Statistical Inference for Uncertain Intelligent Transportation Systems from Discrete Observations

Zhuan Liu and Peng Gao

Abstract—**Uncertain differential equations (UDEs) are a specific type of differential equations that are influenced by Liu processes. The application of UDEs requires careful consideration of statistical inference, and various techniques have been suggested by researchers to calculate the unknown parameters. However, there is limited discussion in the existing literature regarding the asymptotic characteristics of these estimators. This study focuses on the estimation of parameter for uncertain intelligent transportation system described by UDEs based on discrete observations. Initially, we define the least squares estimator, followed by the derivation of consistency and asymptotic distribution of the drift parameter. We then provide the uncertain Hyperbolic model as an illustrative example and present numerical demonstrations to clarify the proposed approach.**

Index Terms—**UDEs; parameter estimation; Liu process; difference equation; implicit Euler scheme**

I. Introduction

Intelligent transportation originated from people's reflections on traditional problems such as traffic congestion, frequent accidents and low efficiency. In the early days, with the rise of computer technology and communication technology, preliminary concepts such as intelligent traffic signal control began to emerge. During the development process, various technologies such as sensors, cameras and satellite positioning are used to collect traffic data. Cloud computing and big data analysis technologies process massive amounts of data to realize functions such as traffic flow prediction and intelligent route planning. At the same time, the development of vehicle networking technology enables communication between vehicles and between vehicles and infrastructure. Therefore, many scholars have studied the intelligent transportation systems. For example, Gong et al. ([10]) surveyed edge intelligence in intelligent transportation systems. Chen et al. ([6]) proposed energy-efficient and regenerative energy recovery schemes for sustainable intelligent transportation system using the Artificial societies, computational experiments, parallel execution framework. Dilek and Dener ([7]) examined computer vision applications in the literature, the machine learning and deep learning methods used in intelligent transportation systems applications.

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Statistical inference holds great significance in modeling stochastic models and has garnered attention from numerous researchers. For example, Zhang et al. ([27]) introduced a numerical technique to identify the structure and estimate line parameters without prior knowledge of voltage angles. Maldonado et al. ([20]) utilized a sequential Bayesian approach to infer parameters in stochastic dynamic load models. Zhang et al. ([28]) delved into the joint estimation of states and parameters in a specific nonlinear bilinear systems category. Ji and Kang ([13]) explored novel methods for on-line parameter estimation in nonlinear systems. Escobar et al. ([9]) proposed various strategies to tackle parameter estimation challenges in stochastic systems operating continuously. Ding ([8]) analyzed the characteristics of two different least squares techniques that effectively handle white and colored noise disturbances using conventional methodologies prevalent in the field. Shin and Park ([22]) utilized a generator-regularized continuous conditional generative adversarial network to estimate uncertain parameters. Amorino et al. ([1]) introduced a contrast function to estimate parameters in a stochastic McKean-Vlasov equation. Mehmood and Raja ([21]) investigated evolutionary heuristics of weighted differential evolution for parameter estimation in the Hammerstein-Wiener model. Brusa et al. ([4]) proposed an evolutionary optimization approach to streamline approximate maximum likelihood estimation in discrete models. In real-world scenarios, factors such as uncertain communication environments like population dynamics with time lag necessitate dealing with time delays. As a result, parameter estimation for stochastic delay differential equations has garnered significant interest in recent decades. Berezansky and Braverman ([3]) discussed the estimation of solutions for delay linear differential equations. Benke and Pap ([2]) examined the convergence properties of the maximum likelihood estimator. Liu and Jia ([17]) utilized the moments method to calculate parameter values from discrete solution observations. Zhu et al. ([29]) explored identifying parameters in a reaction-diffusion rumor propagation system with time delay. Jamilla et al. ([12]) employed a genetic algorithm with multi-parent crossover to estimate parameters in three neutral delay differential equation models with discrete delay. Wei ([23]) derived the consistency and asymptotic distribution of the estimator under condition of two types of small noises.

Stochastic differential equations may not accurately represent certain time-varying systems like stock prices. As a result, Liu ([15]) developed uncertainty theory, which was later refined by Liu ([16]) based on normality, duality, subadditivity, and product axioms. Recent literature has discussed parameter estimation for uncertain differential equations (UDEs). For instance, Li et al. ([14]) presented three methods for parameter estimation in UDEs using discrete observational data. Chen et al. ([5]) applied the method of moments to estimate parameters in an uncertain SIR model, devising a numerical solution algorithm. Liu ([18]) utilized generalized moment estimation techniques. Yang et al. ([25]) applied the α-path method for parameter estimation. Liu and Yang ([19]) proposed moment estimations for unknown parameters using Euler method approximation for high-order UDEs. Wei ([24]) employed a contrast function to derive least squares estimators for an uncertain Vasicek model. Ye and Liu ([26]) suggested a method for determining if an uncertain differential equation adequately fits observed data or not. He et al. ([11]) formulated an algorithm for estimating parameters in a specific uncertain fractional differential equation.

Despite recent advancements in parameter estimation for UDEs, the asymptotic characteristics of the estimators remain unexplored. To address this gap, this study examines parameter estimation for intelligent transportation systems described by UDEs with a small dispersion coefficient from discrete observations. We establish the consistency and asymptotic distribution of the estimator, presenting the uncertain Hyperbolic model as a case study with accompanying simulations. The paper is structured as follows: Section 2 outlines the problem formulation and details the least squares estimator based on the contrast function. Section 3 focuses on determining the asymptotic properties of the least squares estimator. Section 4 includes the uncertain Hyperbolic model as an illustrative example along with simulation results.

