ET-norms and n-roots on EBL-algebras

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Abstract—In this paper, we will define the notion of Et-norms (Et-conorms) as an extension of t-norms (t-conorms). We also introduce some important properties of Et-norms (Et-conorms) on an chain. For example, we characterize strictly monotonicity and continuity of Et-norms (Et-conorms) on an chain. In addition, we introduce some important results between an Et-norm T on an chain L and an EBL-algebra $(L, \vee, \wedge, T, 0)$. On the other hand, we give the notion of n-roots on EBL-algebras and study their main properties. In addition, we define the notion for an EBL-homomorphism to preserve n-roots and prove a necessary and sufficient condition for an EBL-homomorphism to preserve n-roots. Finally, we introduce the notion of strict n-roots and characterize some properties of n-strict EBL-algebras. Particularly, we prove each n-strict EBL-algebra is a BL-algebra.

Index Terms—BL-algebra, EBL-algebra, Et-norm, Et-conorm, n-root, Square root, n-strict EBL-algebra.

I. Introduction

CHANG [2] defined the notion of MV-algebras, which is an algebraic structure of the Lukasiewcz system of many-valued logic. In addition, Mundici [20] proved the categorical equivalent between MV-algebras and unital Abelian lattice-ordered groups. Nowadays, MV-algebras have been applied to graph theory, fuzzy theory, etc.

Hajek [10] defined the notion of BL-algebras. Particularly, for a BL-algebra A and for all $x \in A$, if $x^{--} = x$, then A is an MV-algebra. In addition, Dvurecenskij and Zahiri [3] defined EMV-algebras as an extension of MV-algebras. Similarly, Liu in 2020 [18], defined EBL-algebras, which extended the notion of BL-algebras.

T-norms were original introduced by [21]. It is an important tool in the aspect of fuzzy logic. A t-norm T is a binary operation on $[0,1]$ such that $([0,1], T, \leq)$ is an abelian chain. The paper [14] provided some statements about t-norms and their applications. In a series of three papers [15], [16], [17] ,we known some basic analytical properties of t-norms, general construction methods of t-norms and properties of continuous t-norms. In the paper [7], the authors studied pseudo-t-norms and pseudo-BL-algebras.

In fact, n-roots as a tool for studying algebraic structure is valuable. Nowadays, n-roots has been deeply applied to many aspects. Maltsev [19] defined the notion of R-groups. Over the years, Baumslay [1] studied R-groups, divisible groups and divisible R-groups.

Square roots on MV-algebras were originally introduced by Hohle [12]. In this paper, Hohle provided a classification method of MV-algebras with square roots. In addition, Dvnrecenskij and Zahiri in [4] and [5] studied square roots on EMV-algebras and pseudo MV-algebras, respectively.

Recently, Dvurecenskij, Zahiri and Shenavaei [6] studied n-roots on MV-algebras. They defined n-roots on MV-algebras and studied their main properties. In addition, they presented the notion of n-strict MV-algebras and established the relationship between n-strict MV-algebras and n-divisible MV-algebras. Finally, they proved each MV-algebra with an n-root as a product of an n-strict MV-algebra and a Boolean algebra.

This paper is constructed as follows. In Section 2, we will introduce some notions and results of EBL-algebras and t-norms (t-conorms). In Section 3, we define the notion of Et-norms (Et-conorms) and φ-operators (φ'-operators) on an chain L and study their main properties. In addition, we use an Et-norm T and a φ-operator on L to construct an EBL-algebra $(L, \vee, \wedge, T, 0)$. In Section 4, we define the notion of n-roots on EBL-algebras and introduce some main properties of n-roots on EBL-algebras. We also extend the notion of strict n-roots and study some main properties of n-strict EBL-algebras.

II. Preliminaries

In this section, we will introduce some notions and results on t-norms and EBL-algebras. In the following, we will denote $I(A) = \{x \in A | x \odot x = x\}$.

Definition II.1. [18] An EBL-algebra is defined as an algebraic structure $(A, \vee, \wedge, \odot, 0)$ that satisfies the following axioms:
(EBL1) $(A, \vee, \wedge, 0)$ forms a distributive lattice with minimal element 0;
(EBL2) $(A, \odot, 0)$ forms a commutative semigroup;
(EBL3) Let $m, n \in I(A)$ and $m \leq n$. For all $x, y \in [m, n]$, there exists $x \overset{m,n}{\to} y$, defined as $x \overset{m,n}{\to} y = \vee\{z \in [m, n] | x \odot z \leq y\}$. In addition, the interval $[0, b]$ equipped with the operations $\vee, \wedge, \odot, \overset{b}{\to}$ and bounds $0, b$ forms a BL-algebra;
(EBL4) For all $x \in A$, there exists an idempotent element a satisfying $x \leq a$.

Unless otherwise specified, the following A is expressed as a EBL-algebra. Let a be an arbitrary idempotent element of A. For all $x, y \in [0, a]$, we define a partial ordering relation $\leq$ as follows:

$$x \leq y \Leftrightarrow x \overset{a}{\to} y = a.$$

additionally, we define the following operations: $x^{-a} = x \overset{a}{\to} 0$, $x^{--a} = (x \overset{a}{\to} 0) \overset{a}{\to} 0$, $x^{---a} = ((x \overset{a}{\to} 0) \overset{a}{\to} 0) \overset{a}{\to} 0$, $x^0 = a, x^1 = x, ..., x^n = x^{n-1} \odot x$. In the following, for an algebraic structure A, we will denote a as an arbitrary idempotent element of A.

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Proposition II.2. [18] For all $x, y, z \in [0, a]$, we have the following properties:
(EBLP1) $a \xrightarrow{a} x = x$; $x \xrightarrow{a} x = a$;
(EBLP2) $x \xrightarrow{a} (y \xrightarrow{a} z) = (x \odot y) \xrightarrow{a} z$;
(EBLP3) $x \leq x^{--a}$; $x^{---a} = x$;
(EBLP4) $(x \vee y) \xrightarrow{a} z = (x \xrightarrow{a} z) \wedge (y \xrightarrow{a} z)$; $(x \wedge y) \xrightarrow{a} z = (x \xrightarrow{a} z) \vee (y \xrightarrow{a} z)$;
(EBLP5) If $x \leq y$, then $(y \xrightarrow{a} z) \leq (x \xrightarrow{a} z)$ and $(z \xrightarrow{a} x) \leq (z \xrightarrow{a} y)$;
(EBLP6) $x \odot x^{-a} = 0$; $x \odot y = 0$ iff $x \leq y^{-a}$;
(EBLP7) If $x \leq y$, then $(x \odot z) \leq (y \odot z)$;
(EBLP8) $(x \odot y)^{-a} = x \xrightarrow{a} y^{-a}$;
(EBLP9) $(x \wedge y)^{-a} = x^{-a} \vee y^{-a}$; $(x \vee y)^{-a} = x^{-a} \wedge y^{-a}$;
(EBLP10) $(x \xrightarrow{a} y)^{--a} = x^{--a} \xrightarrow{a} y^{--a}$; $(x \wedge y)^{--a} = x^{--a} \wedge y^{--a}$; $(x \vee y)^{--a} = x^{--a} \vee y^{--a}$; $(x \odot y)^{--a} = x^{--a} \odot y^{--a}$;
(EBLP11) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$; $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$;
(EBLP12) $x \wedge y = x \odot (x \xrightarrow{a} y)$;
(EBLP13) $x \odot y \leq z \Leftrightarrow x \leq (y \xrightarrow{a} z)$;
(EBLP14) $(x \xrightarrow{a} y) \vee (y \xrightarrow{a} x) = a$. In particularly, $x \vee (x \xrightarrow{a} 0) = a$.

