1.8, 1.8)$,
$C_2 = \mathrm{diag}(2, 2, 2)$,
$K_1 = K_2 = 0.65I$, $L = 100$, $\gamma_1 = 0.1$, $\gamma_2 = 0.3$,

$$W_1 = \begin{bmatrix} 0 & 0 & 1 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 2.4 & -2.4 & 0 \\ 2.4 & -2.4 & 0 \\ 2.4 & 0 & -2.4 \end{bmatrix}$$

$$W_1^* = \begin{bmatrix} \gamma_1 & 0 & 0 \\ \gamma_1 & 0 & 0 \\ 0 & 0 & -\gamma_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$W_2^* = \begin{bmatrix} -\gamma_2 & -\gamma_2 & 0 & 0 & 0 \\ 0 & 0 & \gamma_2 & \gamma_2 & 0 \\ 0 & 0.3 & 0 & 0 & -\gamma_2 \end{bmatrix}$$

When $\bar{\tau} = \bar{\sigma} = \bar{\tau}' = \bar{\sigma}' = 0.4$, $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 3$, feasible solutions are obtained by solving the LMIs (34) and (35) using the MATLAB toolbox YALMIP as follows

$$J_1 = \begin{bmatrix} 7.2280 & -0.3610 & 1.4951 & 0.4995 & -0.9653 \\ -0.3610 & 8.3553 & 2.3601 & 1.2862 & -0.2790 \\ 1.4951 & 2.3601 & 8.6715 & 0.8269 & -0.5334 \\ 0.4995 & 1.2862 & 0.8269 & 9.2097 & -0.5960 \\ -0.9653 & -0.2790 & -0.5334 & -0.5960 & 8.1313 \end{bmatrix}$$

$$J_2 = \begin{bmatrix} 13.9802 & 2.6467 & 4.1458 & 3.3775 & -0.7600 \\ 2.6467 & 16.8137 & 5.1989 & 4.0631 & -0.8795 \\ 4.1458 & 5.1989 & 16.5531 & 4.9409 & -1.0252 \\ 3.3775 & 4.0631 & 4.9409 & 16.5521 & -1.0036 \\ -0.7600 & -0.8795 & -1.0252 & -1.0036 & 11.8595 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} 9.7555 & 0.3556 & 1.6919 & 1.1166 & -0.2407 \\ 0.3556 & 10.3407 & 1.7113 & 1.1340 & -0.1795 \\ 1.6919 & 1.7113 & 10.1745 & 1.1499 & 0.0125 \\ 1.1166 & 1.1340 & 1.1499 & 10.7837 & -0.1005 \\ -0.2407 & -0.1795 & 0.0125 & -0.1005 & 9.4409 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} 7.6289 & 1.1243 & 1.6735 & 1.4058 & -0.2091 \\ 1.1243 & 8.8153 & 1.9698 & 1.5190 & -0.3167 \\ 1.6735 & 1.9698 & 8.4293 & 2.0138 & -0.3783 \\ 1.4058 & 1.5190 & 2.0138 & 8.5187 & -0.3645 \\ -0.2091 & -0.3167 & -0.3783 & -0.3645 & 6.5798 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} -5.2991 & -1.4803 & -0.8576 & -1.2057 & -0.2906 \\ -1.8516 & -5.8076 & -0.9995 & -1.2765 & -0.0651 \\ -1.2719 & -1.1073 & -5.3827 & -1.7869 & 0.0079 \\ -1.4550 & -1.1747 & -1.6530 & -5.1603 & -0.0770 \\ 0.0165 & 0.2135 & 0.3365 & 0.2536 & -3.6032 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} -4.4525 & -1.9180 & -1.9060 & -1.9653 & -0.0684 \\ -2.1521 & -6.1130 & -2.1953 & -1.9638 & 0.3329 \\ -2.2533 & -2.4801 & -5.2680 & -3.1160 & 0.4796 \\ -2.1460 & -2.0096 & -3.0456 & -5.1481 & 0.3977 \\ 0.0807 & 0.4693 & 0.6400 & 0.5386 & -2.3075 \end{bmatrix}$$

$$N_1 = \begin{bmatrix} 2.3125 & 0 & 0 & 0 & 0 \\ 0 & 2.3125 & 0 & 0 & 0 \\ 0 & 0 & 2.3125 & 0 & 0 \\ 0 & 0 & 0 & 2.3125 & 0 \\ 0 & 0 & 0 & 0 & 2.3125 \end{bmatrix}$$

$$N_2 = \begin{bmatrix} 3.4027 & 0 & 0 & 0 & 0 \\ 0 & 3.4027 & 0 & 0 & 0 \\ 0 & 0 & 3.4027 & 0 & 0 \\ 0 & 0 & 0 & 3.4027 & 0 \\ 0 & 0 & 0 & 0 & 3.4027 \end{bmatrix}$$

$$\Lambda_1 = \begin{bmatrix} 7.3292 & -0.4411 & 0.0814 & -0.1769 & -0.1995 \\ -0.4411 & 7.3061 & 0.1492 & -0.0410 & -0.0160 \\ 0.0814 & 0.1492 & 7.5480 & -0.3003 & 0.0450 \\ -0.1769 & -0.0410 & -0.3003 & 7.7210 & -0.0165 \\ -0.1995 & -0.0160 & 0.0450 & -0.0165 & 7.9281 \end{bmatrix}$$

