Fuzzy Risk Analysis Based on Ranking Fuzzy Numbers by a Novel Defuzzification Technique with Vagueness in Lower Decision Levels

Peddi Phani Bushan Rao*, Botsa Devaki Rani, Akiri Sridhar

***Abstract*—In real-world applications, the parameters used to describe the decision-making problem are imprecise and vague. Fuzzy modeling of the problem using fuzzy numbers (FNs) can address the inherent vagueness of the parameters effectively. A crucial aspect of decision-making, particularly when uncertainty exists at lower levels, involves ranking FNs. This paper introduces a novel defuzzification technique for ranking Generalized Trapezoidal Fuzzy Numbers with Left and Right Heights (GTFNLRH). For the purpose of ranking, the proposed approach derives a representative value of GTFNLRH. This involves finding a Triangular Fuzzy Quantity (TFQ) using centroids with vagueness at lower decision levels, calculating the value (*VAL*), representing the ill-defined magnitude, determining the ambiguity (*AMB*), quantifying the vagueness within its ill-defined magnitude of the GTFNLRH, and applying *VAL* and *AMB* to the TFQ. Based on these representative values, a novel ranking criterion is established that overcomes the limitations of ranking different FNs observed in some existing fuzzy ranking methods. An application of the novel defuzzification technique is also investigated to assess the fuzzy risk associated with product manufacturing by different companies.**

***Index Terms*—Ambiguity, Centroids, Decision-levels, Fuzzy risk analysis, Fuzzy numbers, Triangular Fuzzy quantity, Value.**

I. Introduction and Literature Review

EVALUATING systems often involve uncertainty stemming from imprecise measurements within decision-making processes. To address this, Zadeh [1] introduced fuzzy sets (FSs) as a powerful framework for representing subjective and imprecise information. When precise parameter values are unavailable due to incomplete information or knowledge gaps, FNs, a specific type of fuzzy set proposed by Zadeh [2], become valuable. FNs find applications in data analysis, engineering, and decision-making.

Ranking FNs is crucial for assessing the inherent uncertainty in decision-making scenarios. It effectively handles imprecise, vague, and ambiguous data in science and engineering systems. However, due to the diverse characteristics of FNs and the inherent subjectivity of ranking methods, the same set of FNs can yield different ranking orders when evaluated using different approaches.

The concept of ranking FNs was first introduced by Jain [3] for decision-making in fuzzy environments, representing imprecise measurements as fuzzy sets. Yager [4] proposed a ranking method based on the centroids of FNs. Subsequently, Dubois & Prade [5] made significant contributions to the field with their work on operations on FNs. Murakami et al. [6] developed a centroid-based algorithm for ordering FNs. While numerous researchers have devised various ranking methods, Bortolan & Degani [7] comprehensively analyzed these approaches. Their study revealed that many existing methods produced inconsistent rankings even in simple scenarios.

Chen [8] proposed a ranking method for FNs based on maximizing and minimizing sets, while Nakamura [9] utilized preference relations for decision-making. Liou & Wang [10] introduced a ranking approach using integral values, and Choobineh & Li [11] developed an index based on alpha-cuts for ordering FNs. Following these contributions, numerous researchers have proposed various ranking methods which includes Fortemps & Roubens [12] ranking FNs using area compensation, Cheng [13] ranking FNs based on the distance between the origin and the centroid of each FN, and Chu & Tsao's [14] ranking FNs based on the area between the centroid point and the origin.

Delgado et al. [15] introduced two key parameters to characterize FNs, *VAL* represents the ill-defined magnitude of the FN, and *AMB*, quantifies the vagueness associated with the FN's value. These parameters are fundamental to the development of fuzzy set theory. This study focuses on GTFNLRH, which has significant applications in areas such as approximate reasoning, fuzzy image processing, fuzzy data analysis, fuzzy clustering, and fuzzy risk analysis.

Key contributions related to GTFNLRH are by Chen et al. [16], who first introduced GTFNLRH for fuzzy risk analysis problems. Pushpinder Singh [17] proposed a ranking method based on a score function derived from areas, Jiang et al. [18], identified shortcomings in Chen et al.'s [16] ranking method and proposed a modified approach using areas and spreads. Rituparna and Bijit [19], utilized GTFNLRH in parametric form and employed *VAL* and *AMB* for ranking, and Barazandeh and Ghazanfari [20] ranked GTFNLRH based on centroids and areas.

Other relevant works include: Sinika & Ramesh [29], who developed an interval-based de-neutrosophication method for the PERT problem in a neutrosophic environment, Hitoshi

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Yano [30], who formulated multiobjective bimatrix games with fuzzy random payoffs, Jana et al. [31], focused on compiling a fuzzy portfolio using large-scale data, and Lee et al. [32], proposed mechanisms for evaluating and mitigating damages under multiple criteria fuzzy behavior.

Risk is typically defined by two key factors: The likelihood of an adverse event occurring, and the severity of the potential consequences. Risk represents the potential for damage or loss associated with human activities. To effectively assess risk, it's crucial to incorporate decision-makers' perspectives. Often, decision-makers express their priorities regarding evaluation criteria using linguistic terms (e.g., "high," "low," "moderate") rather than specific numerical values.

A robust risk assessment model requires the identification of key risk factors within the system or process, evaluating each risk factor's relative importance, and predicting potential challenges. Chen and Chen [21] proposed fuzzy risk analysis based on similarity measures of GTFNs, Chen & Sanguansat [22] analyzed fuzzy risk through similarity measures further, Xu et al. [23] demonstrated fuzzy risk analysis created on new similarity, Chen & Chen [24] established fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and different spreads, and Rao [25] proposed evaluating fuzzy risk based on centroids of centroids.

This work incorporates defuzzification based on the following observations: Chen and Chen [21] demonstrated that the y-coordinate of the centroid of a Generalized Trapezoidal Fuzzy Number (GTrFN) $A=(a,b,c,d;w)$ may not always lie within the interval $[0,w]$, $0\le w\le 1$ of the FN, even though it lies within the interval $[w/3,w/2]$. Many existing ranking methods rely on the y-coordinate of the centroid, particularly when ranking based on the x-coordinate fails. These observations highlight the importance of considering the 'vagueness set $[w/3,w/2]$' in the ranking of FNs. For the GTFNLRH, the y coordinate of the centroid is in $[w_1/3,(w_1^2+w_2^2+w_1w_2)/3(w_1+w_2)]$ for $w_1<w_2$; $0\le w_1\le w_2\le 1$ and $[w_2/3,(w_1^2+w_2^2+w_1w_2)/3(w_1+w_2)]$ for $w_2<w_1$; $0\le w_2\le w_1\le 1$.

This paper introduces a novel defuzzification method by identifying a TFQ within the lower decision-level interval (y-coordinate of the GTFNLRH's centroid), evaluating the TFQ at different decision levels $w_1<w_2$ and $w_2<w_1$ to obtain defuzzified values, and determine the *VAL* and *AMB* of this TFQ to represent the GTFNLRH. This approach differs from previous methods by avoiding the use of reducing functions (as suggested by Delgado et al. [15]) to minimize the impact of decision levels. A ranking criterion is then established using the calculated *VAL* and *AMB*, addressing limitations found in existing GTFNLRH ranking methods. The paper includes a comparative study with other ranking methods to validate the proposed approach, an application of the method to assess fuzzy risk in product engineering, and a discussion of the study's limitations and potential future research directions.

II. PRELIMINARIES

In this section, some fundamental definitions related to the work are presented.

A. Fuzzy Number

An FN is a convex normalized FS A of the real line R with membership function μ_A piecewise continuous, and there is an $y_0\in R$ such that $\mu_A(y_0)=1$.

B. Generalized trapezoidal fuzzy number with different left and right heights [26].

The GTFNLRH denoted by $A=(a,b,c,d;w_1,w_2)$ where $0\le w_1\le 1$ and $0\le w_2\le 1$ and can be seen in Figure 1. If $0\le a\le b\le c\le d\le 1$, then we say A is a standard generalized FN. The membership function $\mu_A(x)$ is defined as follows:

$$\mu_A(x)=\begin{cases}\dfrac{w_1(x-a)}{b-a} & ;a\le x\le b,\\ \dfrac{w_1(c-b)+(w_2-w_1)(x-b)}{c-b} & ;b\le x\le c,\\ \dfrac{w_2(x-d)}{c-d} & ;c\le x\le d,\\ 0 & ;otherwise.\end{cases}$$

If $w_1=w_2=w$, then A is a GTrFN, and membership function as follows:

$$\mu_A(x)=\begin{cases}\dfrac{w(x-a)}{b-a} & ;a\le x\le b,\\ 1 & ;b\le x\le c,\\ \dfrac{w(x-d)}{c-d} & ;c\le x\le d,\\ 0 & ;otherwise.\end{cases}$$

If $w_1=w_2=w$ and $b=c$, then A is a triangular fuzzy number (TFN). If $w_1=w_2=1$, then A is a trapezoidal fuzzy number (TrFN), and if $a=b=c=d$ and $w_1=w_2$, then A is a crisp number.

C. Parametric representation of fuzzy number

A fuzzy number A in parametric form is a pair of functions $[\underline{a}(r),\overline{a}(r)]$; $0\le r\le 1$, satisfying the following conditions:

(i) $\underline{a}(r)$ is an increasing left continuous bounded function on [0,1];

(ii) $\overline{a}(r)$ is a decreasing left continuous bounded function on [0,1];

(iii) $\underline{a}(r)\le\overline{a}(r)$, $0\le r\le 1$.

