IAENG International Journal of Applied Mathematics

Conditions for Linearizing Fourth-Order Ordinary Differential Equations through Point Transformation and Applications

Supaporn Suksern, and Supakit Tammakun

Abstract—**This research encompasses two distinct investigations into the linearization of differential equations. The first study focuses on the reduction of nonlinear fourth-order ordinary differential equations to general linear equations through point transformations. Necessary and sufficient conditions for linearization are derived, accompanied by a detailed procedure for obtaining the linearizing transformations and the coefficients of the resulting linear equations. Illustrative examples are provided to demonstrate the efficacy of the linearization theorems in practical applications. In the second investigation, attention is directed towards the completed linearization problem of fourth-order ordinary differential equations using fiber preserving transformations, building upon the obtained results. A computational program is developed to verify linearity based on the obtained results. Furthermore, the research explores various applications that meet the established linearization criteria, including fourth-order ordinary differential equations, third-order ordinary differential equations under Riccati transformation, and third- and fourth-order partial differential equations under traveling wave solutions.**

Index Terms—**linearization problem, point transformation, fiber preserving, nonlinear ODE.**

I. Introduction

DIFFERENTIAL equations, particularly those that are nonlinear, play a crucial role in modeling complex phenomena across various scientific disciplines. Unlike linear equations, which maintain a straightforward relationship between variables, nonlinear equations introduce complexities due to their non-proportional outputs relative to their inputs. This nonlinearity often results in challenging equations that are not easily solvable using conventional methods.

One powerful approach to tackling nonlinear differential equations is to transform them into linear forms, where well-established solution techniques can be applied. This process, known as linearization, is a specialized aspect of the broader equivalence problem. The equivalence problem asks whether two differential equations, through a series of invertible transformations, can be deemed equivalent belonging to the same class of equations. Solving this problem involves determining the appropriate transformations, identifying invariants, and establishing criteria for equivalence.

Manuscript received September 9, 2024; revised February 10, 2025.

This work was supported in part by Faculty of Science, Naresuan University, Thailand under Grant P2567E010.

S. Suksern is an Associate Professor in the Research center for Academic Excellence in Mathematics, Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand. (corresponding author to provide phone: +6655963244; fax: +6655963201; e-mail: supapornsu@nu.ac.th).

S. Tammakun is a lecturer at the Faculty of Oriental Medicine, Chiang Rai College, Chiang Rai 57000, Thailand. (email: supakit.tammakun@crc.ac.th).

Linearization is a powerful technique used to convert nonlinear ordinary differential equations into linear ones, making them more tractable and easier to solve. The origins of this method date back to the work of Sophus Lie [1], who identified a class of second-order ordinary differential equations that could be reduced to linear forms using point transformations. He further demonstrated that any second-order equation could be transformed into a linear equation through contact transformations.

Following Lie's pioneering work, Liouville [2] and Tresse [3] expanded on the idea by employing relative invariants of equivalence groups under point transformations to solve equivalence problems in second-order equations. Their contributions helped lay the groundwork for modern linearization techniques. More recently, Suksern and Sawatdithep [4] refined these approaches by reducing second-order differential equations to general linear forms and applying their results to a range of nonlinear equations encountered in various scientific fields. Moreover, Sinkala [5] discussed the linearization of second-order ordinary differential equations through point transformations, utilizing symmetries to derive a general solution. This approach allows specific solutions to be obtained from the general solution via suitable transformations based on the symmetries of the equation.

The study of linearization extended to higher-order equations when Bocharov, Sokolov, and Svinolupov [6] tackled third-order ordinary differential equations using point transformations. Grebot [7] also investigated linearization for specific cases of third-order equations, and Ibragimov and Meleshko [8] advanced the field by introducing linearization criteria for both point and contact transformations, focusing on the Laguerre form. Afterwards, Al-Dweik [9] presents the necessary and sufficient conditions for linearization of third-order ordinary differential equations via point transformations, which involves the identically vanishing of specific relative invariants, ensuring equivalence to the normal simplest form. Later, Al-Dweik, Mustafa, Mahomed, and Alassar [10] addressed the linearization of third-order ordinary differential equations using the Cartan equivalence method, providing an invariant characterization and a procedure for constructing equivalent canonical forms through point transformations and auxiliary functions. Recent developments have continued to push the boundaries of linearization. Suksern and Sookcharoenpinyo [11] introduced a new procedure for linearizing third-order nonlinear equations, while Ibragimov, Meleshko, and Suksern [12] showed that all fourth-order equations that can be linearized by point transformations belong to the class of equations linear in the third-order derivative, again focusing on the Laguerre form. Afterwards, Lyakhov, Gerdt, and Michels [13] discussed algorithms for checking the linearizability of nonlinear ordinary differential equations, including fourth-order equations, through point transformations. They utilize Lie point symmetry algebra and differential Thomas decomposition to determine linearization conditions and transformations. Later, Dutt and Qadir [14] provided a classification for third-order ordinary differential equations by using a generalization of contact transformations and then extended that work to fourth-order equations using a generalization of the Lie-Bäcklund transformation. They demonstrated that there are at least four classes of fourth-order linearizable ordinary differential equations.

Despite these advancements, the complexity of nonlinear equations poses significant challenges for linearization, particularly for higher-order systems. Many researchers have focused on simpler cases, such as the Laguerre form, but a comprehensive framework that covers more general forms is needed for broader applications. Point transformations provide a promising approach for tackling this challenge, allowing fourth-order nonlinear ordinary differential equations to be transformed into general linear forms. This method has the potential to be applied to a wide range of nonlinear equations found in nature, offering a versatile and effective tool for solving complex differential equations.

II. Formulation of the Linearization Theorems

A. Obtaining Necessary Condition of Linearization

The first purpose of this research is to linearize the fourth-order ordinary differential equations

$$y^{(4)} = f\left(x, y, y', y'', y'''\right) \tag{1}$$

by using the point transformation

$$t = \varphi\left(x, y\right), \quad u = \psi\left(x, y\right). \tag{2}$$

The study starts with the necessary conditions for linearization. We obtained the general form of equation (1) that can be reduced to a linear equation via point transformation (2). The general linear fourth-order ordinary differential equation is written in the form

$$u^{(4)} + \nu(t)u''' + \omega(t)u'' + \alpha(t)u' + \beta(t)u + \gamma(t) = 0. \tag{3}$$

At the end, we attain two classes of equations candidating for linearization.

Let t and u be new independent and dependent variables, respectively. We get the following transformation of the derivatives

$$\begin{aligned}
u'(t) &= \frac{D_x\psi}{D_x\varphi} = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y} = P(x, y, y'),\\
u''(t) &= \frac{D_xP}{D_x\varphi} = \frac{P_x + y'P_y + y''P_{y'}}{\varphi_x + y'\varphi_y}\\
&= \frac{\Delta}{(\varphi_x + y'\varphi_y)^3}[y'' + \frac{1}{\Delta}(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y)y'^3\\
&\quad +...]\\
&= Q(x, y, y', y''),\\
u'''(t) &= \frac{D_xQ}{D_x\varphi} = \frac{Q_x + y'Q_y + y''Q_{y'} + y'''Q_{y''}}{\varphi_x + y'\varphi_y}\\
&= \frac{\Delta}{(\varphi_x + y'\varphi_y)^5}[(\varphi_x + y'\varphi_y)y''' - 3\varphi_y y''^2\\
&\quad +...]\\
&= R(x, y, y', y'', y'''),\\
u^{(4)}(t) &= \frac{D_xR}{D_x\varphi}\\
&= \frac{R_x + y'R_y + y''R_{y'} + y'''R_{y''} + y^{(4)}R_{y'''}}{\varphi_x + y'\varphi_y}\\
&= \frac{\Delta}{(\varphi_x + y'\varphi_y)^7}[(\varphi_x + y'\varphi_y)^2 y^{(4)}\\
&\quad -10(\varphi_x + y'\varphi_y)\varphi_y y'' y''' + ...],
\end{aligned} \tag{4}$$

where

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''} + y^{(4)}\frac{\partial}{\partial y'''}$$

is the total derivative. Here $\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0$ is Jacobian of the change of variables (2). From equation (4), we can see that the transformation (2) with the conditions $\varphi_y = 0$ and $\varphi_y \neq 0$ give two distinctly different candidates for linearization.

For $\varphi_y = 0$, we replace all results in equation (3) and derive the following equation:

$$\begin{aligned}
&y^{(4)} + (A_1y' + A_0)y''' + B_0y''^2 + (C_2y'^2 + C_1y'\\
&\quad + C_0)y'' + D_4y'^4 + D_3y'^3 + D_2y'^2 + D_1y'\\
&\quad + D_0 = 0,
\end{aligned} \tag{5}$$

where $A_i = A_i(x, y), B_i = B_i(x, y), C_i = C_i(x, y)$ and $D_i = D_i(x, y)$ are arbitrary functions of x, y, as illustrated in Appendix, equations (A.1)-(A.11).

For $\varphi_y \neq 0$, we have done in similar way. Setting $r(x, y) = \frac{\varphi_x}{\varphi_y}$, we derive the following equation:

$$\begin{aligned}
&y^{(4)} + \frac{1}{y' + r}(-10y'' + F_2{y'}^2 + F_1y' + F_0)y'''\\
&\quad + \frac{1}{(y' + r)^2}[15{y''}^3 + (H_2{y'}^2 + H_1y' + H_0){y''}^2\\
&\quad + (J_4{y'}^4 + J_3{y'}^3 + J_2{y'}^2 + J_1y' + J_0)y''\\
&\quad + K_7{y'}^7 + K_6{y'}^6 + K_5{y'}^5 + K_4{y'}^4 + K_3{y'}^3\\
&\quad + K_2{y'}^2 + K_1y' + K_0] = 0,
\end{aligned} \tag{6}$$

where $F_i = F_i(x, y), H_i = H_i(x, y), J_i = J_i(x, y)$ and $K_i = K_i(x, y)$ are arbitrary functions of x, y, as illustrated in Appendix, equations (B.1)-(B.19).