Definition II.3. [18] Let $\emptyset \neq I \subseteq A$. I is called an ideal of A if it satisfies the following conditions:
(1) For all $y \in I$, if $x \leq y$, then $x \in I$;
(2) For all $x, y \in I$, we have $(x \xrightarrow{a} 0) \xrightarrow{a} y \in I$.

Definition II.4. [18] Let A_1 and A_2 be two EBL-algebras. A map $f : A_1 \rightarrow A_2$ is an EBL-homomorphism if it satisfies the following conditions:
(1) For all $x, y \in A_1$, $f(x \vee y) = f(x) \vee f(y)$, $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \odot y) = f(x) \odot f(y)$;
(2) $f(0) = 0$;
(3) For all $x, y \in [0, a]$, $f(x \xrightarrow{a} y) = f(x) \xrightarrow{f(a)} f(y)$.

Proposition II.5. [18] Let $a, b \in I(A)$. For all $x \in [0, a]$, if $a \leq b$, then
(1) $x \xrightarrow{a} 0 = (x \xrightarrow{b} 0) \wedge a$;
(2) $x \xrightarrow{b} 0 = (x \xrightarrow{a} 0) \vee (a \xrightarrow{b} 0)$;
(3) $x \xrightarrow{a} 0 \leq x \xrightarrow{b} 0$;
(4) $a \xrightarrow{b} 0 \in I(A)$. Further, $a \xrightarrow{a} 0 = 0$.

Definition II.6. [8] A binary operation T on $[0, 1]$ is called a t-norm if for all $x, y, z \in [0, 1]$, it satisfies the following conditions:
(T1) $T(x, y) = T(y, x)$;
(T2) $T(T(x, y), z) = T(x, T(y, z))$;
(T3) If $x \leq y$, then $T(x, z) \leq T(y, z)$;
(T4) $T(x, 1) = x$.

Definition II.7. [8] A binary operation S on $[0, 1]$ is called a t-conorm if for all $x, y, z \in [0, 1]$, it satisfies the following conditions:
(S1) $S(x, y) = S(y, x)$;
(S2) $S(S(x, y), z) = S(x, S(y, z))$;
(S3) If $x \leq y$, then $S(x, z) \leq S(y, z)$;
(S4) $S(x, 0) = x$.

Lemma II.8. [9] Let T be a t-norm on $[0, 1]$ and φ be a φ-operator connected with T on $[0, 1]$. For all $x, y \in [0, 1]$, $T(x, \varphi(x, y)) = x \wedge y$ iff T is continuous.

Proposition II.9. [18] If $(A, \vee, \wedge, \odot, 0)$ satisfies the following axioms, then it is an EBL-algebra.
(EBL1) $(A, \vee, \wedge, 0)$ forms a lattice with minimal element 0;
(EBL2) $(A, \odot, 0)$ forms a commutative semigroup;
(EBL3′) For all $x \in A$, there exists an idempotent element b satisfying $x \leq b$. In addition, the interval $[0, b]$ equipped with the operations $\vee, \wedge, \odot, \xrightarrow{b}$ and bounds $0, b$ forms a BL-algebra.

Definition II.10. [18] Let I be an ideal of A. I is called maximal ideal if for all $x \in A \setminus I$, $\langle I \cup \{x\} \rangle = A$.

In the following, we will denote N as the set of positive integers.

Definition II.11. [10] An algebraic structure $(A, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 0, 0)$ is BL-algebra if it satisfies the following axioms for all $x, y, z \in A$:
(BL1) $(A, \vee, \wedge, 0, 1)$ forms a bounded lattice;
(BL3) $(A, \odot, 1)$ forms an abelian monoid;
(BL3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$;
(BL4) $x \wedge y = x \odot (x \rightarrow y)$;
(BL5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

Definition II.12. [3] An EMV-algebra is an algebraic structure $(A, \vee, \wedge, \oplus, 0)$ of type $(2, 2, 2, 0)$ satisfies the following conditions:
(EMV1) $(A, \vee, \wedge, 0)$ forms a distributive lattice with minimal element 0;
(EMV2) $(A, \oplus, 0)$ forms a commutative monoid;
(EMV3) For all $x \in A$, there exists $b \in I(A)$ satisfying $x \leq b$, defined by $\lambda_{m,n}(x) = min\{z \in [0, b] | x \oplus z = b\}$. In addition, the interval $[0, b]$ equipped with the operations $\vee, \wedge, \oplus, \lambda_b(x)$ and bounds $0, b$ forms an MV-algebra.

Definition II.13. [18] Let $\emptyset \neq F \subseteq A$. For all $x, y \in A$, F is called a filter if it satisfies the following conditions:
(1) For all $x \in F$, if $x \leq y$, then $y \in F$;
(2) If $x, y \in F$, then $x \odot y \in F$.

III. Et-norms and EBL-algebras

In this section, we will introduce key concepts and results related to Et-norms on an chain L. In addition, we establish a construction between an Et-norm on L and an EBL-algebra $(L, \vee, \wedge, T, 0)$.

Definition III.1. An algebraic structure L is called an chain if it satisfies the following conditions:
(1) Its natural order is total;
(2) Let M be an binary operation on L. For all $x \in L$, there exists $l \in I(L)$ satisfying $x \leq l$, where $I(L) = \{m \in L | M(m, m) = m\}$.

Definition III.2. A binary operation T on an chain L is an Et-norm, if for all $a \in L$ satisfying $T(a, a) = a$, the following conditions hold for all $x, y, z \in [0, a]$:
(ET1) $T(x, y) = T(y, x)$;
(ET2) $T(T(x, y), z) = T(x, T(y, z))$;
(ET3) If $x \leq y$, then $T(x, z) \leq T(y, z)$;
(ET4) $T(x, a) = x$.

Obviously, the following properties hold:
(1) $T(0, x) = 0$ since $T(0, x) \leq T(0, a) = 0$;

(2) $T(x, y) \leq T(x, a) = x$. Similarly, $T(x, y) \leq T(a, y) = y$;
(3) $T(x_1, y_1) \leq T(x_2, y_2)$ iff $x_1 \leq x_2$ and $y_1 \leq y_2$.