D. Value and ambiguity of a fuzzy number [15]

If A is a fuzzy number with a parametric representation $[\underline{a}(r),\overline{a}(r)]$, $r\in[0,1]$ and $s:[0,1]\to[0,1]$is a reducing function, then the value and ambiguity of the fuzzy number A with respect to the reducing function are defined by:

$$Val(A)=\int_0^1 s(r)\{\underline{a}(r)+\overline{a}(r)\}dr$$
$$Amb(A)=\int_0^1 s(r)\{\overline{a}(r)-\underline{a}(r)\}dr$$

where $s(r)$ is the reducing function and $\int_0^1 s(r)dr=0.5$.

E. Arithmetic Operations between GTFNLRH [26]

If $G_1=(\alpha_1,\beta_1,\gamma_1,\delta_1;w_{L1},w_{R1})$ and $G_2=(\alpha_2,\beta_2,\gamma_2,\delta_2;w_{L2},w_{R2})$ are two GTFNLRH with $(\alpha_2,\beta_2,\gamma_2,\delta_2)\ne(0,0,0,0)$, then the arithmetic operations addition, subtraction, multiplication, and division denoted by $\oplus,\ominus,\otimes,\oslash$ respectively of GTFNLRH $G_1$and G_2 are defined as follows:

$G_1\oplus G_2=(\alpha_1,\beta_1,\gamma_1,\delta_1;w_{L1},w_{R1})$
$\oplus\,(\alpha_2,\beta_2,\gamma_2,\delta_2;w_{L2},w_{R2})$

$$= \begin{bmatrix} \alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2, \delta_1 + \delta_2; \\ \min(w_{L1}, w_{L2}), \min(w_{R1}, w_{R2}) \end{bmatrix}$$

$$G_1 \ominus G_2 = (\alpha_1, \beta_1, \gamma_1, \delta_1; w_{L1}, w_{R1}) \ominus (\alpha_2, \beta_2, \gamma_2, \delta_2; w_{L2}, w_{R2})$$
$$= \begin{bmatrix} \alpha_1 - \delta_2, \beta_1 - \gamma_2, \gamma_1 - \beta_2, \delta_1 - \alpha_2; \\ \min(w_{L1}, w_{L2}), \min(w_{R1}, w_{R2}) \end{bmatrix}$$

$$G_1 \otimes G_2 = (\alpha_1, \beta_1, \gamma_1, \delta_1; w_{L1}, w_{R1}) \otimes (\alpha_2, \beta_2, \gamma_2, \delta_2; w_{L2}, w_{R2})$$
$$= \begin{bmatrix} \alpha_1 \times \alpha_2, \beta_1 \times \beta_2, \gamma_1 \times \gamma_2, \delta_1 \times \delta_2; \\ \min(w_{L1}, w_{L2}), \min(w_{R1}, w_{R2}) \end{bmatrix}$$

$$G_1 \oslash G_2 = (\alpha_1, \beta_1, \gamma_1, \delta_1; w_{L1}, w_{R1}) \oslash (\alpha_2, \beta_2, \gamma_2, \delta_2; w_{L2}, w_{R2})$$
$$\left[\frac{\alpha_1}{\delta_2}, \frac{\beta_1}{\gamma_2}, \frac{\gamma_1}{\beta_2}, \frac{\delta_1}{\alpha_2}; \min(w_{L1}, w_{L2}), \min(w_{R1}, w_{R2})\right]$$

III. Proposed Method

Consider a GTFNLRH $J = (a, b, c, d; w_1, w_2)$, shown in $Fig.1$ and $Fig.2$ with $w_1 < w_2$, $w_2 < w_1$. In the first step of defuzzification, the GTFNLRH is geometrically treated as a quadrilateral PQRS, and this quadrilateral PQRS is partitioned into three plane regions, namely triangular regions PQL, MRS, and quadrilateral region LQRM, shown graphically in Fig. 1 and Fig. 2. The centroids of the triangle PQL, triangle MRS, and quadrilateral LQRM are given by $A((a+2b)/3, w_1/3)$, $B((2c+d)/3, w_2/3)$, and $C\left(\frac{w_1(2b+c)+w_2(b+2c)}{3(w_1+w_2)}, \frac{{w_1}^2+{w_2}^2+w_1w_2}{3(w_1+w_2)}\right)$ respectively. In the second step of defuzzification, the centroids A, B, C are joined to get a triangular FQ, ABC with decision levels in the ranges $\left[\frac{w_1}{3}, \frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}\right]$ for the case $w_1 < w_2$ and $\left[\frac{w_2}{3}, \frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}\right]$ for the case $w_2 < w_1$ respectively.

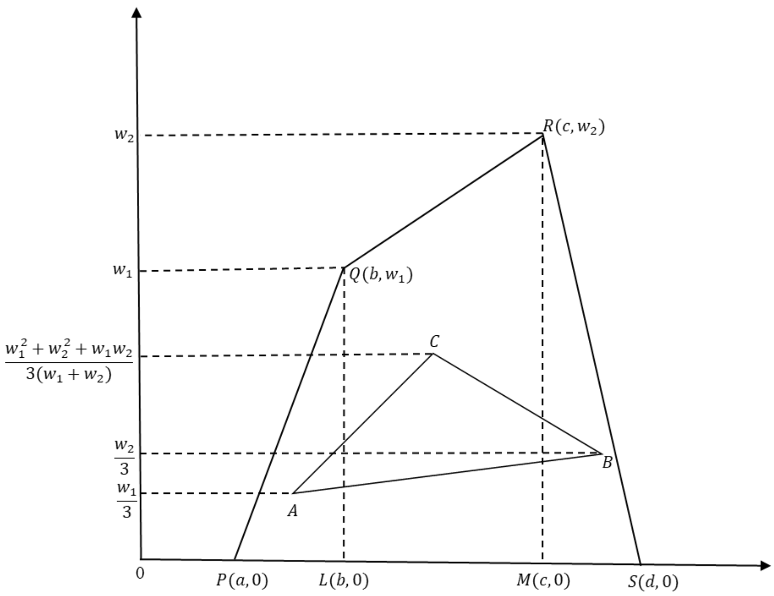

Fig. 1. GTFNLRH and TFQ for $w_1 < w_2$

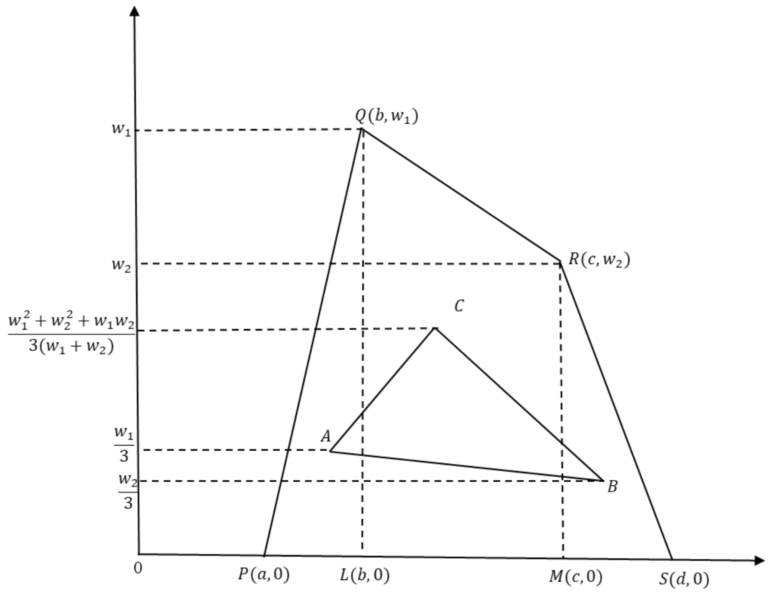

Fig. 2. GTFNLRH and TFQ for $w_2 < w_1$

Definition 1
The TFQ, J^* formed from GTFNLRH, $J=(a,b,c,d;w_1,w_2)$ with decision levels in $\left[\frac{w_1}{3},\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}\right]$, where $0\le w_1\le w_2\le 1$ and $\left[\frac{w_2}{3},\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}\right]$, where $0\le w_2\le w_1\le 1$ is defined as:

$$J^*=\left(\frac{a+2b}{3},\frac{w_1(2b+c)+w_2(b+2c)}{3(w_1+w_2)},\frac{2c+d}{3};\frac{w_1}{3};\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}\right) \quad (1)$$

$$J^*=\left(\frac{a+2b}{3},\frac{w_1(2b+c)+w_2(b+2c)}{3(w_1+w_2)},\frac{2c+d}{3};\frac{w_2}{3};\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}\right) \quad (2)$$

Definition 2
The Membership function of TFQ J^* given by Eq. (1) is defined as:

$$\mu_{J^*}(x)=\begin{cases}\frac{w_1}{3}+\frac{{w_2}^2\left(x-\frac{a+2b}{3}\right)}{w_1(c-a)+w_2(2c-b-a)}, & \frac{a+2b}{3}\le x\le\frac{w_1(2b+c)+w_2(b+2c)}{3(w_1+w_2)},\\ \frac{{w_1}^2+{w_2}^2+w_1w_2}{3(w_1+w_2)}, & x=\frac{w_1(2b+c)+w_2(b+2c)}{3(w_1+w_2)},\\ \frac{w_2}{3}+\frac{{w_1}^2\left(x-\frac{2c+d}{3}\right)}{w_1(2b-c-d.)+w_2(b-d.)}, & \frac{w_1(2b+c.)+w_2(b+2c)}{3(w_1+w_2)}\le x\le\frac{2c+d}{3}.\end{cases} \quad (3)$$