II. Preliminaries

Firstly, we give some definitions about uncertain variables and Liu process.

Definition 1: ([15], [16]) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}:\mathcal{L}\to[0,1]$ is called an uncertain measure if it satisfies the following axioms:
Axiom 1: (Normality Axiom) $\mathcal{M}(\Gamma)=1$ for the universal set Γ.
Axiom 2: (Duality Axiom) $\mathcal{M}(\Lambda)+\mathcal{M}(\Lambda^c)=1$ for any event Λ.
Axiom 3: (Subadditivity Axiom) For every countable sequence of events Λ_1, Λ_2, $\cdots$,

$$\mathcal{M}\{\bigcup_{i=1}^{\infty}\Lambda_i\}\le\sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

Axiom 4: (Product Axiom) Let $(\Gamma_k,\mathcal{L}_k,\mathcal{M}_k)$ be uncertainty spaces for $k=1,2,\cdots$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\{\Pi_{k=1}^{\infty}\Lambda_k\}=\min_{k\ge1}\mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k=1,2,\cdots$.

An uncertain variable ξ is a measurable function from the uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to the set of real numbers.

Definition 2: ([15]) For any real number x, let ξ be an uncertain variable and its uncertainty distribution is defined by

$$\Phi(x)=\mathcal{M}(\xi\le x).$$

In particular, an uncertain variable ξ is called normal if it has an uncertainty distribution

$$\Phi(x)=(1+\exp(\frac{\pi(\mu-x)}{\sqrt{3}\sigma}))^{-1},x\in\Re,$$

denoted by $\mathcal{N}(\mu,\sigma)$. If $\mu=0$, $\sigma=1$, ξ is called a standard normal uncertain variable.

Definition 3: ([16]) An uncertain process C_t is called a Liu process if
(i) $C_0=0$ and almost all sample paths are Lipschitz continuous, (ii) C_t has stationary and independent increments, (iii) the increment $C_{s+t}-C_s$ has a normal uncertainty distribution

$$\Phi_t(x)=(1+\exp(\frac{-\pi x}{\sqrt{3}t}))^{-1},x\in\Re.$$

Moreover, a real-valued function X_t^α is called the α-path of above uncertain differential equation if it solves the corresponding ordinary differential equation

$$dX_t^\alpha=h(t,X_t^\alpha,X_{t-\tau}^\alpha)dt+|w(t,X_t^\alpha,X_{t-\tau}^\alpha)|\Phi^{-1}(\alpha)dt,$$

where

$$\Phi^{-1}(\alpha)=\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha},\quad\alpha\in(0,1).$$

The UDEs considered in this paper is described as follows:

$$\begin{cases}dX_t=a(X_t,\theta)dt+\varepsilon dC_t,\quad t\in[0,1],\\ X_0=x_0,\end{cases}\tag{1}$$

where $\theta\in\Theta$ is an unknown parameter. $\varepsilon\in(0,1]$, C_t is a Liu process. We assume that $\{X_t,t\ge0\}$ is observed at $\{t_i=\frac{i}{n},i=1,2,\cdots,n\}$.

Note that, the contrast function is

$$\rho_{n,\varepsilon}(\theta)=\sum_{i=1}^{n}\frac{|X_{t_i}-X_{t_{i-1}}-a(X_{t_{i-1}},\theta)\Delta t_{i-1}|^2}{\varepsilon^2\Delta t_{i-1}},\tag{2}$$

where $\Delta t_{i-1}=t_i-t_{i-1}=\frac{1}{n}$.

Hence, the least squares estimator could be written as

$$\widehat{\theta}_{n,\varepsilon}=\arg\min_{\theta\in\Theta}\rho_{n,\varepsilon}(\theta).\tag{3}$$

Let

$$\Psi_{n,\varepsilon}(\theta)=\varepsilon^2(\rho_{n,\varepsilon}(\theta)-\rho_{n,\varepsilon}(\theta_0)),\tag{4}$$

where θ_0 is the true parameter value.

We have

$$\widehat{\theta}_{n,\varepsilon}=\arg\min_{\theta\in\Theta}\Psi_{n,\varepsilon}(\theta).\tag{5}$$

III. Main Results and Proofs

Suppose $X^0=(X_t^0,t\ge0)$ is the solution to the given ordinary differential equation:

$$dX_t^0=a(X_t^0,\theta_0)dt,\quad X_0^0=x_0.\tag{6}$$

We introduce some assumptions as follows:

Assumption 1: For $x,y\in\mathbb{R}$, $\theta\in\Theta$, there exists some positive constants K_1 and K_2 satisfying

$$|a(x,\theta)-a(y,\theta)|\le K_1|x-y|,\quad|a(x,\theta)|\le K_2(1+|x|).$$

Assumption 2:

$$\theta\ne\theta_0\Longleftrightarrow a(X_t^0,\theta)\ne a(X_t^0,\theta_0).$$

Assumption 3:

$$|\partial_\theta a(x,\theta)|+|\partial_{\theta\theta}a(x,\theta)|\le L(1+|x|),$$

where $L > 0$ is a constant, $\partial_\theta a(x,\theta)$ and $\partial_{\theta\theta} a(x,\theta)$ denote the once and twice differential of θ respectively.

