The Solution of Self-Adjoint Euler-Cauchy Operator-Differential Equation

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***Abstract*—The topics concerning the solution of differential equations with variable coefficients attracts the interest of numerous academics. We investigate the non-homogeneous third-order self-adjoint Euler-Cauchy operator-differential equation for the first time. The Laplace transform method is applied in this study to determine the general solution of the self-adjoint Euler-Cauchy problem. We demonstrated the effectiveness of this approach in solving the self-adjoint Cauchy-Euler differential equation in the form of the exponential functions.**

***Index Terms*—self-adjoint Euler-Cauchy equations, operator-differential equation, variable coefficients, self-adjoint operator, Laplace transform.**

I. Introduction

THE The Euler-Cauchy equation $t^2y^{''}+aty^{'}+by=0$ [12] has several applications in engineering and physics. The time-harmonic vibrations of a thin elastics rod can be more conveniently described by the differential equation $\frac{d}{dx}\big(E(x)\frac{du}{dx}\big)+\rho\omega^2u=0$ [11], in which $E(x)$ denotes the Young's modulus and equals $\rho\omega^2x^2$, ρ denotes the material density and ω denotes the angular frequency of the vibration. Additionally, Euler-Cauchy differential equations appear when solving second-order partial differential equations (Laplace's equations.) Moreover, in financial mathematics, the Euler-Cauchy DE provides an abstract estimate of the cost of European call options.

More precisely, the solutions of the non-homogeneous self-adjoint Euler-Cauchy DE of the form

$$t^2v^{''}(t)+2Atv^{'}(t)+A^2G(t)=f(t),t\in[0,\infty)$$

were obtained using Laplace transform in our work [5]. The Laplace transform methodology is powerful and effective method to solve both ODEs and PDEs [4], [5]. Moreover, there are many different methods to solve self-adjoint Euler-Cauchy differential equations: differential transform method [6], [15], [18], Sumudu and Elzaki transforms as integral transform methods [16], [17], method of variation of parameters [8], [10], [13] and finally the reduction of the order method [13].

In this paper, for the first time using Laplace transform method to solve third-order non-homogeneous self-adjoint Euler-Cauchy operator-differential equation ;l, with variable coefficients. We approved that the LT technique works well for obtaining the general solutions of the Euler-Cauchy differential equation with variable coefficients. Through this approach we formulated a certain relation between the self-adjoint differential operator and the given Neumann conditions.In the last theorem of this paper we took the right hand side of our differential equations to be equal the bulge function.

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II. Problem formulation

Let $H=L_2[0,\pi]$ be a separable Hilbert space, then in H we would present the following problem

$$\Big(t\frac{\partial}{\partial t}+\Delta\Big)\Big(t\frac{\partial}{\partial t}-\Delta\Big)^2u(t,x)=f(t,x), \quad t\in R_+=[0,+\infty),\ x\in[0,\pi], \tag{1}$$

$$u(t,x)|_{t=0}=u_0(x),\ u_t(t,x)|_{t=0}=u_1(x), \tag{2}$$

on $R_+^2=R_+\times[0,\pi]$ with is a given scalar function $f(t,x)\in L_2(R_+;L_2[0,\pi])$ ([7], [14])

$$\|f\|^2_{L_2(R_+^2;H)}=\int_0^{+\infty}\int_0^{\pi}\|f(t,x)\|^2dxdt<+\infty,$$

and the unknown function $u(t,x)$ in the space $W_2^3(R_+^2;H)$ in which

$$W_2^3(R_+^2;H)=\Big\{u:u(t,x)\in L_2(R_+^2;H), \frac{d^3u}{dt^3}\in L_2(R_+^2;H),u(0,x)=u_0(x),\ u_t(0,x)=u_1(x)\Big\}.$$

