Estimation of Coefficient Bounds for a Class of Meromorphic Bi-univalent Functions Defined by Generalized Bazilevic Functions

Zongtao Li and Dong Guo

***Abstract*—Let $A^* = \{z : z \in C, 1 < |z| < +\infty\}$, and consider the class $M_\Sigma(\mu, \lambda, \phi)$ of meromorphic bi-univalent functions defined in A^*. This work focuses on deriving estimates for the coefficients $|b_0|$, $|b_1|$ and $|b_2|$ of functions in $M_\Sigma(\mu, \lambda, \phi)$, utilizing the properties of meromorphic functions. The findings presented here refine or extend certain results established by earlier researchers.**

***Index Terms*—analytic function, Meromorphic function, Bi-univalent function, Coefficient bound.**

I. Introduction

LET $A = \{z : z \in C, |z| < 1\}$ and $A^0 = \{z : z \in C, 0 < |z| < 1\}$. The set H comprises all analytic functions defined in A that can be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z + a_2 z^2 + a_3 z^3 + \cdots. \quad (1)$$

where a_n are complex coefficients. An analytic function of this form is said to be normalized. If, in addition, such a function is univalent, it is referred to as a normalized univalent analytic function. The class of all normalized univalent analytic functions is denoted by S.

In[1], Srivastava et al. introduced the subclasses $S^*(\phi)$ and $C(\phi)$. For $z \in A, let\ C_n$ be real numbers with $C_1 > 0$, and define

$$\phi(z) = 1 + \sum_{n=1}^{\infty} C_n z^n = 1 + C_1 z + C_2 z^2 + \cdots \quad (2)$$

where $\phi(z)$ maps A onto the right half-plane and is symmetric with respect to the real axis. The subclass $S^*(\phi)$ is defined as

$$S^*(\phi) = \{f : \frac{zf'(z)}{f(z)} \prec \phi(z), f(z) \in H\},$$

and the subclass $C^*(\phi)$ is defined as

$$C(\phi) = \{f : \frac{1 + zf''(z)}{f'(z)} \prec \phi(z), f(z) \in H\}.$$

In [2], Liu investigated the subclass $B(\lambda, \alpha, \sigma, \beta)$ of analytic functions f which satisfied

$$(1-\lambda)(\frac{f(z)}{g(z)})^\alpha + \lambda \frac{zf'(z)}{g(z)}(\frac{f(z)}{g(z)})^\alpha \prec (\frac{1+\beta z}{1-\beta z})^\alpha, f(z) \in H.$$

where parameters $\lambda \geq 0, \alpha \geq 0, 0 < |\beta| \leq 1$ and $\sigma > 0$. when the parameters take specific values, for instance, $\lambda = 1, \beta = 1$, and $\delta = 1$, the class reduces to the well-known class of Bazilevič functions.

The famous Koebe 1/4-theorem asserts that if f is an analytic univalent function in A, then the image $f(A)$ contains the disc $A_{1/4}$ where $A_{1/4}$ represents the open disc centered at the origin with radius $\frac{1}{4}$.

The inverse function of f, denoted by $h = f^{-1}$, is defined by the following relations:

$$h(f(z)) = z, for z \in A,$$

and

$$f(h(w)) = w, for w \in A_{r_0},$$

where $A_{r_0} = \{w : |w| < r_0(f), r_0(f) \geq \frac{1}{4}\}$.

The calculation yields

$$h(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ..., a_n) w^n$$

$$= w - a_2 w^2 + (2a^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots,$$

where K_{n-1}^{-n} is the polynomial defined by the variables $a_2, a_3, ..., a_n$ as introduced in [3].

In [5], Lewin conducted a comprehensive investigation into bi-univalent functions, establishing that for $z \in A$, and $f \in \sigma$, both $f(z)$ and its inverse $h(w)$ are univalent. Building upon Lewin's foundational research, numerous scholars (see [[6]-[10]]) have subsequently explored and derived the initial coefficient bounds for this class of functions, significantly advancing our understanding in this field.

Let $A^* = \{z : z \in C, 1 < |z| < \infty\}$. The class Σ consists of all meromorphic univalent functions g defined on A^* that admit a Laurent series expansion of the form

$$g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}, for z \in A^*. \quad (3)$$

If $g \in \Sigma$, then g possesses an inverse $h = g^{-1}$, which can be expressed as

$$h(w) = w + \sum_{n=0}^{\infty} \frac{c_n}{w^n}, for W \in A_M^*, \quad (4)$$

where $A_M^* = \{w : M < |w| < \infty, M > 0\}$ is the domain h. The inverse function h satisfies the following relations:

$$h(g(z)) = z, for z \in A^*$$

and

$$g(h(w)) = w, for w \in A_M^*.$$

Manuscript received May 04, 2022; revised February 14, 2025. This work was supported by the natural Science Foundation of Anhui Provincial Department of Education (KJ2018A0833) and the Foundation of Guangzhou Civil Aviation College (22x0418,24X4412).