Definition III.3. A binary operation S on an chain L is an Et-conorm, if for all $a \in L$ satisfying $S(a, a) = a$, the following conditions hold for all $x, y, z \in [0, a]$:
(ES1) $S(x, y) = S(y, x)$;
(ES2) $S(S(x, y), z) = S(x, S(y, z))$;
(ES3) if $x \leq y$, then $S(x, z) \leq S(y, z)$;
(ES4) $S(x, 0) = x$.

Similarly, we have:
(1) $S(x, a) = a$ since $a = S(0, a) \leq S(x, a)$;
(2) $x = S(x, 0) \leq S(x, y)$. Similarly, $y = S(0, y) \leq S(x, y)$;
(3) $S(x_1, y_1) \leq S(x_2, y_2)$ iff $x_1 \leq x_2$ and $y_1 \leq y_2$.

In the following, unless stated otherwise, we will denote L as an chain and denote a as an arbitrary element on L such that $T(a, a) = a$ (or $S(a, a) = a$) in this section.

Example III.4. [13] The two basic Et-norms on L together with their dual Et-conorms are as follow:
(1) For all $x, y \in [0, a]$, we define

$$T_M(x, y) = \min(x, y);$$
$$S_M(x, y) = \max(x, y).$$

(2) For all $x, y \in [0, a]$, we define

$$T_W(x, y) = \begin{cases} \min(x, y), & \max(x, y) = a, \\ 0, & x \neq a \text{ and } y \neq a. \end{cases}$$

$$S_W(x, y) = \begin{cases} \max(x, y), & \min(x, y) = 0, \\ a, & x \neq 0 \text{ and } y \neq 0. \end{cases}$$

Proposition III.5. Let T be an Et-norm on L. For all $x, y \in L$, the following statements hold:
(1) $T_W \leq T \leq T_M$;
(2) $S_M \leq S \leq S_W$.

Proof: (1) For all $x, y \in [0, a]$, if $x, y \neq a$, we have $T_W(x, y) = 0 \leq T(x, y)$; If $x = a$ or $y = a$, we have $T_W(x, y) = T(x, y)$. So $T_W(x, y) \leq T(x, y)$. In addition, from $T(x, y) \leq x$ and $T(x, y) \leq y$, it follows that $T(x, y) \leq T_M(x, y)$.
(2) The proof is similar to (1). ■

Definition III.6. Let T be an Et-norm on L and S be an Et-conorm on L. For all $n \in N$, we define x_T^n by

$$x_T^n = \begin{cases} x, & n = 1, \\ T(\overbrace{x, x, \cdots x}^{n-times}), & n > 1. \end{cases}$$

and define x_S^n by

$$x_S^n = \begin{cases} x, & n = 1, \\ S(\overbrace{x, x, \cdots x}^{n-times}), & n > 1. \end{cases}$$

In the following, we will give some notions of Et-norms on L, which are similar to the notions in [13].

- An Et-norm T is said to be continuous, if for every pair of convergent sequences $(x_n)_{n \in N}$ and $(y_n)_{n \in N}$ $\in [0, a]^N$, the following holds:
$$T(\lim_{n \to \infty} x_n, \lim_{n \to \infty} y_n) = \lim_{n \to \infty} T(x_n, y_n).$$
- An Et-norm T is called strictly monotone, if for all $x \in (0, a)$ and $y < z$, the following inequality holds:
$$T(x, y) < T(x, z).$$
An Et-norm T is strict if it is both continuous and strictly monotone.
- An Et-norm T is called Archimedean, if for all $x, y \in [0, a]$ and $n \in N$, the following holds:
$$x_T^{(n)} < y.$$
- An element $x \in (0, a)$ is nilpotent on Et-norm T, if there exists $n \in N$ satisfying $x_T^{(n)} = 0$.
- An Et-norm T is nilpotent, if it is continuous and x is nilpotent for all $x \in (0, a)$.
- An element $x \in (0, a)$ is called a zero divisor on Et-norm T, if there exists $y \in (0, a)$ satisfying $T(x, y) = 0$.

Similarly, we will give some notions of Et-conorms on L.

- An Et-conorm S is said to be continuous, if for every pair of convergent sequences $(x_n)_{n \in N}$, $(y_n)_{n \in N} \in [0, a]^N$, the following holds:
$$S(\lim_{n \to \infty} x_n, \lim_{n \to \infty} y_n) = \lim_{n \to \infty} S(x_n, y_n).$$
- An Et-conorm S is called strictly monotone, if for all $x \in (0, a)$ and $y < z$, the following inequality holds:
$$S(x, y) < S(x, z)$$
An Et-conorm S is strict if it is both continuous and strictly monotone.
- An Et-conorm S is co-Archimedean, if for each $x, y \in [0, a]$ and $n \in N$, we have $x_S^{(n)} > y$.
- An element $x \in (0, a)$ is co-nilpotent on Et-conorm S, if there exists $n \in N$ satisfying $x_S^{(n)} = a$.
- An Et-conorm S is co-nilpotent, if it is continuous and x is co-nilpotent for each $x \in (0, a)$.
- An element $x \in (0, a)$ is called a co-zero divisor on Et-conorm S, if there exists $y \in (0, a)$ satisfying $S(x, y) = a$.

Remark III.7. (1) Let T be a strictly monotone Et-norm on L. For all $x \in (0, a)$, we have $T(x, x) < T(x, a) = x$, which implies $T(x, x) < x$.

(2) For all $0 \neq x \in L$, an Et-norm $T : L^2 \to L$ is strictly monotone iff $T(x, y) = T(x, z) \Rightarrow y = z$.

(3) For each $x \in (0, a)$, if an Et-norm $T : L^2 \to L$ has no zero divisors, then $T(x, x) > 0$.

(4) Let S be a strictly monotone Et-conorm on L. For all $x \in (0, a)$, we have $x = S(x, 0) < S(x, x)$, which implies $S(x, x) > x$.

(5) For all $0 \neq x \in L$, an Et-conorm $S : L^2 \to L$ is strictly monotone iff $S(x, y) = S(x, z) \Rightarrow y = z$.

(6) For each $x \in (0, a)$, if an Et-conorm $S : L^2 \to L$ has no co-zero divisors, then $S(x, x) < a$.

Proposition III.8. For all $x, y \in (0, a)$, an Et-norm $T : L^2 \to L$ is Archimedean iff $\lim_{n \to \infty} x_T^{(n)} = 0$.

Proof: Let T be an Archimedean Et-norm. There exists $m \in N$ satisfying $x_T^{(m)} < y$. From $x_T^{(n)} \leq x_T^{(m)}$

$(n > m)$, which follows $\lim_{n\to\infty} x_T^{(n)} = 0$. Conversely, it is obvious. ■

Proposition III.9. For all $x, y \in (0, a)$, an Et-conorm $S : L^2 \to L$ is co-Archimedean iff $\lim_{n\to\infty} x_S^{(n)} = a$.