In this problem, in $L_2[o,\pi]$, We provide a self-adjoint operator $A\equiv-\Delta$, where Δ is the Laplacian differential operator that involved in a variety of physical and engineering problems, like as heat conduction or diffusion in viscoelastic substances ($A=A^*\geq\lambda_0E$, where λ_0 is the lower spectral boundary and E is the identity operator [3].) Thus problem (1)-(2) would convert to third-order non-homogeneous self-adjoint Euler-Cauchy operator differential with variable coefficients as follow:

$$t^3\frac{d^3G(t)}{dt^3}+At^2\frac{d^2G(t)}{dt^2}-A^2t\frac{dG(t)}{dt}-A^3G(t)=f(t), \quad t\in R_+[0,+\infty) \tag{3}$$

$$G(t)|_{t=0}=\varphi,\quad v^{'}(t)|_{t=0}=\psi \tag{4}$$

in which $G(t)\in W_2^3(R_+;H)$,

$$W_2^3(R_+;H)=\Big\{G(t)\in L_2(R_+;H), \frac{d^3G(t)}{dt^3}\in L_2(R_+;H),A^3G(t)\in L_2(R_+;H)\Big\},$$

with the norm [14], [2]

$$\|G(t)\|^2_{W_2^3(R_+;H)}=\left\|\frac{d^3G(t)}{dt^3}\right\|^2_{L_2(R_+;H}+\left\|A^3G(t)\right\|^2_{L_2(R_+;H}.$$

Moreover, equation (3) can be rewritten in an operator-differential form as follow:

$$P_0G(t) = f(t),$$

where $P_0 = \Big(t\frac{\partial}{\partial t} + \Delta\Big)\Big(t\frac{\partial}{\partial t} - \Delta\Big)^2$ acting in $W_2^3(R_+;H)$ onto $L_2(R_+;H)$.

III. Methodology and results

We say that a continuous function $f(t)$ has its Laplace transform $F(s)$,

$$\mathcal{L}(f(t)) = \int_0^{+\infty} f(t)e^{-st}dt = F(s),$$

The Laplace transform of $f(t)$ is said to be exists if the integral converges for some values os s. other words the Laplace transform of $f(t)$ exists if $f(t)$ is piecewise continuous and should be of exponential order ($\lim_{t\to\infty} e^{-st}f(t) = 0$).
Therefor, the frequency domain function $F(s)$ can be converted to its corresponding time domain equivalent $f(t)$ using the Laplace Inverse $\mathcal{L}^{-1}$ and is given by:

$$\mathcal{L}^{-1}(F(s)) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\alpha-iT}^{\alpha+iT} F(s)e^{st}ds$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha+i\xi)t}F(\alpha+i\xi)d\xi.$$

Remark 1. $\mathcal{L}\Big(t^k f(t)\Big) = (-1)^k F^{(k)}(s).$

Applying Laplace transform for the third-order self-adjoint Euler-Cauchy operator-differential equation (3), yields

$$\mathcal{L}\Big(t^3\frac{d^3G(t)}{dt^3}\Big) + \mathcal{L}\Big(At^2\frac{d^2G(t)}{dt^2}\Big) - \Big(A^2t\frac{dG(t)}{dt}\Big) - \mathcal{L}\big(A^3G(t)\big) = \mathcal{L}\big(f(t)\big).$$

Using Laplace transform's property $\mathcal{L}(tf(t)) = -\frac{d}{ds}F(s)$, we get

$$\mathcal{L}(tv^{'}(t)) = -s\frac{dV(s)}{ds} - V(s),$$

$$\mathcal{L}(t^2v^{''}(t)) = s^2\frac{d^2V(s)}{ds^2} + 4s\frac{dV(s)}{ds} + 2V(s),$$

$$\mathcal{L}(t^3v^{'''}(t)) = -s^3\frac{d^3V(s)}{ds^3} - 9s^2\frac{d^2V(s)}{ds^2} - 18s\frac{dV(s)}{ds} - 6V(s)$$

$$s^3\frac{d^3V(s)}{ds^3} + (9s^2 - s^2A)\frac{d^2V(s)}{ds^2} + (18s - 4sA - sA^2)\frac{dV(s)}{ds} + (A^3 - A^2 - 2A + 6E)V(s) = 0, \quad (5)$$

where $V(s)$ is the Laplace transform of $G(t)$ and E denotes the unit operator.
According to [19], by the Frobenius Method, the self-adjoint Euler-Cauchy (3) has a basis of the $a_0x^{r_1}$, $b_0x^{r_2}$, and $c_0x^{r_3}$, a_0, b_0, c_0 are constants and

$$r^3 + (6E - A)r^2 + (-A^2 - 3A + 11E)r + (-A^2 - 2A + A^3 + 6E) = 0.$$

Next, we want to apply the J. L. Lagrange method to lessen the order of the equation (5).