$$\Lambda_2 = \begin{bmatrix} 25.0499 & 11.0239 & 12.3984 & 11.8912 & -0.3849 \\ 11.0239 & 34.8026 & 14.5091 & 11.9886 & -2.6848 \\ 12.3984 & 14.5091 & 30.8677 & 18.0024 & -3.4466 \\ 11.8912 & 11.9886 & 18.0024 & 30.0667 & -2.9312 \\ -0.3849 & -2.6848 & -3.4466 & -2.9312 & 13.5172 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} 0.7421 & -0.3734 & 0.1047 & -0.1710 & -0.2822 \\ -0.3734 & 0.8371 & 0.2752 & 0.0281 & 0.0237 \\ 0.1047 & 0.2752 & 0.8990 & -0.2628 & -0.0716 \\ -0.1710 & 0.0281 & -0.2628 & 1.1297 & -0.0928 \\ -0.2822 & 0.0237 & -0.0716 & -0.0928 & 1.2632 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 13.2514 & -0.9036 & 2.9773 & 0.7180 & -2.4834 \\ -0.9036 & 16.1015 & 5.1400 & 2.7518 & -0.6383 \\ 2.9773 & 5.1400 & 16.4631 & 1.4325 & -1.4272 \\ 0.7180 & 2.7518 & 1.4325 & 17.6073 & -1.5025 \\ -2.4834 & -0.6383 & -1.4272 & -1.5025 & 15.6923 \end{bmatrix}$$

$$Q_3=\begin{bmatrix} 0.7946 & -0.2783 & -0.0297 & -0.1721 & -0.1249 \\ -0.2783 & 0.6885 & -0.0055 & -0.0765 & 0.0316 \\ -0.0297 & -0.0055 & 0.8322 & -0.3199 & 0.0557 \\ -0.1721 & -0.0765 & -0.3199 & 0.9554 & 0.0170 \\ -0.1249 & 0.0316 & 0.0557 & 0.0170 & 1.1923 \end{bmatrix}$$

$$Q_4=\begin{bmatrix} 15.6371 & 4.0355 & 6.5243 & 5.2312 & -1.2106 \\ 4.0355 & 20.2651 & 8.1885 & 6.2976 & -1.4248 \\ 6.5243 & 8.1885 & 19.8344 & 7.7985 & -1.6443 \\ 5.2312 & 6.2976 & 7.7985 & 19.8121 & -1.6160 \\ -1.2106 & -1.4248 & -1.6443 & -1.6160 & 12.4402 \end{bmatrix}$$

$$Q_5=\begin{bmatrix} 1.6363 & -0.4965 & -0.0927 & -0.3493 & -0.2727 \\ -0.4965 & 1.3656 & -0.0547 & -0.1289 & 0.0756 \\ -0.0927 & -0.0547 & 1.7733 & -0.6789 & 0.1473 \\ -0.3493 & -0.1289 & -0.6789 & 1.9507 & 0.0561 \\ -0.2727 & 0.0756 & 0.1473 & 0.0561 & 2.4855 \end{bmatrix}$$

$$J_3=\begin{bmatrix} 114.2547 & -113.8970 & -7.6670 \\ -113.9870 & 121.4814 & 0.5420 \\ -7.6670 & 0.5420 & 8.1165 \end{bmatrix}$$

$$J_4=\begin{bmatrix} 1.1002 & -0.5812 & -0.5888 \\ -0.5812 & 1.1843 & -0.4962 \\ -0.5888 & -0.4962 & 1.2078 \end{bmatrix}$$

$$R_3=\begin{bmatrix} 2.5211 & -0.7915 & -0.6733 \\ -0.7915 & 2.4549 & -0.6962 \\ -0.6733 & -0.6962 & 2.4249 \end{bmatrix}$$

$$R_4=\begin{bmatrix} 0.0461 & -0.0073 & -0.0070 \\ -0.0073 & 0.0469 & -0.0066 \\ -0.0070 & -0.0066 & 0.0471 \end{bmatrix}$$

$$G_3=\begin{bmatrix} -1.6867 & 0.5396 & 0.4992 \\ 0.5250 & -1.6510 & 0.4743 \\ 0.4804 & 0.4796 & -1.6635 \end{bmatrix}$$

$$G_4=\begin{bmatrix} -0.0328 & -0.0026 & -0.0022 \\ 0.0034 & -0.0306 & 0.0015 \\ 0.0019 & -0.0002 & -0.0310 \end{bmatrix}$$

$$N_3=\begin{bmatrix} 0.3066 & 0 & 0 \\ 0 & 0.3066 & 0 \\ 0 & 0 & 0.3066 \end{bmatrix}$$

$$N_4=\begin{bmatrix} 0.0103 & 0 & 0 \\ 0 & 0.0103 & 0 \\ 0 & 0 & 0.0103 \end{bmatrix}$$

$$A_3=\begin{bmatrix} 4.2340 & -2.2952 & -2.4446 \\ -2.2952 & 4.6262 & -2.0432 \\ -2.4446 & -2.0432 & 4.8299 \end{bmatrix}$$

$$A_4=\begin{bmatrix} 0.1492 & 0.0004 & 0.0063 \\ 0.0004 & 0.1190 & -0.0065 \\ 0.0063 & -0.0065 & 0.1261 \end{bmatrix}$$

$$Q_6=\begin{bmatrix} 0.3617 & -0.2030 & -0.1996 \\ -0.2030 & 0.3889 & -0.1641 \\ -0.1996 & -0.1641 & 0.3894 \end{bmatrix}$$

