A Family of Quasi-quartic Trigonometric Bézier Curves with Two Shape Parameters

Yuanpeng Zhu and Yixin Liu

Abstract—**This paper presents a family of quasi-quartic trigonometric Bézier curves generated using six novel quasi-quartic trigonometric Bézier-like functions characterized by two parameters. These functions exhibit properties similar to those of conventional quartic Bernstein basis functions. The proposed curves can fit the control polygon more closely than various other Bézier-like curves and the conventional quartic Bézier curve when the control points are fixed. This is achieved with ideally low algorithmic complexity, making it computationally efficient. Besides, adjustments only to the shape parameters allow for linear control over the configuration of the quasi-quartic trigonometric Bézier curve due to the ease of obtaining the curve's derivative. Furthermore, when multiple quasi-quartic trigonometric Bézier curves are combined, the resulting composite curve achieves $C^2 \cap FC^3$ continuity automatically, provided that the C^2 continuity condition is satisfied. This ensures smooth transitions between curve segments without additional constraints.**

Index Terms—**trigonometric Bézier curves, trigonometric Bézier functions, shape parametric, geometric continuity.**

I. Introduction

BÉZIER and B-spline curves have been widely used, especially in interactive curve and surface design. However, the shape of a Bézier curve is entirely determined by its control points. While non-uniform rational B-spline (NURBS) curves can address this issue to some extent, their rational forms may be unstable, and calculating their derivatives can be challenging. In computer graphics and computer-aided geometric design (CAGD), parametric curves and surfaces are commonly expressed as a linear combination of basis functions and control points.

Some methods for generating curves using trigonometric Bézier-like functions with shape parameters have been proposed, to improve the shape and enhance the fitting accuracy of curves. α-basis [1] with shape parameters for the space spanned by $\{1,\ t,\ cost,\ sint,\ tcost,\ tsint\}$, is suitable for the design of curve that retains its shape. Some quadratic and cubic trigonometric spline curves with single or multiple shape parameters were shown in [2], [3] and [4]. C-Bézier basis, which can help to accurately represent high order polynomial curves and circular arc, for the space which is spanned by $\{1,\ t,\ t^2,...,\ t^{n-2},\ sint,\ cost\}$, was constructed by using integration in [5]. A practical cubic trigonometric Bézier curve similar to the conventional cubic Bézier curve was given in [6], which has two shape parameters. In their subsequent work, [7] conducted an analysis on the characteristics of the shape of a cubic trigonometric Bézier curve with a single shape parameter, exploring the relationship between the shape and the parameter. A trigonometric Bézier curve characterized by two parameters created using five new trigonometric blending functions was introduced by [8], and a generalization of this curve was shown by [9]. A trigonometric Bézier curve similar to the quintic Bernstein curve constructed by six quartic trigonometric polynomial hybrid functions was introduced in [10]. Six new trigonometric basis functions were presented to get a new quasi-quintic trigonometric Bézier curve [11]. [12] demonstrated the total positivity of the quasi-cubic trigonometric Bézier basis. A new rational cubic trigonometric curve was introduced by [13]. [14] proposed a family of cubic trigonometric non-uniform spline basis functions which have one local shape parameter. A quintic trigonometric Bézier curve was given by [15], which has two shape parameters. [16] presented an alpha-B-spline curve similar to the standard cubic uniform B-spline curve. A new quasi-quintic trigonometric Bézier curve was proposed by [17]. An extended cubic Bézier curve having four parameters was constructed in [18]. [19] introduced a new generalized blended trigonometric Bézier curve with one shape parameter.

This paper introduces a family of quasi-quartic trigonometric Bézier curves generated using six novel quasi-quartic trigonometric Bézier-like functions, characterized by two parameters. The new family of quasi-quartic Bézier curves offer several notable advantages:

(A) Enhanced fitting accuracy with low complexity. The proposed curve fits the control polygon more closely than many other Bézier curves, including the conventional quintic Bézier curve and those presented in [1] and [5], while maintaining ideally low algorithmic complexity. Although it performs slightly less effectively than the quasi-quintic trigonometric Bézier curve [17] in terms of fitting accuracy, it significantly reduces the computational complexity and accelerates the operation.

(B) Linear shape control and ease of derivative calculation. The shape of the curve can be controlled linearly, and its derivative is straightforward to compute. This feature enhances the flexibility and efficiency of curve manipulation.

(C) Automatic high-order continuity. When multiple quasi-quartic trigonometric Bézier curves are combined to form a composite curve, the resulting curve achieves $C^2 \cap FC^3$ continuity automatically, provided that the C^2 continuity condition is satisfied. This ensures smooth and continuous transitions between curve segments without additional constraints.

The remainder of the paper is organized as follows. In section II, we introduce the quasi-quartic trigonometric Bézier-like functions and discuss their characteristics. Section III

Manuscript received October 26, 2024; revised March 12, 2025. The research was funded by the Guangdong Basic and Applied Basic Research Foundation (Nos. 2024A1515010413, 2025A1515011263).

Yuanpeng Zhu is a professor at the Department of Mathematics, South China University of Technology, Guangzhou, 510641, China. (e-mail: ypzhu@scut.edu.cn).

Yixin Liu is a postgraduate student at the Department of Mathematics, South China University of Technology, Guangzhou, 510641, China. (corresponding author to provide e-mail: 202320129608@mail.scut.edu.cn).

constructs the associated quasi-quartic trigonometric Bézier curve and provides a comparison with other Bézier curves. Section IV demonstrates how to manipulate the shape of our proposed curve. Section V details the conditions for C^1, C^2 and $C^2 \cap FC^3$ continuity when connecting quasi-quartic trigonometric Bézier curves and also includes examples of designing parametric curves. Finally, Section VI presents the conclusion.