Zongtao Li is an associate professor of the Department of Mathematics Teaching, Guangzhou Civil Aviation College, Guangzhou, 510403, China (corresponding author; e-mail: lizt2046@163.com).

Dong Guo is a professor of the School of Mathematical Sciences, Yangzhou Polytechnic College, Yangzhou, 225009, China (e-mail: gd791217@163.com).

We say that g is the bi-univalent meromorphic function if $g \in \Sigma, h \in \Sigma$. The class of the meromorphic bi-univalent functions is denoted by M_Σ. For $g \in M_\Sigma$, the calculation yields the following expression for the function h:

$$h(w) = w - b_0 - \sum_{n=1}^{\infty} \frac{1}{n} K_{n+1}^n (b_1, b_2, ..., b_n) w^n$$

$$= w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} - \frac{b_3 + 2b_0 b_2 + b_0^2 b_1 + b_1^2}{w^3} \cdots, \quad (5)$$

where K_{n+1}^n is the polynomial defined by the variables $b_0, b_1, b_2, ..., b_n$ in [4].

In 2013, Hamidi et al. [11] conducted an investigation into the subclass $B\Sigma(\alpha, \lambda) \subset M_\Sigma$, resulting in the estimation of the coefficient $|a_n|$. Following this work, significant attention has been devoted to the study of various subclasses of meromorphic bi-univalent functions, with numerous researchers establishing initial coefficient bounds for these classes (see [12]–[18]).

Definition 1.1: A functin $g(z)$ of the form (2) is said to $g(z) \in M_\Sigma(\mu, \lambda, \Phi)$, if λ and μ are non-negative and the following subordination conditions hold:.

$$(1-\lambda)(\frac{g(z)}{z})^\mu + \lambda \frac{zg'(z)}{g(z)} (\frac{g(z)}{z})^\mu \prec \Phi(z), z \in A^*, \quad (6)$$

and

$$(1-\lambda)(\frac{h(w)}{w})^\mu + \lambda \frac{wh'(w)}{h(w)} (\frac{h(w)}{w})^\mu \prec \Phi(w), w \in A_M^*, \quad (7)$$

where $h(w) = g^{-1}(w)$ is the inverse of g given by (5) and $\Phi(z) = \phi(\frac{1}{z})$ is defined by (2) with the seres expansion

$$\Phi(z) = \phi(\frac{1}{z}) = 1 + \sum_{n=1}^{\infty} \frac{C_n}{z^n}. \quad (8)$$

By varying the parameters associated with the aforementioned definition, it is possible to ascertain that

(1) $M_\Sigma(\mu, \lambda, \frac{z+(1-2\alpha)}{z-1}) = M_\Sigma(\lambda, \mu, \alpha)$ (see [10,11]);
(2) $M_\Sigma(\mu, \lambda, (\frac{z+1}{z-1})^\alpha) = \widetilde{\Sigma}_M^*(\alpha, \mu, \lambda)$ (see [13]);
(3) $M_\Sigma(1, \lambda, \frac{z+(1-2\alpha)}{z-1}) = B\Sigma(\alpha; \lambda)$ (see [14]);
(4) $M_\Sigma(\beta, 1, (\frac{z+1}{z-1})^\alpha) = B(\alpha; \beta)$ (see [15]);
(5) $M_\Sigma(0, 1, \frac{z+(1-2\alpha)}{z-1}) = \Sigma_\beta^\alpha (0 \le \alpha < 1)$ (see[16]);
(6) $M_\Sigma(0, 1, (\frac{z+1}{z-1})^\alpha) = \Sigma^*(\alpha) (0 \le \alpha < 1)$ (see [16]);
(7) $M_\Sigma(\beta, 1, (\frac{z+1}{z-1})^\alpha) = \Sigma^B(\beta, \alpha) (0 \le \alpha < 1)$ (see[16]);
(8) $M_\Sigma(0, 0, (\frac{z+1}{z-1})^\alpha) = \tilde{\tilde{\Sigma}}_*(\alpha) (0 \le \alpha < 1)$ (see [16]);
(9) $M_\Sigma(0, 1, \Phi) = S_{\Sigma'}(\phi)$ (see[17]).

The coefficient estimates of $|b_0|$, $|b_1|$ and $|b_2|$ derived in previous studies were suboptimal, primarily due to the limited constraints imposed in their derivations. In this work, we rigorously address these limitations by incorporating more comprehensive restrictive conditions, thereby refining and improving upon several existing results.

Consider the class$P$consisting of functions$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, which are analytic in the domain$A$and fulfill the requirement that the real part of$p(z)$ is positive, i.e., $Rep(z) > 0$. In [19], Goodman proved the following result.