Theorem 2.1: Any fourth-order ordinary differential equation linearizable by a point transformation has to be one of the forms either equation (5) or (6).

B. Obtaining Sufficient Conditions of Linearization, Linearizing Transformation and Coefficients of Linear Equation

B.1 The First Class of Linearizable Equations

In this case $\varphi_y = 0$, the transformation (2) becomes a fiber preserving transformation, that is

$$t = \varphi(x), \quad u = \psi(x, y). \tag{7}$$

For obtaining the sufficient conditions, one has to solve the compatibility problem. Consider the representations of the coefficients A_i, B_i, C_i and D_i through the unknown functions φ and ψ. We first rewrite the expressions (A.1) for A_1 in the form

$$\psi_{yy} = (\psi_y A_1)/4. \tag{8}$$

One can determine ν from equation (A.2) as

$$\nu = (6\varphi_{xx}\psi_y - 4\varphi_x\psi_{xy} + \varphi_x\psi_y A_0)/(\varphi_x^2\psi_y). \tag{9}$$

Since $\varphi = \varphi(x)$ we have $\nu_y = 0$ yields

$$A_{0y} = A_{1x}. \tag{10}$$

From equations (A.3), (A.4), (A.5), (A.7) and (A.8) one gets the conditions

$$B_0 =(3A_1)/4, \tag{11}$$
$$A_{1y} =-(3A_1^2 - 8C_2)/12, \tag{12}$$
$$A_{1x} =-(3A_0A_1 - 4C_1)/12, \tag{13}$$
$$C_{2y} =-(A_1C_2 - 24D_4)/4, \tag{14}$$
$$C_{1y} =-(A_1C_1 - 12D_3)/4. \tag{15}$$

One can determine ω from equation (A.6) as the following

$$\begin{aligned}\omega =&(4\varphi_{xxx}\varphi_x\psi_y^2 + 3\varphi_{xx}^2\psi_y^2 - 12\varphi_{xx}\varphi_x\psi_{xy}\psi_y \\ &+ 3\varphi_{xx}\varphi_x\psi_y^2A_0 + 12\varphi_x^2\psi_{xy}^2 - 3\varphi_x^2\psi_{xy}\psi_yA_0 \\ &- 6\varphi_x^2\psi_{xxx}\psi_y + \varphi_x^2\psi_y^2C_0)/(\varphi_x^4\psi_y^2).\end{aligned} \tag{16}$$

Since $\varphi = \varphi(x)$ we have $\omega_y = 0$ yields

$$C_{0y} = -((3A_0A_1 - 4C_1)A_0 - 16C_{1x} + 12A_{0x}A_1)/32. \tag{17}$$

From equations (A.9) one gets the condition

$$\begin{aligned}C_{1x} =&(12A_{0x}A_1 + 3A_0^2A_1 - 4A_0C_1 - 8A_1C_0 \\ &+ 32D_2)/16.\end{aligned} \tag{18}$$

One can determine α from equation (A.10) as the following

$$\begin{aligned}\alpha =&(\varphi_{xxxx}\psi_y^3 - 4\varphi_{xxx}\psi_{xy}\psi_y^2 + \varphi_{xxx}\psi_y^3A_0 \\ &+ 12\varphi_{xx}\psi_{xy}^2\psi_y - 3\varphi_{xx}\psi_{xy}\psi_y^2A_0 \\ &- 6\varphi_{xx}\psi_{xxx}\psi_y^2 + \varphi_{xx}\psi_y^3C_0 - 24\varphi_x\psi_{xy}^3 \\ &+ 6\varphi_x\psi_{xy}^2\psi_yA_0 + 24\varphi_x\psi_{xy}\psi_{xxx}\psi_y \\ &- 2\varphi_x\psi_{xy}\psi_y^2C_0 - 4\varphi_x\psi_{xxxy}\psi_y^2 \\ &- 3\varphi_x\psi_{xxx}\psi_y^2A_0 + \varphi_x\psi_y^3D_1)/(\varphi_x^4\psi_y^3).\end{aligned} \tag{19}$$

Since $\varphi = \varphi(x)$ we have $\alpha_y = 0$ yields

$$\begin{aligned}D_{1y} =&(36A_{0x}A_0A_1 - 48A_{0x}C_1 - 48C_{0x}A_1 \\ &+ 192D_{2x} + 9A_0^3A_1 - 12A_0^2C_1 \\ &- 36A_0A_1C_0 + 48A_0D_2 + 32C_0C_1)/288.\end{aligned} \tag{20}$$

One can determine β from equation (A.11) as the following

$$\begin{aligned}\beta =&-(\varphi_x^4\psi_y^3\gamma - 24\psi_{xy}^3\psi_x + 12\psi_{xy}^2\psi_{xx}\psi_y \\ &+ 6\psi_{xy}^2\psi_x\psi_yA_0 - 4\psi_{xy}\psi_{xxx}\psi_y^2 \\ &+ 24\psi_{xy}\psi_{xxx}\psi_x\psi_y - 3\psi_{xy}\psi_{xx}\psi_y^2A_0 \\ &- 2\psi_{xy}\psi_x\psi_y^2C_0 + \psi_{xxxx}\psi_y^3 - 4\psi_{xxxy}\psi_x\psi_y^2 \\ &+ \psi_{xxx}\psi_y^3A_0 - 6\psi_{xxx}\psi_{xx}\psi_y^2 - 3\psi_{xxx}\psi_x\psi_y^2A_0 \\ &+ \psi_{xx}\psi_y^3C_0 + \psi_x\psi_y^3D_1 - \psi_y^4D_0)/(\varphi_x^4\psi_y^3\psi).\end{aligned} \tag{21}$$

Since $\varphi = \varphi(x)$ we have $\beta_y = 0$ yields

$$\begin{aligned}\gamma =&((4((3\psi_xA_0 + C_0\psi + 6\psi_{xx})\psi_y - 6\psi_{xxx}\psi)\psi_{xxx} \\ &+ ((4\psi_y - A_1\psi)D_0 - 4\psi_xD_1 - 4\psi_{xx}C_0)\psi_y^2 \\ &- 4\psi_{xxx}\psi_y^2A_0 + 4(4\psi_x + A_0\psi)\psi_{xxxy}\psi_y \\ &- 4\psi_{xxxx}\psi_y^2 + 4\psi_{xxxxy}\psi_y\psi)\psi_y^2 + 4(((2\psi_xC_0 \\ &+ D_1\psi + 3\psi_{xx}A_0)\psi_y - 6(4\psi_x + A_0\psi)\psi_{xxx} \\ &+ 4\psi_{xxx}\psi_y - 8\psi_{xxxy}\psi)\psi_y^2 - 2(((3\psi_xA_0 + C_0\psi \\ &+ 6\psi_{xx})\psi_y - 18\psi_{xxx}\psi)\psi_y - 3((4\psi_x + A_0\psi)\psi_y \\ &- 4\psi_{xy}\psi)\psi_{xy})\psi_{xy})\psi_{xy} - 4D_{0y}\psi_y^4\psi)/(4\varphi_x^4\psi_y^4).\end{aligned} \tag{22}$$

Since $\varphi = \varphi(x)$ we have $\gamma_y = 0$ yields

$$\begin{aligned}D_{0yy} =&(36A_{0xx}A_0A_1 - 48A_{0xx}C_1 - 18A_{0x}A_0^2A_1 \\ &+ 24A_{0x}A_0C_1 + 48A_{0x}A_1C_0 - 192A_{0x}D_2 \\ &- 48C_{0xx}A_1 - 32C_{0x}C_1 - 288D_{0y}A_1 \\ &+ 192D_{2xx} + 96D_{2x}A_0 - 9A_0^4A_1 + 12A_0^3C_1 \\ &+ 48A_0^2A_1C_0 - 48A_0^2D_2 - 72A_0A_1D_1 \\ &- 48A_0C_0C_1 + 72A_1^2D_0 - 40A_1C_0^2 \\ &+ 160C_0D_2 + 96C_1D_1 - 192C_2D_0)/1152.\end{aligned} \tag{23}$$

All obtained results can be summarized in the following theorems.

Theorem 2.2: Sufficient conditions for equation (5) to be linearizable via the fiber preserving transformation (7) are equations (10), (11), (12), (13), (14), (15), (17), (18), (20) and (23).

Corollary 2.3: Provided that the sufficient conditions in Theorem 2.2 are satisfied, the transformation (7) mapping equation (5) to a linear equation (3) is obtained by solving the compatible system of equation $\varphi_y = 0$ and equation (8) for functions $\varphi(x)$ and $\psi(x, y)$. Finally, the coefficients ν, ω, α, β and γ of the resulting linear equation (3) are given by equations (9), (16), (19), (21) and (22).

The following is an example of how to apply the theorem that has been develop.