II. The quasi-quartic trigonometric Bézier-like functions

Definition II.1. *The quasi-quartic trigonometric Bézier-like functions are defined as follows, for* $\theta \in [0, \frac{\pi}{2}]$,

$$\begin{aligned}
&f_0(\theta) = (1-\sin\theta)^4, \\
&f_1(\theta) = 4(1-\sin\theta)^3 [\sin\theta + (1-k)\cos\theta + k - 1], \\
&f_2(\theta) = 4(1-\sin\theta)^2 (1-\cos\theta) [(1+k)\sin\theta + 5\cos\theta \\
&\qquad -k-1] + \frac{17}{4}(\sin\theta + \cos\theta - 1)^4, \\
&f_3(\theta) = 4(1-\cos\theta)^2 (1-\sin\theta) [(1+l)\cos\theta + 5\sin\theta \\
&\qquad -l-1] + \frac{17}{4}(\cos\theta + \sin\theta - 1)^4, \\
&f_4(\theta) = 4(1-\cos\theta)^3 [\cos\theta + (1-l)\sin\theta + l - 1], \\
&f_5(\theta) = (1-\cos\theta)^4,
\end{aligned}$$

in which k, l *are shape parameters and* $0 \le k, l \le 4$.

Some characteristics of the quasi-quartic trigonometric Bézier-like functions are similar to those of the conventional quartic Bernstein basis functions:
(A) Non-negativity: $f_i(\theta) \ge 0, i = 0, 1, \ldots, 5$.
(B) Unity partition: $\sum_{i=0}^{5} f_i(\theta) \equiv 1$.
(C) Symmetry: $f_i(\theta; k, l) = f_{5-i}\left(\frac{\pi}{2} - \theta; l, k\right), i = 0, 1, 2$.

Proof: (A) Indeed, the value range of k and l is obtained by letting $f_i(\theta) \ge 0, i = 0, 1, \ldots, 5$, when $\theta \in [0, \frac{\pi}{2}]$.

It is easy to know that $f_i(\theta) \ge 0, i = 0, 5$. Let $f_i(\theta) \ge 0, i = 1, 2$, we have

$$\begin{aligned}
&1 - \cos\theta - \sin\theta \le k(1-\cos\theta), \\
&(1-\sin\theta)^3 (1-\cos\theta) k \le \frac{17}{16}(\sin\theta + \cos\theta - 1)^4 + \\
&\qquad (1-\sin\theta)^2 (1-\cos\theta)(5\cos\theta + \sin\theta - 1),
\end{aligned}$$

then $0 \le k \le 4$. Similarly we have $0 \le l \le 4$.
(B) By performing some operations, there are

$$\begin{aligned}
&f_0(\theta) + f_1(\theta) + f_2(\theta) = \frac{13}{2}\cos 2\theta + \sin 4\theta \\
&\quad -\frac{5}{2}\sqrt{2}\sin\left(3\theta + \frac{\pi}{4}\right) - \frac{7}{2}\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right) + 1/2, \\
&f_3(\theta) + f_4(\theta) + f_5(\theta) = -\frac{13}{2}\cos 2\theta - \sin 4\theta \\
&\quad +\frac{5}{2}\sqrt{2}\sin\left(3\theta + \frac{\pi}{4}\right) + \frac{7}{2}\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right) + 1/2.
\end{aligned}$$

Therefore, $\sum_{i=0}^{5} f_i(\theta) = 1$.
(C) After some simple computations, there is

$$\begin{aligned}
f_0(\theta; k, l) &= (1-\sin\theta)^4 \\
&= \left[1 - \cos\left(\frac{\pi}{2} - \theta\right)\right]^4 \\
&= f_5\left(\frac{\pi}{2} - \theta; l, k\right).
\end{aligned}$$

Similarly, we have $f_1(\theta; k, l) = f_4\left(\frac{\pi}{2} - \theta; l, k\right)$ and $f_2(\theta; k, l) = f_3\left(\frac{\pi}{2} - \theta; l, k\right)$. ■

Fig. 1 illustrates various quasi-quartic trigonometric Bézier-like functions generated by adjusting the shape parameters k and l. It provides a visual demonstration of the three characteristics mentioned earlier.

III. The quasi-quartic trigonometric Bézier curve

Definition III.1. *Let* Q_i $(i = 0, 1, \ldots, 5)$ *be points in either* R^2 *or* R^3. *Subsequently, a quasi-quartic trigonometric Bézier curve can be formulated as*

$$r(\theta) = \sum_{i=0}^{5} Q_i f_i(\theta), \quad \theta \in [0, \frac{\pi}{2}], k, l \in [0, 4].$$

The quasi-quartic trigonometric Bézier curve has characteristics as follows, similar to the conventional quartic Bézier curve:
(A) Characteristic of end-points.

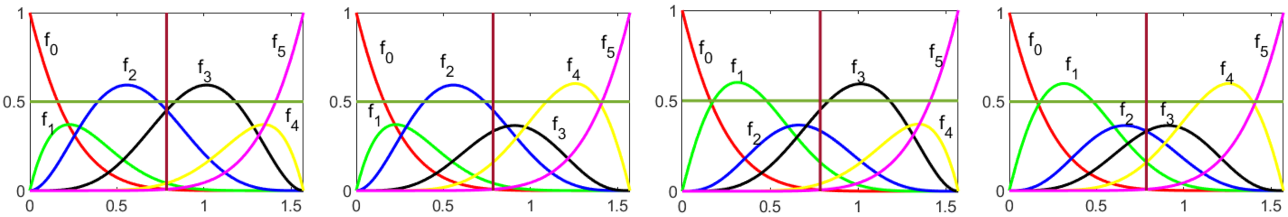
$$\begin{cases}
r(0) = Q_0, \\
r\left(\frac{\pi}{2}\right) = Q_5, \\
r'(0) = 4(Q_1 - Q_0), \\
r'\left(\frac{\pi}{2}\right) = 4(Q_5 - Q_4), \\
r''(0) = 12Q_0 + 4Q_1(k-7) - 4Q_2(k-4), \\
r''\left(\frac{\pi}{2}\right) = 12Q_5 + 4Q_4(l-7) - 4Q_3(l-4), \\
r'''(0) = -20Q_0 - 4Q_1(9k-26) + 12Q_2(3k-7), \\
r'''\left(\frac{\pi}{2}\right) = 20Q_5 + 4Q_4(9l-26) - 12Q_3(3l-7).
\end{cases} \tag{1}$$