Now, we give some important lemmas.

Denote

$$R_t^{n,\varepsilon} = X_{\frac{[nt]}{n}}, \tag{7}$$

where $[nt]$ is the integer part of nt.

Lemma 1: As $\varepsilon \to 0$, $n \to \infty$,

$$\sup_{0\le t\le 1} |X_t - X_t^0| \xrightarrow{P} 0.$$

Proof: Note that

$$X_t - X_t^0 = \int_0^t (a(X_s,\theta_0) - a(X_s^0,\theta_0))ds + \varepsilon C_t. \tag{8}$$

Then, by using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & |X_t - X_t^0|^2 \\ \le\ & 2|\int_0^t (a(X_s,\theta_0) - a(X_s^0,\theta_0))ds|^2 + 2\varepsilon^2 |C_t|^2 \\ \le\ & 2t\int_0^t |a(X_s,\theta_0) - a(X_s^0,\theta_0)|^2 ds + 2\varepsilon^2 \sup_{0\le t\le 1}|C_t|^2 \\ \le\ & 2K_1^2 t\int_0^t |X_s - X_s^0|^2 ds + 2\varepsilon^2 \sup_{0\le t\le 1}|C_t|^2. \end{aligned}$$

Thus, with the help of Gronwall's inequality, we have

$$|X_t - X_t^0|^2 \le 2\varepsilon^2 e^{2K_1^2} \sup_{0\le t\le 1}|C_t|^2. \tag{9}$$

Hence,

$$\sup_{0\le t\le 1}|X_t - X_t^0| \le \sqrt{2}\varepsilon e^{K_1^2} \sup_{0\le t\le 1}|C_t|. \tag{10}$$

Therefore,

$$\sup_{0\le t\le 1}|X_t - X_t^0| \xrightarrow{P} 0, \tag{11}$$

as $\varepsilon \to 0$ and $n \to \infty$. ■

Since $\frac{[nt]}{n} \to t$ as $n \to \infty$, according to Lemma 1, we have

$$R_t^{n,\varepsilon} \xrightarrow{P} X_t^0. \tag{12}$$

Lemma 2: As $\varepsilon \to 0$ and $n \to \infty$,

$$\frac{1}{n}\sum_{i=1}^n a(X_{t_{i-1}},\theta) \xrightarrow{P} \int_0^1 a(X_t^0,\theta)dt.$$

Proof: Since

$$\begin{aligned} & \sup_\theta |\frac{1}{n}\sum_{i=1}^n a(X_{t_{i-1}},\theta) - \int_0^1 a(X_t^0,\theta)dt| \\ & = \sup_\theta |\int_0^1 a(R_t^{n,\varepsilon},\theta)dt - \int_0^1 a(X_t^0,\theta)dt| \\ & \le \sup_\theta \int_0^1 |a(R_t^{n,\varepsilon},\theta) - a(X_t^0,\theta)|dt \\ & \le K_1 \int_0^1 |R_t^{n,\varepsilon} - X_t^0|dt \\ & \le K_1 \sup_{0\le t\le 1}|R_t^{n,\varepsilon} - X_t^0|, \end{aligned}$$

we have

$$\sup_\theta |\frac{1}{n}\sum_{i=1}^n a(X_{t_{i-1}},\theta) - \int_0^1 a(X_t^0,\theta)dt| \xrightarrow{P} 0, \tag{13}$$

as $\varepsilon \to 0$ and $n \to \infty$.

Therefore,

$$\frac{1}{n}\sum_{i=1}^n a(X_{t_{i-1}},\theta) \xrightarrow{P} \int_0^1 a(X_t^0,\theta)dt, \tag{14}$$

as $\varepsilon \to 0$ and $n \to \infty$. ■

Lemma 3: As $\varepsilon \to 0$ and $n \to \infty$,

$$\sum_{i=1}^n a(X_{t_{i-1}},\theta)(C_{t_i} - C_{t_{i-1}}) \xrightarrow{P} \int_0^1 a(X_t^0,\theta)dC_t.$$

Proof: Note that

$$\sum_{i=1}^n a(X_{t_{i-1}},\theta)(C_{t_i} - C_{t_{i-1}}) = \int_0^1 a(R_t^{n,\varepsilon},\theta)dC_t. \tag{15}$$

Then, for each sample γ and any given $\eta > 0$,

$$\begin{aligned} & P(|\int_0^1 a(R_t^{n,\varepsilon}(\gamma),\theta)dC_t - \int_0^1 a(X_t^0(\gamma),\theta)dC_t| > \eta) \\ & \le \frac{K(\gamma)}{\eta}\int_0^1 \mathbb{E}[|a(R_t^{n,\varepsilon}(\gamma),\theta) - a(X_t^0(\gamma),\theta)|]dt \\ & \le \frac{K(\gamma)K_1}{\eta}\int_0^1 \mathbb{E}[|R_t^{n,\varepsilon}(\gamma) - X_t^0(\gamma)|]dt \\ & \to 0, \end{aligned}$$

where $K(\gamma)$ is the Lipschitz constant.