Lemma 1.1: If $n \ge 1$ be a fixed integer, $z \in A$, $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in P$, then $|p_n| \le 2$. This inequality is sharp.

II. Main results and proof

We first estimate the bounds on the coefficients $|b_0|, |b_1|$ and $|b_2|$ for functions in the class $M_\Sigma(\mu, \lambda, \Phi)$.

Theorem 2.1: Let $g(z) \in M_\Sigma(\mu, \lambda, \Phi)$. Then the following coefficient bounds hold:

(i) The coefficient b_0 satisfies

$$|b_0| \le \min\{\frac{C_1}{|\lambda - \mu|}, \frac{\sqrt{2C_1 + 2|C_2 - C_1|}}{\sqrt{|(2\lambda - \mu)(1-\mu)|}},$$

$$\frac{C_1\sqrt{2C_1}}{\sqrt{|(2\lambda - \mu)(1-\mu)C_1^2 - 2(C_2 - C_1)(\lambda - \mu)^2|}}\}; \quad (9)$$

(ii) The coefficient b_1 satisfies

$$|b_1| \le \frac{C_1}{|2\lambda - \mu|}; \quad (10)$$

(iii) The coefficient b_2 satisfies

$$|b_2| \le \frac{C_1 + 2|C_2 - C_1| + |C_1 - 2C_2 + C_3| + \frac{|(3\lambda-\mu)(\mu-1)(\mu-2)C_1^3|}{3(\lambda-\mu)^3}}{|3\lambda - \mu|}. \quad (11)$$

Proof: Let $z \in A^0$. We define the functions:G and H as follows:

$$G(z) = g(\frac{1}{z}) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$$

and

$$H(w) = h(\frac{1}{w}) = \frac{1}{w} + \sum_{n=0}^{\infty} b_n w^n.$$

Consequently, (6) and (7) can be equivalently expressed as

$$(1-\lambda)(zG(z))^\mu - \lambda \frac{zG'(z)}{G(z)} (zG(z))^\mu \prec \phi(z), (z \in A) \quad (12)$$

and

$$(1-\lambda)(zH(w))^\mu + \lambda \frac{wH'(w)}{H(w)} (\frac{H(w)}{w})^\mu \prec \phi(w). (w \in A) \quad (13)$$

Since $g \in M_\Sigma(\mu, \lambda, \Phi)$, the definition of subordination implies the existence of two Schwarz functions $u, v : A \to A$ satisfying

$$(1-\lambda)(zG(z))^\mu - \lambda \frac{zG'(z)}{G(z)} (zg(z))^\mu = \phi(u(z)), \quad z \in A, \quad (14)$$

and

$$(1-\lambda)(zH(w))^\mu - \lambda \frac{wH'(w)}{H(w)} (wH(w))^\mu = \phi(v(w)), \quad w \in A. \quad (15)$$

Using the Schwartz functions u and v, we define the functions p and q in P as follows:

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \quad (16)$$

and

$$q(w) = \frac{1+v(w)}{1-v(w)} = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots. \quad (17)$$

By expanding (16) and (17), we derive the series expansions for u and v as

$$u(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}[p_1z+(p_2-\frac{p_1^2}{2})z^2+(p_3-p_1p_2+\frac{1}{4}p_1^3)z^3+\cdots], \quad (18)$$

$$v(w)=\frac{v(w)-1}{v(w)+1}=\frac{1}{2}[q_1w+(q_2-\frac{q_1^2}{2})w^2+(q_3-q_1q_2+\frac{1}{4}q_1^3)w^3+\cdots]. \quad (19)$$

Based on (14), (15), (18) and (19), we derive the following expansions:

$$(1-\lambda)(zG(z))^\mu-\lambda\frac{zG'(z)}{G(z)}(zg(z))^\mu=1+\sum_{n=0}^{\infty}F_{n+1}(b_0,b_1,...,b_n)z^{n+1}$$

which expands to

$$=1-(\lambda-\mu)b_0z-\frac{2(2\lambda-\mu)b_1+(2\lambda-\mu)(\mu-1)b_0^2}{2}z^2-\frac{6(3\lambda-\mu)b_2+6(3\lambda-\mu)(\mu-1)b_0b_1+(3\lambda-\mu)(\mu-1)(\mu-2)b_0^3}{6}z^3+\cdots, \quad (20)$$

and

$$(1-\lambda)(zH(w))^\mu-\lambda\frac{wH'(w)}{H(w)}(wH(w))^\mu=1+\sum_{n=0}^{\infty}F_{n+1}(c_0,c_1,...,c_n)w^{n+1}$$

which expands to

$$=1+(\lambda-\mu)b_0w+\frac{2(2\lambda-\mu)b_1-(2\lambda-\mu)(\mu-1)b_0^2}{2}w^2+\frac{6(3\lambda-\mu)b_2-6(3\lambda-\mu)(\mu-2)b_0b_1+(3\lambda-\mu)(\mu-1)(\mu-2)b_0^3}{6}w^3+\cdots. \quad (21)$$