(B) Convex hull characteristic. Due to the non-negativity and unity partition mentioned before, the quasi-quartic trigonometric Bézier curve must be located within the convex hull of the control polygon constructed by the control points $\{Q_0, Q_1, \ldots, Q_5\}$.
(C) Symmetry. Control points $\{Q_0, Q_1, \ldots, Q_5\}$ and $\{Q_5, Q_4, \ldots, Q_0\}$ define the same quasi-quartic trigonometric Bézier curve, i.e.,

$$\begin{aligned}
&r(\theta; k, l; Q_0, Q_1, Q_2, Q_3, Q_4, Q_5) \\
&= r\left(\frac{\pi}{2} - \theta; l, k; Q_5, Q_4, Q_3, Q_2, Q_1, Q_0\right),
\end{aligned}$$

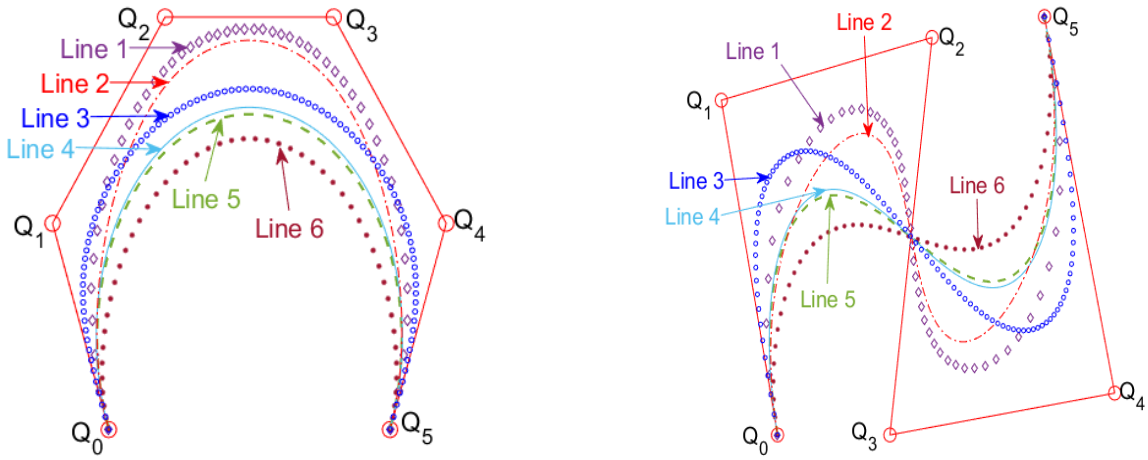
where $0 \le \theta \le \frac{\pi}{2}$, $0 \le k, l \le 4$.

Fig. 2 displays a comparison of five different curves. These include the quasi-quintic trigonometric Bézier curve [17] (Line 1), our quasi-quartic trigonometric Bézier curve with varying parameter values (Line 2 and Line 3), the conventional quintic Bézier curve (Line 4), the C-Bézier curve [5] (Line 5), and the α-Bézier curve [1] (Line 6). As shown in the figure, our curve fits the control polygon more closely than the other curves, with the exception of Line 1. This indicates that our curve is superior in preserving the characteristics of the control polygon compared to the other curves, except for the quasi-quintic trigonometric Bézier curve [17]. Moreover, while our curve performs slightly less effectively than Line 1 in terms of fitting accuracy, it significantly reduces the computational complexity and accelerates the operation.



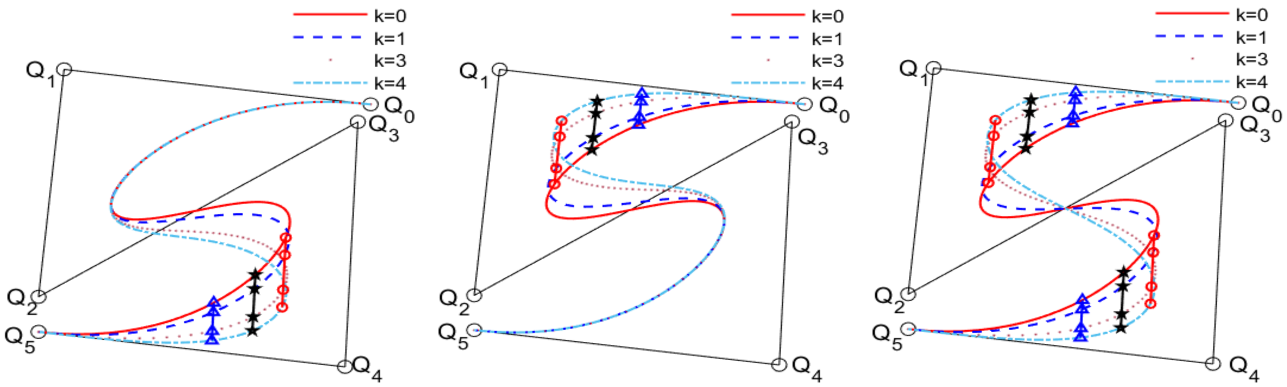
(a) $k=0$, $l=0$ (b) $k=0$, $l=4$ (c) $k=4$, $l=0$ (d) $k=4$, $l=4$

Fig. 1 Different quasi-quartic trigonometric Bézier-like functions generated by adjusting the shape parameters



(a) $k=l=0$ (Line 2), $k=l=4$ (Line 3) (b) $k=l=0$ (Line 2), $k=l=4$ (Line 3)

Fig. 2 The quasi-quartic trigonometric Bézier curve and other Bézier-like curves



(a) $k=0$, $l=0,1,3,4$ (b) $k=0,1,3,4$, $l=0$ (c) $k=l=0,1,3,4$

Fig. 3 Different quasi-quartic trigonometric Bézier curves generated by adjusting the shape parameters

IV. Shape control of the quasi-quartic trigonometric Bézier curve

Carry out the partial differentiation of the quasi-quartic trigonometric Bézier curve $r(\theta)$ with regard to the parameters k and l, then there are

$$\frac{\partial r(\theta)}{\partial k} = 4(\sin\theta - 1)^3 (\cos\theta - 1)(Q_1 - Q_2), \quad (2)$$