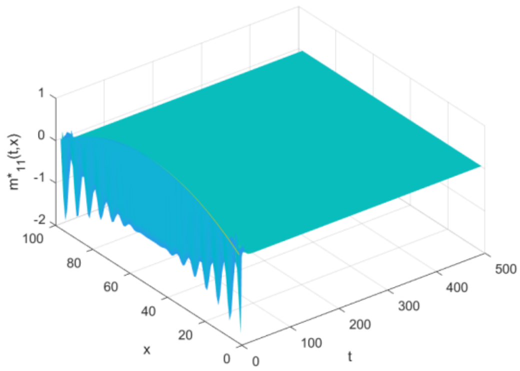
$$Q_7=\begin{bmatrix} 9.1716 & -4.7118 & -5.4090 \\ -4.7118 & 9.3881 & -4.0764 \\ -5.4090 & -4.0764 & 10.2703 \end{bmatrix}$$

$$Q_8=\begin{bmatrix} 0.0164 & -0.0092 & -0.0093 \\ -0.0092 & 0.0190 & -0.0080 \\ -0.0093 & -0.0080 & 0.0190 \end{bmatrix}$$

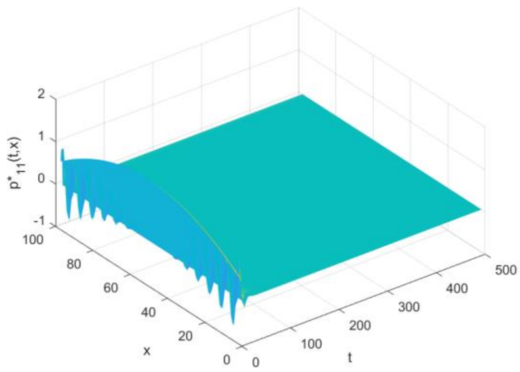
$$Q_9=\begin{bmatrix} 1.8350 & -1.0066 & -0.9609 \\ -1.0066 & 1.9775 & -0.8287 \\ -0.9609 & -0.8287 & 1.9585 \end{bmatrix}$$

$$Q_{10}=\begin{bmatrix} 2.4071 & -1.1878 & -1.5939 \\ -1.1878 & 2.6156 & -1.2284 \\ -1.5939 & -1.2284 & 3.0582 \end{bmatrix}$$

Using the MATLAB toolbox YALMIP, feasible solutions can be obtained; From the numerical simulations mentioned above, it can be seen that our theory has eliminated the restriction of the upper bound being less than 1 for the time-delay derivative. After that, we have plotted the trajectories for Eq. (1) and Eq. (2), as shown below

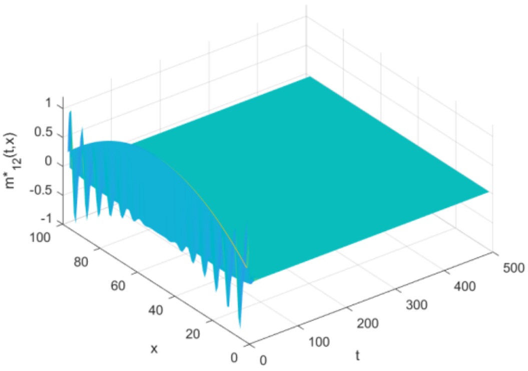


(a)

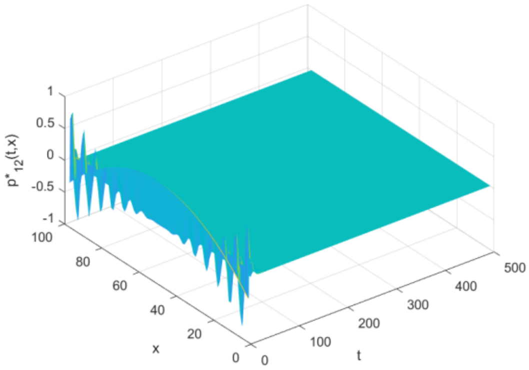


(b)

Fig. 1 The trajectories of $m_{11}^{*}(t,x)$ and $p_{11}^{*}(t,x)$, (a) $m_{11}^{*}(t,x)$, (b) $p_{11}^{*}(t,x)$

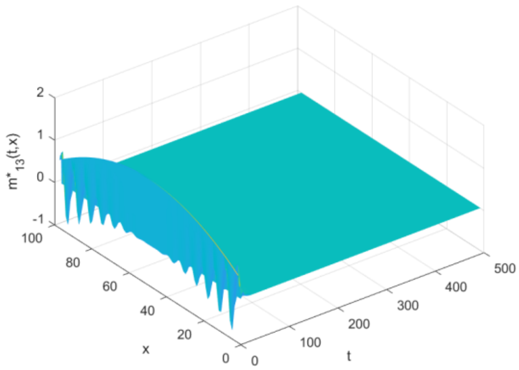


(a)

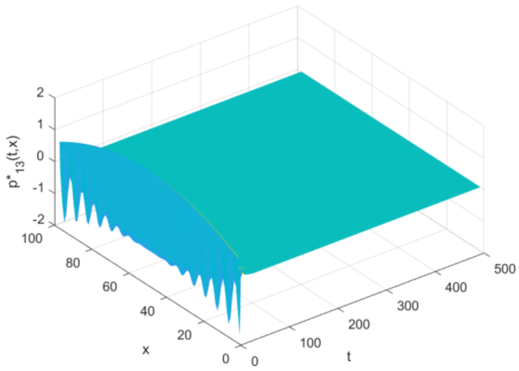


(b)

Fig. 2 The trajectories of $m_{12}^{*}(t,x)$ and $p_{12}^{*}(t,x)$, (a) $m_{12}^{*}(t,x)$, (b) $p_{12}^{*}(t,x)$