Example 2.4: Consider the nonlinear ordinary differential equation [12]

$$\begin{aligned}&x^2y(2y^{(4)} + y) + 8x^2y'y''' + 16xyy''' \\ &+6x^2y''^2 + 48xy'y'' + 24yy'' + 24y'^2 = 0.\end{aligned} \tag{24}$$

It is an equation of the form (5) in Theorem 2.1 with the coefficients

$$A_1 = \tfrac{4}{y},\ A_0 = \tfrac{8}{x},\ B_0 = \tfrac{3}{y},\ C_2 = 0,$$
$$C_1 = \tfrac{24}{xy},\ C_0 = \tfrac{12}{x^2},\ D_4 = 0,\ D_3 = 0,$$
$$D_2 = \tfrac{12}{x^2y},\ D_1 = 0,\ D_0 = \tfrac{y}{2}.$$

One can check that these coefficients obey the conditions in Theorem 2.2. Hence, an equation (24) is linearizable via a fiber preserving transformation. Applying Corollary 2.3, the linearizing transformation is found by solving the following equations

$$\varphi_y = 0, \quad \psi_{yy} = \frac{\psi_y}{y}. \tag{25}$$

One can find the particular solution for equations in (25) as

$$\varphi = x, \quad \psi = x^2y^2.$$

So that, one obtains the linearizing transformation

$$t = x, \quad u = x^2y^2. \tag{26}$$

From Corollary 2.3, the coefficients ν, ω, α, β and γ of the resulting linear equation (3) are

$$\nu = 0,\ \omega = 0,\ \alpha = 0,\ \beta = 1,\ \gamma = 0.$$

Hence, the nonlinear equation (24) can be mapped by transformation (26) into the linear equation

$$u^{(4)} + u = 0. \tag{27}$$

The solution of equation (27) is

$$\begin{aligned} u(t) =& (C_0 \cos\frac{t}{\sqrt{2}} + C_1 \sin\frac{t}{\sqrt{2}})e^{\frac{t}{\sqrt{2}}} \\ &+ (C_2 \cos\frac{t}{\sqrt{2}} + C_3 \sin\frac{t}{\sqrt{2}})e^{-\frac{t}{\sqrt{2}}}, \end{aligned} \tag{28}$$

where C_0, C_1, C_2 and C_3 are arbitrary constants. Substituting the transformation (26) into equation (28), we obtain the nonlinear solution is

$$\begin{aligned} x^2y^2 =& (C_0 \cos\frac{x}{\sqrt{2}} + C_1 \sin\frac{x}{\sqrt{2}})e^{\frac{x}{\sqrt{2}}} \\ &+ (C_2 \cos\frac{x}{\sqrt{2}} + C_3 \sin\frac{x}{\sqrt{2}})e^{-\frac{x}{\sqrt{2}}}. \end{aligned}$$

B.2 The Second Class of Linearizable Equations

In this case, the problem is formulated as follows. Given the coefficients F_i, H_i, J_i and K_i of equation (6), find the necessary and sufficient conditions for integrability of the overdetermined system of equations (B.1)-(B.19) for the unknown functions $\varphi(x, y)$ and $\psi(x, y)$.

Recall that according to our notations, the following equations hold

$$\varphi_x = r\varphi_y, \tag{29}$$

$$\psi_x = (\varphi_y\psi_y r - \Delta)/\varphi_y, \tag{30}$$

and

$$\nu_x = (\varphi_x\nu_y)/\varphi_y, \tag{31}$$

$$\omega_x = (\varphi_x\omega_y)/\varphi_y, \tag{32}$$

$$\alpha_x = (\varphi_x\alpha_y)/\varphi_y, \tag{33}$$

$$\beta_x = (\varphi_x\beta_y)/\varphi_y, \tag{34}$$

$$\gamma_x = (\varphi_x\gamma_y)/\varphi_y. \tag{35}$$

One can determine ν from equation (B.1) as the following

$$\nu = ((10\varphi_{yy} + \varphi_y F_2)\Delta - 4\varphi_y\Delta_y)/(\varphi_y^2\Delta). \tag{36}$$

Substituting ν into equation (31), one obtains the derivative

$$\begin{aligned} \Delta_{xy} =& (4((\Delta_{yy}\Delta - \Delta_y^2)r + \Delta_x\Delta_y) + (4\Delta_y \\ &- F_2\Delta)r_y\Delta + 10r_{yy}\Delta^2 - F_{2y}r\Delta^2 \\ &+ F_{2x}\Delta^2)/(4\Delta). \end{aligned} \tag{37}$$

Equation (B.2) provides the derivative

$$\Delta_x = (20r_y\Delta + 4\Delta_y r + F_1\Delta - 2F_2r\Delta)/4. \tag{38}$$

Substituting equation (38) into equation (37), one gets the condition

$$r_{yy} = -(F_{1y} - F_{2x} - F_{2y}r - r_yF_2)/10. \tag{39}$$

From equations (B.3), (B.4), (B.5) and (B.6), one obtains the following conditions

$$r_x = (10r_yr - F_0 + F_1r - F_2r^2)/10, \tag{40}$$

$$H_2 = -3F_2, \tag{41}$$

$$H_1 = -3(5F_1 - 2F_2r)/4, \tag{42}$$

$$H_0 = -3(6F_0 - F_1r)/4. \tag{43}$$

One can determine ω from equation (B.7) as equation (C.1). Substituting ω from equation (C.1) into equation (32), one obtains the condition (C.2). Equations (B.8), (B.9), (B.10) and (B.11) provide conditions (C.3), (C.4), (C.5), (C.6). One can determine γ and α from equations (B.12)-(B.13) as equations (C.7)-(C.8). Substituting α from equation (C.8) into equation (33), and γ from equation (C.7) into equation (35), one obtains conditions (C.9) and (C.10). Substituting β from equation (C.10) into equation (34), one obtains the condition (C.11). From equation (B.14), one obtains condition (C.12). Substituting the relation J_{4x} from equation (C.12) into J_{4xyy} in equation (C.11), one obtains condition (C.13). From equations (B.15), (B.16), (B.17), (B.18) and (B.19), one obtains the following conditions (C.14)-(C.18).

Theorem 2.5: Sufficient conditions for equation (6) to be linearizable via the point transformation (2) are equations (39), (40), (41), (42), (43), (C.2), (C.3), (C.4), (C.5), (C.6), (C.9), (C.12), (C.13), (C.14), (C.15), (C.16), (C.17), (C.18).

Corollary 2.6: Provided that the sufficient conditions in Theorem 2.5 are satisfied, the transformation (2) mapping equation (6) to a linear equation (3) is obtained by solving the compatible system of equations (29), (30) and (38) for the functions $\varphi(x, y)$ and $\psi(x, y)$. Finally, the coefficients $\nu, \omega, \alpha, \beta$ and γ of the resulting linear equation (3) are given by equations (36), (C.1), (C.7), (C.8), (C.10).

Example 2.7: Consider the nonlinear ordinary differential equation

$$\begin{aligned} &16y'^7y + 112y'^6y + 336y'^5y - 8y'^4y'' \\ &+560y'^4y - 32y'^3y'' + 560y'^3y - 48y'^2y'' \\ &+y'^2y^{(4)} + 336y'^2y - 10y'y''y''' - 32y'y'' \\ &+2y'y^{(4)} + 112y'y + 15y''^3 - 10y''y''' \\ &-8y'' + y^{(4)} + 16y = 0. \end{aligned} \tag{44}$$

It is an equation of the form (6) in Theorem 2.1 with the coefficients

$$\begin{aligned} &r = 1,\ F_2 = 0,\ F_1 = 0,\ F_0 = 0, \\ &H_2 = 0,\ H_1 = 0,\ H_0 = 0,\ J_4 = -8, \\ &J_3 = -32,\ J_2 = -48,\ J_1 = -32, \\ &J_0 = -8,\ K_7 = 16y,\ K_6 = 112y, \\ &K_5 = 336y,\ K_4 = 560y,\ K_3 = 560y, \\ &K_2 = 336y,\ K_1 = 112y,\ K_0 = 16y. \end{aligned} \tag{45}$$

One can check that these coefficients obey the conditions in Theorem 2.5. Hence an equation (44) is linearizable via a point transformation. Applying Corollary 2.6, the linearizing transformation is found by solving the following equations

$$\varphi_x = \varphi_y, \quad \psi_x = \frac{\varphi_y\psi_y - \Delta}{\varphi_y}, \quad \Delta_x = 0. \tag{46}$$

One can find the particular solution for equations in (46) as

$$\varphi = x + y, \quad \psi = y, \quad \Delta = 1.$$

So that, one obtains the linearizing transformation

$$t = x + y, \ u = y. \tag{47}$$

From Corollary 2.6, the coefficients ν, ω, α, β and γ of the resulting linear equation (3) are

$$\nu = 0, \quad \omega = 8, \quad \alpha = 0, \quad \beta = 16, \quad \gamma = 0.$$

Hence, the nonlinear equation (44) can be mapped by transformation (47) into the linear equation

$$u^{(4)} + 8u'' + 16u = 0. \tag{48}$$

The solution of equation (48) is

$$u = C_0 \cos 2t + C_1 \sin 2t + tC_2 \cos 2t + tC_3 \sin 2t, \tag{49}$$

where C_0, C_1, C_2 and C_3 arbitrary constants. Substituting the transformation (47) into equation (49), we get the nonlinear solution

$$\begin{aligned} y =& C_0 \cos 2(x+y) + C_1 \sin 2(x+y) \\ &+ (x+y)C_2 \cos 2(x+y) \\ &+ (x+y)C_3 \sin 2(x+y). \end{aligned}$$

III. Some Applications

The second purpose is to find some applications which satisfy the obtained theorems in section II. The obtained results are as follows.