Lemma 2. *The equation (5) has the form of a basis v_1 and*

$$v_1(t)\int\Big(\int\frac{s^{A-9E}}{{v_1}^3}ds\Big)ds,\ v_1(t)\Bigg(\int\Big(\int\frac{s^{A-9E}}{{v_1}^3}ds\Big)ds\Bigg)^2$$

of solutions.

Proof: *Assuming that there exists a solution $v_1(t) \in W_2^3(R_+;H)$ of equation (5) and we want to find a basis. Now Let's find the other solutions$v_2(t)$ and $v_3(t)$. Substitute*

$$G(t) = v_2(t) = uv_1(t), v_3(t) = u^2v_1(t),$$
$$v^{'}(t) = u^{'}v_1(t) + uv_1^{'}(t),$$
$$v^{''}(t) = u^{''}v_1(t) + 2u^{'}v_1^{'}(t) + uv_1^{''}(t),$$
$$v^{'''}(t) = u^{'''}v_1(t) + 3u^{''}v_1^{'}(t) + 3u^{'}v_1^{''}(t) + uv_1^{'''}(t)$$

into the equation (5), we get

$$u^{'''}\{s^3v_1(t)\} + u^{''}\{3s^3v_1^{'}(t) + (9s^2 - s^2A)v_1(t)\} + u^{'}\{3s^3v_1^{''}(t) + (18s^2 - 2s^2A)v_1^{'}(t) + (18s - 4sA - sA^2)v_1(t)\} + u\{s^3v_1^{'''}(t) + (9s^2 - s^2A)v_1^{''}(t) + (18s - 4sA - sA^2)v_1^{'}(t) + (A^3 - A^2 - 2A + 6E)v_1(t)\} = 0.$$

Upon dividing by $s^3v_1(t)$ for

$$s^3v_1^{'''}(t) + (9s^2 - s^2A)v_1^{''}(t) + (18s - 4sA - sA^2)v_1^{'}(t) + (A^3 - A^2 - 2A + 6E)v_1(t) = 0,$$

we obtain

$$u^{'''} + u^{''}\Big\{3\frac{v_1^{'}(t)}{v_1(t)} + \frac{9E - A}{s}\Big\} + u^{'}\Big\{3\frac{v_1^{''}(t)}{v_1(t)} + \frac{18E - 2A}{s}\frac{v_1^{'}(t)}{v_1(t)} + \frac{18E - 4A - A^2}{s^2}\Big\} = 0.$$

Let

$$3s^3v_1^{''}(t) + (18E - 2A)s^2v_1^{'}(t) + (18E - 4A - A^2)sv_1(t) = 0$$

has auxiliary equation

$$3s^3\lambda^2 + (18E - 2A)s^2\lambda + (18E - 4A - A^2)s = 0.$$

Then put $u^{''} = G(t)$, we get:

$$v^{'}(t) + G(t)\Big(\frac{3v_1^{'}(t)}{v_1(t)} + \frac{9E - A}{s}\Big) = 0.$$

Through a few simple calculations, yields

$$u = \int\Big(\int\frac{s^{A-9E}}{{v_1}^3}ds\Big)ds$$

and also

$$v_2(t) = v_1(t)\int\Big(\int\frac{s^{A-9E}}{{v_1}^3}ds\Big)ds,$$
$$v_3(t) = v_1(t)\Bigg(\int\Big(\int\frac{s^{A-9E}}{{v_1}^3}ds\Big)ds\Bigg)^2.$$

As a result of $u > 0$*, the quotient* $\frac{v_3(t)}{v_2(t)} = \frac{v_2(t)}{v_1(t)} = u = \int\big(\int v ds\big)ds$ *cannot be constant, so that* $v_1(t)$*,* $v_2(t)$ *and* $v_3(t)$ *form a basis of solutions.* ■