Additionally, we have the expansions for $\phi(u(z))$ and $\phi(v(w))$:

$$\phi(u(z))=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}C_kD_n^k(p_1,p_2,...,p_n)z^n$$

which expands to

$$=1+\frac{C_1p_1}{2}z+(\frac{C_1p_2}{2}+\frac{C_2-C_1}{4}p_1^2)z^2+[\frac{C_1p_3}{2}+\frac{C_2-C_1}{2}p_1p_2+\frac{C_1-2C_2+C_3}{8}p_1^3]z^3+\cdots, \quad (22)$$

and

$$\phi(v(w))=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}C_kD_n^k(q_1,q_2,...,q_n)w^n$$

which expands to

$$=1+\frac{C_1q_1}{2}w+(\frac{C_1q_2}{2}+\frac{C_2-C_1}{4}q_1^2)w^2+[\frac{C_1q_3}{2}+\frac{C_2-C_1}{2}q_1q_2+\frac{C_1-2C_2+C_3}{8}q_1^3]w^3+\cdots. \quad (23)$$

Here F_{n+1} denotes the Faber polynomial of degree $n+1$ as defined in [12] and for $k\le n$,

$$D_n^k(p_1,p_2,...,p_n)=\sum\frac{k!p_1^{\mu_1}...p_n^{\mu_n}}{\mu_1!...\mu_n!},$$

where the non-negative integers $\mu_1,...,\mu_n$ satisfying the following conditions(see[20]):

$$\sum_{i=1}^{n}\mu_i=k,\sum_{i=1}^{n}\mu_i=n.$$

Consequently, the following relations can be derived by comparing the initial coefficients.

$$-(\lambda-\mu)b_0=\frac{C_1p_1}{2}, \quad (24)$$

$$-\frac{1}{2}[2(2\lambda-\mu)b_1+(2\lambda-\mu)(\mu-1)b_0^2]=\frac{1}{2}C_1p_2+\frac{C_2-C_1}{4}p_1^2, \quad (25)$$

$$-[(3\lambda-\mu)b_2+(3\lambda-\mu)(\mu-1)b_0b_1+\frac{1}{6}(3\lambda-\mu)(\mu-1)(\mu-2)b_0^3]=\frac{C_1p_3}{2}+\frac{C_2-C_1}{2}p_1p_2+\frac{C_1-2C_2+C_3}{8}p_1^3, \quad (26)$$

$$(\lambda-\mu)b_0=\frac{C_1q_1}{2}, \quad (27)$$

$$\frac{1}{2}[2(2\lambda-\mu)b_1-(2\lambda-\mu)(\mu-1)b_0^2]=\frac{1}{2}C_1q_2+\frac{C_2-C_1}{4}q_1^2. \quad (28)$$

$$(3\lambda-\mu)b_2-(3\lambda-\mu)(\mu-1)b_0b_1+\frac{1}{6}(3\lambda-\mu)(\mu-1)(\mu-2)b_0^3=\frac{C_1q_3}{2}+\frac{C_2-C_1}{2}q_1q_2+\frac{C_1-2C_2+C_3}{8}q_1^3. \quad (29)$$

(i) From (24) and (27), we obtain the following relations:

$$p_1=-q_1 \quad (30)$$

and

$$b_0^2=\frac{C_1^2(p_1^2+q_1^2)}{8(\lambda-\mu)^2}. \quad (31)$$

Applying Lemma 1.1 in (31), we obtain

$$|b_0|\le\frac{C_1}{|\lambda-\mu|}. \quad (32)$$

From (25) and (28), we get

$$(2\lambda-\mu)(1-\mu)b_0^2=\frac{C_1}{2}(p_2+q_2)+\frac{C_2-C_1}{4}(p_1^2+q_1^2). \quad (33)$$

Applying Lemma 1.1 in (33), we obtain

$$|b_0|\le\frac{\sqrt{2C_1+2|C_2-C_1|}}{\sqrt{|(2\lambda-\mu)(1-\mu)|}}. \quad (34)$$

From (30), (31) and (33), we obtain

$$p_1^2=\frac{2C_1(\lambda-\mu)^2(p_2+q_2)}{(2\lambda-\mu)(1-\mu)C_1^2-2(C_2-C_1)(\lambda-\mu)^2}. \quad (35)$$

From (31) and (35), we obtain

$$b_0^2=\frac{C_1^3(p_2+q_2)}{2(2\lambda-\mu)(1-\mu)C_1^2-4(C_2-C_1)(\lambda-\mu)^2}. \quad (36)$$

Applying Lemma 1.1 in (36) , we obtain

$$|b_0|\le\frac{C_1\sqrt{2C_1}}{\sqrt{|(2\lambda-\mu)(1-\mu)C_1^2-2(C_2-C_1)(\lambda-\mu)^2|}}. \quad (37)$$

Combined with (32), (34) and (37), we get the same value of $|b_0|$ as Equation (7).