A. Linearization for Some Interesting Third-Order Ordinary Differential Equations Under the Riccati Transformation

Example 3.1: Equation in the article [15]
The third-order member of the Riccati hierarchy is given by Euler et al [15] as

$$y''' + 8yy'' + 6y'^2 + 24y^2y' + 8y^4 = 0. \tag{50}$$

Applying [16] one checks that the equation cannot be linearized by a point transformation or contact transformation or generalized Sundman transformation. Under the Riccati transformation $y = \frac{a\omega'}{\omega}$ equation (50) become

$$\begin{aligned} &\omega^{(4)}\omega^3 + 8\omega'''\omega' a\omega^2 - 4\omega'''\omega'\omega^2 + 6\omega''^2 a\omega^2 \\ &-3\omega''^2\omega^2 + 24\omega''\omega'^2 a^2\omega - 36\omega''\omega'^2 a\omega + 12\omega''\omega'^2\omega \\ &+8\omega'^4 a^3 - 24\omega'^4 a^2 + 22\omega'^4 a - 6\omega'^4 = 0. \end{aligned} \tag{51}$$

It is an equation of the form (5) in Theorem 2.1 with the coefficients

$$\begin{aligned} &A_1 = \frac{4(2a-1)}{\omega}, \ A_0 = 0, \ B_0 = \frac{3(2a-1)}{\omega}, \\ &C_2 = \frac{12(2a^2-3a+1)}{\omega^2}, \ C_1 = 0, \ C_0 = 0, \\ &D_4 = \frac{2(4a^3-12a^2+11a-3)}{\omega^3}, \ D_3 = 0, \\ &D_2 = 0, \ D_1 = 0, \ D_0 = 0. \end{aligned}$$

One can check that these coefficients obey the conditions in Theorem 2.2. Hence, an equation (50) is linearizable via a fiber preserving transformation. Applying Corollary 2.3, the linearizing transformation is found by solving the following equations

$$\varphi_\omega = 0, \quad \psi_{\omega\omega} = \frac{\psi_\omega(2a-1)}{\omega}. \tag{52}$$

One can find the particular solution for equations in (52) as

$$\varphi = x, \quad \psi = \omega^{2a}.$$

So that, one obtains the linearizing transformation

$$t = x, \quad u = \omega^{2a}. \tag{53}$$

From Corollary 2.3, the coefficients $\tilde{\nu}, \tilde{\omega}, \tilde{\alpha}, \tilde{\beta}$ and $\tilde{\gamma}$ of the resulting linear equation (3) are

$$\tilde{\nu} = 0, \ \tilde{\omega} = 0, \ \tilde{\alpha} = 0, \ \tilde{\beta} = 0, \ \tilde{\gamma} = 0.$$

Hence, the nonlinear equation (51) can be mapped by transformation (53) into the linear equation

$$u^{(4)} = 0.$$

So that,

$$u = C_0 + C_1 t + C_2 t^2 + C_3 t^3, \tag{54}$$

where C_0, C_1, C_2 and C_3 are arbitrary constants. Substituting equation (53) into equation (54), we get

$$\omega^{2a} = C_0 + C_1 x + C_2 x^2 + C_3 x^3.$$

So that,

$$\omega = (C_0 + C_1 x + C_2 x^2 + C_3 x^3)^{\frac{1}{2a}}.$$

Hence, the original nonlinear solution is

$$y = \frac{1}{2}[\frac{C_1 + 2C_2 x + 3C_3 x^2}{C_0 + C_1 x + C_2 x^2 + C_3 x^3}].$$

Example 3.2: Equation in the article [17]

- **The significance of the problem**

In [17], Euler and Euler considered the equation

$$y''' - a_1 yy'' - a_2 y'^2 - a_3 y^2 y' - a_4 y^4 = 0, \tag{55}$$

where a_i are constants. They found that this equation can be linearized to $u''' = 0$ under a generalized sundman transformation.

- **Applying the obtained theorems to the problem**

Let us consider the nonlinear third-order ordinary differential equation (55), under the Riccati transformation $y = \frac{a\omega'}{\omega}$, equation (55) becomes

$$\begin{aligned} &\omega^{(4)}\omega^3 - \omega'''\omega' a a_1 \omega^2 - 4\omega'''\omega'\omega^2 - \omega''^2 a a_2 \omega^2 \\ &- 3\omega''^2\omega^2 - \omega''\omega'^2 a^2 a_3 \omega + 3\omega''\omega'^2 a a_1 \omega \\ &+ 2\omega''\omega'^2 a a_2 \omega + 12\omega''\omega'^2\omega - \omega'^4 a^3 a_4 \\ &+ \omega'^4 a^2 a_3 - 2\omega'^4 a a_1 - \omega'^4 a a_2 - 6\omega'^4 = 0. \end{aligned} \tag{56}$$

It is an equation of the form (5) in Theorem 2.1 with the coefficients

$$\begin{aligned} &A_1 = -\frac{4(aa_2+3)}{3\omega}, \ A_0 = 0, \ B_0 = -\frac{aa_2+3}{\omega}, \\ &C_2 = \frac{2(a^2{a_2}^2 + 9aa_2 + 18)}{3\omega^2}, \ C_1 = 0, \ C_0 = 0, \\ &D_4 = -\frac{a^3{a_2}^3 + 18a^2{a_2}^2 + 99aa_2 + 162}{27\omega^3}, \\ &D_3 = 0, \ D_2 = 0, \ D_1 = 0, \ D_0 = 0. \end{aligned}$$

One can check that the equations (10), (13), (15), (17), (18), (20) and (23) in Theorem 2.2 are satisfied. Now, the equations (11), (12) and (14) are satisfied when the followings conditions hold, that is

$$a_1 = \frac{4a_2}{3},\ a_3 = -\frac{2{a_2}^2}{3},\ a_4 = \frac{a_2^3}{27}.$$

Applying Corollary 2.3, the linearizing transformation is found by solving the following equations

$$\varphi_\omega = 0, \quad \psi_{\omega\omega} = -\frac{(aa_2+3)\psi_\omega}{3\omega}. \tag{57}$$

One can find the particular solution for equations in (57) as

$$\varphi = x, \quad \psi = \omega^{-\frac{aa_2}{3}}.$$

So that, one obtains the linearizing transformation

$$t = x, \quad u = \omega^{-\frac{aa_2}{3}}. \tag{58}$$

From Corollary 2.3, the coefficients $\tilde{\nu}, \tilde{\omega}, \tilde{\alpha}, \tilde{\beta}$ and $\tilde{\gamma}$ of the resulting linear equation (3) are

$$\tilde{\nu} = 0,\ \tilde{\omega} = 0,\ \tilde{\alpha} = 0,\ \tilde{\beta} = 0,\ \tilde{\gamma} = 0.$$

Hence, the nonlinear equation (56) can be mapped by transformation (58) into the linear equation

$$u^{(4)} = 0.$$

So that,

$$u = C_0 + C_1 t + C_2 t^2 + C_3 t^3, \tag{59}$$

where C_0, C_1, C_2 and C_3 are arbitrary constants. Substituting equation (58) into equation (59), we get

$$\omega^{-\frac{aa_2}{3}} = C_0 + C_1 x + C_2 x^2 + C_3 x^3.$$

So that,

$$\omega = (C_0 + C_1 x + C_2 x^2 + C_3 x^3)^{-\frac{3}{aa_2}}.$$

Hence, the original nonlinear solution is

$$y = -\frac{3}{a_2}\left[\frac{C_1 + 2C_2 x + 3C_3 x^2}{C_0 + C_1 x + C_2 x^2 + C_3 x^3}\right].$$

Example 3.3: Equation in the article [18]

- **The significance of the problem**

In [18], Guha, Choudhury, and Khanra proved that Painlevé-type differential equations of the third-order in the polynomial class must take the form

$$\begin{aligned} &y''' + (K_1 y + K_5(x))y'' + K_2 y'^2 + (K_3 y^2 \\ &+ K_6(x)y + K_7(x))y' + K_4 y^4 + K_8(x)y^3 \\ &+ K_9(x)y^2 + K_{10}(x)y + K_{11}(x) = 0, \end{aligned} \tag{60}$$

where $K_i, i = 1, ..., 4$ are certain rational or algebraic numbers and $K_j, j = 5, ..., 11$ are locally analytic functions of the complex variable x.

- **Applying the obtained theorems to the problem**

Let us consider the nonlinear third-order ordinary differential equation (60), under the Riccati transformation $y = \frac{a\omega'}{\omega}$, equation (60) become

$$\begin{aligned} &\omega^{(4)} a\omega^3 + \omega'''\omega' a^2\omega^2 K_1 - 4\omega'''\omega' a\omega^2 \\ &+ \omega''' a\omega^3 K_5 + \omega''^2 a^2\omega^2 K_2 - 3\omega''^2 a\omega^2 \\ &+ \omega''\omega'^2 a^3\omega K_3 - 3\omega''\omega'^2 a^2\omega K_1 - 2\omega''\omega'^2 a^2\omega K_2 \\ &+ 12\omega''\omega'^2 a\omega + \omega''\omega' a^2\omega^2 K_6 - 3\omega''\omega' a\omega^2 K_5 \\ &+ \omega'' a\omega^3 K_7 + \omega'^4 a^4 K_4 - \omega'^4 a^3 K_3 + 2\omega'^4 a^2 K_1 \\ &+ \omega'^4 a^2 K_2 - 6\omega'^4 a + \omega'^3 a^3\omega K_8 - \omega'^3 a^2\omega K_6 \\ &+ 2\omega'^3 a\omega K_5 + \omega'^2 a^2\omega^2 K_9 - \omega'^2 a\omega^2 K_7 \\ &+ \omega' a\omega^3 K_{10} + \omega^4 K_{11} = 0. \end{aligned} \tag{61}$$