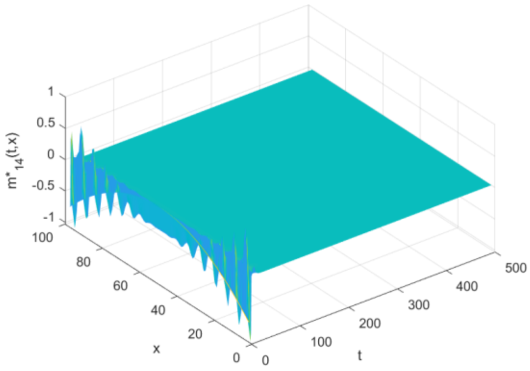


(a)

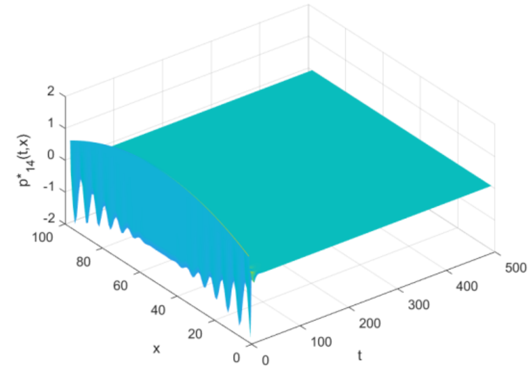


(b)

Fig. 3 The trajectories of $m_{13}^{*}(t,x)$ and $p_{13}^{*}(t,x)$, (a) $m_{13}^{*}(t,x)$, (b) $p_{13}^{*}(t,x)$

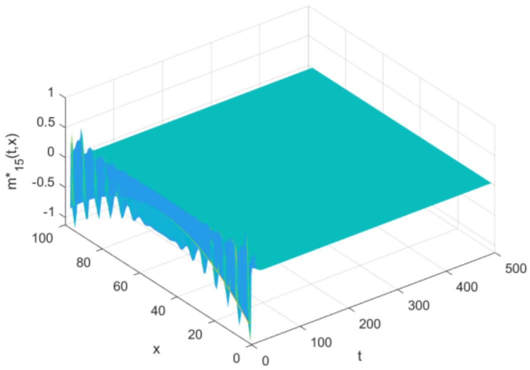


(a)

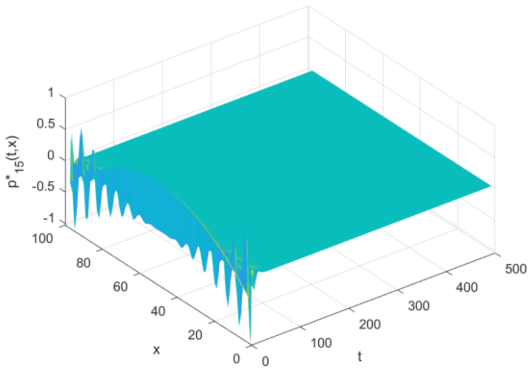


(b)

Fig. 4 The trajectories of $m_{14}^{*}(t,x)$ and $p_{14}^{*}(t,x)$, (a) $m_{14}^{*}(t,x)$, (b) $p_{14}^{*}(t,x)$

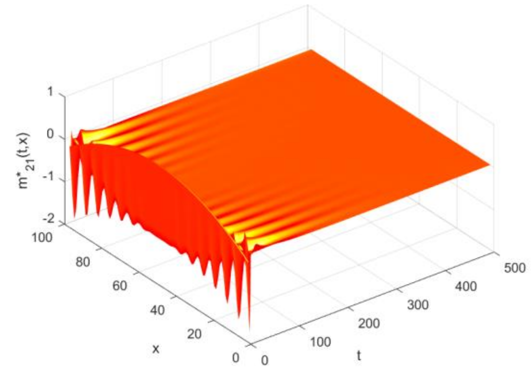


(a)

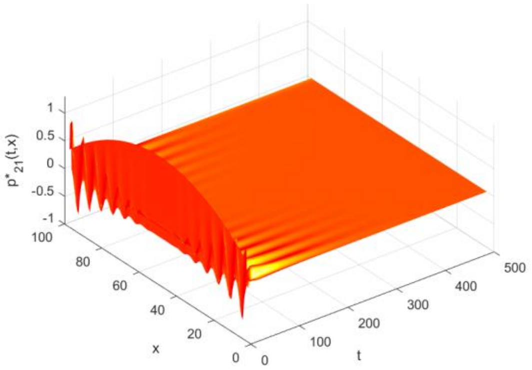


(b)

Fig. 5 The trajectories of $m_{15}^{*}(t,x)$ and $p_{15}^{*}(t,x)$, (a) $m_{15}^{*}(t,x)$, (b) $p_{15}^{*}(t,x)$

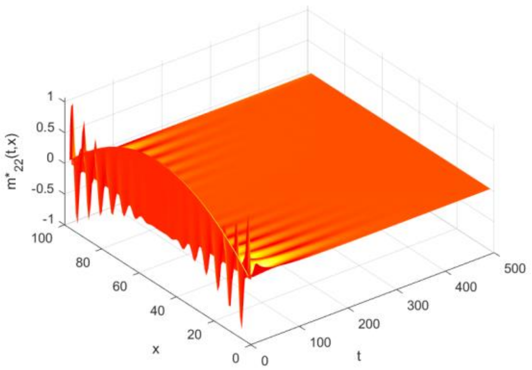


(a)

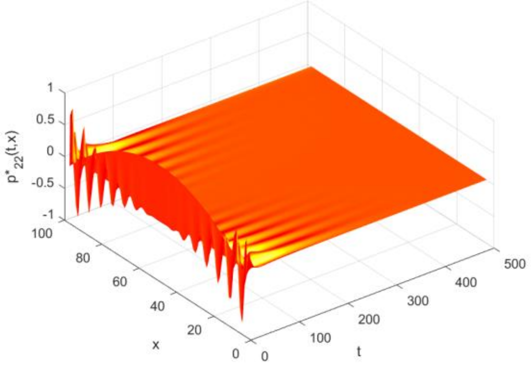


(b)