(ii) From (25) and (28), we obtain

$$2(2\lambda-\mu)b_1=-\frac{C_1(p_2-q_2)}{2}. \quad (38)$$

Applying Lemma 1.1 in (38) once again, we get

$$|b_1| \leq \frac{C_1}{|2\lambda-\mu|}. \tag{39}$$

Multiplying (25) by (28), we get

$$-4(2\lambda-\mu)^2 b_1^2 = \frac{(C_2-C_1)^2}{4} p_1^2 q_1^2 + \frac{(C_2-C_1)C_1}{2} \cdot$$
$$(p_1^2 q_2 + q_1^2 p_2) + C_1^2 p_2 q_2 - (2\lambda-\mu)^2(1-\mu)^2 b_0^4. \tag{40}$$

Substituting (30), (31) into (40), and applying Lemma 1.1, we can obtain

$$|b_1| \leq \frac{1}{|2\lambda-\mu|} \cdot$$
$$\sqrt{(C_2-C_1)^2 + 2|C_2-C_1|C_1 + C_1^2 + \frac{(2\lambda-\mu)^2(1-\mu)^2 C_1^4}{4(\lambda-\mu)^4}}. \tag{41}$$

Substituting (30), (33) into (40), from Lemma 1.1, we can obtain

$$|b_1| \leq \frac{1}{|2\lambda-\mu|} \sqrt{2(C_2-C_1)^2 + 4C_1|C_2-C_1| + 2C_1^2}. \tag{42}$$

Substituting (36) into (40), from Lemma 1.1, we can obtain

$$|b_1| \leq \frac{1}{|2\lambda-\mu|} \cdot$$
$$\sqrt{(C_2-C_1)^2 + 2|C_2-C_1|C_1 + C_1^2 + \frac{(2\lambda-\mu)^2(1-\mu)^2 C_1^6}{[(2\lambda-\mu)(1-\mu)C_1^2 - 2(C_2-C_1)(\lambda-\mu)^2]^2}}. \tag{43}$$

Adding the square of (31) to the square of (28), we obtain

$$4(2\lambda-\mu)^2 b_1^2 = \frac{C_1^2}{2}(p_2^2+q_2^2) + \frac{(C_2-C_1)^2}{8}(p_1^4+q_1^4)$$
$$+\frac{(C_2-C_1)C_1}{2}(p_1^2 p_2 + q_1^2 q_2) - (2\lambda-\mu)^2(1-\mu)^2 b_0^4. \tag{44}$$

From (40) and Lemma 1.1, we get

$$|-4(2\lambda-\mu)^2 b_1^2| \leq |\frac{(C_2-C_1)^2}{4} p_1^2 q_1^2| + |\frac{(C_2-C_1)C_1}{2}(p_1^2 q_2$$
$$+q_1^2 p_2)| + |B_1^2 p_2 q_2| + |(2\lambda-\mu)^2(1-\mu)^2 b_0^4| \leq$$
$$4(C_2-C_1)^2 + 8|C_2-C_1|C_1 + 4C_1^2 + (2\lambda-\mu)^2(1-\mu)^2 b_0^4. \tag{45}$$

From (44) and Lemma 1.1, we get

$$|4(2\lambda-\mu)^2 b_1^2| \leq |\frac{C_1^2}{2}(p_2^2+q_2^2)| + |\frac{(C_2-C_1)^2}{8}(p_1^4+q_1^4)| +$$
$$|\frac{(C_2-C_1)C_1}{2}(p_1^2 p_2 + q_1^2 q_2)| + |(2\lambda-\mu)^2(1-\mu)^2 b_0^4| \leq 4(C_2$$
$$-C_1)^2 + 8|(C_2-C_1)|C_1 + 4C_1^2 + (2\lambda-\mu)^2(1-\mu)^2 b_0^4. \tag{46}$$