Definition 3
The Parametric form for the TFQ, J^* given by Eq. (1) with decision level $\delta\in\left[\frac{w_1}{3},\frac{{w_1}^2+{w_2}^2+w_1w_2}{3(w_1+w_2)}\right]$, where $w_1<w_2$ and $0\le w_1\le w_2\le 1$ is defined as:

$$\left[\underline{J}^*(\delta),\overline{J}^*(\delta)\right]=\begin{cases}\left[\frac{a+2b}{3}+\left(\frac{w_1(c-a)+w_2(2c-b-a)}{{w_2}^2}\right)\left(\delta-\frac{w_1}{3}\right),0\right], & if\ \frac{w_1}{3}\le\delta\le\frac{w_2}{3}\\ \left[\begin{array}{l}\frac{a+2b}{3}+\left(\frac{w_1(c-a)+w_2(2c-b-a)}{{w_2}^2}\right)\left(\delta-\frac{w_1}{3}\right),\\ \frac{2c+d}{3}+\left(\frac{w_1(2b-c-d)+w_2(b-d.)}{{w_1}^2}\right)\left(\delta-\frac{w_2}{3}\right)\end{array}\right], & if\ \frac{w_2}{3}\le\delta\le\frac{{w_1}^2+{w_2}^2+w_1w_2}{3(w_1+w_2)}\end{cases} \quad (4)$$

and satisfy the following conditions:
(i) $\underline{J}^*(\delta)$ is an increasing left continuous bounded function on $\left[\frac{w_1}{3},\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}\right]$,
(ii) $\overline{J}^*(\delta)$ is a decreasing left continuous bounded function on $\left[\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)},\frac{w_2}{3}\right]$,
(iii) $\underline{J}^*(\delta)\le\overline{J}^*(\delta)$ for $\frac{w_1}{3}\le\delta\le\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}$.

Definition 4
For two arbitrary FNs in parametric form
$A=\left[\underline{a}(r)\overline{a}(r)\right]$, and $B=\left[\underline{b}(r)\overline{b}(r)\right]$, $0\le r\le 1$, with corresponding TFQ in parametric form
$A^*=\left[\underline{A}^*(\delta),\overline{A}^*(\delta)\right]$ and $B^*=\left[\underline{B}^*(\delta),\overline{B}^*(\delta)\right]$, $\delta\in\left[\frac{w_1}{3},\frac{{w_1}^2+{w_2}^2+w_1w_2}{3(w_1+w_2)}\right]$, the addition (A^*+B^*) and scalar multiplication by c are defined as follows:

$\left(\underline{A^*+B^*}\right)\delta=\underline{A}^*(\delta)+\underline{B}^*(\delta)$
$\left(\overline{A^*+B^*}\right)\delta=\overline{A}^*(\delta)+\overline{B}^*(\delta)$
$\left(\underline{cA^*}\right)(\delta)=c\underline{A}^*(\delta)$ for $c\ge 0$
$\left(\underline{cA^*}\right)(\delta)=c\overline{A}^*(\delta)$ for $c<0$
$\left(\overline{cA^*}\right)(\delta)=c\,\overline{A}^*(\delta)$ for $c\ge 0$
$\left(\overline{cA^*}\right)(\delta)=c\,\underline{A}^*(\delta)$ for $c<0$.

Definition 5
The *VAL* of the GTFNLRH, $J=(a,b,c,d;w_1,w_2)$ where $w_1<w_2$ is defined by using Eq. (4).

$$VAL(J)=\int_{\frac{w_1}{3}}^{\frac{w_2}{3}}\left[\underline{J}^*(\delta)+\overline{J}^*(\delta)\right]d\,\delta+\int_{\frac{w_2}{3}}^{\frac{{w_1}^2+{w_2}^2+w_1w_2}{3(w_1+w_2)}}\left[\underline{J}^*(\delta)+\overline{J}^*(\delta)\right]d\delta \quad (5)$$

On integrating Eq. (5), we get

$$VAL(J)=\frac{1}{18(w_1+w_2)^2}\left[\begin{array}{c}{w_1}^3(2b+3c+d)+{w_2}^3(a+3b+2c)\\+w_1{w_2}^2(a+4b+c)+{w_1}^2w_2(b+4c+d)\end{array}\right] \quad (6)$$

Definition 6
The *AMB* of the GTFNLRH, $J=(a,b,c,d;w_1,w_2)$ where $w_1<w_2$ is defined by using Eq. (4).

$$AMB(J)=\int_{\frac{w_1}{3}}^{\frac{w_2}{3}}\left[\overline{J}^*(\delta)-\underline{J}^*(\delta)\right]d\,\delta+\int_{\frac{w_2}{3}}^{\frac{{w_1}^2+{w_2}^2+w_1w_2}{3(w_1+w_2)}}\left[\overline{J}^*(\delta)-\underline{J}^*(\delta)\right]d\delta \quad (7)$$

Integrating Eq. (7), we get

$$AMB(J)=\frac{1}{18(w_1+w_2)^2}\left[\begin{array}{c}{w_1}^3(2b+3c+d)-{w_2}^3(a+3b+2c\\-w_1{w_2}^2(a+4b+c)+{w_1}^2w_2(b+4c+d)\end{array}\right] \quad (8)$$

Definition 7
The Parametric form for the TFQ, J^* given by Eq. (2) with decision level $\delta\in\left[\frac{w_2}{3},\frac{{w_1}^2+{w_2}^2+w_1w_2}{3(w_1+w_2)}\right]$, where $w_2<w_1$ and $0\le w_2\le w_1\le 1$ is defined as:

$$\left[\underline{J}^*(\delta),\overline{J}^*(\delta)\right]=\begin{cases}\left[0,\frac{2c+d}{3}+\left(\frac{w_1(2b-c-d)+w_2(b-d)}{{w_1}^2}\right)\left(\delta-\frac{w_2}{3}\right)\right], & if\ \frac{w_2}{3}\le\delta\le\frac{w_1}{3}\\ \left[\begin{array}{l}\frac{a+2b}{3}+\left(\frac{w_1(c-a)+w_2(2c-b-a)}{{w_2}^2}\right)\left(\delta-\frac{w_1}{3}\right),\\ \frac{2c+d}{3}+\left(\frac{w_1(2b-c-d)+w_2(b-d.)}{{w_1}^2}\right)\left(\delta-\frac{w_2}{3}\right)\end{array}\right], & if\ \frac{w_1}{3}\le\delta\le\frac{{w_1}^2+{w_2}^2+w_1w_2}{3(w_1+w_2)}\end{cases} \quad (9)$$

and satisfy the following conditions:
(i) $\underline{J}^*(\delta)$ is an increasing left continuous bounded function on $\left[\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)},\frac{w_1}{3}\right]$,
(ii) $\overline{J}^*(\delta)$ is a decreasing left continuous bounded function on $\left[\frac{w_2}{3},\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}\right]$,
(iii) $\underline{J}^*(\delta)\le\overline{J}^*(\delta)$ for $\frac{w_2}{3}\le\delta\le\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}$.

Definition 8
For two arbitrary FNs in parametric form
$A = [\underline{a}(r)\overline{a}(r)]$, and $B = [\underline{b}(r)\overline{b}(r)]$, $0 \le r \le 1$, with corresponding TFQ in parametric form
$A^* = \left[\underline{A}^*(\delta), \overline{A}^*(\delta)\right]$ and $B^* = \left[\underline{B}^*(\delta), \overline{B}^*(\delta)\right]$, $\delta \in \left[\frac{w_2}{3}, \frac{{w_1}^2+{w_2}^2+w_1w_2}{3(w_1+w_2)}\right]$, the addition $(A^* + B^*)$ and scalar multiplication by c are defined as follows:

$$\left(\underline{A^* + B^*}\right)\delta = \underline{A}^*(\delta) + \underline{B}^*(\delta)$$
$$\left(\overline{A^* + B^*}\right)\delta = \overline{A}^*(\delta) + \overline{B}^*(\delta)$$
$$\left(\underline{cA^*}\right)(\delta) = c\underline{A}^*(\delta) \text{ for } c \ge 0$$
$$\left(\underline{cA^*}\right)(\delta) = c\overline{A}^*(\delta) \text{ for } c < 0$$
$$\left(\overline{cA^*}\right)(\delta) = c\,\overline{A}^*(\delta) \text{ for } c \ge 0$$
$$\left(\overline{cA^*}\right)(\delta) = c\,\underline{A}^*(\delta) \text{ for } c < 0.$$

Definition 9
The *VAL* of the GTFNLRH, $J = (a, b, c, d; w_1, w_2)$ where $w_2 < w_1$ is defined by using Eq. (9).

$$VAL(J) = \int_{\frac{w_2}{3}}^{\frac{w_1}{3}} \left[\underline{J}^*(\delta) + \overline{J}^*(\delta)\right] d\,\delta + \int_{\frac{w_1}{3}}^{\frac{{w_1}^2+{w_2}^2+w_1w_2}{3(w_1+w_2)}} \left[\underline{J}^*(\delta) + \overline{J}^*(\delta)\right] d\delta \tag{10}$$

On integrating Eq. (10), we get

$$VAL(J) = \frac{1}{18(w_1+w_2)^2}\begin{bmatrix} {w_1}^3(2b+3c+d) + {w_2}^3(a+3b+2c) \\ +w_1{w_2}^2(a+4b+c) + {w_1}^2w_2(b+4c+d) \end{bmatrix} \tag{11}$$

Definition 10
The *AMB* of the GTFNLRH, $J = (a, b, c, d; w_1, w_2)$, where $w_1 < w_2$ is defined by using Eq. (9).