Therefore,

$$\sum_{i=1}^n a(X_{t_{i-1}},\theta)(C_{t_i} - C_{t_{i-1}}) \xrightarrow{P} \int_0^1 a(X_t^0,\theta)dC_t, \tag{16}$$

as $\varepsilon \to 0$ and $n \to \infty$. ■

Theorem 1: As $\varepsilon \to 0$ and $n \to \infty$,

$$\widehat{\theta}_{n,\varepsilon} \xrightarrow{P} \theta_0.$$

Proof: Note that

$$\begin{aligned} & \varepsilon^2(\rho_{n,\varepsilon}(\theta) - \rho_{n,\varepsilon}(\theta_0)) \\ & = \sum_{i=1}^n -(X_{t_i} - X_{t_{i-1}} - \frac{1}{n}a(X_{t_{i-1}},\theta) + X_{t_i} - X_{t_{i-1}} \\ & -\frac{1}{n}a(X_{t_{i-1}},\theta_0))(a(X_{t_{i-1}},\theta) - a(X_{t_{i-1}},\theta_0)) \\ & = -2\sum_{i=1}^n (a(X_{t_{i-1}},\theta) - a(X_{t_{i-1}},\theta_0))(X_{t_i} - X_{t_{i-1}} \\ & -\frac{1}{n}a(X_{t_{i-1}},\theta_0)) \\ & +\frac{1}{n}\sum_{i=1}^n (a(X_{t_{i-1}},\theta) - a(X_{t_{i-1}},\theta_0))^2. \end{aligned}$$

Since

$$X_{t_i} - X_{t_{i-1}} = \int_{t_{i-1}}^{t_i} a(X_s,\theta_0)ds + \varepsilon(C_{t_i} - C_{t_{i-1}}), \tag{17}$$

we have

$$\begin{aligned}
&\sum_{i=1}^{n}(a(X_{t_{i-1}},\theta)-a(X_{t_{i-1}},\theta_0))\\
&(X_{t_i}-X_{t_{i-1}}-\frac{1}{n}a(X_{t_{i-1}},\theta_0))\\
&=\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(a(X_{t_{i-1}},\theta)-a(X_{t_{i-1}},\theta_0))\\
&(a(X_s,\theta_0)-a(X_{t_{i-1}},\theta_0))ds\\
&+\varepsilon\sum_{i=1}^{n}(a(X_{t_{i-1}},\theta)-a(X_{t_{i-1}},\theta_0))(C_{t_i}-C_{t_{i-1}})\\
&=\int_0^1(a(R_s^{n,\varepsilon},\theta)-a(R_s^{n,\varepsilon},\theta_0))\\
&(a(X_s,\theta_0)-a(R_s^{n,\varepsilon},\theta_0))ds\\
&+\varepsilon\int_0^1(a(R_s^{n,\varepsilon},\theta)-a(R_s^{n,\varepsilon},\theta_0)dC_s.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\sup_\theta|\int_0^1(a(R_s^{n,\varepsilon},\theta)-a(R_s^{n,\varepsilon},\theta_0))\\
&(a(X_s,\theta_0)-a(R_s^{n,\varepsilon},\theta_0))ds|\\
&\leq\int_0^1\sup_\theta(|a(R_s^{n,\varepsilon},\theta)|+|a(R_s^{n,\varepsilon},\theta_0)|)\\
&K_1|X_s-R_s^{n,\varepsilon}|ds\\
&\leq 2K_1K_2\int_0^1(1+|R_s^{n,\varepsilon}|)|X_s-R_s^{n,\varepsilon}|ds\\
&\leq 2K_1K_2\int_0^1(1+|R_s^{n,\varepsilon}|)(|X_s-X_s^0|\\
&+|R_s^{n,\varepsilon}-X_s^0|)ds\\
&\leq 2K_1K_2(1+\sup_{0\leq s\leq1}|X_s|)(\sup_{0\leq s\leq1}|X_s-X_s^0|\\
&+\sup_{0\leq s\leq1}|R_s^{n,\varepsilon}-X_s^0|).
\end{aligned}$$

As $X_s\xrightarrow{P}X_s^0$ and $R_s^{n,\varepsilon}\xrightarrow{P}X_s^0$, we have

$$\begin{aligned}
&\int_0^1(a(R_s^{n,\varepsilon},\theta)-a(R_s^{n,\varepsilon},\theta_0))\\
&(a(X_s,\theta_0)-a(R_s^{n,\varepsilon},\theta_0))ds\\
&\xrightarrow{P}0,
\end{aligned}$$

as $\varepsilon\to0$ and $n\to\infty$.

For each sample γ and any given $\eta>0$,

$$\begin{aligned}
&P(|\varepsilon\int_0^1(a(R_s^{n,\varepsilon},\theta)-a(R_s^{n,\varepsilon},\theta_0))dC_s|>\eta)\\
&\leq\frac{\varepsilon K(\gamma)}{\eta}\int_0^1\mathbb{E}[|a(R_s^{n,\varepsilon},\theta)-a(R_s^{n,\varepsilon},\theta_0)|]ds\\
&\leq\frac{\varepsilon K(\gamma)}{\eta}\int_0^1\mathbb{E}[2K_2(1+|R_s^{n,\varepsilon}|)]ds\to0,
\end{aligned}\tag{18}$$

as $\varepsilon\to0$ and $n\to\infty$.

Hence,

$$\varepsilon\int_0^1(a(R_s^{n,\varepsilon},\theta)-a(R_s^{n,\varepsilon},\theta_0))dC_s\xrightarrow{P}0,\tag{19}$$

as $\varepsilon\to0$ and $n\to\infty$.