Theorem 3. *The operator* $P_0 : W_2^3(R_+; H) \to L_2(R_+; H)$ *is an isomorphism and the general solution of problem* (3)-(4) *will be*

$$G(t) = \int_0^{+\infty} G(t-\tau)f(\tau)d\tau + \varphi e^{Am^{-1}t} + Am^{-1}\varphi\, te^{Am^{-1}t} - \frac{1}{2}A^2m^{-2}\varphi\, t^2e^{Am^{-1}t} + \psi te^{Am^{-1}t}.$$

Proof: By Applying Laplace transform to equation (3)*, we have*

$$\big(-A-(sm+1)E\big)\big(A-(sm+1)E\big)^2V(s) = F(s)$$

where $V(s), F(s)$ *are Laplace transforms of* $G(t), f(t)$ *respectively and* $m = \frac{d}{ds}$*.*
Additionally, since $\big(-A-(sm+1)E\big)\big(A-(sm+1)E\big)^2$ *is invertible polynomial operator, thus*

$$\begin{aligned}
&V(s) = \big(-A-(sm+1)E\big)^{-1}\big(A-(sm+1)E\big)^{-2}F(s),\\
&G(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\delta-iT}^{\delta+iT}\big(-A-(sm+1)E\big)^{-1}.\\
&\big(A-(sm+1)E\big)^{-2}F(s)e^{st}dt\\
&= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\delta-iT}^{\delta+iT}\big(-A-(sm+1)E\big)^{-1}.\\
&\big(A-(sm+1)E\big)^{-2}\left(\int_0^{+\infty} f(\tau)e^{-st}dt\right)e^{st}dt\\
&= \int_0^{+\infty}\Big(\frac{1}{2\pi i}\int_{\delta-iT}^{\delta+iT}\big(-A-(sm+1)E\big)^{-1}.\\
&\big(A-(sm+1)E\big)^{-2}e^{s(t-\tau)}dt\Big)f(\tau)d\tau\\
&= \int_0^{+\infty} G(t-\tau)f(\tau)d\tau.
\end{aligned}$$

We will now show that $G(t) \in W_2^3(R_+; H)$*.*
Based on Parseval's Theorem in the frequency domain, we obtain

$$\begin{aligned}
&\|G(t)\|^2_{W_2^3(R_+;H)} = \left\|\frac{d^3G(t)}{dt^3}\right\|^2_{L_2(R_+;H)} + \|A^3G(t)\|^2_{L_2}\\
&= \big\|(-m^3s^3 - 9m^2s^2 - 18ms - 6)V(s)\big\|^2_{L_2(R_+;H)}\\
&+ \|A^3V(s)\|^2_{L_2(R_+;H)}\\
&= \left\|\frac{-m^3s^3 - 9m^2s^2 - 18ms - 6}{\big(-A-(sm+1)E\big)\big(A-(sm+1)E\big)^2}F(s)\right\|^2_{L_2}\\
&+ \left\|A^3\frac{\big(-A-(sm+1)E\big)^{-1}}{\big(A-(sm+1)E\big)^{-2}}F(s)\right\|^2_{L_2(R_+;H)}\\
&\le \sup_{s\in R}\left\|\frac{-m^3s^3 - 9m^2s^2 - 18ms - 6}{\big(-A-(sm+1)E\big)\big(A-(sm+1)E\big)^2}\right\|^2_{H\to H}\\
&.\|F(s)\|^2_{L_2(R_+;H)}\\
&+ \sup_{s\in R}\left\|\frac{A^3}{\big(-A-(sm+1)E\big)\big(A-(sm+1)E\big)^2}\right\|^2_{H\to H}\\
&.\|F(s)\|^2_{L_2(R_+;H)}
\end{aligned} \tag{6}$$