$$AMB(J) = \int_{\frac{w_2}{3}}^{\frac{w_1}{3}} \left[\overline{J}^*(\delta) - \underline{J}^*(\delta)\right] d\,\delta + \int_{\frac{w_1}{3}}^{\frac{{w_1}^2+{w_2}^2+w_1w_2}{3(w_1+w_2)}} \left[\overline{J}^*(\delta) - \underline{J}^*(\delta)\right] d\delta \tag{12}$$

Integrating Eq. (12), we get

$$AMB(J) = \frac{1}{18(w_1+w_2)^2}\begin{bmatrix} {w_1}^3(2b+3c+d) - {w_2}^3(a+3b+2c \\ -w_1{w_2}^2(a+4b+c) + {w_1}^2w_2(b+4c+d) \end{bmatrix} \tag{13}$$

Definition 11
If $w_1 = w_2 = w$, then GTFNLRH will become a GTFN and the *VAL* and *AMB* in decision level $\left[\frac{w}{3}, \frac{w}{2}\right]$ are given by:

$$VAL(J) = \frac{w}{36}(a + 5b + 5c + d) \tag{14}$$
$$AMB(J) = \frac{w}{36}(-a - 2b + 2c + d) \tag{15}$$

Definition 12
If $J_1 = (a_1, b_1, c_1, d_1; w_1, w_2)$ and $J_2 = (a_2, b_2, c_2, d_2; w_1, w_2)$ are two GTFNLRH, then the following ranking order is defined based on the definition of *VAL* and *AMB* given by Eqns. (6), (8), (11), and (13).
For both cases $w_1 \ne w_2$ and $w_1 = w_2$
(1) If $VAL(J_1) \succ VAL(J_2)$, then $J_1 \succ J_2$
(2) If $VAL(J_1) \prec VAL(J_2)$, then $J_1 \prec J_2$
(3) If $VAL(J_1) = VAL(J_2)$, then
 (a) if $AMB(J_1) \succ AMB(J_2)$, then $J_1 \prec J_2$,
 (b) if $AMB(J_1) \prec AMB(J_2)$, then $J_1 \succ J_2$,
 (c) if $AMB(J_1) = AMB(J_2)$, then $J_1 \sim J_2$.

I. Propositions and Reasonable Properties

This section presents propositions related to *VAL*, and *AMB* of the GTFNLRH with reference to TFQ in lower decision levels. The reasonable properties for ranking FNs the proposed approach satisfy are also presented.

Proposition 1
If $J = (a, b, c, d; w_1, w_2)$ is a GTFNLRH, and J^* is the TFQ of J with parametric representation $\left[\underline{J}^*(\delta), \overline{J}^*(\delta)\right]$, then $Val(kJ) = k\,VAL(J)$ and $AMB(kJ) = k\,AMB(J)$ for scalar k.

Proof. Case (i) for $w_1 < w_2$
By using Eq. (5), we get

$$VAL(kJ) = \int_{\frac{w_1}{3}}^{\frac{w_2}{3}} \left[k\underline{J}^*(\delta) + k\overline{J}^*(\delta)\right] d\,\delta + \int_{\frac{w_2}{3}}^{\frac{{w_1}^2+{w_2}^2+w_1w_2}{3(w_1+w_2)}} \left[k\underline{J}^*(\delta) + k\overline{J}^*(\delta)\right] d\delta = kVAL(J)$$

By using Eq. (7), we get

$$AMB(kJ) = \int_{\frac{w_1}{3}}^{\frac{w_2}{3}} \left[k\overline{J}^*(\delta) - k\underline{J}^*(\delta)\right] d\,\delta + \int_{\frac{w_2}{3}}^{\frac{{w_1}^2+{w_2}^2+w_1w_2}{3(w_1+w_2)}} \left[k\overline{J}^*(\delta) - k\underline{J}^*(\delta)\right] d\delta = kAMB(J)$$

Case(ii) for $w_2 < w_1$
By using Eq. (10), we get

$$Val(kJ) = \int_{\frac{w_2}{3}}^{\frac{w_1}{3}} \left[k\underline{J}^*(\delta) + k\overline{J}^*(\delta)\right] d\,\delta + \int_{\frac{w_1}{3}}^{\frac{{w_1}^2+{w_2}^2+w_1w_2}{3(w_1+w_2)}} \left[k\underline{J}^*(\delta) + k\overline{J}^*(\delta)\right] d\delta = kVAL(J)$$

By using Eq. (12), we get

$$AMB(kJ) = \int_{\frac{w_2}{3}}^{\frac{w_1}{3}} [kJ^*(\delta) - kJ^*(\delta)] d\,\delta + \int_{\frac{w_1}{3}}^{\frac{{w_1}^2+{w_2}^2+w_1w_2}{3(w_1+w_2)}} \left[k\overline{J}^*(\delta) - kJ^*(\delta)\right] d\delta = kAMB(J)$$

For $k = -1$, $VAL(-J) = -VAL(J)$, and $AMB(-J) = -AMB(J)$ for $w_1 < w_2$, and $w_2 < w_1$.

Proposition 2
Case (i) If $J_1 = (a_1, b_1, c_1, d_1; w_1, w_2)$ and $J_2 = (a_2, b_2, c_2, d_2; w_1, w_2)$ are two GTFNLRH, and J_1^*, J_2^* are the TFQ of J_1, J_2 with parametric representation $\left[\underline{J_1}^*(\delta), \overline{J_1}^*(\delta)\right]$, $\left[\underline{J_2}^*(\delta), \overline{J_2}^*(\delta)\right]$ respectively, then $VAL(J_1 + J_2) = VAL(J_1) + VAL(J_2)$
$AMB(J_1 + J_2) = AMB(J_1) + AMB(J_2)$

Proof. Case (i) for $w_1 < w_2$
By using Eq. (5), we get

$$VAL(J_1 + J_2) = \int_{\frac{w_1}{3}}^{\frac{w_2}{3}} \begin{bmatrix} \underline{J_1^*(\delta) + J_2^*(\delta)} + \\ \overline{J_1^*(\delta) + J_2^*(\delta)} \end{bmatrix} d\delta + \int_{\frac{w_2}{3}}^{\frac{{w_1}^2+{w_2}^2+w_1w_2}{3(w_1+w_2)}} \begin{bmatrix} \underline{J_1^*(\delta) + J_2^*(\delta)} + \\ \overline{J_1^*(\delta) + J_2^*(\delta)} \end{bmatrix} d\delta$$

$$= \left\{ \begin{array}{c} \int_{\frac{w_1}{3}}^{\frac{w_2}{3}} \left[\underline{J_1}^*(\delta) + \overline{J_1}^*(\delta) \right] d\,\delta \\ + \int_{\frac{w_2}{3}}^{\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}} \left[\underline{J_1}^*(\delta) + \overline{J_1}^*(\delta) \right] d\delta \end{array} \right\}.$$

$$+ \left\{ \begin{array}{c} \int_{\frac{w_1}{3}}^{\frac{w_2}{3}} \left[\underline{J_2}^*(\delta) + \overline{J_2}^*(\delta) \right] d\,\delta \\ + \int_{\frac{w_2}{3}}^{\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}} \left[\underline{J_2}^*(\delta) + \overline{J_2}^*(\delta) \right] d\delta \end{array} \right\}$$

$= VAL(J_1) + VAL(J_2)$.

By using the Eq. (7), we get

$$AMB(J_1+J_2) = \int_{\frac{w_1}{3}}^{\frac{w_2}{3}} \left[\begin{array}{c} \overline{J_1^*(\delta) + J_2^*(\delta)} \\ -\left(\underline{J_1^*(\delta) + J_2^*(\delta)} \right) \end{array} \right] d\,\delta + \int_{\frac{w_2}{3}}^{\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}} \left[\begin{array}{c} \overline{J_1^*(\delta) + J_2^*(\delta)} \\ -\left(\underline{J_1^*(\delta) + J_2^*(\delta)} \right) \end{array} \right] d\delta$$

$$= \left\{ \begin{array}{c} \int_{\frac{w_1}{3}}^{\frac{w_2}{3}} \left[\overline{J_1}^*(\delta) - \underline{J_1}^*(\delta) \right] d\,\delta \\ + \int_{\frac{w_2}{3}}^{\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}} \left[\overline{J_1}^*(\delta) - \underline{J_1}^*(\delta) \right] d\delta \end{array} \right\}$$

$$+ \left\{ \begin{array}{c} \int_{\frac{w_1}{3}}^{\frac{w_2}{3}} \left[\underline{J_2}^*(\delta) - \overline{J_2}^*(\delta) \right] d\,\delta \\ + \int_{\frac{w_2}{3}}^{\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}} \left[\underline{J_2}^*(\delta) - \overline{J_2}^*(\delta) \right] d\delta \end{array} \right\}$$

$= AMB\ (J_1) + AMB(J_2)$.

Case (ii) for $w_2 < w_1$

By using Eq. (10), we get

$$VAL(J_1+J_2) = \int_{\frac{w_2}{3}}^{\frac{w_1}{3}} \left[\begin{array}{c} \underline{J_1^*(\delta) + J_2^*(\delta)} \\ + \overline{J_1^*(\delta) + J_2^*(\delta)} \end{array} \right] d\,\delta + \int_{\frac{w_1}{3}}^{\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}} \left[\begin{array}{c} \underline{J_1^*(\delta) + J_2^*(\delta)} \\ + \overline{J_1^*(\delta) + J_2^*(\delta)} \end{array} \right] d\delta$$

$$= \left\{ \begin{array}{c} \int_{\frac{w_2}{3}}^{\frac{w_1}{3}} \left[\underline{J_1}^*(\delta) + \overline{J_1}^*(\delta) \right] d\,\delta \\ + \int_{\frac{w_1}{3}}^{\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}} \left[\underline{J_1}^*(\delta) + \overline{J_1}^*(\delta) \right] d\delta \end{array} \right\}$$

$$+ \left\{ \begin{array}{c} \int_{\frac{w_2}{3}}^{\frac{w_1}{3}} \left[\underline{J_2}^*(\delta) + \overline{J_2}^*(\delta) \right] d\,\delta \\ + \int_{\frac{w_1}{3}}^{\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}} \left[\underline{J_2}^*(\delta) + \overline{J_2}^*(\delta) \right] d\delta \end{array} \right\}$$

$= VAL\ (J_1) + VAL(J_2)$.