Proof: Let S be an co-Archimedean Et-conorm. There exists $m \in N$ satisfying $x_S^{(m)} > y$. From $x_S^{(n)} \geq x_S^{(m)}$ $(n > m)$, which follows $\lim_{n\to\infty} x_S^{(n)} = a$. Conversely, it is obvious. ■

In the following, we will characterize some properties of continuous Archimedean Et-norms and continuous co-Archimedean Et-conorms.

Theorem III.10. Let T be a continuous Archimedean Et-norm on L. The following statements are equivalent:
(1) There exists a nilpotent element of T;
(2) T is nilpotent;
(3) T has zero divisors;
(4) T is not strict.

Proof: (1) ⇒ (2): Let y be a nilpotent element of T. There exists $m \in N$ satisfying $y_T^{(m)} = 0$. Since T is Archimedean, there exists $N_1 \in N$ satisfying $x_T^{(N_1)} < y$ for all $x \in L$. So $x_T^{(N_1 m)} < y_T^{(m)} = 0$. Hence, we have that x is a niplotent element.
(2) ⇒ (1): It is obvious.
(1) ⇒ (3): Let x be a nilpotent element of T. There exists the smallest m satisfying $x_T^{(m)} = 0$. That is, $T(x_T^{(m-1)}, x) = 0$.
(3) ⇒ (1): Let x be a zero divisor of T. There exists $y \in (0, a)$ satisfying $T(x, y) = 0$. Assume that $x \leq y$. Then $T(x, x) \leq T(x, y) = 0$. That is, $x_T^{(2)} = 0$. Hence, we have that x is a niplotent element of T.
(3) ⇒ (4): Let x be a zero divisor of T. There exists $y \in (0, a)$ satisfying $T(x, y) = 0$. (i) If $x \neq y$, we assume that $x < y$. Then $T(x, x) \leq T(x, y) = 0$. So T is not strict; (ii) If $x = y$, then $T(y, y) = T(x, y) = 0$, which follows $T(0, y) \leq T(y, y) = 0$. Hence, T is not strict.
(4) ⇒ (3): Let T be an not strict Et-norm on L. There exist $u, v, w \in [0, a]$, $u > 0$, $v < w$ satisfying $T(u, v) = T(u, w)$. Since $T(v, w) \leq v < w = T(a, w)$ and continuity of T, there exists $z \in [v, a]$ satisfying $v = T(z, w)$. Then $T(u, w) = T(u, v) = T(u, T(w, z)) = T(T(u, w), z)$. In addition, by mathematical induction, for all $n \in N$, $T(u, w) = T(T(u, w), z_T^{(n)})$. Finally, by continuity of T again, $T(u, w) = \lim_{n\to\infty} T(T(u, w), z_T^{(n)}) = T(T(u, w), \lim_{n\to\infty} z_T^{(n)}) = T(T(u, w), 0) = 0$. Hence, u and w are zero divisors. ■

Theorem III.11. Let S be a continuous co-Archimedean Et-conorm on L. The following statements are equivalent:
(1) There exists a co-nilpotent element of S;
(2) S is co-nilpotent;
(3) S has co-zero divisors;
(4) S is not strict.

Proof: (1) ⇒ (2): Let y be a co-nilpotent element of S. There exists $m \in N$ satisfying $y_S^{(m)} = a$. Since S is co-Archimedean, there exists $N_1 \in N$ satisfying $x_S^{(N_1)} > y$ for all $x \in L$. So $x_S^{(N_1 m)} > y_S^{(m)} = a$.
(2) ⇒ (1): It is obvious.
(1) ⇒ (3): Let x be a co-nilpotent element of S. There exists the smallest m such that $x_S^{(m)} = a$. That is, $S(x_S^{(m-1)}, x) = a$.
(3) ⇒ (1): Let x be a co-zero divisor of S. There exists $y \in (0, a)$ satisfying $S(x, y) = a$. Assume that $y \leq x$. Then $a = S(x, y) \leq S(x, x)$. That is, $x_S^{(2)} = a$. So we have that x is a co-niplotent element of S.
(3) ⇒ (4): Let x be a co-zero divisor of S. There exists $y \in (0, a)$ such that $S(x, y) = a$. (i) If $x \neq y$, we assume that $x < y$. We have $a = S(x, y) \leq S(y, y)$. So S is not strict; (ii) If $x = y$, $S(y, y) = S(x, y) = a$, which follows $S(y, y) \leq S(y, a) = a$.
(4) ⇒ (3): Let S be an not strict Et-conorm on L. There exist $u, v, w \in [0, a]$, $u < a$, $v > w$ satisfying $S(u, v) = S(u, w)$. Since $S(v, w) \geq v > w = S(w, 0)$ and continuity of S, there exists $z \in [0, v]$ satisfying $v = S(z, w)$. Then $S(u, w) = S(u, v) = S(u, S(w, z)) = S(S(u, w), z)$. In addition, by mathematical induction, for all $n \in N$, $S(u, w) = S(S(u, w), z_S^{(n)})$. Finally, by continuity of S again, $S(u, w) = \lim_{n\to\infty} S(S(u, w), z_S^{(n)}) = S(S(u, w), \lim_{n\to\infty} z_S^{(n)}) = S(S(u, w), a) = a$. Hence, u and w are co-zero divisors. ■

Definition III.12. A map $F : L^2 \to L$ is called an Et-submorm, if for each $x, y \in [0, a]$, it satisfies the following conditions:
(F1) $F(x, y) = F(y, x)$;
(F2) $F(F(x, y), z) = F(x, F(y, z))$;
(F3) if $x \leq y$, then $F(x, z) \leq F(y, z)$;
(F4) $F(x, y) \leq min(x, y)$.

Obviously, each Et-norm is an Et-subnorm, but not vice versa. For example, for $x, y \in [0, a]$, the map $f : L^2 \to L, (x, y) \mapsto 0$ is an Et-subnorm but not an Et-norm.

Definition III.13. A map $F' : L^2 \to L$ is called an Et-subcomorm, if for each $x, y \in [0, a]$, it satisfies the following conditions:
$(F'1)$ $F'(x, y) = F'(y, x)$;
$(F'2)$ $F'(F'(x, y), z) = F'(x, F'(y, z))$;
$(F'3)$ if $x \leq y$, then $F'(x, z) \leq F'(y, z)$;
$(F'4)$ $F'(x, y) \geq max(x, y)$.

Similarly, each Et-conorm is an Et-subconorm, but not vice versa. For example, for $x, y \in [0, a]$, the map $f' : L^2 \to L, (x, y) \mapsto a$ is an Et-subconorm but not an Et-conorm.

Proposition III.14. Let $F : L^2 \to L$ be an Et-subnorm. For all $x, y \in [0, b]$, the map $T : L^2 \to L$ defined by

$$T(x, y) = \begin{cases} F(x, y), & \text{if } (x, y) \in [0, b), \\ min(x, y), & x = b \text{ or } y = b. \end{cases}$$

is an Et-norm, where $b \in I(L)$ satisfying $F(b, b) = b$.