It is an equation of the form (5) in Theorem 2.1 with the coefficients

$$\begin{aligned} &A_1 = \frac{4(aK_2-3)}{3\omega},\ A_0 = K_5,\ B_0 = \frac{aK_2-3}{\omega}, \\ &C_2 = \frac{2(a^2{K_2}^2 - 9aK_2 + 18)}{3\omega^2},\ C_1 = \frac{K_5(aK_2-3)}{\omega}, \\ &C_0 = K_7,\ D_4 = \frac{a^3{K_2}^3 - 18a^2{K_2}^2 + 99aK_2 - 162}{27\omega^3}, \\ &D_3 = \frac{K_5(a^2{K_2}^2 - 9aK_2 - 3)}{3\omega},\ D_2 = \frac{K_7(aK_2-3)}{3\omega}, \\ &D_1 = K_{10},\ D_0 = \frac{K_{11}\omega}{a}. \end{aligned}$$

One can check that the equations (10), (16), (18), (20) and (23) in Theorem 2.2 are satisfied. Now, the equations (11) - (15) and (17) are satisfied when the followings conditions hold, that is

$$\begin{aligned} &K_1 = \frac{4K_2}{3},\ K_3 = \frac{2{K_2}^2}{3},\ K_4 = \frac{{K_2}^3}{27}, \\ &K_6 = K_2K_5,\ K_8 = \frac{{K_2}^2 K_5}{9},\ K_9 = \frac{K_2K_7}{3}. \end{aligned}$$

Applying Corollary 2.3, the linearizing transformation is found by solving the following equations

$$\varphi_\omega = 0, \quad \psi_{\omega\omega} = \frac{\psi_\omega(aK_2-3)}{3\omega}. \tag{62}$$

One can find the particular solution for equations in (62) as

$$\varphi = x, \quad \psi = \omega^{\left(\frac{aK_2}{3}\right)}.$$

So that, one obtains the linearizing transformation

$$t = x, \quad u = \omega^{\left(\frac{aK_2}{3}\right)}. \tag{63}$$

From Corollary 2.3, the coefficients $\tilde{\nu}, \tilde{\omega}, \tilde{\alpha}, \tilde{\beta}$ and $\tilde{\gamma}$ of the resulting linear equation (3) are

$$\tilde{\nu} = K_5,\ \tilde{\omega} = K_7,\ \tilde{\alpha} = K_{10},\ \tilde{\beta} = \frac{K_2K_{11}}{3},\ \tilde{\gamma} = 0.$$

Hence, the nonlinear equation (61) can be mapped by transformation (63) into the linear equation

$$u^{(4)} + K_5 u''' + K_7 u'' + K_{10} u' + \frac{K_2K_{11}}{3} u = 0. \tag{64}$$

Example 3.4: Equation in the article [19]

- **The significance of the problem**

In [19], Ablowitz, Chakravarty, and Halburd studied a general class of Chazy equation, defined as

$$y''' - 2yy'' + 3y'^2 - \frac{4(6y' - y^2)^2}{36 - n^2} = 0. \tag{65}$$

This equation was first written down and solved by Chazy [20]-[22] and is known today as the generalized Chazy equation. Clarkson and Olver [23] showed that a necessary condition for equation (65) to possess the Painlevé property is that the coefficient of the right hand side must be some $\alpha = \frac{4}{36-n^2}$, provided that $n \neq 6$. It has been further shown that the cases n = 2, 3, 4 and 5, correspond to the dihedral triangle, tetrahedral, octahedral and icosahedral symmetry classes.

In [18], Guha, Choudhury, and Khanra considered equation (65) the case n = 2. The third-order Riccati equation is equivalent to

$$y''' - 2yy'' + 3y'^2 - \frac{(6y' - y^2)^2}{8} = 0. \tag{66}$$

- **Applying the obtained theorems to the problem**

Let us consider the nonlinear third-order ordinary differential equation (66), under the Riccati transformation $y = \frac{a\omega'}{\omega}$, equation (66) becomes

$$\begin{aligned} &8\omega^{(4)}\omega^3 - 16\omega'''\omega' a\omega^2 - 32\omega'''\omega'\omega^2 \\ &- 12\omega''^2 a\omega^2 - 24\omega''^2\omega^2 + 12\omega''\omega'^2 a^2\omega \\ &+ 72\omega''\omega'^2 a\omega + 96\omega''\omega'^2\omega - \omega'^4 a^3 \\ &- 12\omega'^4 a^2 - 44\omega'^4 a - 48\omega'^4 = 0. \end{aligned} \tag{67}$$

It is an equation of the form (5) in Theorem 2.1 with the coefficients

$$\begin{aligned} &A_1 = -\frac{2(a+2)}{\omega},\ A_0 = 0,\ B_0 = -\frac{3(a+2)}{2\omega}, \\ &C_2 = \frac{3(a^2+6a+8)}{2\omega^2},\ C_1 = 0,\ C_0 = 0, \\ &D_4 = -\frac{a^3+12a^2+44a+48}{8\omega^3},\ D_3 = 0, \\ &D_2 = 0,\ D_1 = 0,\ D_0 = 0. \end{aligned}$$

One can check that these coefficients obey the conditions in Theorem 2.2. Hence an equation (66) is linearizable via a fiber preserving transformation. Applying Corollary 2.3, the linearizing transformation is found by solving the following equations

$$\varphi_\omega = 0, \quad \psi_{\omega\omega} = -\frac{\psi_\omega(a+2)}{2\omega}. \tag{68}$$

One can find the particular solution for equations in (68) as

$$\varphi = x, \quad \psi = \omega^{-\frac{a}{2}}.$$

So that, one obtains the linearizing transformation

$$t = x, \quad u = \omega^{-\frac{a}{2}}. \tag{69}$$

From Corollary 2.3, the coefficients $\tilde{\nu}, \tilde{\omega}, \tilde{\alpha}, \tilde{\beta}$ and $\tilde{\gamma}$ of the resulting linear equation (3) are

$$\tilde{\nu} = 0,\ \tilde{\omega} = 0,\ \tilde{\alpha} = 0,\ \tilde{\beta} = 0,\ \tilde{\gamma} = 0.$$

Hence, the nonlinear equation (67) can be mapped by transformation (69) into the linear equation

$$u^{(4)} = 0.$$

So that,

$$u = C_0 + C_1 t + C_2 t^2 + C_3 t^3, \tag{70}$$

where C_0, C_1, C_2 and C_3 are arbitrary constants. Substituting equation (69) into equation (70), we get

$$\omega^{-\frac{a}{2}} = C_0 + C_1 x + C_2 x^2 + C_3 x^3.$$

So that,

$$\omega = (C_0 + C_1 x + C_2 x^2 + C_3 x^3)^{-\frac{2}{a}}.$$

Hence, the original nonlinear solution is

$$y = -\frac{2(C_1 + 2C_2 x + 3C_3 x^2)}{(C_0 + C_1 x + C_2 x^2 + C_3 x^3)}.$$

B. Linearization for Some Interesting Fourth-Order Partial Differential Equations Under the Travelling Wave Solutions

Travelling waves are observed in many areas of science such as a result of a chemical reaction in combustion [24] and the impulses that are apparent in nerve fibres [25]. Travelling wave solutions are derived from solving the corresponding partial differential equations. These solutions are in the form

$$u(x,t) = H(z) \text{ ,where } z = x - Dt.$$

Here, the spatial and time domains are represented as x and t, with the velocity of the wave given as D.

Example 3.5: Equation in the article [26]

- **The significance of the problem**

The symmetry reductions of a class of nonlinear fourth-order partial differential equation given by

$$\begin{aligned} u_{tt} =& (\kappa u + \gamma u^2)_{xx} + \nu u u_{xxxx} + \mu u_{xxtt} \\ &+ \alpha u_x u_{xxx} + \beta u^2{}_{xx}, \end{aligned} \tag{71}$$

where $\alpha, \beta, \gamma, \mu, \nu$ and κ are arbitrary constants. This equation maybe thought of as fourth-order analogue of a generalization of the Camassa-Holm equation, in which there has been considerable interest recently. Furthermore, this equation is a Boussinesq-type equation which arises as a model of vibrations of harmonic mass-spring chain.

- **Applying the obtained theorems to the problem**

Let us consider the nonlinear fourth-order partial differential equation (71), Of particular interest amoung solutions of equation (71) are travelling wave solutions:

$$u(x,t) = H(x - Dt),$$

where D is a constant phase velocity and the argument $x - Dt$ is a phase of the wave.

Substituting the representation of a solution into equation (71), one finds

$$\begin{aligned} &(\nu H + \mu D^2)H^{(4)} + \alpha H' H''' + \beta H''^2 \\ &+ (2\gamma H + \kappa - D^2)H'' + 2\gamma H'^2 = 0. \end{aligned} \tag{72}$$

It is an equation of the form (5) in Theorem 2.1 with the coefficients

$$\begin{aligned} &A_1 = \frac{\alpha}{\nu H + \mu D^2},\ A_0 = 0,\ B_0 = \frac{\beta}{\nu H + \mu D^2}, \\ &C_2 = 0,\ C_1 = 0,\ C_0 = \frac{2\gamma H + \kappa - D^2}{\nu H + \mu D^2},\ D_4 = 0, \\ &D_3 = 0,\ D_2 = \frac{2\gamma}{\nu H + \mu D^2},\ D_1 = 0,\ D_0 = 0. \end{aligned}$$

From Theorem 2.2 equation (72) is linearizable if and only if

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \nu = 0 \tag{73}$$
$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \kappa = D^2 \tag{74}$$
$$\alpha = 4\nu, \quad \beta = 3\nu, \quad \nu \neq 0, \quad \kappa = \frac{(2\gamma\mu + \nu)D^2}{\nu}. \tag{75}$$

In the cases of equations (73) and (74), these relations make equation (72) becomes linear equation. Consider the case of equation (75).