By using the Eq. (7), we get

$$AMB(J_1+J_2) = \int_{\frac{w_1}{3}}^{\frac{w_2}{3}} \left[\begin{array}{c} \overline{J_1^*(\delta) + J_2^*(\delta)} \\ -\left(\underline{J_1^*(\delta) + J_2^*(\delta)} \right) \end{array} \right] d\,\delta + \int_{\frac{w_2}{3}}^{\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}} \left[\begin{array}{c} \overline{J_1^*(\delta) + J_2^*(\delta)} \\ -\left(\underline{J_1^*(\delta) + J_2^*(\delta)} \right) \end{array} \right] d\delta$$

$$= \left\{ \begin{array}{c} \int_{\frac{w_1}{3}}^{\frac{w_2}{3}} \left[\overline{J_1}^*(\delta) - \underline{J_1}^*(\delta) \right] d\,\delta \\ + \int_{\frac{w_2}{3}}^{\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}} \left[\overline{J_1}^*(\delta) - \underline{J_1}^*(\delta) \right] d\delta \end{array} \right\}$$

$$+ \left\{ \begin{array}{c} \int_{\frac{w_1}{3}}^{\frac{w_2}{3}} \left[\underline{J_2}^*(\delta) - \overline{J_2}^*(\delta) \right] d\,\delta \\ + \int_{\frac{w_2}{3}}^{\frac{w_1^2+w_2^2+w_1w_2}{3(w_1+w_2)}} \left[\underline{J_2}^*(\delta) - \overline{J_2}^*(\delta) \right] d\delta \end{array} \right\}$$

$= AMB\ (J_1) + AMB(J_2)$.

Proposition 3

If J_1 and J_2 are two GTFNLRH, then

$VAL(J_1 - J_2) = VAL(J_1) - VAL(J_2)$

$AMB(J_1 - J_2) = AMB(J_1) - AMB(J_2)$.

Proof. This is a consequence of Propositions 1 and 2.

$VAL(J_1 - J_2) = VAL(J_1 + (-J_2))$

$= VAL\ (J_1) + VAL(-J_2) = VAL(J_1) - VAL(J_2)$

$AMB(J_1 - J_2) = AMB(J_1 + (-J_2)$

$= AMB\ (J_1) + AMB(-J_2) = AMB(J_1) - AMB(J_2)$

Proposition 4

If $J = (0,0,0,0; w_1, w_2)$ then $VAL(J) = 0$, and $AMB(J) = 0$ for both $w_1 < w_2$ and $w_2 < w_1$.

Proof. The proof is a direct consequence of Eq. (5), Eq. (7), Eq. (10) & Eq. (14).

Proposition 5

If $J = (x, x, x, x; w, w)$ is a crisp number, then $VAL(J) = 1/3$ and $AMB(J) = 0$.

Proof. The proof is a direct consequence of Eq. (13), & Eq. (14) by substituting $a = b = c = d = x$.

Proposition 6

If J_1, J_2 and J_3 are three arbitrary GTFNLRH, such that $J_1 \prec J_2$ and $J_2 \prec J_3$ then, $J_1 \prec J_3$ provides order by the values.

Proof. By using Definition 12, for $J_1 \prec J_2, VAL(J_1) < VAL(J_2)$ and for $J_2 \prec J_3, VAL(J_2) < VAL(J_3)$.

This implies that $VAL\ (J_1) < VAL\ (J_3)$, and hence, $J_1 \prec J_3$.

Reasonable Properties (Wang & Kerre, [33])

Wang and Kerre [33] proposed some axioms that serve as reasonable properties for the ordering of fuzzy quantities. Let $\mathcal{A}$ be a finite subset of the set of all fuzzy quantities $\mathcal{F}$ and let $\mathcal{R}$ be the proposed ranking approach. The proposed ranking approach satisfies the following axioms:

Axiom 1: If $A \in \mathcal{A}$, then $A \succcurlyeq A$ by the ranking approach $\mathcal{R}$ on $\mathcal{A}$.

Axiom 2: If $(A, B) \in \mathcal{A}^2$, and $A \succcurlyeq B, B \succcurlyeq A$ by the ranking approach $\mathcal{R}$ on $\mathcal{A}$, then $A \sim B$ on $\mathcal{A}$.

Axiom 3: If $(A, B, C) \in \mathcal{A}^3$, and $A \succcurlyeq B, B \succcurlyeq C$ by the ranking approach $\mathcal{R}$ on $\mathcal{A}$, then $A \succcurlyeq C$ on $\mathcal{A}$.

Proof: By using Definition (12),

for $A \succcurlyeq B, VAL(A) \geq VAL(B)$,

and for $B \succcurlyeq C, VAL(B) \geq VAL(C)$.

This implies $Val(A) \geq VAL(C)$, and hence, $A \succcurlyeq C$.

Axiom 4: Let $A, B, A + C, B + C$ be elements of $\mathcal{F}$. If $A \succcurlyeq B$ by $\mathcal{R}$, then $A + C \succcurlyeq B + C$ by $\mathcal{R}$.

Proof: Let $A \succcurlyeq B$,

then by Definition (12), $VAL(A) \geq VAL(B)$,

$\Longrightarrow Val(A) + VAL(C) \geq VAL(B) + VAL(C)$

$\Longrightarrow Val(A+C) \geq VAL(B+C)$ (By Proposition 3)
$\Longrightarrow (A+C) \geq (B+C)$ (By Definition 12).

II. Comparative Study

In this section, a comparative study is carried out with six sets of GTFNLRH and six sets of the most frequently used triangular and trapezoidal FNs by decision makers.

A. *Comparative study for GTFNLRH*

This section presents a comparative study of the suggested approach with the methods presented in [17, 18,19, 20] using six sets of GTFNLRH cited from different works in the existing literature on ranking FNs.

Set 1: Consider two GTFNLRH with different cores cited from the work of Pushpinder Singh [17], shown in Figure 3.

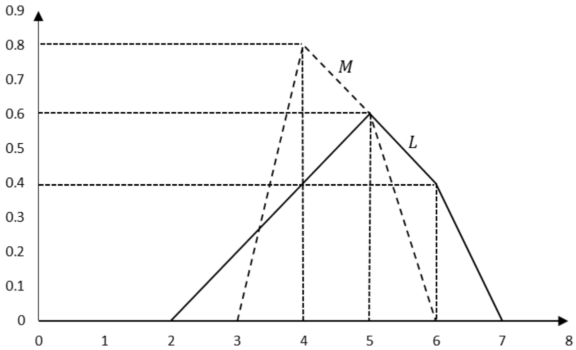

Fig. 3. $L = (2, 5, 6, 7; 0.6, 0.4)$
$M = (3, 4, 5, 6; 0.8, 0.6)$.

Using (10), $VAL(L)$ = 0.9604, and $VAL(M)$ = 1.0963.
As $VAL(L) < VAL(M) \Longrightarrow L \prec M$. This result is consistent with the result of Pushpinder Singh [17].

Set 2: Consider two GTFNLRH, cited from the work of Rituparna et al. [19], shown in Figure 4.

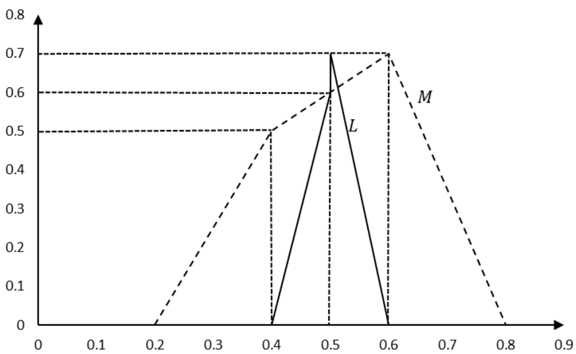

Fig. 4. $L = (0.4, 0.5, 0.5, 0.6; 0.6, 0.7)$
$M = (0.2, 0.4, 0.6, 0.8; 0.5, 0.6)$

Using (5), $VAL(L)$ = 0.1084, and $VAL(M)$ = 0.0899.
As $VAL(M) < VAL(L) \Longrightarrow M \prec L$. This result coincides with the result of Rituparna et al. [19].

Set 3: Consider three GTFNLRH, cited from the work of Barazandeh et al. [20], shown in Figure 5.
Using (5), $VAL(L)$ = 0.09256, $VAL(M)$ = 0.12186, and $VAL(N)$ = 0.153657.
As $VAL(L) < VAL(M) < VAL(N) \Longrightarrow L \prec M \prec N$. This result is consistent with the result of Barazandeh et al. [20].

Set 4: Consider a GTrFN and two GTFNLRH, taken from Jiang et al. [18], shown in Figure 6.