From the spectral decomposition of operator A *for* $sm+1 \ge 0$*, we have*

$$\begin{aligned}
&\left\|\frac{-m^3s^3 - 9m^2s^2 - 18ms - 6}{\big(-A-(sm+1)E\big)\big(A-(sm+1)E\big)^2}\right\|^2_{H\to H}\\
&\le \sup_{\mu\in\sigma(A)}\left|\frac{(ms+1)^3}{\mu^3 + (ms+1)^3}\right| \le 1,
\end{aligned} \tag{7}$$

$$\begin{aligned}
&\left\|\frac{A^3}{\big(-A-(sm+1)E\big)\big(A-(sm+1)E\big)^2}\right\|^2_{H\to H}\\
&\le \sup_{\mu\in\sigma(A)}\left|\frac{\mu^3}{\mu^3 + (ms+1)^3}\right| \le 1,
\end{aligned} \tag{8}$$

from (7) *and* (8) *into* (6) *where* $\sigma(A)$ *is the spectrum of operator* A*, yields*

$$\|G(t)\|^2_{W_2^3(R_+;H)} \le 2\|F(s)\|^2_{L_2(R_+;H)} = 2\|f(t)\|^2_{L_2(R_+;H)},$$

Consequently, $G(t) \in W_2^3(R_+; H)$*.*

For the homogeneous solution, take the LT to the third-order homogeneous self-adjoint Euler-Cauchy DE, yields

$$\mathcal{L}\Big(t^3\frac{d^3G(t)}{dt^3}\Big) + \mathcal{L}\Big(At^2\frac{d^2G(t)}{dt^2}\Big) - \Big(A^2t\frac{dG(t)}{dt}\Big) - \mathcal{L}\big(A^3G(t)\big) = 0,$$

$$-\frac{d^3}{ds^3}(s^3V(s) - s^2\varphi - s\psi) + A\frac{d^2}{ds^2}(s^2V(s) - s\varphi - \psi) + A^2\frac{d}{ds}(sV(s) - \varphi) - A^3V(s) = 0.$$

Put $\frac{d}{ds}$*, where* $V(s) = \mathcal{L}(G(t))$*:*

$$-m^3(s^3V(s) - s^2\varphi - s\psi) + Am^2(s^2V(s) - s\varphi - \psi) + A^2m(sV(s) - \varphi) - A^3V(s) = 0,$$

and then solve for $V(s)$ *i.e.,*

$$V(s) = \frac{m^3s^2\varphi + m^3s\psi - Am^2s\varphi - Am^2\psi - A^2m\varphi}{m^3s^3 - Am^2s^2 - A^2ms + A^3},$$

$ms \neq A$.
for all values of t, the homogeneous solution $G(t)$ take the form

$$\begin{aligned} &G(t) \\ &= \mathcal{L}^{-1}\Big[\frac{m^3s^2\varphi + m^3s\psi - Am^2s\varphi - Am^2\psi - A^2m\varphi}{m^3s^3 - Am^2s^2 - A^2ms + A^3}\Big] \\ &= \mathcal{L}^{-1}\Big[\frac{s^2\varphi + s\psi - Am^{-1}s\varphi - Am^{-1}\psi - A^2m^{-2}\varphi}{s^3 - Am^{-1}s^2 - A^2m^{-2}s + A^3m^{-3}}\Big] \\ &= \mathcal{L}^{-1}\Big[\frac{\varphi}{(sE - Am^{-1})} + \frac{Am^{-1}\varphi}{(sE - Am^{-1})^2} \\ &- \frac{A^2m^{-2}\varphi}{(sE - Am^{-1})^3} + \frac{\psi}{(sE - Am^{-1})^2}\Big]. \end{aligned}$$

Consequently,

$$\begin{aligned} G(t) &= \varphi e^{Am^{-1}t} + Am^{-1}\varphi\, te^{Am^{-1}t} \\ &- \frac{1}{2}A^2m^{-2}\varphi\, t^2e^{Am^{-1}t} + \psi te^{Am^{-1}t}. \end{aligned}$$

■

Theorem 4. *For the initial value problem* (3)-(4) *of a self-adjoint operator-differential equation we have:*
(i) The Neumann boundary conditions can be expressed by

$$\varphi = (2s - 18sA^{-2} + 4sA^{-1})V(s),$$
$$\psi = (18s^2A^{-2} - 4s^2A^{-1})V(s).$$

(ii)

$$\begin{aligned} V(s) =&(3A - 6E)\ln s \\ &- \frac{3(A-E)^2}{2s^2}\mathcal{L}\Big[\frac{G(t)}{t}\Big] - \frac{3}{s}F(s) \end{aligned}$$