Fig. 6 The trajectories of $m_{21}^{*}(t,x)$ and $p_{21}^{*}(t,x)$, (a) $m_{21}^{*}(t,x)$, (b) $p_{21}^{*}(t,x)$

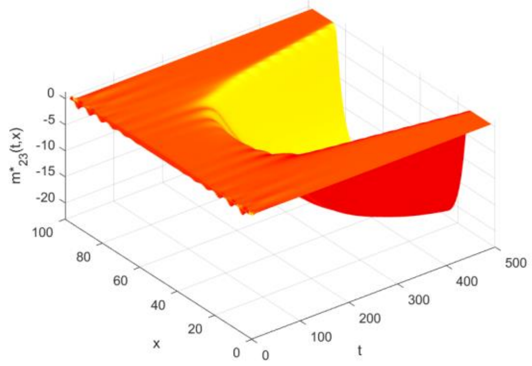


(a)

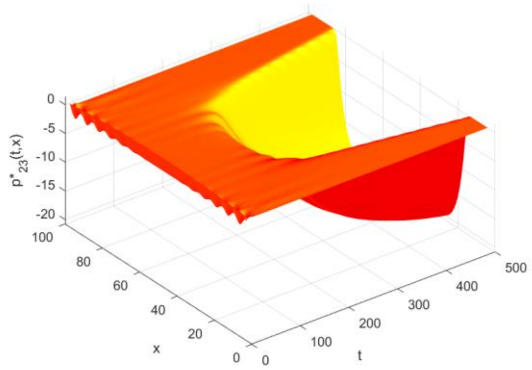


(b)

Fig. 7 The trajectories of $m_{22}^{*}(t,x)$ and $p_{22}^{*}(t,x)$, (a) $m_{22}^{*}(t,x)$, (b) $p_{22}^{*}(t,x)$



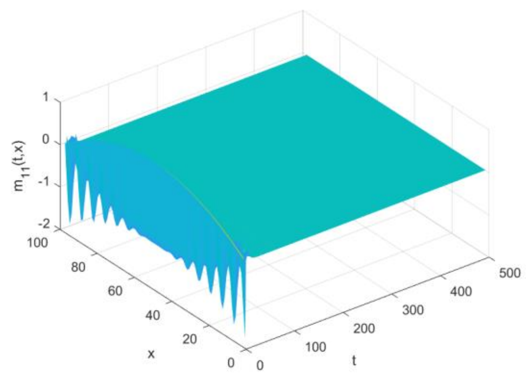
(a)



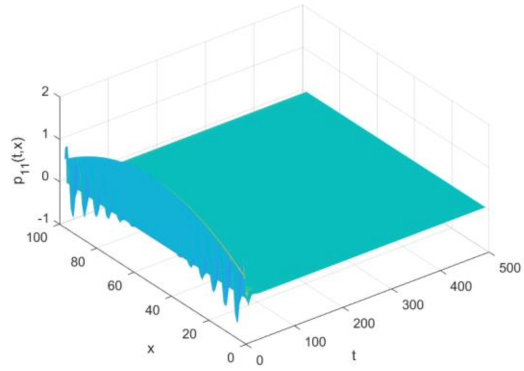
(b)

Fig. 8 The trajectories of $m_{23}^{*}(t,x)$ and $p_{23}^{*}(t,x)$, (a) $m_{23}^{*}(t,x)$, (b) $p_{23}^{*}(t,x)$

As can be seen from Figs. 1-8, in the absence of interaction terms, Eq. (1) is stable, while Eq. (2) is unstable. This manuscript also presents the asymptotic stability analysis for mRNA and protein levels, as delineated within the framework of Dirichlet boundary conditions, across Figs. 9-16.

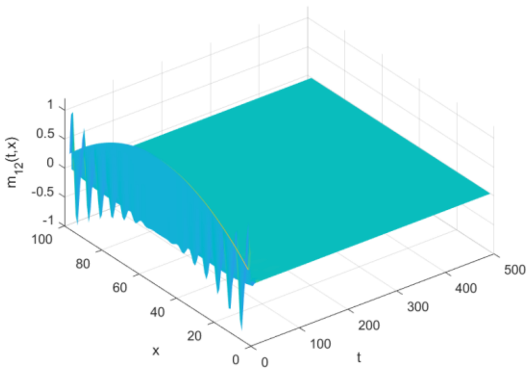


(a)

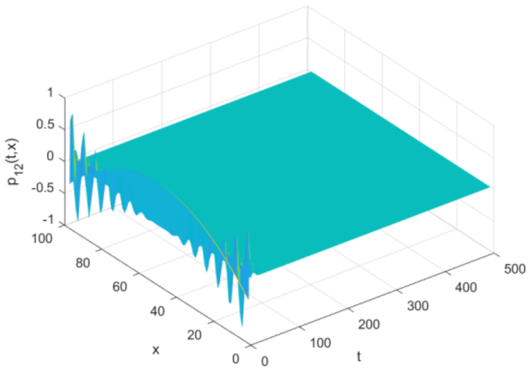


(b)

Fig. 9 The trajectories of $m_{11}(t,x)$ and $p_{11}(t,x)$, (a) $m_{11}(t,x)$, (b) $p_{11}(t,x)$