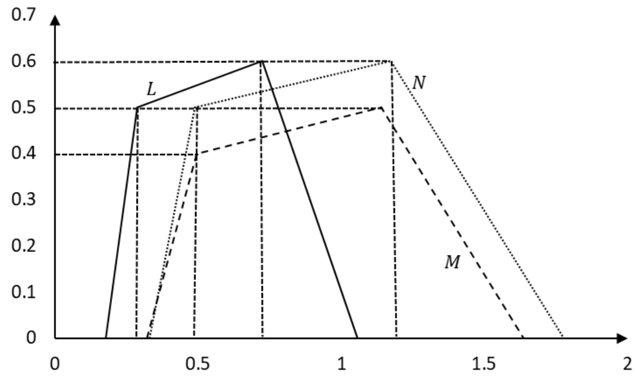

Fig. 5. $L = (0.1765, 0.2860, 0.7244, 1.0574; 0.5, 0.6)$
$M = (0.3221, 0.4949, 1.1392, 1.6373; 0.4\ 0.5)$
$N = (0.3290, 0.4890, 1.1737, 1.7787; 0.5, 0.6)$

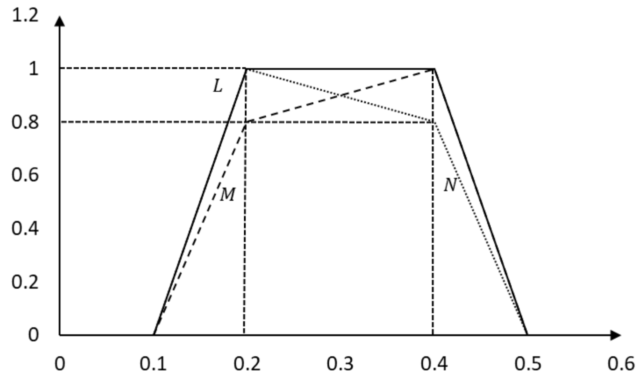

Fig. 6. $L = (0.1, 0.2, 0.4, 0.5; 1, 1)$
$M = (0.1, 0.2, 0.4, 0.5; 0.8, 1)$
$N = (0.1, 0.2, 0.4, 0.5; 1, 0.8)$

Using (13), (5), and (10), we get $VAL(L)$ = 0.1, $VAL(M)$ = 0.087229, and $VAL(N)$ = 0.094993.
As $VAL(M) < VAL(N) < VAL(L) \Longrightarrow M \prec N \prec L$.
As L is a normal FN, one must prefer L over M and N.
However, the ranking orders of Jiang et al. [18] and Barazandeh et al. [20] are $N \prec L \prec M$ is inconsistent with human intuition. They prefer N over L.

Set 5: Consider the images of GTFNLRH of Set 4, taken from Jiang et al. [18], shown in Figure 7.
Using (13), (5), and (10), we get $VAL(L)$ = −0.1, $VAL(M)$ = −0.08722, and $VAL(N)$ = −0.09499.
As $VAL(M) > VAL(N) > VAL(L) \Longrightarrow M > N > L$.
This result is consistent with Set 4 and satisfies the symmetry between Set 4 and Set 5.

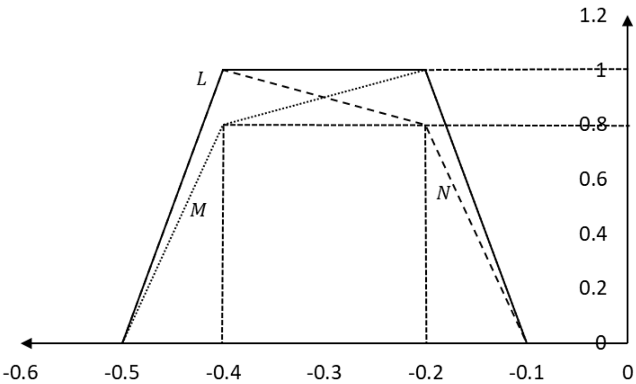

Fig. 7. $L = (-0.5, -0.4, -0.2, -0.1; 1, 1)$
$M = (-0.5, -0.4, -0.2, -0.1; 1, 0.8)$
$N = (-0.5, -0.4, -0.2, -0.1; 0.8, 1)$

Set 6: Consider two GTFNLRH, taken from Pushpinder Singh [17], shown in Figure 8.
Using (10), $VAL(L)$ = 0.4582, and $VAL(M)$ = 0.4061.
As $VAL(M) < VAL(L) \Longrightarrow M \prec L$.
However, Pushpinder Singh's [17] ranking result is inconsistent with human intuition.

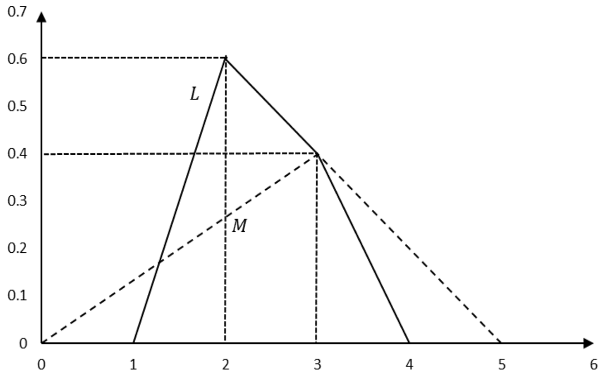

Fig. 8. $L = (1, 2, 3, 4; 0.6, 0.4)$
$M = (0, 3, 4, 5; 0.4, 0.2)$.

B. Comparative study on frequently cited trapezoidal and triangular FNs

This section demonstrates a comparative study of the proposed approach with six sets of frequently cited trapezoidal and triangular FNs taken from [5]. A comparative study of the proposed method is carried out against the methods presented in [4, 14, 16, 22, 24, 28], and the results are presented in Table I.
Set 1: A = (0.1,0.3,0.3,0.5), B = (0.3,0.5,0.5,0.7)
Set 2: A = (0.1,0.2,0.4,0.5), B = (0.1,0.3,0.3,0.5)
Set 3: A = (0.1,0.3,0.3,0.5), B = (0.2,0.3,0.3,0.4)
Set 4: A =(0.1,0.3,0.3,0.5;0.8,0.8),B =(0.1,0.3,0.3,0.5)
Set 5: A = (0.1,0.2,0.4,0.5), B = (1,1,1,1)
Set 6: A = (0.1,0.3,0.3,0.5), B = (-0.5,-0.3,-0.3,-0.1)
The conclusions derived from Table I are:
Set 1: The ranking order is consistent with all fuzzy ranking methods in Table I.
Set 2 and Set 3: The FNs, A and B, have the same VAL. Therefore, the ranking order is decided by using AMB. B's AMB is smaller than A's AMB, so we prefer B to A. This result is consistent with Chen and Chen’s [24] method and Wu et al.’s [27] method, as the spread of B is smaller than the spread of A. Yager’s [4], Chu and Tsao’s [14], Chen and Sanguansat’s [22], and Chen et al.’s [16] methods failed to discriminate FNs, A and B.
Set 4: The ranking order is consistent with all fuzzy ranking methods mentioned in Table I except Yager’s [4] method. Although the centroids of FNs A and B differ, Yager [4] failed to discriminate FNs.
Set 5: The ranking order is consistent with all fuzzy ranking methods mentioned in Table I except Yager’s [4] and Chu and Tsao’s [14] methods. Yager [4] and Chu and Tsao [14] failed to discriminate FNs A and B, as FN B is a crisp number.
Set 6: The ranking order is consistent with all fuzzy ranking methods in Table I.

III. Application-Fuzzy Risk Analysis

This section outlines a fuzzy risk assessment procedure to quantify the risk associated with product manufacturing by different companies. The evaluation criteria are expressed as linguistic terms represented by GTFNLRH. The risk analysis algorithm assesses both the likelihood of an adverse event and the severity of its potential consequences, and the ranking function developed in Section III is utilized to evaluate risk, magnitude of loss, and the likelihood of failure for individual components.
Fuzzy risk analysis was proposed by Schmucker [28]. Consider n companies $\tilde{F}_i$, $1 \leq i \leq n$, producing the products $\tilde{A}_i$, $1 \leq i \leq n$ and each product is made with p sub-products $\tilde{A}_{ik}$, $1 \leq k \leq n$. To evaluate the failure probability $\tilde{P}_i$of the product $\tilde{A}_i$, we use the assessing terms $\tilde{P}_{ik}$ and $\tilde{Q}_{ik}$ called “likelihood of failure” and “magnitude of loss” of the sub-products $\tilde{A}_{ik}$, both are GTFNs, represented as linguistic terms. Chen et al. [16] used a linguistic term set comprising nine members to represent linguistic values. Each member of the linguistic term set corresponds to a GTFN, as shown in Table II. Figure 9. Shows the structure of fuzzy risk analysis proposed by Schmucker [28].

The set of rules for evaluating fuzzy risk by the proposed ranking method is presented in the following steps:

Step 1: Compute the failure probability $\tilde{P}_i$ for product $\tilde{A}_i$ for $1 \leq i \leq n$ which is a GTFNLRH. By using the fuzzy weighted average method (Chen et al.[24] and Schmucker [28]) and GTFNs, arithmetic operations $\oplus, \otimes$ and $\oslash$ defined in section II, the assessing terms $\tilde{P}_{ik}$ and $\tilde{Q}_{ik}$ of the sub-product $\tilde{A}_i$ are aggregated to obtain $\tilde{P}_i$, as shown below: $\tilde{P}_i = \left[\sum_{i=1}^{n} \tilde{P}_{ik} \otimes \tilde{Q}_{ik}\right] \oslash \ \sum_{i=1}^{n} \tilde{Q}_{ik}$
i.e., $\tilde{P}_i = \left(p_{i1}, p_{i2}, p_{i3}, p_{i4}; w_{L\tilde{P}_i}, w_{R\tilde{P}_i}\right)$ (16)
where $\tilde{P}_i$ is a GTFLRH and $1 \leq i \leq n$.