Case $\alpha = 4\nu$, $\beta = 3\nu$, $\nu \neq 0$, $\kappa = \frac{(2\gamma\mu+\nu)D^2}{\nu}$.

Applying Corollary 2.3, the linearizing transformation is found by solving the following equation

$$\varphi_H = 0, \quad \psi_{HH} = \frac{\psi_H \nu}{D^2\mu + H\nu}. \tag{76}$$

One can find the particular solution for equations in (76) as

$$\varphi = x - Dt, \quad \psi = 2D^2\mu H + \nu H^2.$$

So that, one obtains the linearizing transformation

$$\tilde{t} = x - Dt, \quad \tilde{u} = 2D^2\mu H + \nu H^2. \tag{77}$$

From Corollary 2.3, the coefficients $\tilde{\nu}$, $\tilde{\omega}$, $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ of the resulting linear equation (3) are

$$\tilde{\nu} = 0,\ \tilde{\omega} = \frac{2\gamma}{\nu},\ \tilde{\alpha} = 0,\ \tilde{\beta} = 0,\ \tilde{\gamma} = 0.$$

Hence, the nonlinear equation (72) can be mapped by transformation (77) into the linear equation

$$\tilde{u}^{(4)} + \frac{2\gamma}{\nu}\tilde{u}'' = 0. \tag{78}$$

where γ , ν are arbitrary constants.

• Case $\frac{2\gamma}{\nu}$ = 0, the solution of equation (78) is

$$\tilde{u}(\tilde{t}) = C_0 + C_1\tilde{t} + C_2\tilde{t}^2 + C_3\tilde{t}^3, \tag{79}$$

where C_0, C_1, C_2 and C_3 are arbitrary constants. Substituting equation (77) into equation (79), we get the solution of ordinary differential equation

$$\begin{aligned} 2D^2\mu H + \nu H^2 &= C_0 + C_1(x - Dt) + C_2(x - Dt)^2 \\ &\quad + C_3(x - Dt)^3. \end{aligned}$$

So that, the solution of partial differential equation (71) is

$$\begin{aligned} 2D^2\mu u + \nu u^2 &= C_0 + C_1(x - Dt) + C_2(x - Dt)^2 \\ &\quad + C_3(x - Dt)^3. \end{aligned}$$

• Case $\frac{2\gamma}{\nu} > 0$, the solution of equation (78) is

$$\tilde{u}(\tilde{t}) = C_0 + C_1\tilde{t} + C_2\cos(\sqrt{\frac{2\gamma}{\nu}}\tilde{t}) + C_3\sin(\sqrt{\frac{2\gamma}{\nu}}\tilde{t}), \tag{80}$$

where C_0, C_1, C_2 and C_3 are arbitrary constants. Substituting equation (77) into equation (80), we get the solution of ordinary differential equation

$$\begin{aligned} 2D^2\mu H + \nu H^2 &= C_0 + C_1(x - Dt) \\ &\quad + C_2\cos(\sqrt{\frac{2\gamma}{\nu}}(x - Dt)) \\ &\quad + C_3\sin(\sqrt{\frac{2\gamma}{\nu}}(x - Dt)). \end{aligned}$$

So that, the solution of partial differential equation (71) is

$$\begin{aligned} 2D^2\mu u + \nu u^2 &= C_0 + C_1(x - Dt) \\ &\quad + C_2\cos(\sqrt{\frac{2\gamma}{\nu}}(x - Dt)) \\ &\quad + C_3\sin(\sqrt{\frac{2\gamma}{\nu}}(x - Dt)). \end{aligned}$$

• Case $\frac{2\gamma}{\nu} < 0$, the solution of equation (78) is

$$\tilde{u}(\tilde{t}) = C_0 + C_1\tilde{t} + C_2 e^{\sqrt{\frac{2\gamma}{\nu}}\tilde{t}} + C_3 e^{-\sqrt{\frac{2\gamma}{\nu}}\tilde{t}}, \tag{81}$$

where C_0, C_1, C_2 and C_3 are arbitrary constants. Substituting equation (77) into equation (81), we get the solution of ordinary differential equation

$$\begin{aligned} 2D^2\mu H + \nu H^2 &= C_0 + C_1(x - Dt) + C_2 e^{\sqrt{\frac{2\gamma}{\nu}}(x-Dt)} \\ &\quad + C_3 e^{-\sqrt{\frac{2\gamma}{\nu}}(x-Dt)}. \end{aligned}$$

So that, the solution of partial differential equation (71) is

$$\begin{aligned} 2D^2\mu u + \nu u^2 &= C_0 + C_1(x - Dt) + C_2 e^{\sqrt{\frac{2\gamma}{\nu}}(x-Dt)} \\ &\quad + C_3 e^{-\sqrt{\frac{2\gamma}{\nu}}(x-Dt)}. \end{aligned}$$

C. Linearization for Some Interesting Third-Order Partial Differential Equations Under the Travelling Wave Solutions

Example 3.6: Equation in the article [27] and [28]

- **The significance of the problem**

Members of the class of evolutionary PDEs

$$m_t + Au_x m + Bum_x + Cuu_x + Du_{xxt} = Ku_x,$$

where $m = u - \alpha^2 u_{xx}$ is the Helmholtz operator acting on the dependent variable u, function of the spatial variable x and time t, and A, B, C, D, K are constants, have recently attracted intense interest from both a mathematical and physical perspective, following the derivation of a member of this class (also known as the Camassa-Holm equation), in the context of shallow-water wave dynamics.

For $A = 0, B = 1, C = 0, D = 0$, and $K = 0$, this is a regularized Burgers equation

$$u_t + uu_x = \alpha^2(u_{xxt} + uu_{xxx}). \tag{82}$$

In [27] Bhat and Fetecau showed that solutions of this equation converge strongly to physically relevant weak solutions of the Hopf equation $u_t + uu_x = 0$ as $\alpha \to 0$, provided the initial data $u(x, 0)$ are in a suitable function space. Thus, equation (82) has been proposed as an alternative to Burgers equation $u_t + uu_x = 0$ in this respect.

In [28] Camassa, Chiu, Lee, and Sheu employed a two-step iterative scheme for solving a class of PDEs involving the Helmholtz operator. They investigated solution properties of members of this class of PDEs.

- **Applying the obtained theorems to the problem**

Let us consider the nonlinear third-order partial differential equation (82), Let $u = w_t$, then equation (82) become

$$w_{tt} + w_t w_{xt} = \alpha^2(w_{xxty} + w_t w_{xxxt}). \tag{83}$$

Of particular interest amoung solutions of equation (83) are travelling wave solutions:

$$w(x, t) = H(x - Dt),$$

where D is a constant phase velocity and the argument $x-Dt$ is a phase of the wave.

Substituting the representation of a solution into equation (83), one finds

$$D^2H'' + D^2H'H'' - \alpha^2(D^2H^{(4)} + D^2H'H^{(4)}) = 0. \quad (84)$$

It is an equation of the form (5) in Theorem 2.1 with the coefficients

$$A_1 = A_0 = B_0 = C_2 = C_1 = 0,\ C_0 = -\tfrac{1}{\alpha^2},$$
$$D_4 = D_3 = D_2 = D_1 = D_0 = 0.$$

One can check that these coefficients obey the conditions in Theorem 2.2. Hence, an equation (83) is linearizable via a fiber preserving transformations. Applying Corollary 2.3, the linearizing transformation is found by solving the equation

$$\varphi_H = 0, \quad \psi_{HH} = 0. \quad (85)$$

One can find the particular solution for equations in (85) as

$$\varphi = x - Dt, \quad \psi = H.$$

So that, one obtains the linearizing transformation

$$\tilde{t} = x - Dt, \quad \tilde{u} = H. \quad (86)$$

From Corollary 2.3, the coefficients $\tilde{\nu}$, $\tilde{\omega}$, $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ of the resulting linear equation (3) are

$$\tilde{\nu} = 0,\ \tilde{\omega} = -\frac{1}{\alpha^2},\ \tilde{\alpha} = 0,\ \tilde{\beta} = 0,\ \tilde{\gamma} = 0.$$

Hence, the nonlinear equation (83) can be mapped by transformation (86) into the linear equation

$$\tilde{u}^{(4)} - \frac{1}{\alpha^2}\tilde{u}'' = 0. \quad (87)$$

The solution of equation (87) is

$$\tilde{u}(\tilde{t}) = C_1 + C_2 t + C_3 e^{-\frac{1}{\alpha}\tilde{t}} + C_4 e^{\frac{1}{\alpha}\tilde{t}}, \quad (88)$$

where C_1, C_2, C_3 and C_4 are arbitrary constants. Substituting equation (86) into equation (88), we get the solution of ordinary differential equation

$$H = C_1 + C_2(x - Dt) + C_3 e^{-\frac{1}{\alpha}(x-Dt)} + C_4 e^{\frac{1}{\alpha}(x-Dt)}.$$

So that, the solution of partial differential equation (83) is

$$w = C_1 + C_2(x - Dt) + C_3 e^{-\frac{1}{\alpha}(x-Dt)} + C_4 e^{\frac{1}{\alpha}(x-Dt)}.$$

Hence, the solution of nonlinear equation (82) is $u = w_t$, i.e.

$$u = -DC_2 + \frac{D}{\alpha}C_3 e^{-\frac{1}{\alpha}(x-Dt)} - \frac{D}{\alpha}C_4 e^{\frac{1}{\alpha}(x-Dt)}.$$

IV. Conclusion

In summary, if a fourth-order ordinary differential equation is not in one of the forms specified in Theorem 2.1, it definitely cannot be linearized by the point transformation. The form that satisfies corresponding conditions in either Theorem 2.2 or Theorem 2.5 is linearizable via the point transformation. The original solution can be attained by applying the transformations derived from Corollary 2.3 and Corollary 2.6. This method has been proven to be effective for various fourth-order ordinary differential equations in literature, as well as some third-order ordinary differential equations and fourth-order partial differential equations under specific conditions.