Proof: It is obvious. ■

Proposition III.15. Let $F' : L^2 \to L$ be an Et-subconorm. For all $x, y \in [0, b]$, the map $S : L^2 \to L$ defined by

$$S(x, y) = \begin{cases} F'(x, y), & \text{if } (x, y) \in (0, b], \\ max(x, y), & x = 0 \text{ or } y = 0. \end{cases}$$

is an Et-conorm, where $b \in I(L)$ satisfying $F'(b,b) = b$.

Proof: It is obvious. ■

Definition III.16. A binary operation φ on L is called a φ-operator connected with a given Et-norm $T : L^2 \to L$, if for each $x, y, z \in [0,a]$, it satisfies the following conditions:
(1) If $y \le z$, then $\varphi(x,y) \le \varphi(x,z)$;
(2) $T(\varphi(x,y),x) \le y$;
(3) $y \le \varphi(x, T(y,x))$.

Definition III.17. A binary operation φ' on L is called a φ'-operator connected with a given Et-conorm $S : L^2 \to L$, if for each $x, y, z \in [0,a]$, it satisfies the following conditions:
(1) If $y \le z$, then $\varphi'(x,y) \le \varphi'(x,z)$;
(2) $S(\varphi'(x,y),x) \ge y$;
(3) $\varphi'(x, S(y,x)) \le y$.

In the following, we will give some equivalent conditions of a φ-operator connected with an Et-norm T and a φ'-operator connected with an Et-conorm S.

Theorem III.18. Let T be an Et-norm on L and φ be a φ-operator connected with T. For all $x, y, z \in [0,a]$, the following conditions are equivalent:
(1) (i) If $y \le z$, then $\varphi(x,y) \le \varphi(x,z)$;
(ii) $T(\varphi(x,y),x) \le y$;
(iii) $y \le \varphi(x, T(y,x))$;
(2) $\varphi(x,y) = sup\{z \in [0,a] | T(z,x) \le y\}$;
(3) $T(z,x) \le y \Leftrightarrow z \le \varphi(x,y)$.

Proof: (1) $\Rightarrow$ (2): By (ii), $\varphi(x,y) \in \{z \in [0,a] | T(z,x) \le y\}$, which implies $\varphi(x,y) \le sup\{z \in [0,a] | T(z,x) \le y\}$. Let $\varphi(x,y) < sup\{z \in [0,a] | T(z,x) \le y\} = z_0$. We have $T(z_0,x) \le y$, which follows $z_0 \le \varphi(x, T(z_0,x)) \le \varphi(x,y) < z_0$, which is a contradiction.
(2) $\Rightarrow$ (3): It is obvious.
(3) $\Rightarrow$ (1): By (3), $T(\varphi(x,y),x) \le y$ iff $\varphi(x,y) \le \varphi(x,y)$. So (ii) holds. By (3) again, $y \le \varphi(x, T(y,x))$ iff $T(y,x) \le T(y,x)$. So (iii) holds. In addition, if $y \le z$, by (ii), we have $T(\varphi(x,y),x) \le y \le z$, which implies $T(\varphi(x,y),x) \le z$. By (3) again, $\varphi(x,y) \le \varphi(x,z)$. So (i) holds. ■

Theorem III.19. Let S be an Et-conorm on L and φ' be a φ'-operator connected with S. For all $x, y, z \in [0,a]$, the following conditions are equivalent:
(1) (i) If $y \le z$, then $\varphi'(x,y) \le \varphi'(x,z)$;
(ii) $S(\varphi'(x,y),x) \ge y$;
(iii) $\varphi'(x, S(y,x)) \le y$;
(2) $\varphi'(x,y) = inf\{z \in [0,a] | S(z,x) \ge y\}$;
(3) $S(z,x) \ge y \Leftrightarrow z \ge \varphi'(x,y)$.

Proof: (1) $\Rightarrow$ (2): By (ii), $\varphi'(x,y) \in \{z \in [0,a] | S(z,x) \ge y\}$, which implies $\varphi'(x,y) \ge inf\{z \in [0,a] | S(z,x) \ge y\}$. Let $\varphi'(x,y) > inf\{z \in [0,a] | S(z,x) \ge y\} = z_0$. Then $S(z_0,x) \ge y$, which follows $z_0 \ge \varphi'(x, S(z_0,x)) \ge \varphi'(x,y) > z_0$, which is a contradiction.
(2) $\Rightarrow$ (3): It is obvious.
(3) $\Rightarrow$ (1): By (3), $S(\varphi(x,y),x) \ge y$ iff $\varphi'(x,y) \ge \varphi'(x,y)$. So (ii) holds. By (3) again, $y \ge \varphi'(x, S(y,x))$ iff $S(y,x) \ge S(y,x)$. So (iii) holds. In addition, if $y \ge z$, by (ii), $S(\varphi'(x,y),x) \ge y \ge z$, which implies $S(\varphi'(x,y),x) \ge z$. By (3) again, $\varphi'(x,y) \ge \varphi'(x,z)$. So (i) holds. ■

In the following propositions, we will study the relationship between Et-norms (Et-conorm) and EBL-algebras (EMV-algebras).

Proposition III.20. Let $(L, \vee, \wedge, \odot, 0)$ be an EBL-algebra. We have that $\odot$ is an Et-norm on L.

Proof: Let b be an arbitrary element on L. By the notion of EBL-algebras, for all $x, y \in [0,b]$, we have that $\odot$ satisfies the following conditions:
(i) $x \odot y = y \odot x$;
(ii) $(x \odot y) \odot z = x \odot (y \odot z)$;
(iii) if $y \le z$, then $x \odot y \le x \odot z$;
(iv) $x \odot b = x$.
Hence, $\odot$ is an Et-norm on L. ■

Proposition III.21. Let $(L, \vee, \wedge, \oplus, 0)$ be an EMV-algebra. We have that $\oplus$ is an Et-conorm on L.

Proof: Let b be an arbitrary element on L. By the notion of EBL-algebras, for all $x, y \in [0,b]$, we have that $\oplus$ satisfies the following conditions:
(i) $x \oplus y = y \oplus x$;
(ii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
(iii) if $y \le z$, then $x \oplus y \le x \oplus z$;
(iv) $x \oplus 0 = x$.
Hence, $\oplus$ is an Et-conorm on L. ■

Proposition III.22. Let T be a continuous Et-norm on L and φ be a φ-operator connected with T. We have that $(L, \vee, \wedge, T, 0)$ is an EBL-algebra.