Therefore,

$$\begin{aligned}
&\sum_{i=1}^{n}(a(X_{t_{i-1}},\theta)-a(X_{t_{i-1}},\theta_0))\\
&(X_{t_i}-X_{t_{i-1}}-\frac{1}{n}a(X_{t_{i-1}},\theta_0))\xrightarrow{P}0,
\end{aligned}$$

as $\varepsilon\to0$ and $n\to\infty$.

According to Lemma 2, we obtain

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}(a(X_{t_{i-1}},\theta)-a(X_{t_{i-1}},\theta_0))^2\\
&\xrightarrow{P}\int_0^1(a(X_t^0,\theta)-a(X_t^0,\theta_0))^2dt,
\end{aligned}$$

as $\varepsilon\to0$ and $n\to\infty$.

Then, we have

$$\varepsilon^2(\rho_{n,\varepsilon}(\theta)-\rho_{n,\varepsilon}(\theta_0))\xrightarrow{P}\int_0^1(a(X_t^0,\theta)-a(X_t^0,\theta_0))^2dt,\tag{20}$$

as $\varepsilon\to0$ and $n\to\infty$.

By using Assumption 2 and the continuity of X_t^0, for each $\delta>0$, we get

$$\begin{aligned}
&\inf_{|\theta-\theta_0|>\delta}\int_0^1(a(X_t^0,\theta)-a(X_t^0,\theta_0))^2dt\\
&>\int_0^1(a(X_t^0,\theta_0)-a(X_t^0,\theta_0))^2dt=0.
\end{aligned}$$

Therefore,

$$\widehat{\theta}_{n,\varepsilon}\xrightarrow{P}\theta_0,\tag{21}$$

as $\varepsilon\to0$ and $n\to\infty$.

■

Theorem 2: As $\varepsilon\to0$, $n\to\infty$ and $n\varepsilon\to\infty$,

$$\varepsilon^{-1}(\widehat{\theta}_{n,\varepsilon}-\theta_0)\xrightarrow{d}\frac{\int_0^1\partial_\theta a(X_t^0,\theta_0)dC_t}{\int_0^1(\partial_\theta a(X_t^0,\theta_0))^2dt}$$

Proof: Since

$$\begin{aligned}
&\frac{d\varepsilon^2(\rho_{n,\varepsilon}(\theta)-\rho_{n,\varepsilon}(\theta_0))}{d\theta}\\
&=-2\sum_{i=1}^{n}\partial_\theta a(X_{t_{i-1}},\theta)(X_{t_i}-X_{t_{i-1}}\\
&-a(X_{t_{i-1}},\theta)\Delta t_{i-1}),
\end{aligned}$$

and

$$\begin{aligned}
&\varepsilon^{-1}\sum_{i=1}^{n}\partial_\theta a(X_{t_{i-1}},\theta_0)(X_{t_i}-X_{t_{i-1}}\\
&-a(X_{t_{i-1}},\theta_0)\Delta t_{i-1})\\
&=\varepsilon^{-1}\sum_{i=1}^{n}\partial_\theta a(X_{t_{i-1}},\theta_0)\\
&\int_{t_{i-1}}^{t_i}(a(X_s,\theta_0)-a(X_{t_{i-1}},\theta_0))ds\\
&+\sum_{i=1}^{n}\partial_\theta a(X_{t_{i-1}},\theta_0)(C_{t_i}-C_{t_{i-1}}).
\end{aligned}$$

According to Lemma 3, we obtain

$$\sum_{i=1}^{n}\partial_\theta a(X_{t_{i-1}},\theta_0)(C_{t_i}-C_{t_{i-1}})\xrightarrow{P}\int_0^1\partial_\theta a(X_t^0,\theta_0)dC_t,\tag{22}$$

as $\varepsilon \to 0$, $n \to \infty$.

For $s \in [t_{i-1}, t_i]$, we have

$$\begin{aligned} X_s - X_{t_{i-1}} &= \int_{t_{i-1}}^{s} a(X_v, \theta_0)dv + \varepsilon(C_s - C_{t_{i-1}}) \\ &= \int_{t_{i-1}}^{s} (a(X_v, \theta_0) - a(X_{t_{i-1}}, \theta_0))dv \\ &\quad + a(X_{t_{i-1}}, \theta_0)(s - t_{i-1}) + \varepsilon(C_s - C_{t_{i-1}}). \end{aligned}$$

By using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |X_s - X_{t_{i-1}}|^2 &\le 2|\int_{t_{i-1}}^{s} (a(X_v, \theta_0) - a(X_{t_{i-1}}, \theta_0))dv|^2 \\ &\quad + 2(|a(X_{t_{i-1}}, \theta_0)|(s - t_{i-1}) \\ &\quad + \varepsilon|C_s - C_{t_{i-1}}|)^2 \\ &\le \frac{2K_1^2}{n} \int_{t_{i-1}}^{s} |X_v - X_{t_{i-1}}|^2 dv \\ &\quad + 2(\frac{1}{n}|a(X_{t_{i-1}}, \theta_0)| \\ &\quad + \varepsilon \sup_{t_{i-1} \le s \le t_i} |C_s - C_{t_{i-1}}|)^2. \end{aligned}$$

With the help of Gronwall's inequality, we have

$$\begin{aligned} &|X_s - X_{t_{i-1}}|^2 \le 2(\frac{1}{n}|a(X_{t_{i-1}}, \theta_0) \\ &+ \varepsilon \sup_{t_{i-1} \le s \le t_i} |C_s - C_{t_{i-1}}|)^2 e^{\frac{2K_1^2}{n}(s - t_{i-1})}. \end{aligned}$$