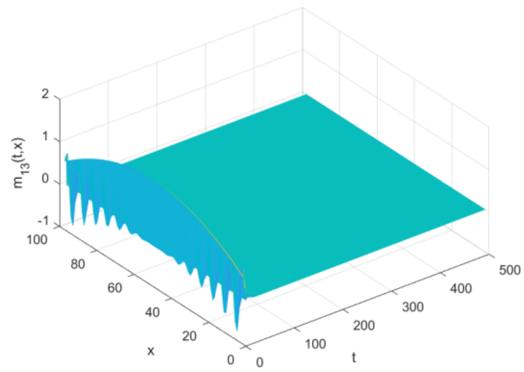


(a)

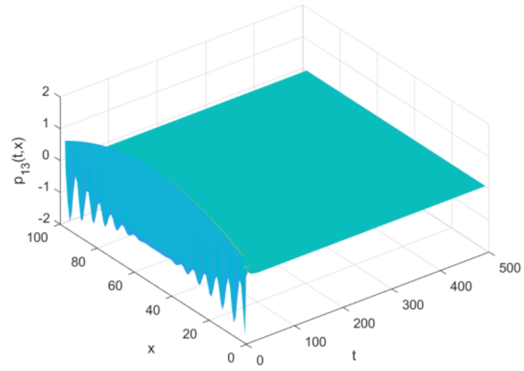


(b)

Fig.10 The trajectories of $m_{12}(t,x)$ and $p_{12}(t,x)$, (a) $m_{12}(t,x)$, (b) $p_{12}(t,x)$

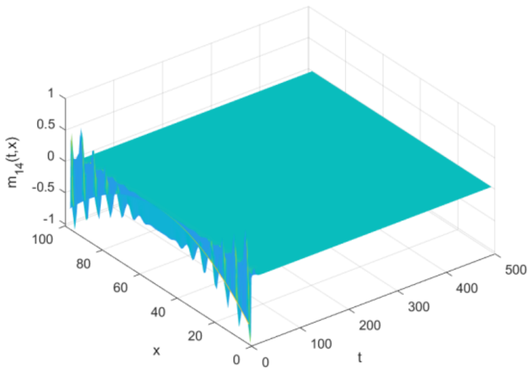


(a)

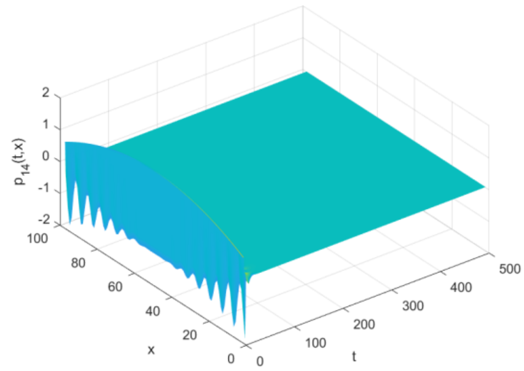


(b)

Fig. 11 The trajectories of $m_{13}(t,x)$ and $p_{13}(t,x)$, (a) $m_{13}(t,x)$, (b) $p_{13}(t,x)$

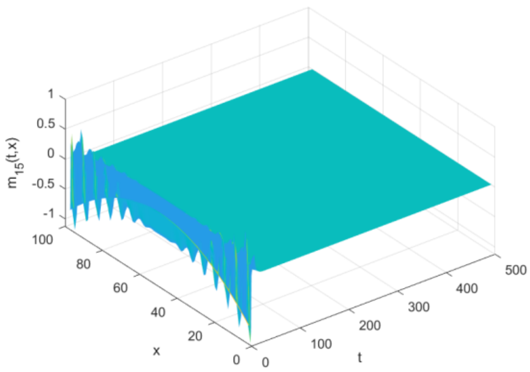


(a)

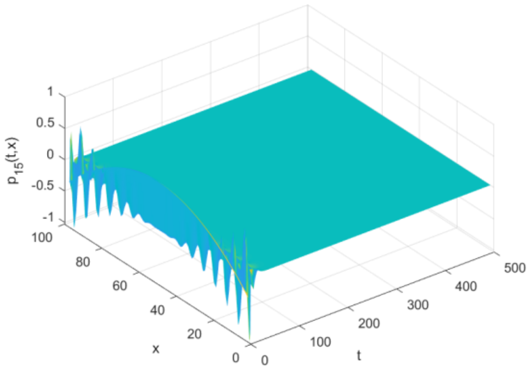


(b)

Fig. 12 The trajectories of $m_{14}(t,x)$ and $p_{14}(t,x)$, (a) $m_{14}(t,x)$, (b) $p_{14}(t,x)$

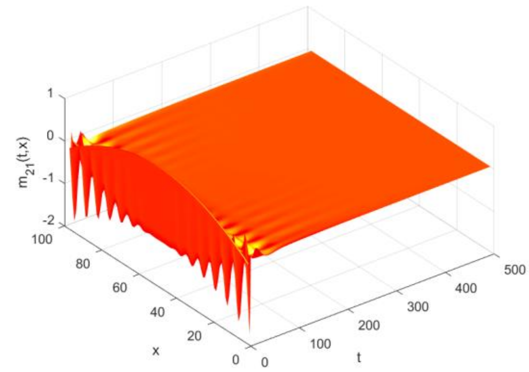


(a)

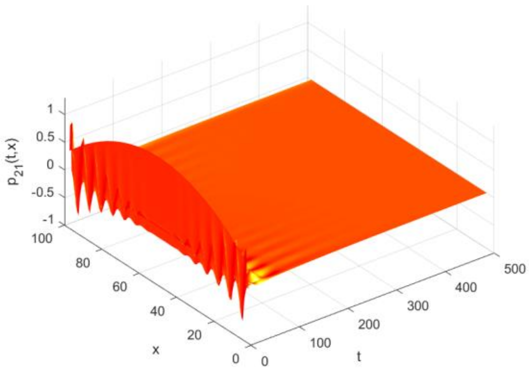


(b)