Proof: By Proposition II.9, $(L, \vee, \wedge, T, 0)$ satisfies the following conditions:
(1) $(L, \vee, \wedge, 0)$ forms a distributive lattice;
(2) $(L, T, 0)$ forms a commutative semigroup;
(3) Let b be the biggest element that satisfies $T(b,b) = b$ on L. Then the interval $[0,b]$ equipped with the operations $\vee, \wedge, \odot, \xrightarrow{b}$ and bounds $0, b$ forms a BL-algebra, where $x \le y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y$. In fact, it satisfies the following conditions:
(i) $(L, \vee, \wedge, 0, b)$ forms a bounded lattice;
(ii) (L, T, b) forms an abelian monoid;
(iii) for $x, y, z \in [0,b]$, $T(x,y) \le z$ iff $x \le \varphi(y,z)$;
(iv) For $x, y \in [0,b]$, by Lemma II.8, $x \wedge y = T(x, \varphi(x,y))$;
(v) For all $x, y \in [0,b]$, by Theorem III.18 (2), $\varphi(x,y) \vee \varphi(y,x) = b$.
Hence, $(L, \vee, \wedge, T, 0)$ is an EBL-algebra. ■

IV. n-roots of EBL-algebras

In this section, we aim to introduce the notion and some results of n-roots on EBL-algebras. In addition, we characterize some properties of n-strict EBL-algebras.

Definition IV.1. Let $n \in N$. A map $r : A \to A$ is called an n-root if for all $x \in A$, it satisfies the following conditions:
(R1) $r(x)^n = \overbrace{r(x) \odot r(x) \odot \cdots r(x)}^{n-times} = x$;
(R2) For all $y \in A$, $y^n \le x$ implies that $y \le r(x)$.

A 2-root can be called a square root. In addition, we define $r(x)^0 = x$ and $r^n = r \circ r^{n-1}$.

Remark IV.2. Let r be an n-root on A.
(1) r is a one-to-one map. For some $x_1, x_2 \in A$, if $r(x_1) = r(x_2)$, by (R1), we have $x_1 = (r(x_1))^n = (r(x_2))^n = x_2$.
(2) If there exists an n-root s on A, then $r(x) = s(x)$. Since $(r(x))^n \leq x$, by (R2), we have $r(x) \leq s(x)$. Similarly, we have $s(x) \leq r(x)$. Hence, $r(x) = s(x)$.
(3) Let $m \in N$. For all $x \in A$, $(r^m(x))^n = (r(r^{m-1}(x)))^n = r^{m-1}(x)$. Similarly, $(r^m(x))^{n^t} = (r^{m-1}(x))^{n^{(t-1)}} = \cdots = r^{m-t}(x)$.
(4) We define a map l: $A \to A$ by $l := r^n$. Then l is an n^n-root. For all $x \in A$, by (3), $(l(x))^{n^n} = (r^n(x))^{n^n} = r^{n-n}(x) = x$. So (R1) holds. In addition, let $y \in A$ satisfying $(y)^{n^n} \leq x$. By (R1), $(y)^{n^n} \leq x = (l(x))^{n^n} = (r^n(x))^{n^n}$. It implies that $y \leq r^n(x) = l(x)$, so (R2) holds.
(5) Let s be an m-root on A. For all $x \in A$, if $m \leq n$, then $s(x) \leq r(x)$. Since $(s(x))^n \leq (s(x))^m = x$, by (R2), $s(x) \leq r(x)$.

Proposition IV.3. Let r be an n-root on A and $a \in I(A)$. For all $x, y \in [0, a]$, the following statements hold:
(1) If $x \leq y$, then $r(x) \leq r(y)$;
(2) $x \leq x \vee r(0) \leq r(x)$;
(3) If $r(a) \leq a$, then $r(a) = a$;
(4) $r(x) \odot r(y) \leq r(x \odot y)$;
(5) $x \wedge y \leq r(x) \odot r(y)$. In addition, if $r(0) \geq a$, then $a = 0$;
(6) $x = r(x)$ iff $r(x) \in I(A)$;
(7) $x \leq r(x^n) \leq r(x)$;
(8) $x \wedge x^{-a} \leq r(0)$;
(9) If $r(y) \in [0, a]$, $r(y) \in \{x \wedge (x^{n-1} \xrightarrow{a} y) : x \in A\}$;
(10) $(r(x^n))^n = (r(x)^n)^n$;
(11) If $r(x), r(y) \leq a$, then $r(x) \xrightarrow{a} r(y) = r(x \xrightarrow{a} y) \wedge a$;
(12) $(r(x) \wedge r(x^{-a}))^2 \leq r(0)$ for each $x \in A$.

Proof: (1) By $r(x)^n = x \leq y$ and (R2), $r(x) \leq r(y)$.
(2) Clearly, $r(0)^n = 0 \leq x$. By (R2), $r(0) \leq r(x)$. So $x \vee r(0) \leq r(x)$.
(3) Since $r(a)^n = a \leq r(a)$, we have $r(a) = a$.
(4) By $(r(x) \odot r(y))^n = r(x)^n \odot r(y)^n = x \odot y$ and (R2), we have $r(x) \odot r(y) \leq r(x \odot y)$.
(5) By (1), we have $x \wedge y = r(x \wedge y)^n \leq r(x \wedge y) \odot r(x \wedge y) \leq r(x) \odot r(y)$. In addition, $a = a^n \leq r(0)^n = 0$.
(6) If $r(x) = x$, we have $r(x) = x = r(x)^n \leq r(x)^2 \leq r(x)$. It implies that $r(x)^2 = r(x)$. Conversely, if $r(x) \in I(A)$, we have $x = r(x)^n = r(x)$.
(7) Since $x^n \leq x^n$, we have $x \leq r(x^n)$. In addition, by (1), we have $r(x^n) \leq r(x)$.
(8) By $(x \wedge x^{-a})^n \leq (x \wedge x^{-a})^2 \leq x \odot x^{-a} = 0$ and (R2), we have $x \wedge x^{-a} \leq r(0)$.
(9) By $r(y)^n = y$, we have $r(y) \leq r(y)^{n-1} \xrightarrow{a} y$, which follows
$r(y) = r(y) \wedge (r(y)^{n-1} \xrightarrow{a} y) \in \{x \wedge (x^{n-1} \xrightarrow{a} y) : x \in A\}$.
(10) By (R1), $(r(x^n))^n = x^n = (r(x)^n)^n$.
(11) We have

$$
\begin{aligned}
& x \odot (r(x) \xrightarrow{a} r(y))^n \\
&= r(x)^n \odot (r(x) \xrightarrow{a} r(y))^n \\
&= (r(x) \wedge r(y))^n \\
&\leq r(y)^n \\
&= y.
\end{aligned}
$$