Step 2: Using (10) of the proposed ranking method, evaluate the *VAL* of each GTFLRH, $\tilde{P}_i$ for $w_1 \neq w_2$ and $1 \leq i \leq n$

$$Val(\tilde{P}_i) = \frac{1}{18(w_{L\tilde{P}_i}+w_{R\tilde{P}_i})^2}\left[w_{L\tilde{P}_i}{}^3(2p_{i2}+3p_{i3}+p_{i4}) + w_{R\tilde{P}_i}{}^3(p_{i1}+3p_{i2}+2p_{i3}) + w_{L\tilde{P}_i}w_{R\tilde{P}_i}{}^2(p_{i1}+4p_{i2}+p_{i3}) + w_{L\tilde{P}_i}{}^2 w_{R\tilde{P}_i}(p_{i2}+4p_{i3}+p_{i4})\right] \quad (17)$$

Step 3: The higher the *VAL* $\tilde{P}_i$, the more likelihood of failure $\tilde{P}_i$ of product $\tilde{A}_i$ manufactured by company $\tilde{F}_i$.

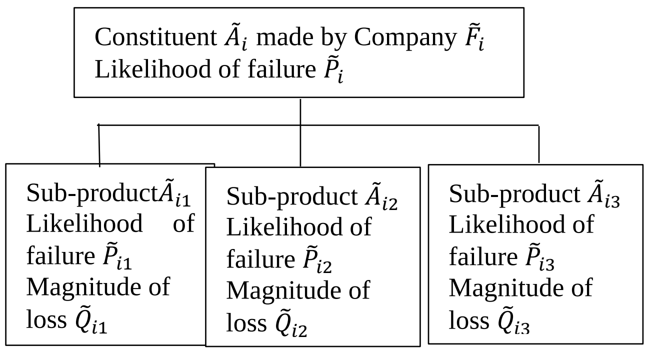

Fig. 9. Structure of Fuzzy Risk Analysis [28]

Numerical Example

Let there be three companies $\tilde{F}_1$, $\tilde{F}_2$ and $\tilde{F}_3$, producing the components $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$ respectively. For each $\tilde{A}_i$ there are three sub-products $\tilde{A}_{i1}$, $\tilde{A}_{i2}$ and $\tilde{A}_{i3}$, $1 \leq i \leq 3$.

The magnitude of loss $\tilde{Q}_{ik}$, the likelihood of failure $\tilde{P}_{ik}$, for sub-products $\tilde{A}_{ik}$ (Chen et al.[24]) made by company $\tilde{F}_i$, $1 \le i \le 3$ for $1 \le k \le 3$ are shown in Table III. The proposed fuzzy risk analysis approach is presented in the following steps:

Step 1: Using (16), the likelihood of failure $\tilde{P}_{ik}$ of product $\tilde{A}_i$ made by company $\tilde{F}_i$ is obtained by grouping the estimated items $\tilde{P}_{ik}$ and $\tilde{Q}_{ik}$ of the sub-products $\tilde{A}_{ik}$, shown in Table III, where $1 \le i \le 3$ and $1 \le k \le 3$.

$$\tilde{P}_1 = \begin{bmatrix} (\tilde{P}_{11} \otimes \tilde{Q}_{11} \oplus \tilde{P}_{12} \otimes \tilde{Q}_{12} \oplus \tilde{P}_{13} \otimes \tilde{Q}_{13}) \\ \oslash (\tilde{Q}_{11} \oplus \tilde{Q}_{12} \oplus \tilde{Q}_{13}) \end{bmatrix}$$

$$= (0.1765, 0.2860, 07244, 1.0574; 0.5, 0.6)$$

$$\tilde{P}_2 = \begin{bmatrix} (\tilde{P}_{21} \otimes \tilde{Q}_{21} \oplus \tilde{P}_{22} \otimes \tilde{Q}_{22} \oplus \tilde{P}_{23} \otimes \tilde{Q}_{23}) \\ \oslash (\tilde{Q}_{21} \oplus \tilde{Q}_{22} \oplus \tilde{Q}_{23}) \end{bmatrix}$$

$$= (0.3221, 0.4949, 1.1392, 1.6373; 0.4, 0.5)$$

$$\tilde{P}_3 = \begin{bmatrix} (\tilde{P}_{31} \otimes \tilde{Q}_{31} \oplus \tilde{P}_{32} \otimes \tilde{Q}_{32} \oplus \tilde{P}_{33} \otimes \tilde{Q}_{33}) \\ \oslash (\tilde{Q}_{31} \oplus \tilde{Q}_{32} \oplus \tilde{Q}_{33}) \end{bmatrix}$$

$$= (0.3290, 0.4890, 1.1737, 1.7787; .5, 0.6)$$

Step 2: Using (16), the VAL $\tilde{P}_i$, $1 \le i \le 3$ for each GTFNLRH $\tilde{P}_i$, are $VAL(\tilde{P}_1) = 0.0564$, $VAL(\tilde{P}_2) = 0.1218$, and $VAL(\tilde{P}_3) = 0.1530$.

Step 3: As $VAL(\tilde{P}_3) > VAL(\tilde{P}_2) > VAL(\tilde{P}_1)$, the ranking order of the GTFNLRH $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3$ is $\tilde{P}_3 \succ \tilde{P}_2 \succ \tilde{P}_1$. Therefore, the order of the risk of companies $\tilde{F}_{1,} \tilde{F}_2$ and $\tilde{F}_3$ is $\tilde{F}_3 \succ \tilde{F}_2 \succ \tilde{F}_1$. This implies that the product $\tilde{A}_3$ made by the company $\tilde{F}_3$ has the highest probability of failure, and $\tilde{F}_1$ has the lowest probability of failure. The proposed method is consistent with the result of Chen et al. [24].

TABLE I

COMPARATIVE STUDY OF THE PROPOSED METHOD WITH EXISTING FUZZY RANKING METHODS

Methods	Set 1		Set 2		Set 3	
	A	B	A	B	A	B
Yager [4]	0.3000	0.5000	0.3000	0.3000	0.3000	0.3000
Chu & Tsao [4]	0.1500	0.2500	0.1500	0.1500	0.1500	0.1500
Chen & Chen [24]	0.2579	0.4298	0.2573	0.2579	0.2579	0.2774
Chen & Sanguansat [22]	0.3000	0.5000	0.3000	0.3000	0.3000	0.3000
Chen et al. [16]	0.2553	0.4444	0.2553	0.2553	0.2553	0.2553
Wu et al. [27]	0.5906	07014	0.5884	0.5906	0.5906	0.6006
Proposed Method	0.1000	0.1667	0.0222	0.0111	0.0111	0.0055
Methods	Set 1		Set 2		Set 3	
	A	B	A	B	A	B
Yager [4]	0.3000	0.3000	0.3000	#	0.3000	-0.3000
Chu & Tsao [4]	0.1200	0.1500	0.1500	#	0.1500	-0.1500
Chen & Chen [24]	0.2063	0.2579	0.2537	1.0000	0.2579	-0.2579
Chen & Sanguansat [22]	0.2824	0.3000	0.3000	1.0000	0.3000	-0.3000
Chen et al. [16]	0.2462	0.2553	0.2553	1.0000	0.2553	-0.2553
Wu et al. [27]	0.5332	0.5906	0.5884	1.0000	0.5906	-0.5906
Proposed Method	0.0800	0.1000	0.1000	0.3333	0.1000	-0.1000

#: Method cannot be applied for the FN

TABLE III

THE MAGNITUDE OF LOSS $\tilde{Q}_{ik}$ AND THE LIKELIHOOD OF FAILURE $\tilde{P}_{ik}$ OF THE SUB-PRODUCT [24]

Company	Sub-products	Magnitude of loss $\tilde{Q}_{ik}$	Likelihood of failure $\tilde{P}_{ik}$
$\tilde{F}_1$	$\tilde{A}_{11}$	$\tilde{Q}_{11} = (0.04, 0.1, 0.18, 0.23; 0.8, 0.9)$	$\tilde{P}_{11} = (0.17, 0.22, 0.36, 0.42; 0.9, 0.9)$
	$\tilde{A}_{12}$	$\tilde{Q}_{12} = (0.58, 063, 0.80, 0.86; 0.65, 0.7)$	$\tilde{P}_{12} = (0.32, 0.41, 0.58, 0.65; 0.9, 0.7)$
	$\tilde{A}_{13}$	$\tilde{Q}_{13} = (0, 0, 0, 0; 0.5, 0.6)$	$\tilde{P}_{13} = (0.58, 0.63, 0.8, 0.86; 0.8, 0.9)$
$\tilde{F}_2$	$\tilde{A}_{21}$	$\tilde{Q}_{21} = (0.04, 0.1, 0.18, 0.23; 0.8, 0.7)$	$\tilde{P}_{21} = (0.93, 0.98, 1, 1; 0.85, 0.8)$
	$\tilde{A}_{22}$	$\tilde{Q}_{22} = (0.58, 063, 0.80, 0.86; 1, 0.5)$	$\tilde{P}_{22} = (0.58, 0.63, 0.8, 0.86; 0.9, 0.9)$
	$\tilde{A}_{23}$	$\tilde{Q}_{23} = (0, 0, 0.02, 0.07; 0.4, 0.8)$	$\tilde{P}_{23} = (0.32, 0.41, 0.58, 0.65; 0.7, 0.9)$
$\tilde{F}_3$	$\tilde{A}_{31}$	$\tilde{Q}_{31} = (0.04, 0.1, 0.18, 0.23; 1, 1)$	$\tilde{P}_{31} = (0.17, 0.22, 0.36, 0.42; 0.95, 0.95)$
	$\tilde{A}_{32}$	$\tilde{Q}_{32} = (0.58, 063, 0.80, 0.86; 0.8, 0.8)$	$\tilde{P}_{32} = (0.72, 0.78, 0.92, 0.97; 0.5, 0.6)$
	$\tilde{A}_{33}$	$\tilde{Q}_{33} = (0, 0, 0.07, 0.02; 0.9, 0.7)$	$\tilde{P}_{33} = (0.58, 0.63, 0.8, 0.86; 1, 1)$