It can be demonstrated that the two sides of equations (45) and (46) are identical. Now, substituting (31), (33) and (36) into (46) respectively yields (41), (42) and (43). Consequently, by applying the aforementioned methodology to (39), (41) and (43), we arrive at the following results:

$$|b_1| \leq \min\{\frac{C_1}{|2\lambda-\mu|}, \frac{1}{|2\lambda-\mu|} \cdot$$
$$\sqrt{(C_2-C_1)^2 + 2|C_2-C_1|C_1 + C_1^2 + \frac{(2\lambda-\mu)^2(1-\mu)^2 C_1^4}{4(\lambda-\mu)^4}}, \frac{1}{|2\lambda-\mu|} \cdot$$
$$\sqrt{|(C_2-C_1)^2| + 2|C_2-C_1|C_1 + C_1^2 + \frac{(2\lambda-\mu)^2(1-\mu)^2 B_1^6}{[(2\lambda-\mu)(1-\mu)C_1^2 - 2(C_2-C_1)(\lambda-\mu)^2]^2}},$$
$$\frac{1}{2|\lambda-\mu|}\sqrt{2(C_2-C_1)^2 + 4|C_2-C_1|C_1 + 2C_1^2} = \frac{C_1}{|2\lambda-\mu|}.$$

(iii) Subtracting (26) from (29), we have

$$2(3\lambda-\mu)b_2 + \frac{1}{3}(3\lambda-\mu)(\mu-1)(\mu-2)b_0^3 =$$
$$\frac{C_1}{2}(q_3-p_3) + \frac{C_2-C_1}{2}(q_1 q_2 - p_1 p_2) + \frac{C_1-2C_2+C_3}{8}(q_1^3-p_1^3). \tag{47}$$

From (24), we have

$$b_0 = -\frac{C_1 p_1}{2(\lambda-\mu)}. \tag{48}$$

Considering (30) and (48), we obtain from (47) that

$$2(3\lambda-\mu)b_2 = \frac{C_1}{2}(q_3-p_3) - \frac{C_2-C_1}{2}(p_2+q_2)p_1$$
$$-\frac{C_1-2C_2+C_3}{4}p_1^3 + \frac{(3\lambda-\mu)(\mu-1)(\mu-2)C_1^3 p_1^3}{24(\lambda-\mu)^3}. \tag{49}$$

Applying Lemma 1.1 to the above equation and performing a simple calculation yields the result of (iii).

III. Main corollaries

By varying μ, λ and ϕ in $M_\Sigma(\mu,\lambda,\Phi)$, we can derive some interesting results that build upon the insights of existing research. Some of these findings confirm previous conclusions, while others correct some previous research errors.

By setting $\lambda = 1$ in Theorem 2.1, we obtain the following corollary.

Corollary 3.1: ([11]) Let $g(z) \in M_\Sigma(\mu, 1, \Phi)$. Then

$$|b_0| \leq min\{\frac{C_1}{|1-\mu|}, \frac{\sqrt{2C_1 + 2|C_2-C_1|}}{\sqrt{|(2-\mu)(1-\mu)|}},$$
$$\frac{C_1\sqrt{2C_1}}{\sqrt{|(2-\mu)(1-\mu)B_1^2 - 2(C_2-C_1)(1-\mu)^2|}}\}.$$
$$|b_1| \leq \frac{C_1}{|2-\mu|}.$$

Let $\Phi(z) = \frac{z+(1-2\alpha)}{z-1} = 1 + \frac{2(1-\alpha)}{z} + \frac{2(1-\alpha)}{z^2} + \cdots for 0 \leq \alpha < 1$. Then $C_1 = C_2 = 2(1-\alpha)$. From Theorem 2.1, we derive the following result.

Corollary 3.2: Let $g(z) \in M_\Sigma(\mu, \lambda, \frac{z+(1-2\alpha)}{z-1})$. Then,

$$|b_0| \leq \begin{cases} \frac{2\sqrt{1-\alpha}}{\sqrt{|(2\lambda-\mu)(1-\mu)|}}, & 0 \leq \alpha < 1 - \frac{(\lambda-\mu)^2}{|(2\lambda-\mu)(1-\mu)|}; \\ \frac{2(1-\alpha)}{|\lambda-\mu|}, & 0 \leq 1 - \frac{(\lambda-\mu)^2}{|(2\lambda-\mu)(1-\mu)|} \leq \alpha < 1, \end{cases}$$

and

$$|b_1| \leq \frac{2(1-\alpha)}{|2\lambda-\mu|}.$$

From Corollary 3.2, we further obtain the following result.

Corollary 3.3: Let $g(z) \in M_\Sigma(1, \lambda, \frac{z+(1-2\alpha)}{z-1})$. Then $|b_0| \leq \frac{2(1-\alpha)}{|\lambda-1|}$ and $|b_1| \leq \frac{2(1-\alpha)}{|2\lambda-1|}$.

Remark 3.1: Two coefficient estimates for Corollary 3.2 are the same as those for Theorem 3.2 in [12]. Also, the conclusion of Corollary 3.3 is the same as that of Theorem

1.2 in [14].