It implies that $(r(x) \xrightarrow{a} r(y))^n \leq x \xrightarrow{a} y$. By (R2), $r(x) \xrightarrow{a} r(y) \leq r(x \xrightarrow{a} y)$. Hence, $r(x) \xrightarrow{a} r(y) \leq r(x \xrightarrow{a} y) \wedge a$. Conversely, by (1) and (4), $r(x) \odot r(x \xrightarrow{a} y) \leq r(x \odot (x \xrightarrow{a} y)) = r(x \wedge y) \leq r(y)$. In addition,

$$
\begin{aligned}
& r(x) \odot (r(x \xrightarrow{a} y) \wedge a) \\
&= (r(x) \odot r(x \xrightarrow{a} y)) \wedge (r(x) \odot a) \\
&= r(x) \odot r(x \xrightarrow{a} y) \\
&\leq r(y).
\end{aligned}
$$

It implies that $r(x \xrightarrow{a} y) \wedge a \leq r(x) \xrightarrow{a} r(y)$.
(12) We have

$$
\begin{aligned}
& ((r(x) \wedge r(x^{-a}))^2)^n \\
&= (r(x) \wedge r(x^{-a}))^n \odot (r(x) \wedge r(x^{-a}))^n \\
&\leq r(x)^n \odot r(x^{-a})^n \\
&= x \odot x^{-a} \\
&= 0.
\end{aligned}
$$

It implies that $(r(x) \wedge r(x^{-a}))^2 \leq r(0)$. ■

Proposition IV.4. Let A be an EBL-algebra and $a \in I(A)$. For all $x, y \in [0, a]$, we have $x \odot y \leq (x \odot x) \vee (y \odot y)$.

Proof: We have

$$
\begin{aligned}
& x \odot y \\
&= (x \odot y) \odot a \\
&= (x \odot y) \odot ((x \xrightarrow{a} y) \vee (y \xrightarrow{a} x)) \\
&= ((x \odot y) \odot (x \xrightarrow{a} y)) \vee ((x \odot y) \odot (y \xrightarrow{a} x)) \\
&\leq (x \odot x) \vee (y \odot y).
\end{aligned}
$$

■

Proposition IV.5. Let r be a square-root on A and $a \in I(A)$. For all $x, y \in [0, a]$, the following statements hold:
(1) $r(x) \odot r(y) \leq x \vee y$;
(2) $r(x) \wedge r(y) = r(x \wedge y)$;
(3) If $r(x), r(y) \leq a$ and $y \leq r(x) \odot r(y)$, then $y \leq x$;
(4) $x \wedge (x \xrightarrow{a} 0) \leq r(0)$.

Proof: (1) By Proposition IV.4, we have $r(x) \odot r(y) \leq (r(x) \odot r(x)) \vee (r(y) \odot r(y)) = x \vee y$.
(2) By Proposition IV.3 (1), $r(x \wedge y) \leq r(x) \wedge r(y)$. In addition,

$$
\begin{aligned}
& (r(x) \wedge r(y)) \odot (r(x) \wedge r(y)) \\
&= (r(x) \odot r(x)) \wedge (r(x) \odot r(y)) \wedge (r(y) \odot r(x)) \wedge (r(y) \odot r(y)) \\
&= x \wedge y \wedge (r(x) \odot r(y)) \\
&\leq x \wedge y.
\end{aligned}
$$

Hence, $r(x) \wedge r(y) \leq r(x \wedge y)$.
(3) Let $r(x), r(y) \leq a$. Then

$$
\begin{aligned}
& y \\
&= y \wedge (r(x) \odot r(y)) \\
&= (r(y) \odot r(y)) \wedge (r(x) \odot r(y)) \\
&= r(y) \odot (r(x) \wedge r(y)) \\
&= r(y) \odot (r(y) \odot (r(y) \xrightarrow{a} r(x))) \\
&\leq r(x) \odot (r(y) \odot (r(y) \xrightarrow{a} r(x))) \\
&= r(x) \odot (r(x) \wedge r(y)) \\
&\leq r(x) \odot r(x) \\
&= x.
\end{aligned}
$$

(4) We have

$$
\begin{aligned}
& (x \wedge (x \xrightarrow{a} 0)) \odot (x \wedge (x \xrightarrow{a} 0)) \\
&= ((x \wedge (x \xrightarrow{a} 0)) \odot x) \wedge ((x \wedge (x \xrightarrow{a} 0)) \odot (x \xrightarrow{a} 0)) \\
&\leq ((x \xrightarrow{a} 0) \odot x) \wedge ((x \xrightarrow{a} 0) \odot x) \\
&= 0.
\end{aligned}
$$

It implies that $x \wedge (x \xrightarrow{a} 0) \leq r(0)$. ■

Proposition IV.6. Let r be an n-root on A and $a \in I(A)$. For all $x, y \in [0, a]$, we define $r_a : [0, a] \to [0, a]$ by $r_a(x) = r(x) \wedge a$. If $r(x) \leq a$, then r_a is an n-root.

Proof: Clearly, $r_a(x)^n = (r(x) \wedge a)^n = x$. In addition, if $y^n \le x$, we have $y \le r(x)$, which follows $y \le r(x) \wedge a = r_a(x)$. ■

Proposition IV.7. Let r be an n-root on A. For all $X \subseteq A$, we have $\bigwedge r(X) = r(\bigwedge X)$.

Proof: For all $x \in X$, by Proposition IV.3 (1), we have $r(\bigwedge X) \le r(x)$. So $r(\bigwedge X) \le \bigwedge r(x)$. Assume that $y \in A$ such that $y \le r(x)$. We have $y^n \le r(x)^n = x$. In addition, by (R2), we have $y^n \le \bigwedge X$ and $y \le r(\bigwedge X)$. So $\bigwedge r(x) \le r(\bigwedge X)$. Hence, $\bigwedge r(X) = r(\bigwedge X)$. ■

In the following, we will prove that $s \circ r : A \to A$ is an mn-root on the EBL-algebra A, where s is an m-root and r is an n-root on A.

Proposition IV.8. Let s be an m-root and r be an n-root on A. Then $s \circ r$ is an mn-root on A.

Proof: For all $x \in A$, we have $(s \circ r(x))^{mn} = (s(r(x))^m)^n = r(x)^n = x$. So (R1) holds. In addition, if $y \in A$ such that $y^{mn} \le x$, we have $y^m \le r(x)$ and $y \le s(r(x))$. So (R2) holds. Hence, $s \circ r : A \to A$ is an mn-root on A. ■

In the following, we will give some properties of n-roots connected with EBL-homomorphisms.

Proposition IV.9. Let r be an n-root on A_1 and $f : A_1 \to A_2$ be an EBL-homomorphism. We define $t : f(A_1) \to f(A_1)$ by $t(f(x)) = f(r(x))$. Then t is an n-root on $f(A_1)$.