Then,

$$\begin{aligned} &\sup_{t_{i-1} \le s \le t_i} |X_s - X_{t_{i-1}}| \le \sqrt{2}(\frac{1}{n}|a(X_{t_{i-1}}, \theta_0) \\ &+ \varepsilon \sup_{t_{i-1} \le s \le t_i} |C_s - C_{t_{i-1}}|) e^{\frac{K_1^2}{n^2}}. \end{aligned}$$

Hence,

$$\begin{aligned} &|\varepsilon^{-1} \sum_{i=1}^{n} \partial_\theta a(X_{t_{i-1}}, \theta_0) \\ &\int_{t_{i-1}}^{t_i} (a(X_s, \theta_0) - a(X_{t_{i-1}}, \theta_0))ds| \\ &\le \varepsilon^{-1} \sum_{i=1}^{n} |\partial_\theta a(X_{t_{i-1}}, \theta_0)| \\ &|\int_{t_{i-1}}^{t_i} (a(X_s, \theta_0) - a(X_{t_{i-1}}, \theta_0))ds| \\ &\le \varepsilon^{-1} K_1 \sum_{i=1}^{n} |\partial_\theta a(X_{t_{i-1}}, \theta_0)| \int_{t_{i-1}}^{t_i} |X_s - X_{t_{i-1}}| ds \\ &\le \frac{K_1}{n\varepsilon} \sum_{i=1}^{n} |\partial_\theta a(X_{t_{i-1}}, \theta_0)| \sup_{t_{i-1} \le s \le t_i} |X_s - X_{t_{i-1}}| \\ &\le \frac{\sqrt{2}K_1 e^{\frac{K_1^2}{n^2}}}{n\varepsilon} \frac{1}{n} \sum_{i=1}^{n} |\partial_\theta a(X_{t_{i-1}}, \theta_0)||a(X_{t_{i-1}}, \theta_0)| \\ &+ \frac{\sqrt{2}K_1 e^{\frac{K_1^2}{n^2}}}{n} \sum_{i=1}^{n} |\partial_\theta a(X_{t_{i-1}}, \theta_0)| \\ &\sup_{t_{i-1} \le s \le t_i} |C_s - C_{t_{i-1}}|. \end{aligned}$$

Since

$$\frac{1}{n} \sum_{i=1}^{n} |\partial_\theta a(X_{t_{i-1}}, \theta_0)||a(X_{t_{i-1}}, \theta_0)| \le LK_2(1 + \sup_{0 \le t \le 1} |X_t|). \tag{23}$$

When $\varepsilon \to 0$, $n \to \infty$ and $n\varepsilon \to \infty$, we have

$$\frac{\sqrt{2}K_1 e^{\frac{K_1^2}{n^2}}}{n\varepsilon} \frac{1}{n} \sum_{i=1}^{n} |\partial_\theta a(X_{t_{i-1}}, \theta_0)||a(X_{t_{i-1}}, \theta_0)| \xrightarrow{P} 0. \tag{24}$$

As

$$\frac{1}{n} \sum_{i=1}^{n} \sup_{t_{i-1} \le s \le t_i} |C_s - C_{t_{i-1}}| \xrightarrow{P} 0, \tag{25}$$

we have

$$\begin{aligned} &\frac{\sqrt{2}K_1 e^{\frac{K_1^2}{n^2}}}{n} \sum_{i=1}^{n} |\partial_\theta a(X_{t_{i-1}}, \theta_0)| \sup_{t_{i-1} \le s \le t_i} |C_s - C_{t_{i-1}}| \\ &\le \sqrt{2}K_1 e^{\frac{K_1^2}{n^2}} L(1 + \sup_{0 \le t \le 1} |X_t|) \\ &\frac{1}{n} \sum_{i=1}^{n} \sup_{t_{i-1} \le s \le t_i} |C_s - C_{t_{i-1}}| \\ &\xrightarrow{P} 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &\varepsilon^{-1} \sum_{i=1}^{n} \partial_\theta a(X_{t_{i-1}}, \theta_0)(X_{t_i} - X_{t_{i-1}} \\ &- a(X_{t_{i-1}}, \theta_0)\Delta t_{i-1}) \\ &\xrightarrow{P} \int_0^1 \partial_\theta a(X_t^0, \theta_0) dC_t. \end{aligned}$$

Then,

$$\varepsilon^{-1} \partial_\theta \Psi_{n,\varepsilon}(\theta_0) \xrightarrow{P} \int_0^1 \partial_\theta a(X_t^0, \theta_0) dC_t. \tag{26}$$

Note that

$$\begin{aligned} \partial_\theta \Psi_{n,\varepsilon}(\theta_0) &= \partial_\theta \Psi_{n,\varepsilon}(\widehat{\theta}_{n,\varepsilon}) \\ &\quad + \partial_{\theta\theta} \Psi_{n,\varepsilon}(\widehat{\theta}_{n,\varepsilon} + \delta(\theta_0 - \widehat{\theta}_{n,\varepsilon}))(\theta_0 - \widehat{\theta}_{n,\varepsilon}), \end{aligned}$$

where $\delta \in [0, 1]$.