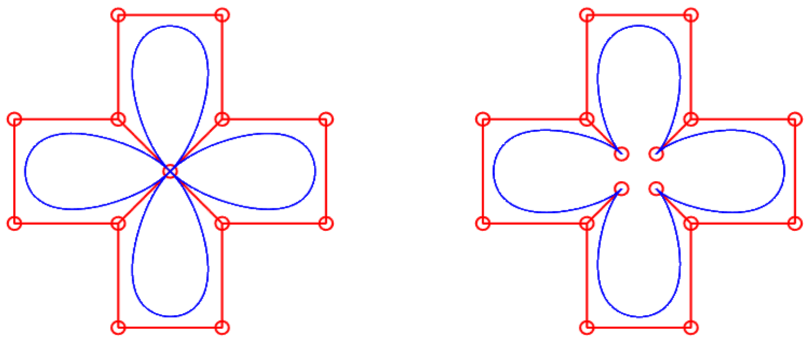
$$\frac{\partial r(\theta)}{\partial l} = 4(\cos\theta - 1)^3 (\sin\theta - 1)(Q_4 - Q_3). \quad (3)$$

Therefore, $\frac{\partial r(\theta)}{\partial k}$ and $\frac{\partial r(\theta)}{\partial l}$ are independent of k and l, which means the derivatives are easy to compute. This also indicates that when $\theta \in \left[0, \frac{\pi}{2}\right]$ and the control points Q_i $(i=0,1,\ldots,5)$ are fixed, the corresponding point on the curve will move in the directions given by (2) and (3) linearly as k and l are changed, respectively.

Fig. 3 (a) displays the quasi-quartic trigonometric Bézier curves when $k=0$ and $l=0,1,3,4$. It is clear that if k is fixed and l varies from $l=0$ to $l=4$, the curve moves parallel to the boundary vector $\overrightarrow{Q_3Q_4}$. Relatively, Fig. 3 (b) displays the curves when $k=0,1,3,4$ and $l=0$. Similarly, if l is fixed and k varies from $k=0$ to $k=4$, the curve moves parallel to the boundary vector $\overrightarrow{Q_2Q_1}$. The curve moves towards the control points Q_1 and Q_4 as k and l change from 0 to 4 simultaneously, as Fig. 3 (c) shows.

V. Connect several quasi-quartic trigonometric Bézier curves

We can creat more complex curves by merging several quasi-quartic trigonometric Bézier curves. To construct a C^0



(a) Closed curve ($k = l = 0$) (b) Open curve ($k = l = 0$)

Fig. 4 A closed and an open C^0 continuous quasi-quartic trigonometric Bézier curve

continuous curve, we can connect two curve pieces at a single point by ensuring that the starting point of the following curve coincides with the endpoint of the previous one. Fig. 4 shows examples of both a closed C^0 continuous curve and an open C^0 continuous curve, each constructed by merging four curve pieces.

The following discussion focuses on the connection conditions required to achieve C^1 and $C^2 \cap FC^3$ continuity when connecting two of our curves. Denote the two curves as

$$r_1(\theta; k_1, l_1) = \sum_{i=0}^{5} Q_i f_i(k_1, l_1), \tag{4}$$

$$r_2(\theta; k_2, l_2) = \sum_{i=0}^{5} T_i f_i(k_2, l_2), \tag{5}$$

where k_1, k_2, l_1, l_2 are shape parameters and $0 \le k_1, k_2, l_1, l_2 \le 4$. For knots $v_1 < v_2 < v_3$, the curve $r(v)$ connected by (4) and (5) is defined as

$$r(v) = \begin{cases} r_1\left(\frac{\pi}{2} \cdot \frac{v - v_1}{h_1}; k_1, l_1\right), & v \in [v_1, v_2], \\ r_2\left(\frac{\pi}{2} \cdot \frac{v - v_2}{h_2}; k_2, l_2\right), & v \in [v_2, v_3], \end{cases}$$

where $h_i = v_{i+1} - v_i, i = 1, 2$. We obtain the following theorems by direct calculation.

Theorem V.1. *$r(v)$ is C^1 continuous at the knot v_2, for $k_i, l_i \in [0, 4]$, $i = 1, 2$, when the condition*

$$Q_5 = T_0 = \frac{h_1 T_1 + h_2 Q_4}{h_1 + h_2} \tag{6}$$

holds. $r(v)$ is C^2 continuous at the knot v_2, for $k_2 \in [0, 4)$, $k_1, l_1, l_2 \in [0, 4]$, when the condition (6) and the condition

$$\begin{aligned} T_2 = & \frac{h_2^2(l_1 - 4)Q_3 - \left[h_1 h_2(k_2 - 7) + h_2^2(l_1 - 7)\right]Q_4}{h_1^2(k_2 - 4)} \\ & + \frac{\left[\left(h_1 h_2 + h_1^2\right)(k_2 - 7) - 3h_2^2 + 3h_1^2\right]Q_5}{h_1^2(k_2 - 4)} \end{aligned} \tag{7}$$

hold simultaneously.

Proof: Based on the end-point characteristic of the Bézier-like functions shown by (1), do some simple operations then there are

$$\begin{aligned} r\left(v_2^-\right) &= Q_5, \\ r\left(v_2^+\right) &= T_0, \\ r'\left(v_2^-\right) &= \frac{2\pi}{h_1}(Q_5 - Q_4), \\ r'\left(v_2^+\right) &= \frac{2\pi}{h_2}(T_1 - T_0), \\ r''\left(v_2^-\right) &= \frac{\pi^2}{h_1^2}\left[3Q_5 + (l_1 - 7)Q_4 - (l_1 - 4)Q_3\right], \\ r''\left(v_2^+\right) &= \frac{\pi^2}{h_2^2}\left[3T_0 + (k_2 - 7)T_1 - (k_2 - 4)T_2\right]. \end{aligned}$$

Therefore, when the condition (6) holds, $r(v)$ is C^1 continuous at v_2, since $r\left(v_2^-\right) = r\left(v_2^+\right)$ and $r'\left(v_2^-\right) = r'\left(v_2^+\right)$ hold. Besides, when the condition (7) holds at the same time, $r(v)$ is C^2 continuous at v_2, since $r''\left(v_2^-\right) = r''\left(v_2^+\right)$ also holds. These indicate the theorem. ■

It is interesting that if $r(v)$ is known to be C^1 continuous at the knot v_2, the determination of T_1 is solely influenced by Q_5 and Q_4, irrespective of the two shape parameters.