Proof: For all $x \in A_1$, $t(f(x))^n = f(r(x))^n = f(r(x)^n) = f(x)$. So (R1) holds. In addition, let $x, y \in A_1$ such that $f(y)^n \le f(x)$. Then $f(y^n \xrightarrow{a} x) = f(y^n) \xrightarrow{f(a)} f(x) = f(a)$. In addition,

$$\begin{aligned} & y^n \xrightarrow{a} x \\ & \le r_a(y^n \xrightarrow{a} x) \\ & = r_a(y^n) \xrightarrow{a} r_a(x) \\ & \le r_a(y)^n \xrightarrow{a} r_a(x) \\ & = y \xrightarrow{a} r_a(x). \end{aligned}$$

So $f(a) = f(y^n \xrightarrow{a} x) \le f(y \xrightarrow{a} r_a(x)) \le f(a)$. Then $f(y \xrightarrow{a} r_a(x)) = f(a)$ and $f(y) \xrightarrow{f(a)} f(r_a(x)) = f(a)$. That is, $f(y) \le f(r_a(x)) \le f(r(x)) = t(f(x))$. Hence, (R2) holds. Finally, we prove that $f(x) = f(y)$ implies that $t(f(x)) = t(f(y))$. In fact, $f(r(x))^n = f(r(x)^n) = f(x) = f(y)$. By (R2), $t(f(x)) = f(r(x)) \le f(r(y)) = t(f(y))$. Similarly, $t(f(y)) = f(r(y)) \le f(r(x)) = t(f(x))$. ■

Corollary IV.10. Let I be an ideal of A and r be an n-root on A. We define a map $t : A/I \to A/I$ by $t(x/I) = r(x)/I$. Then t is an n-root on A/I.

Proof: For all $x \in A$, there exists an EBL-homomorphism $f : A \to A/I$ defined by $x \to x/I$. Then $t(f(x)) = t(x/I) = r(x)/I = f(r(x))$. By Proposition IV.9, we have that $t : A/I \to A/I$ is an n-root on A/I. ■

Let r_1: $A_1 \to A_1$ and r_2: $A_2 \to A_2$ be two n-roots. If $f : A_1 \to A_2$ is an EBL-homomorphism, for all $x \in A_1$, $f(r_1(x))^n = f(x)$ and $f(r_1(x)) \le r_2(f(x))$.

Definition IV.11. Let $f : A_1 \to A_2$ be an EBL-homomorphism, r_1: $A_1 \to A_1$ and r_2: $A_2 \to A_2$ be two n-roots. For all $x \in A_1$, we say f preserves n-roots if $f(r_1(x)) = r_2(f(x))$.

Theorem IV.12. Let $f : A_1 \to A_2$ be an EBL-homomorphism, r_1: $A_1 \to A_1$ and r_2: $A_2 \to A_2$ be two n-roots. We have f preserves n-roots iff $f(A_1)$ is closed under r_2.

Proof: Let f preserves n-roots. For all $f(x) \in f(A_1)$, we have $r_2(f(x)) = f(r_1(x)) \in f(A_1)$. So $f(A_1)$ is closed under r_2. Conversely, let $f(A_1)$ be closed under r_2. By Proposition IV.9, we have $t : f(A_1) \to f(A_1)$ defined by $t(f(x)) = f(r_1(x))$ is an n-root on $f(A_1)$. That is, $r_2|_{f(A_1)} = t$ is an n-root on $f(A_1)$. Hence, $r_2(f(x)) = t(f(x)) = f(r_1(x))$. ■

Definition IV.13. An EBL-algebra $(A, \vee, \wedge, \odot, 0)$ with an n-root $r : A \to A$ is called strict if, for all $b \ge r(0)$, we have $r_b(0)^{n-1} = r_b(0)^{-b}$, where $r_b(0)^{-b} = r_b(0) \xrightarrow{b} 0$.

In the following, we will characterize some properties of n-strict EBL-algebras.

Theorem IV.14. Each n-strict EBL-algebra has a top element.

Proof: Let r be a strict n-root on EBL-algebra A and $a \in I(A)$ such that $r(0) \le a$. We have $r_a(0)^{n-1} = r(0)^{n-1} \wedge a = r(0)^{n-1} = r(0) \xrightarrow{a} 0$. In addition, we choose $b \in I(A)$ such that $a \le b$. Similarly, we have $r_b(0)^{n-1} = r(0)^{n-1} \wedge b = r(0)^{n-1} = r(0) \xrightarrow{b} 0$. By Proposition II.5 (ii), we have $r(0)^{n-1} = r(0) \xrightarrow{b} 0 = (r(0) \xrightarrow{a} 0) \vee (a \xrightarrow{b} 0)$. That is, $a \xrightarrow{b} 0 \le r(0)^{n-1}$. Hence, we have $b = a \vee (a \xrightarrow{b} 0) \le a \vee r(0)^{n-1} \le a \vee a = a$. So $a = b$. Hence, a is the top element of A. ■

Corollary IV.15. (i) Let r be an n-root on an EBL-algebra $(A, \vee, \wedge, \odot, 0)$. If $b \in I(A)$ such that $r(0) \le b$ and r_b is strict, for each $a \in I(A)$ satisfying $r(0) \le a < b$, then r_a cannot be strict.
(ii) Each n-strict EBL-algebra is a n-strict BL-algebra.

Proof: (i) If r_a is strict, by Theorem IV.14, we have $a = b$, which is a contradiction.
(ii) It is obvious. ■

Proposition IV.16. Let r be an n-root on A. For all $m, p \in N$ such that $n = mp$, if $r(x^p) = r(x)^p$, then $r(x)^p$ is an m-root of A.

Proof: For all $x \in A$, we have $(r(x)^p)^m = r(x)^{mp} = r(x)^n = x$. In addition, let $y^m \le x$. Then $y^n \le x^p$. From r is an n-root, it follows that $y \le r(x^p) = r(x)^p$. Hence, $r(x)^p$ is an m-root of A. ■

In the following, we will give some properties of n-roots connected with filters.

Proposition IV.17. Let F be a filter of A and r be an n-root on A. For all $x \in F$, $\{r(x) | x \in F\} \subseteq F$.

Proof: By Proposition IV.3 (7), we have $x \le r(x)$. From the definition of filters, it follows that $\{r(x) | x \in F\} \subseteq F$.

■

Proposition IV.18. Let F be a filter of A and r be an n-root on A. For all $x, y \in F$, we have $r(x) \odot r(y) \in F$.

Proof: By Definition II.13, we have $x \odot y \in F$. In addition, from $x \odot y \leq r(x) \odot r(y)$, it follows that $r(x) \odot r(y) \in F$.

■

V. Conclusion

In this paper, we extended the notion of Et-norms (Et-conorms) as an extension of t-norms (t-conorms). We characterize some algebraic properties of Et-norms (Et-norms). In addition, we established an relationship between an Et-norm on chain L and EBL-algebra $(L, \vee, \wedge, T, 0)$. On the other hand, we also defined the notion of an n-root on EBL-algebras and studied their important properties. Some results on EBl-algebras are obvious. Particularly, we proved that each n-strict EBL-algebra is a BL-algebra.

In the following study, there are many valued topics can be studied: (1) Does there exists a classification of EBL-algebras with n-roots? (2) Is it possible to extend Et-norms to partial order lattice structure?

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