Appendix

A. The coefficients of equation (5)

$$\begin{aligned}
A_1 =& (4\psi_{yy})/\psi_y, && \text{(A.1)}\\
A_0 =& -((6\varphi_{xx} - \varphi_x^2\nu)\psi_y - 4\varphi_x\psi_{xy})/(\varphi_x\psi_y), && \text{(A.2)}\\
B_0 =& (3\psi_{yy})/\psi_y, && \text{(A.3)}\\
C_2 =& (6\psi_{yyy})/\psi_y, && \text{(A.4)}\\
C_1 =& 3((\varphi_x\psi_{yy}\nu + 4\psi_{xyy})\varphi_x - 6\varphi_{xx}\psi_{yy})/(\varphi_x\psi_y), && \text{(A.5)}\\
C_0 =& ((15\varphi_{xx}^2 + \varphi_x^4\omega - 4\varphi_{xxx}\varphi_x)\psi_y + 3((\varphi_x\psi_{xy}\nu\\
&+ 2\psi_{xxy})\varphi_x - (\varphi_x\psi_y\nu + 6\psi_{xy})\varphi_{xx})\varphi_x)\\
&/(\varphi_x^2\psi_y), && \text{(A.6)}\\
D_4 =& \psi_{yyyy}/\psi_y, && \text{(A.7)}\\
D_3 =& ((\varphi_x\psi_{yyy}\nu + 4\psi_{xyyy})\varphi_x - 6\varphi_{xx}\psi_{yyy})\\
&/(\varphi_x\psi_y), && \text{(A.8)}\\
D_2 =& -(3((\varphi_x\psi_{yy}\nu + 6\psi_{xyy})\varphi_x - 5\varphi_{xx}\psi_{yy})\varphi_{xx}\\
&- (\varphi_x^2\psi_{yy}\omega + 3\varphi_x\psi_{xyy}\nu + 6\psi_{xxyy})\varphi_x^2\\
&+ 4\varphi_{xxx}\varphi_x\psi_{yy})/(\varphi_x^2\psi_y), && \text{(A.9)}\\
D_1 =& ((3((\varphi_x\psi_y\nu + 10\psi_{xy})\varphi_x - 5\varphi_{xx}\psi_y)\varphi_{xx}\\
&- (\varphi_x^2\psi_y\omega + 6\varphi_x\psi_{xy}\nu + 18\psi_{xxy})\varphi_x^2)\varphi_{xx}\\
&+ (\varphi_x^3\psi_y\alpha + 2\varphi_x^2\psi_{xy}\omega + 3\varphi_x\psi_{xxy}\nu\\
&+ 4\psi_{xxxy})\varphi_x^3 - ((\varphi_x\psi_y\nu + 8\psi_{xy})\varphi_x\\
&- 10\varphi_{xx}\psi_y)\varphi_{xxx}\varphi_x - \varphi_{xxxx}\varphi_x^2\psi_y)\\
&/(\varphi_x^3\psi_y), && \text{(A.10)}\\
D_0 =& (((\varphi_x^2\psi_{xx}\omega + \varphi_x\psi_{xxx}\nu + \psi_{xxxx}\\
&+ (\beta\psi + \gamma)\varphi_x^4)\varphi_x^2 + 3(\varphi_x\psi_x\nu\\
&+ 5\psi_{xx})\varphi_{xx}^2 - ((\varphi_x\psi_x\omega + 3\psi_{xx}\nu)\varphi_x\\
&+ 6\psi_{xxx})\varphi_{xx}\varphi_x + ((10\varphi_{xx} - \varphi_x^2\nu)\psi_x\\
&- 4\varphi_x\psi_{xx})\varphi_{xxx})\varphi_x(15\varphi_{xx}^3 - \varphi_x^6\alpha\\
&+ \varphi_{xxxx}\varphi_x^2)\psi_x)/(\varphi_x^3\psi_y). && \text{(A.11)}
\end{aligned}$$

B. The coefficients of equation (6)

$$\begin{aligned}
F_2 =& ((\varphi_y\nu\Delta + 4\Delta_y)\varphi_y - 10\varphi_{yy}\Delta)/(\varphi_y\Delta), && \text{(B.1)}\\
F_1 =& (2((2(\Delta_x + \Delta_y r - 5r_y\Delta) + \varphi_y\nu r\Delta)\varphi_y\\
&- 10\varphi_{yy}r\Delta))/(\varphi_y\Delta), && \text{(B.2)}\\
F_0 =& (-((2((5r_y\Delta - 2\Delta_x)r + 5r_x\Delta)\\
&- \varphi_y\nu r^2\Delta)\varphi_y + 10\varphi_{yy}r^2\Delta))/(\varphi_y\Delta), && \text{(B.3)}\\
H_2 =& 3(10\varphi_{yy}\Delta - \varphi_y^2\nu\Delta - 4\varphi_y\Delta_y)/(\varphi_y\Delta), && \text{(B.4)}\\
H_1 =& -3((5\Delta_x + 3\Delta_y r - 25r_y\Delta\\
&+ 2\varphi_y\nu r\Delta)\varphi_y - 20\varphi_{yy}r\Delta)/(\varphi_y\Delta), && \text{(B.5)}\\
H_0 =& -3((((\varphi_y\nu r^2 - 15r_x - 10r_y r)\Delta + (5\Delta_x\\
&- \Delta_y r)r)\varphi_y - 10\varphi_{yy}r^2\Delta)/(\varphi_y\Delta), && \text{(B.6)}\\
J_4 =& -(10\varphi_{yyy}\varphi_y\Delta - 45\varphi_{yy}^2\Delta + 6\varphi_{yy}\varphi_y^2\nu\Delta\\
&+ 30\varphi_{yy}\varphi_y\Delta_y - \varphi_y^4\omega\Delta - 3\varphi_y^3\Delta_y\nu\\
&- 6\varphi_y^2\Delta_{yy})/(\varphi_y^2\Delta), && \text{(B.7)}
\end{aligned}$$

$$\begin{aligned}
J_3 =&((6(2(\Delta_{xy}+\Delta_{yy}r-5r_y\Delta_y)\\
&-5r_{yy}\Delta)+(3(\Delta_x+3\Delta_y r-4r_y\Delta)\nu\\
&+4\varphi_y\omega r\Delta)\varphi_y)\varphi_y^2-6((5(\Delta_x+3\Delta_y r\\
&-4r_y\Delta)+4\varphi_y\nu r\Delta)\varphi_y-30\varphi_{yy}r\Delta)\varphi_{yy}\\
&-40\varphi_{yyy}\varphi_y r\Delta)/(\varphi_y^2\Delta),
\end{aligned}\tag{B.8}$$

$$\begin{aligned}
J_2 =&3((2(\Delta_{xx}+\Delta_{yy}r^2+4\Delta_{xy}r-5(2\Delta_x\\
&+3\Delta_y r-5r_y\Delta)r_y-10r_{yy}r\Delta-5r_x\Delta_y\\
&-5r_{xy}\Delta)+(((3(\Delta_x+\Delta_y r)-10r_y\Delta)r\\
&-2r_x\Delta)\nu+2\varphi_y\omega r^2\Delta)\varphi_y)\varphi_y^2\\
&-2((5((3(\Delta_x+\Delta_y r)-10r_y\Delta)r\\
&-2r_x\Delta)+6\varphi_y\nu r^2\Delta)\varphi_y-45\varphi_{yy}r^2\Delta)\varphi_{yy}\\
&-20\varphi_{yyy}\varphi_y r^2\Delta)/(\varphi_y^2\Delta),
\end{aligned}\tag{B.9}$$

$$\begin{aligned}
J_1 =&-((2((5(3(3\Delta_x+\Delta_y r)-14r_y\Delta)r_y\\
&-6(\Delta_{xy}r+\Delta_{xx})+20r_{yy}r\Delta)r\\
&+5(3(\Delta_x+\Delta_y r)-16r_y\Delta)r_x\\
&+5r_{xx}\Delta+20r_{xy}r\Delta)-(3((3\Delta_x\\
&+\Delta_y r-8r_y\Delta)r-4r_x\Delta)\nu\\
&+4\varphi_y\omega r^2\Delta)\varphi_y r)\varphi_y^2+6((5((3\Delta_x\\
&+\Delta_y r-8r_y\Delta)r-4r_x\Delta)\\
&+4\varphi_y\nu r^2\Delta)\varphi_y-30\varphi_{yy}r^2\Delta)\varphi_{yy}r\\
&+40\varphi_{yyy}\varphi_y r^3\Delta)/(\varphi_y^2\Delta),
\end{aligned}\tag{B.10}$$

$$\begin{aligned}
J_0 =&-(((2((5r_{yy}r\Delta-3\Delta_{xx})r+5r_{xx}\Delta\\
&+5r_{xy}r\Delta)-5(7r_y\Delta-6\Delta_x)r_y r)r\\
&-5(2(7r_y\Delta-3\Delta_x)r+9r_x\Delta)r_x\\
&+(3((2r_y\Delta-\Delta_x)r+2r_x\Delta)\nu\\
&-\varphi_y\omega r^2\Delta)\varphi_y r^2)\varphi_y^2-(3(2(5((2r_y\Delta\\
&-\Delta_x)r+2r_x\Delta)-\varphi_y\nu r^2\Delta)\varphi_y\\
&+15\varphi_{yy}r^2\Delta)\varphi_{yy}-10\varphi_{yyy}\varphi_y r^2\Delta)r^2)\\
&/(\varphi_y^2\Delta),
\end{aligned}\tag{B.11}$$