Corollary 3.4: Let $g(z) \in M_\Sigma(0, 1, \frac{z+(1-2\alpha)}{z-1}) = \Sigma_\beta^\alpha \quad (0 \le \alpha < 1)$.
Then

$$|b_0| \le \begin{cases} \sqrt{2(1-\alpha)}, & 0 \le \alpha < \frac{1}{2}; \\ 2(1-\alpha), & \frac{1}{2} \le \alpha < 1, \end{cases}$$

and

$$|b_1| \le 1 - \alpha.$$

Remark 3.2: The estimates of $|b_0|$ in Corollary 3.4 is better than that given by Theorem 1 in [16]. $|b_1|$ has the same situation.

Let $\Phi(z) = (\frac{z+1}{z-1})^\alpha = 1 + \frac{2\alpha}{z} + \frac{2\alpha^2}{z^2} + \frac{4\alpha^3+2\alpha}{3}\frac{1}{z^3} + \cdots for 0 \le \alpha \le 1$. Then $C_1 = 2\alpha, C_2 = 2\alpha^2$. From Theorem 2.1, we derive the following result.

Corollary 3.5: Let $g(z) \in M_\Sigma(\mu, \lambda, (\frac{z+1}{z-1})^\alpha) = \widetilde{\Sigma}_M^*(\alpha, \mu, \lambda)$. Then

$$|b_0| \le \min\{\frac{2\alpha}{|\lambda-\mu|}, \frac{2\sqrt{2\alpha-\alpha 2}}{\sqrt{|(2\lambda-\mu)(1-\mu)|}}, \frac{2\alpha}{\sqrt{|(2\lambda-\mu)(1-\mu)\alpha+(1-\alpha)(\lambda-\mu)^2|}}\}$$

and

$$|b_1| \le \frac{2\alpha}{|2\lambda - \mu|}.$$

Remark 3.3: The estimate of the coefficient $|b_0|$ in Corollaries 3.5 improves upon the result given in [13].

Corollary 3.6: $g(z) \in M_\Sigma(\beta, 1, (\frac{z+1}{z-1})^\alpha) = B(\alpha; \beta)$. Then

$$|b_0| \le \min\{\frac{2\alpha}{|1-\beta|}, \frac{2\sqrt{-\alpha 2+2\alpha}}{\sqrt{|(2-\beta)(1-\beta)|}}, \frac{2\alpha}{\sqrt{|(2-\beta)(1-\beta)\alpha+(1-\alpha)(1-\beta)^2|}}\},$$

$$|b_1| \le \frac{2\alpha}{|2-\beta|}.$$

By setting $\beta = 0$ in Corollary 3.6, we obtain the following result.

Corollary 3.7: Let $0 \le \alpha < 1$, $g(z) \in M_\Sigma(0, 1, (\frac{z+1}{z-1})^\alpha) = \Sigma^*(\alpha)$. Then

$$|b_0| \le \min\{2\alpha, \sqrt{-2\alpha 2 + 4\alpha}, \frac{2\alpha}{\sqrt{1+\alpha}}\} = \frac{2\alpha}{\sqrt{1+\alpha}}$$

and

$$|b_1| \le \alpha.$$

Remark 3.4: The estimates for $|b_0|$ and $|b_1|$ given in Corollary 3.6 improve upon those given in Theorem 2 in [15] and Theorem 3 in [16]. Specifically,Theorem 2 obtained by Halim in [14] states that If $g \in \Sigma^*(\alpha)$ with $0 < \alpha \le 1$, then $|b_0| \le 2\alpha$ and $|b_1| \le \sqrt{5}\alpha^2$. However, Corollary 3.7 shows that the estimate for $|b_1|$ is incorrect, and the correct bound is $|b_1| \le \alpha$. Thus, the coefficient estimates in Corollary 3.7 are superior to those in Theorem 2 of [16].

Corollary 3.8: Let $g(z) \in M_\Sigma(0, 1, \Phi) = S_{\Sigma'}(\Phi)$. Then

$$|b_0| \le C_1, \quad |b_1| \le \frac{C_1}{2},$$

and

$$|b_2| \le \frac{1}{3}(C_1 + 2|C_2 - C_1| + |C_1 - 2C_2 + C_3| + C_1^3).$$

Remark 3.5: The estimates for $|b_0|$ and $|b_2|$ in Corollary 3.8 agree with the bounds given by Murugusundaramoorthy et al. [[17], Theorem 2.4(i), (iii)], However, the estimate for $|b_1|$ in Corolly is sharper than the result in Theorem 2.4(ii) of [17]. This demonstrates that the results in Corollary 3.8 are superior to those in Theorem 2.4 of [17].

IV. Conclusion

This paper first determines an accurate estimate of coefficients for the meromorphic real part function. This estimate was used as a lemma to study a more extensive class of Bazilevič functions and obtain more accurate estimates of the initial coefficients for this class of functions. This paper not only optimises the conclusions of relevant papers but also corrects an erroneous result in Halim's paper (see Theorem 2 in [16]).