(iii) We'll express the general solution $G(t)$ as follows:

$$\begin{aligned} G(t) &= \varphi e^{Am^{-1}t} + Am^{-1}\varphi\, te^{Am^{-1}t} \\ &- \frac{1}{2}A^2m^{-2}\varphi\, t^2e^{Am^{-1}t} + \psi te^{Am^{-1}t} \\ &+ L^{-1}\Big[\frac{F(s)}{m^3s^3 - Am^2s^2 - A^2ms + A^3}\Big]. \end{aligned}$$

Proof: (i) Applying LT to equation (3), *yields*

$$\begin{aligned} &-\frac{d^3}{ds^3}(s^3V(s) - s^2\varphi - s\psi) + A\frac{d^2}{ds^2}(s^2V(s) - s\varphi - \psi) \\ &+ A^2\frac{d}{ds}(sV(s) - \varphi) - A^3V(s) = F(s) \end{aligned} \tag{9}$$

From equation (5) *and equation* (9), *we get*

$$-s^3V(s) + s^2\varphi + s\psi = s^3V(s),$$
$$s\varphi + \psi = 2s^2V(s)$$
$$A^2sV(s) - A^2\varphi = 18sV(s) - 4sAV(s) - sA^2V(s).$$

Hence,

$$\varphi = (2s - 18sA^{-2} + 4sA^{-1})V(s),$$
$$\psi = (18s^2A^{-2} - 4s^2A^{-1})V(s).$$

(ii) two times Integrate equation (5) *w.r.t s from s to ∞, we get*

$$\begin{aligned} &\frac{1}{20}s^5\frac{dV(s)}{ds} + \Big(\frac{3}{4}s^4 - \frac{1}{12}s^4A\Big) + \Big(3s^3 - \frac{2}{3}s^3A - \frac{1}{6}s^3A^2\Big) \\ &+ (A^3 - A^2 - 2A + 6E)\int_0^{+\infty}\mathcal{L}\Big[\frac{G(t)}{t}\Big]ds = \frac{1}{2}s^2F(s), \end{aligned}$$

$$\begin{aligned} \frac{dV(s)}{ds} &= \frac{5A/3 - 15}{s} - \frac{60 - 40A/3 - 10A^2/3}{s^2} \\ &- \frac{20(A^3 - A^2 - 2A + 6E)}{s^5}\int_0^{+\infty}\mathcal{L}\Big[\frac{G(t)}{t}\Big]ds + \frac{10F(s)}{s^3}. \end{aligned}$$

Integrating the above equation with respect to s, we get

$$\begin{aligned} V(s) &= (5A/3 - 15)\ln s + \frac{60 - 40A/3 - 10A^2/3}{s} \\ &+ \frac{5(A^3 - A^2 - 2A + 6E)}{s^4}\int_0^{+\infty}\mathcal{L}\Big[\frac{G(t)}{t}\Big]ds - \frac{5F(s)}{s^2}. \end{aligned}$$

(iii) This follows directly from Theorem 3.1. ■

Consider a continuous-time signal $h(t)$, a discrete-time signal $h(k)$, $k \in N$ can be obtained by taking samples of $h(t)$ at equal intervals of T. Since the frequency domain representation of the equation (5) is

$$\begin{aligned} &V(s) = \frac{m^3s^2\varphi + m^3s\psi - Am^2s\varphi - Am^2\psi - A^2m\varphi}{m^3s^3 - Am^2s^2 - A^2ms + A^3}, \\ &(d/ds = m, ms \neq A) \end{aligned} \tag{$**$}$$

Evidently, the equation

$$(**) \approx Th^*(t) = T\sum_{k=0}^{\infty}h(k)\delta(t - kT),$$

is a weighted impulses, where $h^*(t)$. Alternatively, the final result implies that the third-order self-adjoint Euler-Cauchy equation is oscillatory on T if it has a complex characteristic root.

On the other hand, the result shows that if the self-adjoint Euler-Cauchy equation has a complex characteristic root, it is oscillatory on T.