Since $\partial_\theta \Psi_{n,\varepsilon}(\widehat{\theta}_{n,\varepsilon})$=0, we get

$$\partial_\theta \Psi_{n,\varepsilon}(\theta_0) = \partial_{\theta\theta} \Psi_{n,\varepsilon}(\widehat{\theta}_{n,\varepsilon} + \delta(\theta_0 - \widehat{\theta}_{n,\varepsilon}))(\theta_0 - \widehat{\theta}_{n,\varepsilon}), \tag{27}$$

Note that

$$\begin{aligned} &\partial_{\theta\theta} \Psi_{n,\varepsilon}(\theta_0) \\ &= -2 \sum_{i=1}^{n} [\partial_{\theta\theta} a(X_{t_{i-1}}, \theta_0)(X_{t_i} - X_{t_{i-1}} \\ &- a(X_{t_{i-1}}, \theta_0)\Delta t_{i-1}) \\ &- \Delta t_{i-1}(\partial_\theta a(X_{t_{i-1}}, \theta_0))^2] \\ &= -2 \sum_{i=1}^{n} \partial_{\theta\theta} a(X_{t_{i-1}}, \theta_0)(X_{t_i} - X_{t_{i-1}} \\ &- a(X_{t_{i-1}}, \theta_0)\Delta t_{i-1}) \\ &+ 2 \sum_{i=1}^{n} \Delta t_{i-1}(\partial_\theta a(X_{t_{i-1}}, \theta_0))^2. \end{aligned}$$

According to Lemma 2, we have

$$\sum_{i=1}^{n} \Delta t_{i-1}(\partial_\theta a(X_{t_{i-1}}, \theta_0))^2 \xrightarrow{P} \int_0^1 (\partial_\theta a(X_t^0, \theta_0))^2 ds, \tag{28}$$

as $\varepsilon \to 0$ and $n \to \infty$.

As

$$\sum_{i=1}^{n} \partial_{\theta\theta} a(X_{t_{i-1}}, \theta_0)(X_{t_i} - X_{t_{i-1}} - a(X_{t_{i-1}}, \theta_0)\Delta t_{i-1}) = \sum_{i=1}^{n} \partial_{\theta\theta} a(X_{t_{i-1}}, \theta_0) \int_{t_{i-1}}^{t_i} (a(X_s, \theta_0) - a(X_{t_{i-1}}, \theta_0))ds + \varepsilon \sum_{i=1}^{n} \partial_{\theta\theta} a(X_{t_{i-1}}, \theta_0)(C_{t_i} - C_{t_{i-1}}). \quad (29)$$

By applying Lemma 3, we obtain

$$\varepsilon \sum_{i=1}^{n} \partial_{\theta\theta} a(X_{t_{i-1}}, \theta_0)(C_{t_i} - C_{t_{i-1}}) \xrightarrow{P} 0, \quad (30)$$

as $\varepsilon \to 0$ and $n \to \infty$.

Then, we have

$$\sum_{i=1}^{n} \partial_{\theta\theta} a(X_{t_{i-1}}, \theta_0) \int_{t_{i-1}}^{t_i} (a(X_s, \theta_0) - a(X_{t_{i-1}}, \theta_0))ds \xrightarrow{P} 0, \quad (31)$$

as $\varepsilon \to 0$ and $n \to \infty$.

Hence,

$$\sum_{i=1}^{n} \partial_{\theta\theta} a(X_{t_{i-1}}, \theta_0)(X_{t_i} - X_{t_{i-1}} - a(X_{t_{i-1}}, \theta_0)\Delta t_{i-1}) \xrightarrow{P} 0, \quad (32)$$

as $\varepsilon \to 0$ and $n \to \infty$.

Then,

$$\partial_{\theta\theta} \Psi_{n,\varepsilon}(\theta_0) \xrightarrow{P} \int_0^1 (\partial_\theta a(X_t^0, \theta_0))^2 ds, \quad (33)$$

as $\varepsilon \to 0$ and $n \to \infty$.

With the help of Theorem 1, we get

$$\varepsilon^{-1}(\widehat{\theta}_{n,\varepsilon} - \theta_0) \xrightarrow{d} \frac{\int_0^1 \partial_\theta a(X_t^0, \theta_0) dC_t}{\int_0^1 (\partial_\theta a(X_t^0, \theta_0))^2 dt}, \quad (34)$$

as $\varepsilon \to 0$ and $n \to \infty$.

The proof is complete. ■

IV. Example

Consider the following uncertain Hyperbolic model:

$$\begin{cases} dX_t = \theta \dfrac{X_t}{\sqrt{1 + X_t^2}} dt + \varepsilon dC_t, \quad t \in [0, 1], \\ X_0 = x_0, \end{cases} \quad (35)$$

where $\theta \in \Theta$ is an unknown parameter. $\varepsilon \in (0, 1]$, C_t is a Liu process.

Since

$$|\theta \frac{x}{\sqrt{1 + x^2}} - \theta \frac{y}{\sqrt{1 + y^2}}| \leq 2|\theta||x - y|, \quad (36)$$

$$|\theta \frac{x}{\sqrt{1 + x^2}}| \leq |\theta||1 + x|, \quad (37)$$

$$|\partial_\theta(\theta \frac{x}{\sqrt{1 + x^2}})| + |\partial_{\theta\theta}(\theta \frac{x}{\sqrt{1 + x^2}})| = |\frac{x}{\sqrt{1 + x^2}}| \leq |1 + x|, \quad (38)$$

the coefficients of (46) satisfy Assumptions 1-4.