We often use the connection matrix to describe Frenet continuity (FC) [20], [21]. We now say that $r(v)$ is FC^3 continuous at v_2, if

$$\begin{bmatrix} r\left(v_2^+\right) \\ r'\left(v_2^+\right) \\ r''\left(v_2^+\right) \\ r'''\left(v_2^+\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega_{11} & 0 & 0 \\ 0 & \omega_{21} & \omega_{11}^2 & 0 \\ 0 & \omega_{31} & \omega_{32} & \omega_{11}^3 \end{bmatrix} \begin{bmatrix} r\left(v_2^-\right) \\ r'\left(v_2^-\right) \\ r''\left(v_2^-\right) \\ r'''\left(v_2^-\right) \end{bmatrix}, \omega_{11} > 0.$$

[21] showed that $C^2 \bigcap FC^3$ continuity is a reasonable smoothness property for use in practice. We get a result as follows, based on the end-point characteristic.

Theorem V.2. *$r(v)$ is $C^2 \bigcap FC^3$ continuous at the knot v_2, when the conditions (6) and (7) hold simultaneously.*

Proof: According to the conditions, $r(v)$ is C^2 continuous at v_2 by Theorem V.1. Besides, based on the end-point characteristic of the Bézier-like functions given in (1), there are

$$\begin{aligned} r'''\left(v_2^-\right) = & \left(\frac{\pi}{2h_1}\right)^3 \left[f_3'''\left(\frac{\pi}{2}\right)Q_3 + f_4'''\left(\frac{\pi}{2}\right)Q_4 \right. \\ & \left. + f_5'''\left(\frac{\pi}{2}\right)Q_5\right], \\ r'''\left(v_2^+\right) = & \left(\frac{\pi}{2h_2}\right)^3 \left[f_0'''(0)T_0 + f_1'''(0)T_1 + f_2'''(0)T_2\right]. \end{aligned}$$

Since

$$f_0'''(0) + f_1'''(0) + f_2'''(0) = 0,$$

$$f_3'''\left(\frac{\pi}{2}\right)+f_4'''\left(\frac{\pi}{2}\right)+f_5'''\left(\frac{\pi}{2}\right)=0,$$
$$Q_5-Q_4=\frac{h_1}{2\pi}r'(v_2),$$
$$Q_3-Q_4=\frac{3h_1}{2\pi(l_1-4)}r'(v_2)-\frac{h_1^2}{\pi^2(l_1-4)}r''(v_2),$$
$$T_1-T_0=\frac{h_2}{2\pi}r'(v_2),$$
$$T_2-T_1=-\frac{3h_2}{2\pi(k_2-4)}r'(v_2)-\frac{h_2^2}{\pi^2(k_2-4)}r''(v_2),$$

we have

$$\begin{aligned}
&r'''\left(v_2^-\right)\\
=&\left(\frac{\pi}{2h_1}\right)^3\left[f_3'''\left(\frac{\pi}{2}\right)(Q_3-Q_4)+f_5'''\left(\frac{\pi}{2}\right)(Q_5-Q_4)\right]\\
=&\left(\frac{\pi}{2h_1}\right)^3\left\{f_3'''\left(\frac{\pi}{2}\right)\left[\frac{3h_1}{2\pi(l_1-4)}r'(v_2)\right.\right.\\
&\left.\left.-\frac{h_1^2}{\pi^2(l_1-4)}r''(v_2)\right]+f_5'''\left(\frac{\pi}{2}\right)\frac{h_1}{2\pi}r'(v_2)\right\}\\
=&\left(\frac{\pi}{2h_1}\right)^3\left\{-f_3'''\left(\frac{\pi}{2}\right)\frac{h_1^2}{\pi^2(l_1-4)}r''(v_2)+\right.\\
&\left.\left[f_3'''\left(\frac{\pi}{2}\right)\frac{3h_1}{2\pi(l_1-4)}+f_5'''\left(\frac{\pi}{2}\right)\frac{h_1}{2\pi}\right]r'(v_2)\right\},\\
&r'''\left(v_2^+\right)\\
=&\left(\frac{\pi}{2h_2}\right)^3\left[f_0'''(0)(T_0-T_1)+f_2'''(0)(T_2-T_1)\right]\\
=&\left(\frac{\pi}{2h_2}\right)^3\left\{-f_0'''(0)\frac{h_2}{2\pi}r'(v_2)\right.\\
&\left.-f_2'''(0)\left[\frac{3h_2}{2\pi(k_2-4)}r'(v_2)+\frac{h_2^2}{\pi^2(k_2-4)}r''(v_2)\right]\right\}\\
=&\left(\frac{\pi}{2h_2}\right)^3\left\{-f_2'''(0)\frac{h_2^2}{\pi^2(k_2-4)}r''(v_2)\right.\\
&\left.-\left[f_0'''(0)\frac{h_2}{2\pi}+f_2'''(0)\frac{3h_2}{2\pi(k_2-4)}\right]r'(v_2)\right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&r'''\left(v_2^+\right)-r'''\left(v_2^-\right)\\
=&\left(\frac{\pi}{2h_2}\right)^3\left\{-f_2'''(0)\frac{h_2^2}{\pi^2(k_2-4)}r''(v_2)\right.\\
&\left.-\left[f_0'''(0)\frac{h_2}{2\pi}+f_2'''(0)\frac{3h_2}{2\pi(k_2-4)}\right]r'(v_2)\right\}\\
&-\left(\frac{\pi}{2h_1}\right)^3\left\{-f_3'''\left(\frac{\pi}{2}\right)\frac{h_1^2}{\pi^2(l_1-4)}r''(v_2)+\right.\\
&\left.\left[f_3'''\left(\frac{\pi}{2}\right)\frac{3h_1}{2\pi(l_1-4)}+f_5'''\left(\frac{\pi}{2}\right)\frac{h_1}{2\pi}\right]r'(v_2)\right\}\\
=&\left\{-\left(\frac{\pi}{2h_2}\right)^3f_2'''(0)\frac{h_2^2}{\pi^2(k_2-4)}+\right.\\
&\left.\left(\frac{\pi}{2h_1}\right)^3f_3'''\left(\frac{\pi}{2}\right)\frac{h_1^2}{\pi^2(l_1-4)}\right\}r''(v_2)\\
&+\left\{-\left(\frac{\pi}{2h_2}\right)^3\left[f_0'''(0)\frac{h_2}{2\pi}+f_2'''(0)\frac{3h_2}{2\pi(k_2-4)}\right]\right.\\
&\left.-\left(\frac{\pi}{2h_1}\right)^3\left[f_3'''\left(\frac{\pi}{2}\right)\frac{3h_1}{2\pi(l_1-4)}+f_5'''\left(\frac{\pi}{2}\right)\frac{h_1}{2\pi}\right]\right\}r'(v_2).
\end{aligned}$$