Example 5. *Consider the third-order self-adjoint Euler-Cauchy equation*

$$t^3v'''(t) + At^2v''(t) - A^2tv'(t) - A^3G(t) = 0,\ t \in R_+[0, +\infty) \tag{10}$$

Applying Laplace transform to equation (10), *yields*

$$\begin{aligned} &s^3\frac{d^3V(s)}{ds^3} + (9s^2 - s^2A)\frac{d^2V(s)}{ds^2} \\ &+ (18s - 4sA - sA^2)\frac{dV(s)}{ds} \\ &+ (A^3 - A^2 - 2A + 6E)V(s) = 0. \end{aligned}$$

Suppose the solution of (10) *is of the form $V(s) = s^r$, where the constant $r \in R$. i.e.,*

$$\begin{aligned} &s^r[r(r-1)(r-2) + (9-\mu)r(r-1) \\ &+ (-\mu^2 - 4\mu + 18)r - \mu^2 - 2\mu + \mu^3 + 6] = 0. \end{aligned}$$

Where, $\mu \in \sigma(A)$.
Since $s^r \neq 0$, we get

$$\begin{aligned} &s^r[r(r-1)(r-2) + (9-\mu)r(r-1) + (-\mu^2 - 4\mu + 18)r \\ &- \mu^2 - 2\mu + \mu^3 + 6] = 0. \end{aligned}$$

Rearranging the equation, yields

$$r^3 + (6-\mu)r^2 + (-\mu^2 - 3\mu + 11)r + (-\mu^2 - 2\mu + \mu^3 + 6) = 0.$$

In the case of $\mu = 0$, then

$$r^3 + 6r^2 + 11r + 6 = 0,$$

and so

$$(r+1)(r+2)(r+3) = 0.$$

Thus, we have $r=-1,-2,-3$ *three real roots. Therefore, the solutions of* (10) *are* $V(s)=s^{-1}$, $V(s)=s^{-2}$, $V(s)=s^{-3}$.

In case of the nonhomogeneous self-adjoint Euler-Cauchy with right hand side equals a bulge function which is given by $e^{-(t-l)^2/2}$ is a positive constant. We introduce the following lemma

Lemma 6.

$$\mathcal{L}\Big[e^{-(t-l)^2/2}\Big]=e^{-l^2/2}\Big[\frac{1}{s}+\frac{-1+l^2}{s^3}+\frac{l(s^2-3+l^2)}{s^4}\Big]. \tag{11}$$

See [9].

Theorem 7. *The general solution of the self-adjoint Euler-Cauchy equation with bulge function of the form*

$$\begin{aligned}&t^3v^{'''}+At^2v^{''}-A^2tv^{'}-A^3G(t)=e^{-(t-l)^2/2},\\&t\in R_+=[0,+\infty),\ v(0)=\varphi,\ v^{'}(0)=\psi,\end{aligned} \tag{12}$$

where $G(t)$ *is unknown function can be expressed by*

$$\begin{aligned}G(t)&=\varphi e^{Am^{-1}t}+Am^{-1}\varphi\, te^{Am^{-1}t}\\&-\frac{1}{2}A^2m^{-2}\varphi\, t^2e^{Am^{-1}t}+\psi te^{Am^{-1}t}\\&+\frac{1}{2}e^{-l^2/2}\int_0^t t^2e^{-Am^{-1}t}dt\\&+\frac{l}{2}e^{-l^2/2}\int_0^t\int_0^t t^2e^{-Am^{-1}t}dt\\&+\frac{-1+l^2}{2}e^{-l^2/2}\int_0^t\int_0^t\int_0^t t^2e^{-Am^{-1}t}dt\\&+\frac{l^3-3l}{2}e^{-l^2/2}\int_0^t\int_0^t\int_0^t\int_0^t t^2e^{-Am^{-1}t}dt.\end{aligned}$$

Proof of Theorem 7. can be directly obtained from Theorem 1. and equation (11).