TABLE II

LINGUISTIC TERMS AND THEIR CORRESPONDING GTFNs [16]

Linguistic Terms	GTFNs
Absolutely Low (AL)	(0, 0, 0, 0; 1)
Very Low (VL)	(0, 0, 0.02, 0.07; 1, 1;)
Low (L)	(0.04, 0.10, 0.18, 0.23; 1, 1)
Fairly Low (FL)	(0.17, 0.22, 0.36, 0.42; 1, 1)
Medium (M)	(032, 0.41, 0.58, 0.65; 1, 1)
Fairly High (FH)	(0.58, 0.63, 0.80, 0.86; 1, 1)
High (H)	(0.72, 0.78, 0.92, 0.97; 1, 1)
Very High (VH)	(0.93, 0.98, 1, 1; 1, 1)
Absolutely High (AH)	(1, 1, 1, 1; 1, 1)

VI. CONCLUSIONS

This paper introduces a novel defuzzification technique for ranking GTFNLRH. The technique identifies a TFQ by analyzing the centroids of three distinct regions within the geometric representation of GTFNLRH at a specific lower decision level. Using the *VAL* and *AMB* of the FN, a representative defuzzified value is determined for ranking purposes. A comparative study with existing ranking methods demonstrates the effectiveness of the proposed approach across various FNs. The paper presents a new methodology for evaluating fuzzy risk in product manufacturing by companies. Decision-maker assessments of the likelihood of failure and the magnitude of loss are represented as GTFNLRH. The proposed ranking procedure based on *VAL* and *AMB* is then applied to determine the overall risk for each company. This approach aligns with human intuition by considering the decision-maker's perspective at critical levels. The proposed approach has broad applications, including multi-criteria decision-making and addressing scheduling challenges in project planning where activity times are represented as FNs.

VII. LIMITATIONS AND FUTURE SCOPE

This study focuses on traditional fuzzy sets, where only the degree of membership of an element is considered. The degree of non-membership is not explicitly addressed, which is a limitation of this work. Future research could extend this ranking procedure to other types of fuzzy sets that incorporate the degree of non-membership, such as Intuitionistic Fuzzy Sets and Pythagorean Fuzzy Sets.

REFERENCES

[1] L. A. Zadeh, "Fuzzy sets," Information and Control, vol. 8, no. 3, pp. 338-353, 1965.

[2] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning-I," Information Sciences, vol. 8, pp. 199-249, 1975.

[3] R. Jain, "Decision making in the presence of fuzzy variable," IEEE Transactions on Systems Man and Cybernetics, vol. 6, pp. 698-703, 1976.

[4] R. R. Yager, "Ranking fuzzy subsets over the unit interval," In proceedings of 17th IEEE International Conference on Decision and Control, San Diego, California, pp. 1435-1437, 1978.

[5] D. Dubois, and H. Prade, "Operations on fuzzy numbers," International Journal of Systems Science, vol. 9, no. 6, pp. 613- 626, 1978.

[6] S. Murakami, H. Maeda, and S. Imamura, "Fuzzy decision analysis on the development of centralized regional energy control system," In Proceedings of the ISAC symposium on fuzzy information, knowledge representation and decision analysis, vol 16, no. 3, pp .363-368, 1983.

[7] G. Bortolan, and R. Degani, "A review of some methods for ranking fuzzy subsets," Fuzzy Sets and Systems, vol. 15, no. 1, pp. 1-19, 1985.

[8] S. H. Chen, "Ranking fuzzy numbers with maximizing set an minimizing set," Fuzzy Sets and Systems, vol. 17, pp. 113-122, 1985.

[9] K. Nakamura, "Preference relations on a set of fuzzy utilities as a basis for decision making," Fuzzy Sets and Systems, vol. 20, pp. 147-162, 1986.

[10] T. S. Liou, and M. J. Wang, "Ranking fuzzy numbers with integral value," Fuzzy Sets and Systems, vol. 50, no. 3, pp. 247-255, 1992.

[11] F. Choobineh, and H. Li, "An index for ordering fuzzy numbers," Fuzzy Sets and Systems, vol. 54, no. 3, pp. 287-294, 1993.

[12] P. Fortemps, and M. Roubens, "Ranking and defuzzification methods based on area compensation," Fuzzy Sets and Systems, vol. 82, pp. 319-330, 1996.

[13] C. H. Cheng, "A new approach for ranking fuzzy numbers by distance method," Fuzzy Sets and Systems, vol. 95, pp. 307-317, 1998.

[14] C. H. Chu, and T. Sao, "Ranking fuzzy numbers with an area between the centroid point and original point," Computers and Mathematics with Applications, vol. 43, no. 1, pp. 111-117, 2002.

[15] M. Delgado, M. A. Vila, and W. Voxman, "On a canonical representation of fuzzy numbers," Fuzzy Sets and Systems, vol. 93, no. 1, pp. 125-135, 1998.

[16] S. M. Chen, A. Munif, G. S. Chen, H. C. Liu, and B. C. Kuo, "Fuzzy risk analysis based on ranking generalized fuzzy numbers with different left heights and right heights," Expert System with Applications, vol. 39, no. 7, pp. 6320-6334, 2012.

[17] S. Pushpinder, "A new approach for the ranking of fuzzy sets with different heights," Journal of Applied Research and Technology, vol. 10, pp. 941- 949, 2012.

[18] W. Jiang, Y. Luo, X. Y. Qin, and J. Zhan "An Improved Method to Rank Generalized Fuzzy Numbers with Different Left Heights and Right Heights," Journal of Intelligent and Fuzzy Systems, vol. 28(5), pp. 2343– 2355, 2015.

[19] C. Rituparna, and C. Bijit, "A new method of ranking parametric form of fuzzy numbers using value and ambiguity," Applied Soft Computing, vol. 52, pp. 1154-1168, 2017.

[20] Y. Barazandeh, and B. Ghazanfari, "A novel method for ranking generalized fuzzy numbers with two different heights and its application in fuzzy risk analysis," Iranian Journal of Fuzzy Systems, vol. 18, no. 2, pp. 81-91, 2021.

[21] S. J. Chen, and S. M. Chen, "Fuzzy risk analysis based on similarity measures of generalized fuzzy numbers," IEEE Transactions on Fuzzy Systems, vol.11, pp. 45-56, 2003.

[22] S. M. Chen K, and Sanguansat, "Analysing fuzzy risk based on a new fuzzy ranking method between generalized fuzzy numbers," Expert System with Applications, vol. 38, pp. 2163-2171, 2011.

[23] Z. Xu, S. Shang, W. Qian, and W. Shu, "A method for fuzzy risk analysis based on the new similarity of trapezoidal fuzzy numbers," Expert System with Applications, vol. 37, pp.1920-1927, 2010.

[24] S. M. Chen, and J. H. Chen, "Fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and different spreads," Expert System with Applications, vol. 36, no. 3, pp. 6833 - 6842, 2009.

[25] P. Phani Bushan Rao, "Evaluating fuzzy risk based on a new method of ranking fuzzy numbers using centroid of centroids," International Journal of Mathematics in Operational Research, vol. 14, pp. 451-472, 2019.

[26] W. E. Lee, and S. M. Chen, "Fuzzy risk analysis based on fuzzy numbers with different shapes and different deviations," Expert System with Applications, vol. 34, pp. 2763- 2771, 2008.

[27] X. Wu, F. Liu, H. Xue, Y. Zheng, Y. Shou, and W. Jiang, "Fuzzy risk analysis based on a new method for ranking generalized fuzzy numbers," Iranian Journal of Fuzzy Systems, vol.15, no.3, pp.117-139, 2018.

[28] K. J. Schmucker, *Fuzzy sets, natural language computations, and risk analysis*, MD: Computer Science Press, 1984.

[29] S. Sinika, and G. Ramesh, "Trapezoidal Neutrosophic Program Evaluation and Review Technique Using Interval Arithmetic Operations," IAENG International Journal of Applied Mathematics, vol. 54, no. 3, pp. 324-341, 2024.

[30] Hitoshi Yano, "Multiobjective Fuzzy Random Bimatrix Games and An Equilibrium Solution Concept," Lecture Notes in Engineering and Computer Science: Proceedings of the International MultiConference of Engineers and Computer Scientists, 2021, 20-22 October, 2021, Hong Kong, pp. 228-233.

[31] P. Jana, D. Rosadi, and E. D. Supandi, "Fuzzy Portfolio with Multi objective Approach Using the Treynor Ratio," IAENG International Journal of Computer Science, vol. 51, no. 8, pp. 1086-1093, 2024.

[32] T.C. Lee, Y. C. O.Yang, P. T. Liu, and Y. H. Liao, "Evaluating Mechanism and Axiomatic Processes under Sustainability and Multiple Criteria Fuzzy Behavior," Engineering Letters, vol. 32, no. 8, pp. 1667 1674, 2024.

[33] X. Wang, and E. E. Kerre, “Reasonable properties for the ordering of fuzzy quantities (I),” Fuzzy Sets and Systems, vol. 118, pp. 375-385, 2001.