Fig. 13 The trajectories of $m_{15}(t,x)$ and $p_{15}(t,x)$, (a) $m_{15}(t,x)$, (b) $p_{15}(t,x)$

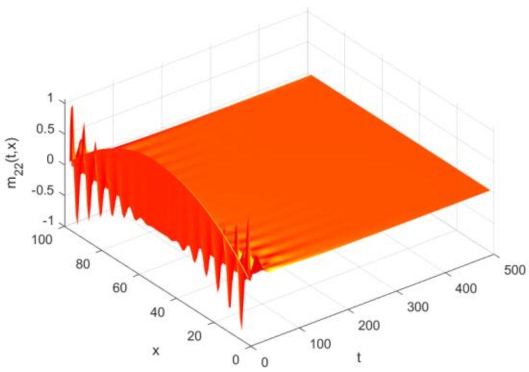


(a)

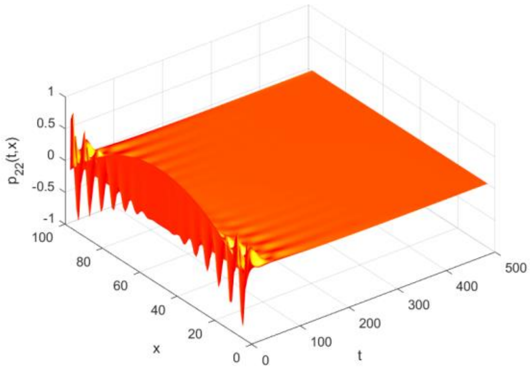


(b)

Fig. 14 The trajectories of $m_{21}(t,x)$ and $p_{21}(t,x)$, (a) $m_{21}(t,x)$, (b) $p_{21}(t,x)$

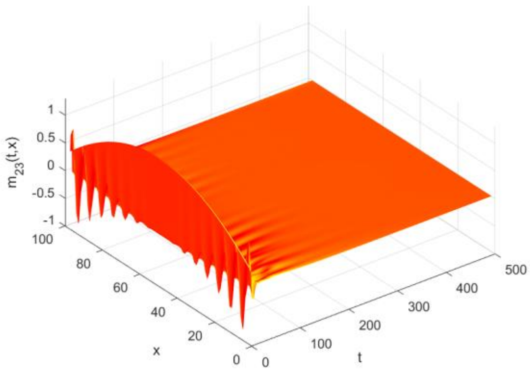


(a)

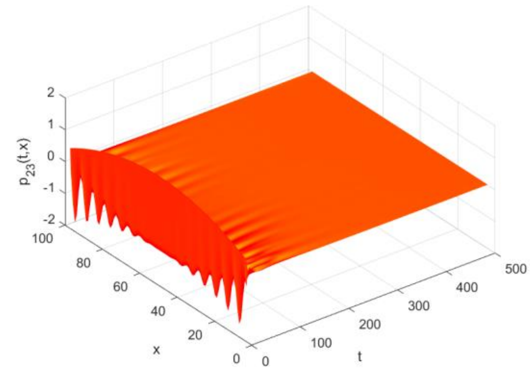


(b)

Fig. 15 The trajectories of $m_{22}(t,x)$ and $p_{22}(t,x)$, (a) $m_{22}(t,x)$, (b) $p_{22}(t,x)$



(a)



(b)

Fig. 16 The trajectories of $m_{23}(t,x)$ and $p_{23}(t,x)$, (a) $m_{23}(t,x)$, (b) $p_{23}(t,x)$

Obviously, due to the presence of interaction terms, both systems are stable at this time. From Figs. 9-16, it is evident that the theoretical framework we have put forward is viable.

Tab 1. Upper bounds on $\bar{\tau} = \bar{\sigma} = \bar{\tau}' = \bar{\sigma}'$ with different μ

Case	0.7	0.93	1.0	2.0
[26, Theorem 1]	∞	--	--	--
[13, Theorem 1]	∞	∞	--	--
Theorem 1	5.3855	0.6267	0.6267	0.6267

(1) When $\mu = 0.7$, the LMI conditions given by [26, Theorem 1], [13, Theorem 1], and Theorem 1 all have feasible solutions.

(2) When $0.93 \le \mu \le 1$, initially there is no feasible solution under the LMI conditions given in [26, Theorem 1], and then there is also no feasible solution under the LMI conditions given in both [26, Theorem 1] and [13, Theorem 1].

(3) When $\mu \ge 1$, a feasible solution exists only under the LMI conditions given in Theorem 1.

From Tab 1, it can be seen that when the derivative of the time delay is greater than 1, a feasible solution exists under the theorem we proposed. It can be intuitively observed that we have expanded the upper bound of the time delay derivative, thereby addressing the problem we are studying.

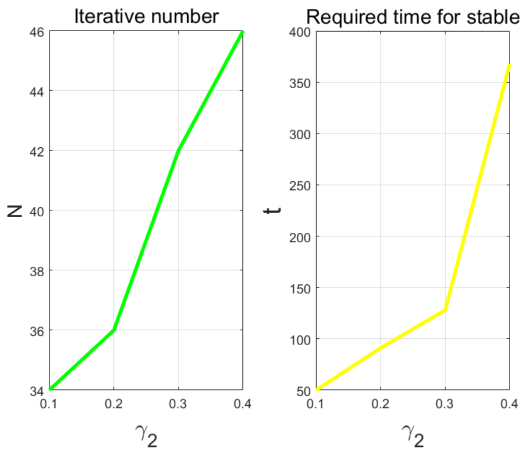


Fig. 17 Iterative number and required time for stable with different γ_2.