Then, we have

$$\widehat{\theta}_{n,\varepsilon} \xrightarrow{P} \theta_0, \quad (39)$$

and

$$\varepsilon^{-1}(\widehat{\theta}_{n,\varepsilon} - \theta_0) \xrightarrow{d} \frac{\int_0^1 \frac{X_t^0}{\sqrt{1+X_t^0}} dC_t}{\int_0^1 \frac{(X_t^0)^2}{1+X_t^0} dt}. \quad (40)$$

It is easy to check that

$$\widehat{\theta}_{n,\varepsilon} = \frac{\sum_{i=1}^{n} \frac{(X_{t_i} - X_{t_{i-1}})X_{t_{i-1}}}{\sqrt{1+X_{t_{i-1}}^2}}}{\frac{1}{n}\sum_{i=1}^{n} \frac{X_{t_{i-1}}^2}{1+X_{t_{i-1}}^2}}. \quad (41)$$

Estimating the parameter $\widehat{\theta}_{n,\varepsilon}$ utilizing the discrete sample $(X_{t_i})_{i=0,1,\ldots,n}$, with the initial value $x_0 = 0.1$. When $\varepsilon = 0.5$ in Table 1, ε equals 0.1 in Table 2 and $\varepsilon = 0.05$ in Table 3, analysis of the simulation data indicates that with sufficient sample size n and sufficiently small ε, the estimated parameter closely approximates the true value.

TABLE I
Least squares estimator simulation results of $\widehat{\theta}_{n,\varepsilon}$, $\varepsilon = 0.5$

True	Average Value		Absolute Error
θ_0	Size n	$\widehat{\theta}_{n,\varepsilon}$	$\lvert\theta_0 - \widehat{\theta}_{n,\varepsilon}\rvert$
	1000	1.1836	0.1836
1	2000	1.1579	0.1579
	5000	1.1228	0.1228

TABLE II
Least squares estimator simulation results of $\widehat{\theta}_{n,\varepsilon}$, $\varepsilon = 0.1$

True	Average Value		Absolute Error
θ_0	Size n	$\widehat{\theta}_{n,\varepsilon}$	$\lvert\theta_0 - \widehat{\theta}_{n,\varepsilon}\rvert$
	1000	1.0469	0.0469
1	2000	1.0253	0.0253
	5000	1.0178	0.0178

Suppose there are 20 sets of observed data presented in Table 4. Next, we calculate the least squares estimator using this information:

$$\widehat{\theta}_{n,\varepsilon} = 2.1817.$$

Thus, let $\varepsilon = 0.8$, the uncertain Hyperbolic model could be written as

$$dX_t = 2.1817 \frac{X_t}{\sqrt{1 + X_t^2}} dt + 0.8 dC_t.$$

TABLE III
LEAST SQUARES ESTIMATOR SIMULATION RESULTS OF $\widehat{\theta}_{n,\varepsilon}$, $\varepsilon = 0.01$

True	Average Value		Absolute Error
θ_0	Size n	$\widehat{\theta}_{n,\varepsilon}$	$\lvert\theta_0 - \widehat{\theta}_{n,\varepsilon}\rvert$
1	10000	1.0063	0.0063
	20000	1.0041	0.0041
	50000	1.0025	0.0025

Hence, the γ-path X_t^γ $(0 < \gamma < 1)$ is the solution for given ordinary differential equation:

$$dX_t^\gamma = 2.1817X_t^\gamma dt + 0.8\frac{\sqrt{3}}{\pi}\ln\frac{\gamma}{1-\gamma}dt.$$

TABLE IV
OBSERVATIONS OF UNCERTAIN HYPERBOLIC MODEL

n	1	2	3	4	5	6	7	8	9	10
t_i	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
X_{t_i}	0.51	1.29	1.12	1.93	1.45	2.03	2.87	3.92	2.58	2.90
n	11	12	13	14	15	16	17	18	19	20
t_i	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
X_{t_i}	2.77	3.67	3.75	3.31	3.53	1.30	2.05	2.87	1.56	2.61

Based on the data shown in Figure 1, every observation is within the range of 0.05-path $X_t^{0.05}$ to 0.93-path $X_t^{0.93}$. As a result, the techniques for determining parameters prove to be successful.

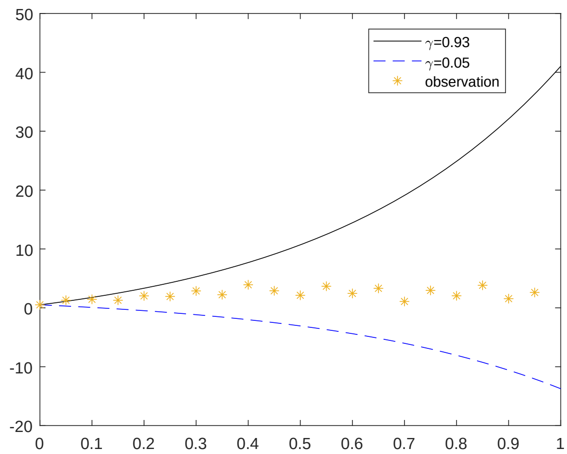

Fig. 1. Observations and γ-path of X_t

V. CONCLUSION

This study explores parameter estimation challenges for general UDEs using discrete observations. Our analysis includes establishing the consistency and asymptotic distribution of the estimator. Future investigations will focus on estimating uncertain delay differential equations.

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