This means that under the condition of smooth connection in C^2 continuity, the gained composite quasi-quartic trigonometric Bézier curve may achieve $C^2\bigcap FC^3$ continuity automatically. ■

The combination of two quasi-quartic trigonometric Bézier curves is shown in Fig. 5. The black dash-dotted line corresponds to $k_1=0=l_1$, the black dashed one corresponds to $k_2=0=l_2$, and the blue solid line corresponds to $k_3=2$ and $l_3=0$ respectively. Fig. 5 (a) shows C^1 continuity at the connection point $Q_5(T_0)$ of the two curves, satisfying condition (6). Fig. 5 (b) demonstrates $C^2\cap FC^3$ continuity at the connection point $Q_5(T_0)$, satisfying both conditions (6) and (7).

Fig. 6, 7 and 8 depict the combination of several pieces of our curves. Designers have the flexibility to make minor adjustments to the shape of the obtained curves by simply modifying the values of the shape parameters k and l.

Fig. 9 (a) and (c) display space $C^2\cap FC^3$ continuous curves constructed by combining three quasi-quartic trigonometric Bézier curves. The related porcupine plots illustrate the normalized curvature along their principal normal, facilitating a clearer observation of the $C^2\cap FC^3$ continuous curves. Additionally, Fig. 9 (b) and (d) show the corresponding torsion curves.

VI. Conclusion

The quasi-quartic trigonometric Bézier curve, constructed using six novel quasi-quartic trigonometric Bézier-like functions characterized by two parameters, offers practical value in the field of Computer-Aided Geometric Design (CAGD). This curve exhibits properties similar to those of conventional quartic Bézier curves but with enhanced fitting capabilities. Specifically, it can fit the control polygon more closely than many other Bézier curves, while maintaining ideally low algorithmic complexity. As a result, it will be more in tune with the shape of the control polygon. Moreover, the shape of the curve can be linearly controlled, and its derivative is straightforward to compute. When multiple quasi-quartic trigonometric Bézier curves are merged, the resulting composite curve can achieve $C^2\cap FC^3$ continuity automatically, provided that the C^2 continuity condition is satisfied. This feature ensures smooth transitions between curve segments without additional constraints. Furthermore, the shape of the curve can be precisely altered by adjusting the shape parameters, even when the control points remain unchanged. This flexibility allows designers to make minor modifications to the curve's shape as needed, enhancing the curve's adaptability and usability in various applications.

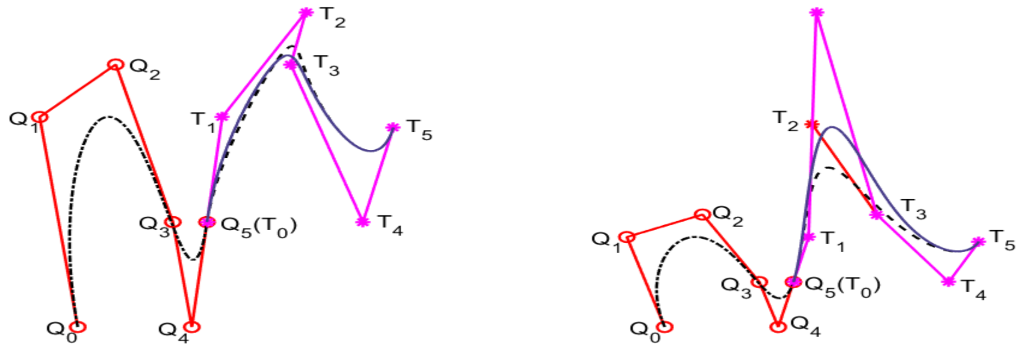
References

[1] E. Mainar, J. M. Pena, and J. Sánchez-Reyes, "Shape preserving alternatives to the rational Bézier model," *Computer aided geometric design*, vol. 18, no. 1, pp. 37–60, 2001.

[2] X. Han, "Quadratic trigonometric polynomial curves with a shape parameter," *Computer Aided Geometric Design*, vol. 19, no. 7, pp. 503–512, 2002.

[3] X. Han, "Cubic trigonometric polynomial curves with a shape parameter," *Computer Aided Geometric Design*, vol. 21, no. 6, pp. 535–548, 2004.

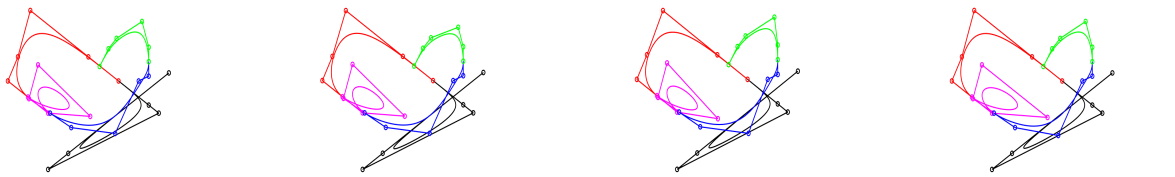
[4] X. Wu, X. Han, and S. Luo, "Quadratic trigonometric spline curves with multiple shape parameters," in *2007 10th IEEE International Conference on Computer-Aided Design and Computer Graphics*. IEEE, 2007, pp. 413–416.