References

[1] H. M. Srivastava, G. Murugusundaramoorthy and K. Vijaya, "Coefficient estimates for some families of bi-Bazilevič functions of the Ma-Minda type involving the Hohlov operator", *Journal of classical Analysis*, vol. 2, no. 2, pp. 167-181, 2013.

[2] M. S. Liu, "The Fekete-Szegö inequality for certain class of analytic functions", *Acta Mathematica Scientia (Series A)*, vol.22, no. 1, pp. 8-14, 2002.

[3] H. Airault and J. Ren, "An algebra of differential operators and generating functions on the set of univalent functions", *Bulletin des Sciences Mathématiques*, vol. 126, no. 5, pp. 343-367, 2002.

[4] H. Airault and A. Bouali, "Differential calculus on the Faber polynomials", *Bull. Sci. Math.*, vol. 130, pp.179-222, 2006.

[5] M. Lewin, "on a Coefficient problem for Bi-univalent Functions", *Proceedings of the American Mathematical Society*, vol. 18, no. 1, pp. 63-68, 1967.

[6] B. Khanl, H. M. Srivastava, M. Tahir, M. Darus, Q. Z. Ahmad and N. Khan, "Applications of a certain q-integral operator to the subclasses of analytic and bi-univalent functions", *AIMS Mathematics*, vol. 6, no. 1, pp. 1024C1039, 2021.

[7] S. Yalcm, W. G. Atshan and H. Z. Hassan, "Coefficients assessment for certain subclasses of bi-univalent functions related with quasi-subordination", *Publ. Inst. Math. (Beograd)(N.S.)*, Vol. 108, no. 122, pp. 155-162, 2020.

[8] A. Akgül, "Certain inequalities for a general class of analytic and bi-univalent functions", *Sahand Communications in Mathematical Analysis*, vol, 14, no. 1, pp. 1-13, 2019.

[9] H. Orhan, N. Magesh and V. K. Balaji, "Fekete-Szegö problem for certain class of Ma-Minda bi-univalent functions", *Afrika Matematika*, vol. 27, pp. 889-897, 2016.

[10] Z. G. Peng and Q. Q. Han, "On the coefficients of several classes of bi-univalent functions", in*Acta Math Scientia*, vol. 34B, no. 1, pp. 228-240, 2016.

[11] S. G. Hamidi, S. A. Halim and J. M. Jahangiri, "Coefficient estimates for a class of meromorphic bi-univalent functions", *C. R. Acad, Sci. Paris, Ser I.*, vol. 351, pp. 349-352, 2013.

[12] S. G. Hamidi, T. Janani, G. Murugusundaramoorthy and G. M. Jahangiri, "Coefficient estimates for certain classes of meromorphic bi-univalent functions", *C. R. Acad. Sci. Paris, Ser. I*, vol. 352, pp. 277-282, 2014.

[13] H. Orhan, N. Magesh and V. K. Balaji, "Initial coefficient bounds for certain classes of meromorphic bi-univalent functions", *Asian-European Journal of Mathematics*, vol. 7. no. 1, (9 pages), DOI: 10.1142/S1793557114500053, 2014.

[14] D. Guo, Z. T. Li and L. P. Xiong, "Coefficient estimates for several classes of meromorphically bi-univalent functions", *Journal of Mathematical Research with Applications*, vol. 38, no. 6, pp. 597-608, 2018.

[15] J. M. Jahangiri and S. G. Hamidi, "Coefficients of meromorphic bi- Bazilevič Functions", *Journal of Complex Analysis*, Vlume2014, Article ID263917,4 pages, http://dx.dol.org/10.1155/2014/263917.

[16] S. A. Halim, S. G. Hamidi and V. Ravichandran, "Coefficient estimates for meromorphic bi-univalent functions", *arXiv:1108.4089v1[math.CV]*, vol. 20, pp. 1-9, 2011.

[17] G. Murugusundaramoorthy, T. Janani and N. E. CHO, "Coefficient estimates of Mocanu-type meromorphic bi-univalent functions of complex order", *Proceedings of the Jangjeon Mathematical Society*, vol. 19, no. 4, pp. 691-700, 2016.

[18] Y. J. Sim and O. S. Kwon, "Certain subclasses of meromorphically bi-univalent functions", *Bull. Malays. Math. Sci.Soc.*, vol. 40, pp. 841-855, 2017.

[19] A. W. Goodman, *Univalent Functions*, Vol. I. Polygonal Publishing House, Washington: New Jersey, 1983.
[20] H. M. Srivastava, S. Khan, Q. Z. Ahmad, N. Khan and S. Hussain,"The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator," *Stud. Univ. Babes-Bolyai Math.*, vol. 63, No. 4, pp.419-436,2018.