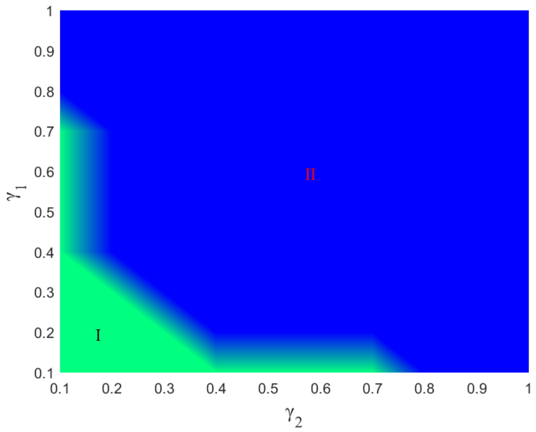


Fig. 18 Determining the stability of Eq. (30) and Eq. (31) under different coupling coefficients.

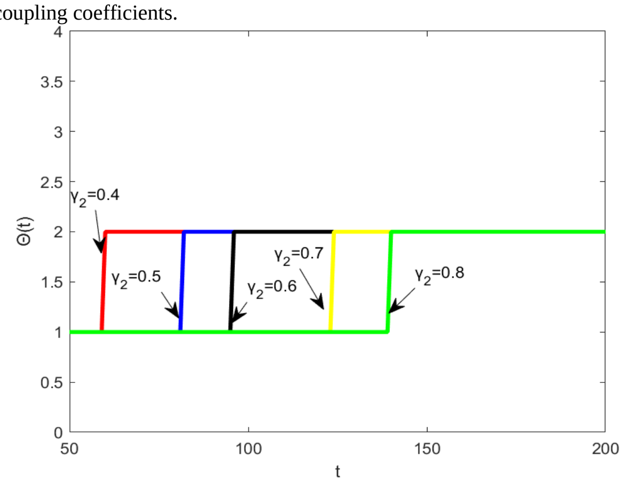


Fig. 19 The evolution of $\Theta(t)$ with γ_2

Since this paper is based on the gene regulatory network with interactions, in order to gain a deeper understanding of the changes in interactions, we studied the stability of the system under different coupling strengths. From Figure 17, it can be observed that when $\gamma_1 = 0.1$, $\gamma_2 \in (0.1, 0.2, 0.3, 0.4)$, as the coupling strength γ_2 increases, the number of iterations required to solve the feasible solution of the LMI grows, and the time for the system to reach stability also becomes longer. Both metrics exhibit the same trend of change as the coupling strength increases, indicating that these two indicators can reflect the system's variation to some extent. Therefore, it can be concluded that within a certain range, an increase in coupling strength γ_1, γ_2 prolongs the stabilization time of the interacting GRN, and γ_1, γ_2 has a significant impact on the system's stabilization time.

As shown in Fig. 18, if the parameter values fall within Region I, the proposed gene regulatory network model is stable. However, if the parameter values fall within Region II, the system becomes unstable.

We defined the criteria for evaluating the stabilization time of the interactive GRN as follows:

$$\gamma(t) = \int_{\Sigma}\Big(w_i \left|m_{1i}(t,x) - m_{1i}(t-\iota,x)\right| + w_i' \left|p_{1i}(t,x) - p_{1i}(t-\iota,x)\right| + w_u \left|m_{2u}(t,x) - m_{2u}(t-\iota,x)\right| + w_u' \left|p_{2u}(t,x) - p_{2u}(t-\iota,x)\right|\Big) dx$$

Where $w_i > 0$, $w_i' > 0\,(i = 1,\dots,n_1)$; $w_u > 0$, $w_u' > 0\,(u = 1,\dots,n_2)$ are the weights of m_{1i}, p_{1i}, m_{2u} and p_{2u}.

$$\begin{cases} \Theta(t) = 1, & \gamma(t) > \iota_0 \\ \Theta(t) = 2, & \gamma(t) \le \iota_0 \end{cases}$$

ι_0 is a small positive value, and $\Theta(t)$ serves as a metric for evaluating stability. If $\Theta(t) = 2$, the interactive gene regulatory network is considered stable; if $\Theta(t) = 1$, the network is deemed unstable. The evolution of $\Theta(t)$ with respect to γ_2 is shown in Figure 19, where $\gamma_2 \in \{0.4, 0.5, 0.6, 0.7, 0.8\}$, $\gamma_1 = 0.1$, $w_i = w_i' = w_u = w_u' = 1/16$, $\iota_0 = 0.1$. It is observed that the stability duration of the interacting GRNs increases as γ_2 increases.

By combining the provided images, we can understand the dynamic behavior of the system more comprehensively from different perspectives.

V. Discussion

Under Dirichlet boundary conditions, this paper explores the stability conditions of interacting GRNs. It focuses on the asymptotic behavior of the trivial solution. The study considers terms that describe both reactions and diffusion. The asymptotic stability criteria were established by constructing a novel Lyapunov functional and utilizing the results of the provided lemmas. Notably, Theorem 1 in this paper removes a restrictive requirement. The condition that the maximum magnitude of the delay derivative must be below 1 is no longer necessary. This change significantly enhances the generality of the theoretical findings. It also improves their applicability in various contexts. Moreover, the effectiveness and practicality of the theoretical results have been visually demonstrated through carefully designed numerical simulation experiments. These simulations not only corroborate the theoretical analysis. They also introduce a fresh perspective. This perspective is valuable for studying the stability of genetic regulatory networks. This study represents significant advancements and breakthroughs in the investigation of stability dynamics in GRNs.

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