(a) A quasi-quartic trigonometric Bézier curve with C^1 continuity

(b) A quasi-quartic trigonometric Bézier curve with $C^2 \cap FC^3$ continuity

Fig. 5 Combine two quasi-quartic trigonometric Bézier curves



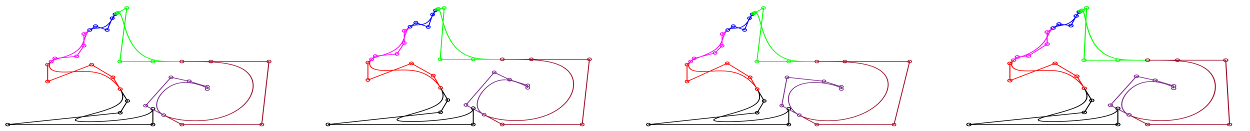
(a) $k_1 = 2.5$, $l_1 = 3.3$, $k_2 = 1$, $l_2 = 0.5$, $k_3 = 0.3$, $l_3 = 2$, $k_4 = 0.7$, $l_4 = 2$, $k_5 = 1$, $l_5 = 2$

(b) $k_1 = 2$, $l_1 = 3$, $k_2 = 1$, $l_2 = 0.5$, $k_3 = 0.3$, $l_3 = 2$, $k_4 = 0.7$, $l_4 = 2$, $k_5 = 0$, $l_5 = 2$

(c) $k_1 = 2.5$, $l_1 = 3.3$, $k_2 = 0.8$, $l_2 = 1$, $k_3 = 0.3$, $l_3 = 2$, $k_4 = 1$, $l_4 = 3$, $k_5 = 1$, $l_5 = 2$

(d) $k_1 = 2.5$, $l_1 = 3.3$, $k_2 = 1$, $l_2 = 0.5$, $k_3 = 1.2$, $l_3 = 2.2$, $k_4 = 0.7$, $l_4 = 2$, $k_5 = 1$, $l_5 = 2$

Fig. 6 Butterfly constructed by combining five quasi-quartic trigonometric Bézier curves



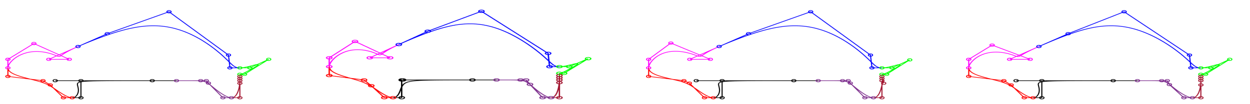
(a) $k_1 = 2$, $l_1 = 1.2$, $k_2 = 1$, $l_2 = 3$, $k_3 = 3$, $l_3 = 1$, $k_4 = 1$, $l_4 = 4$, $k_5 = 3$, $l_5 = 4$, $k_6 = 1$, $l_6 = 2$, $k_7 = 2$, $l_7 = 2$

(b) $k_1 = 2.2$, $l_1 = 1$, $k_2 = 1$, $l_2 = 3$, $k_3 = 3$, $l_3 = 1$, $k_4 = 1.2$, $l_4 = 3$, $k_5 = 3$, $l_5 = 4$, $k_6 = 1$, $l_6 = 2$, $k_7 = 2.2$, $l_7 = 1$

(c) $k_1 = 2$, $l_1 = 1.2$, $k_2 = 1.3$, $l_2 = 3.2$, $k_3 = 3$, $l_3 = 1$, $k_4 = 1$, $l_4 = 4$, $k_5 = 3$, $l_5 = 4$, $k_6 = 1.4$, $l_6 = 2.5$, $k_7 = 2$, $l_7 = 2$

(d) $k_1 = 2$, $l_1 = 1.2$, $k_2 = 1$, $l_2 = 3$, $k_3 = 2$, $l_3 = 1.6$, $k_4 = 1$, $l_4 = 4$, $k_5 = 2$, $l_5 = 3$, $k_6 = 1$, $l_6 = 2$, $k_7 = 2$, $l_7 = 2$

Fig. 7 Dog constructed by combining seven quasi-quartic trigonometric Bézier curves



(a) $k_1 = 2.5$, $l_1 = 1$, $k_2 = 0$, $l_2 = 2$, $k_3 = 1.3$, $l_3 = 2$, $k_4 = 0.5$, $l_4 = 2$, $k_5 = 1$, $l_5 = 4$, $k_6 = 1.3$, $l_6 = 1$, $k_7 = 0.5$, $l_7 = 4$

(b) $k_1 = 2$, $l_1 = 1.3$, $k_2 = 0$, $l_2 = 2$, $k_3 = 1.3$, $l_3 = 2$, $k_4 = 0$, $l_4 = 2.2$, $k_5 = 1$, $l_5 = 4$, $k_6 = 1.3$, $l_6 = 1$, $k_7 = 0$, $l_7 = 3$

(c) $k_1 = 2.5$, $l_1 = 1$, $k_2 = 0.5$, $l_2 = 2.3$, $k_3 = 1.3$, $l_3 = 2$, $k_4 = 0.5$, $l_4 = 2$, $k_5 = 0.8$, $l_5 = 3$, $k_6 = 1.3$, $l_6 = 1$, $k_7 = 0.5$, $l_7 = 4$

(d) $k_1 = 2.5$, $l_1 = 1$, $k_2 = 0$, $l_2 = 2$, $k_3 = 1$, $l_3 = 3$, $k_4 = 0.5$, $l_4 = 2$, $k_5 = 1$, $l_5 = 4$, $k_6 = 2$, $l_6 = 1.3$, $k_7 = 0.5$, $l_7 = 4$

Fig. 8 Turtle constructed by combining seven quasi-quartic trigonometric Bézier curves

[5] Q. Chen and G. Wang, "A class of Bézier-like curves," *Computer Aided Geometric Design*, vol. 20, no. 1, pp. 29–39, 2003.