Another case of the right hand side of (1) is a function involving a Gaussian e^{-at^2}, $a>0$. We introduce the following lemma

Lemma 8.

$$\mathcal{L}\Big[e^{-at^2}\Big]=\frac{1}{2}e^{\frac{as^2}{4}}\,\text{erfc}\Big(\frac{\sqrt{a}s}{2}\Big), \tag{13}$$

where the Error Function $\text{erfc}(x)=1-\frac{2}{d\sqrt{a}}\int_0^x e^{-t^2}dt.$

Proof:

$$\begin{aligned}\mathcal{L}\Big[e^{-at^2}\Big]&=\int_0^\infty e^{-at^2}e^{-st}dt=\int_0^\infty e^{-a(t^2+ts)}dt\\&=\int_0^\infty e^{-a\left(\left(t+\frac{s}{2}\right)^2-\frac{s^2}{4}\right)}dt=e^{\pi\frac{s^2}{4}}\int_0^\infty e^{-a\left(t+\frac{s}{2}\right)^2}dt\\&=e^{\pi\frac{s^2}{4}}\int_{\frac{s}{2}}^\infty e^{-au^2}du+\\&=e^{\pi\frac{s^2}{4}}\left[\int_0^\infty e^{-au^2}du-\int_0^{\frac{s}{2}}e^{-au^2}du\right]\end{aligned}$$

It is known that $\int_0^\infty e^{-au^2}du=\frac{1}{2}$ *and the Error Function* $\text{erf}(x)=\frac{2}{\sqrt{a}}\int_0^x e^{-t^2}dt$ *and erfc* $(x)=1-$ *erf* (x)
By substitution $x=\sqrt{a}u$*, yields*

$$\int_0^{\frac{s}{2}}e^{-au^2}du=\frac{1}{\sqrt{a}}\int_0^{\frac{s\sqrt{a}}{2}}e^{-x^2}dx=\frac{1}{2}\text{erf}\Big(\frac{s\sqrt{a}}{2}\Big).$$

Consequently,

$$\begin{aligned}\mathcal{L}\Big[e^{-at^2}\Big]&=e^{\frac{as^2}{4}}\Big[\frac{1}{2}-\frac{1}{2}\text{erf}\Big(\frac{\sqrt{a}s}{2}\Big)\Big]\\&=\frac{1}{2}e^{\frac{as^2}{4}}\,\text{erfc}\Big(\frac{\sqrt{a}s}{2}\Big).\end{aligned} \tag{14}$$

■

Theorem 9. *The general solution of the self-adjoint Euler-Cauchy equation involving a Gaussian is*

$$\begin{aligned}&t^3G^{'''}+At^2v^{''}-A^2tG^{'}-A^3G(t)=e^{-at^2},\\&t\in[0,+\infty),\ G(0)=g_0,\ G^{'}(0)=g_1,\end{aligned} \tag{15}$$

where $G(t)$ *is unknown function can be expressed by*

$$\begin{aligned}G(t)&=g_0e^{Am^{-1}t}+Am^{-1}g_0\, te^{Am^{-1}t}\\&-\frac{1}{2}A^2m^{-2}g_0\, t^2e^{Am^{-1}t}+g_1te^{Am^{-1}t}\\&+\mathcal{L}^{-1}\left[\frac{e^{\frac{as^2}{4}}\,erfc\big(\frac{\sqrt{a}s}{2}\big)}{2(sE-Am^{-1})^3}\right]\end{aligned}$$

Theorem 10. *The general solution of initial value problem*

$$\begin{aligned}&t^3\Lambda^{'''}+At^2v^{''}-A^2t\Lambda^{'}-A^3\Lambda(t)=\sum_{n=0}^\infty a_nt^n,\\&t\in[0,+\infty),\ \Lambda(0)=\lambda_0,\ \Lambda^{'}(0)=\lambda_1,\end{aligned} \tag{16}$$

is

$$\begin{aligned}\Lambda(t)&=\lambda_0e^{Am^{-1}t}+Am^{-1}\lambda_0\, te^{Am^{-1}t}\\&-\frac{1}{2}A^2m^{-2}\lambda_0\, t^2e^{Am^{-1}t}+\lambda_1te^{Am^{-1}t}\\&+\mathcal{L}^{-1}\Big[(sE-Am^{-1})^{-3}\sum_{n=0}^\infty\frac{n!}{s^{n+1}}\Big].\end{aligned}$$

Proof of Theorem 9 and Theorem 10. can be directly obtained from Theorem 1.

IV. Conclusion

The Laplace transform method is a potent technique that can be used for accurately determining the general solutions of non-homogeneous third-order self-adjoint Euler-Cauchy operator-differential equations with variable coefficients.

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