[6] X.-A. Han, Y. Ma, and X. Huang, "The cubic trigonometric Bézier curve with two shape parameters," *Applied Mathematics Letters*, vol. 22, no. 2, pp. 226–231, 2009.

[7] X.-A. Han, X. Huang, and Y. Ma, "Shape analysis of cubic trigonometric Bézier curves with a shape parameter," *Applied Mathematics and Computation*, vol. 217, no. 6, pp. 2527–2533, 2010.

[8] X. Han and Y. Zhu, "Curve construction based on five trigonometric blending functions," *BIT Numerical Mathematics*, vol. 52, pp. 953–979, 2012.

[9] M. Hoffmann, I. Juhász, and G. Károlyi, "A control point based curve with two exponential shape parameters," *BIT Numerical Mathematics*, vol. 54, pp. 691–710, 2014.

[10] Y. Zhu, X. Han, J. Han *et al.*, "Quartic trigonometric Bézier curves and shape preserving interpolation curves," *Journal of Computational Information Systems*, vol. 8, no. 2, pp. 905–914, 2012.

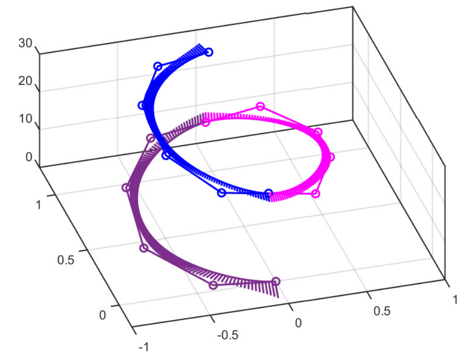
[11] U. Bashir, M. Abbas, M. N. H. Awang, and J. M. Ali, "A class of quasi-quintic trigonometric Bézier curve with two shape parameters," *Science Asia*, vol. 39, no. 2, pp. 11–15, 2013.

[12] X. Han and Y. Zhu, "Total positivity of the cubic trigonometric Bézier basis," *Journal of Applied Mathematics*, vol. 2014, no. 1, p. 198745, 2014.

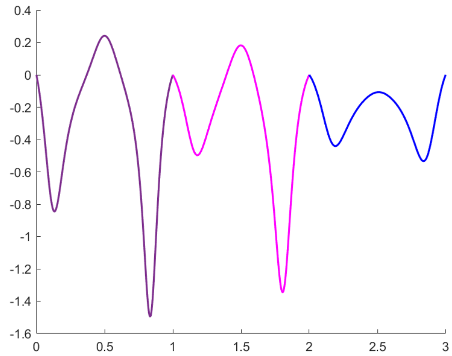
[13] U. Bashir and J. M. Ali, "Rational cubic trigonometric Bézier curve with two shape parameters," *Computational and Applied Mathematics*, vol. 35, pp. 285–300, 2016.

[14] L. Yan, "Cubic trigonometric nonuniform spline curves and surfaces," *Mathematical Problems in Engineering*, vol. 2016, no. 1, p. 7067408, 2016.

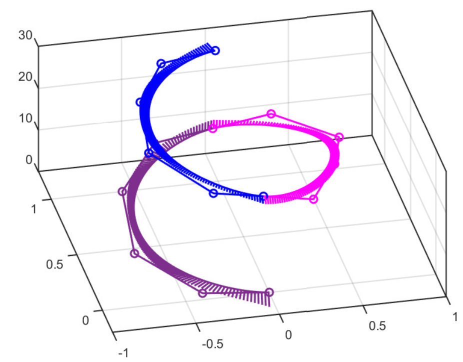
[15] M. Misro, A. Ramli, and J. Ali, "Quintic trigonometric Bézier curve with two shape parameters," *Sains Malaysiana*, vol. 46, no. 5, pp. 825–831, 2017.



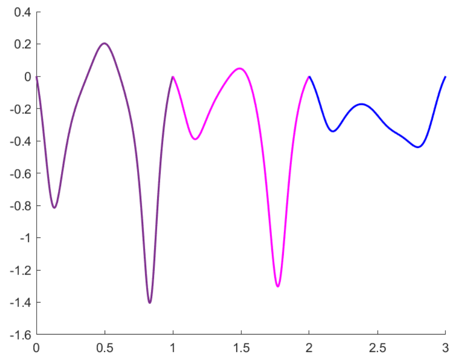
(a) Space $C^2 \cap FC^3$ continuous curve ($k_i = l_i = 0, i = 1, 2, 3$)



(b) The torsion curve of the curve in (a)



(c) Space $C^2 \cap FC^3$ continuous curve ($k_i = l_i = 0.5, i = 1, 2, 3$)



(d) The torsion curve of the curve in (c)

Fig. 9 Different space $C^2 \cap FC^3$ continuous curves generated by adjusting the shape parameters and the corresponding torsion curves

[16] W. Xie and J. Li, "C^3 cubic trigonometric b-spline curves with a real parameter," *Journal of the National Science Foundation of Sri Lanka*, vol. 46, no. 1, 2018.

[17] X. Tan and Y. Zhu, "Quasi-quintic trigonometric Bézier curves with two shape parameters," *Computational and Applied Mathematics*, vol. 38, pp. 1–13, 2019.

[18] L. Yan, M. Jie, and Z. Wei, "A new extension of the cubic Bézier curve," *Journal of Computer-Aided Design & Computer Graphics*, vol. 34, no. 10, pp. 1590–1603, 2022.

[19] H. Kaur and M. R. Goyal, "New generalized blended trigonometric Bézier curves with one shape parameter," *Filomat*, vol. 38, no. 2, pp. 705–725, 2024.

[20] J. Hoschek and D. Lasser, *Fundamentals of computer aided geometric design*. AK Peters, Ltd., 1993.

[21] P. Costantini and C. Manni, "Geometric construction of spline curves with tension properties," *Computer Aided Geometric Design*, vol. 20, no. 8-9, pp. 579–599, 2003.