$$\begin{aligned}
K_7 =&-(\varphi_{yyyy}\varphi_y^2\psi_y-10\varphi_{yyy}\varphi_{yy}\varphi_y\psi_y\\
&+\varphi_{yyy}\varphi_y^3\psi_y\nu+4\varphi_{yyy}\varphi_y^2\psi_{yy}\\
&+15\varphi_{yy}^3\psi_y-3\varphi_{yy}^2\varphi_y^2\psi_y\nu-15\varphi_{yy}^2\varphi_y\psi_{yy}\\
&+\varphi_{yy}\varphi_y^4\psi_y\omega+3\varphi_{yy}\varphi_y^3\psi_{yy}\nu\\
&+6\varphi_{yy}\varphi_y^2\psi_{yyy}-\varphi_y^7\beta\psi-\varphi_y^7\gamma-\varphi_y^6\psi_y\alpha\\
&-\varphi_y^5\psi_{yy}\omega-\varphi_y^4\psi_{yyy}\nu-\varphi_y^3\psi_{yyyy})/(\varphi_y^2\Delta),
\end{aligned}\tag{B.12}$$

$$\begin{aligned}
K_6 =&((7\varphi_y^4\psi_y\alpha r-4\Delta_{yyy}+7(\beta\psi+\gamma)\varphi_y^5 r\\
&+(7\psi_{yy}\omega r-\alpha\Delta)\varphi_y^3+(7\psi_{yyy}\nu r\\
&-2\Delta_y\omega)\varphi_y^2+(7\psi_{yyyy}r-3\Delta_{yy}\nu)\varphi_y)\varphi_y^3\\
&-((7\varphi_y^3\psi_y\omega r+21\varphi_y^2\psi_{yy}\nu r-3\varphi_y^2\omega\Delta\\
&+42\varphi_y\psi_{yyy}r-12\varphi_y\Delta_y\nu-30\Delta_{yy})\varphi_y^2\\
&+105(\varphi_y\psi_y r-\Delta)\varphi_{yy}^2-3(7\varphi_y^2\psi_y\nu r\\
&-30\Delta_y+5(7\psi_{yy}r-\nu\Delta)\varphi_y)\varphi_{yy}\varphi_y)\varphi_{yy}\\
&-((7\varphi_y^2\psi_y\nu r-20\Delta_y+4(7\psi_{yy}r\\
&-\nu\Delta)\varphi_y)\varphi_y-10(7\varphi_y\psi_y r\\
&-6\Delta)\varphi_{yy})\varphi_{yyy}\varphi_y-(7\varphi_y\psi_y r\\
&-5\Delta)\varphi_{yyyy}\varphi_y^2)/(\varphi_y^3\Delta),
\end{aligned}\tag{B.13}$$

$$\begin{aligned}
K_5 =&-(((3((\Delta_{xy}+5\Delta_{yy}r-4r_y\Delta_y-2r_{yy}\Delta)\nu\\
&-7\psi_{yyyy}r^2)+((\Delta_x+11\Delta_y r-3r_y\Delta)\omega\\
&-21\psi_{yyy}\nu r^2-3(7((\beta\psi+\gamma)\varphi_y\\
&+\psi_y\alpha)\varphi_y r+7\psi_{yy}\omega r-2\alpha\Delta)\varphi_y r)\varphi_y)\varphi_y\\
&+2(3(\Delta_{xyy}+3\Delta_{yyy}r-5r_y\Delta_{yy}\\
&-5r_{yy}\Delta_y)-5r_{yyy}\Delta))\varphi_y^3-3(((2((\Delta_x\\
&+11\Delta_y r-3r_y\Delta)\nu-21\psi_{yyy}r^2)\\
&-(7\varphi_y\psi_y\omega r+21\psi_{yy}\nu r-6\omega\Delta)\varphi_y r)\varphi_y\\
&+10(\Delta_{xy}+5\Delta_{yy}r-4r_y\Delta_y-2r_{yy}\Delta))\varphi_y^2\\
&-3((5(\Delta_x+11\Delta_y r-3r_y\Delta)-(7\varphi_y\psi_y\nu r\\
&+35\psi_{yy}r-10\nu\Delta)\varphi_y r)\varphi_y+35(\varphi_y\psi_y r\\
&-2\Delta)\varphi_{yy}r)\varphi_{yy})\varphi_{yy}-((10(\Delta_x+11\Delta_y r\\
&-3r_y\Delta)-3(7\varphi_y\psi_y\nu r+28\psi_{yy}r\\
&-8\nu\Delta)\varphi_y r)\varphi_y+30(7\varphi_y\psi_y r\\
&-12\Delta)\varphi_{yy}r)\varphi_{yyy}\varphi_y+3(7\varphi_y\psi_y r\\
&-10\Delta)\varphi_{yyyy}\varphi_y^2 r)/(\varphi_y^3\Delta),
\end{aligned}\tag{B.14}$$

$$\begin{aligned}
K_4 =&-((((\Delta_{xx}+31\Delta_{yy}r^2+13\Delta_{xy}r-8(\Delta_x\\
&+6\Delta_y r-2r_y\Delta)r_y-26r_{yy}r\Delta-4r_x\Delta_y\\
&-4r_{xy}\Delta)\nu-35\psi_{yyyy}r^3)\varphi_y+2(45r_{yy}r_y\Delta\\
&-10r_{yy}\Delta_x-55r_{yy}\Delta_y r+50r_y^2\Delta_y\\
&-20r_y\Delta_{xy}-50r_y\Delta_{yy}r+11\Delta_{xyy}r\\
&+2\Delta_{xxy}+17\Delta_{yyy}r^2-20r_{yyy}r\Delta\\
&-5r_x\Delta_{yy}-10r_{xy}\Delta_y-5r_{xyy}\Delta)\\
&+(((5(\Delta_x+5\Delta_y r)-14r_y\Delta)r-r_x\Delta)\omega\\
&-35\psi_{yyy}\nu r^3-5(7((\beta\psi+\gamma)\varphi_y\\
&+\psi_y\alpha)\varphi_y r+7\psi_{yy}\omega r-3\alpha\Delta)\varphi_y r^2)\varphi_y^2)\varphi_y^3\\
&-((10(\Delta_{xx}+31\Delta_{yy}r^2+13\Delta_{xy}r\\
&-8(\Delta_x+6\Delta_y r-2r_y\Delta)r_y\\
&-26r_{yy}r\Delta-4r_x\Delta_y-4r_{xy}\Delta)\\
&+(6(((5(\Delta_x+5\Delta_y r)-14r_y\Delta)r-r_x\Delta)\nu\\
&-35\psi_{yyy}r^3)-5(7\varphi_y\psi_y\omega r+21\psi_{yy}\nu r\\
&-9\omega\Delta)\varphi_y r^2)\varphi_y)\varphi_y^2-15((3((5(\Delta_x\\
&+5\Delta_y r)-14r_y\Delta)r-r_x\Delta)-(7\varphi_y\psi_y\nu r\\
&+35\psi_{yy}r-15\nu\Delta)\varphi_y r^2)\varphi_y+35(\varphi_y\psi_y r\\
&-3\Delta)\varphi_{yy}r^2)\varphi_{yy})\varphi_{yy}-5((2((5(\Delta_x\\
&+5\Delta_y r)-14r_y\Delta)r-r_x\Delta)\\
&-(7\varphi_y\psi_y\nu r+28\psi_{yy}r-12\nu\Delta)\varphi_y r^2)\varphi_y\\
&+10(7\varphi_y\psi_y r-18\Delta)\varphi_{yy}r^2)\varphi_{yyy}\varphi_y\\
&+5(7\varphi_y\psi_y r-15\Delta)\varphi_{yyyy}\varphi_y^2 r^2)/(\varphi_y^3\Delta),
\end{aligned}\tag{B.15}$$

$$\begin{aligned}
K_3 =&-((((13\Delta_{xxy}+35\Delta_{yyy}r^2)r+\Delta_{xxx}\\
&+31\Delta_{xyy}r^2-5(3\Delta_{xx}+26\Delta_{yy}r^2\\
&+23\Delta_{xy}r-(15\Delta_x+49\Delta_y r\\
&-25r_y\Delta)r_y)r_y-5(13\Delta_x+32\Delta_y r\\
&-50r_y\Delta)r_{yy}r-65r_{yyy}r^2\Delta-5(3\Delta_{xy}\\
&+5\Delta_{yy}r-16r_y\Delta_y-7r_{yy}\Delta)r_x-5r_{xx}\Delta_y\\
&-5r_{xxy}\Delta-5(3\Delta_x+11\Delta_y r-15r_y\Delta)r_{xy}\\
&-30r_{xyy}r\Delta+(((2(2\Delta_{xx}+17\Delta_{yy}r^2\\
&+11\Delta_{xy}r)-(29\Delta_x+75\Delta_y r-51r_y\Delta)r_y
\end{aligned}$$

$$
\begin{aligned}
&-45r_{yy}r\Delta)r-(3\Delta_x+13\Delta_y r\\
&-13r_y\Delta)r_x-r_{xx}\Delta-14r_{xy}r\Delta)\nu\\
&-35\psi_{yyyy}r^4+(2((5(\Delta_x+3\Delta_y r)\\
&-13r_y\Delta)r-2r_x\Delta)\omega-35\psi_{yyy}\nu r^3\\
&-5(7((\beta\psi+\gamma)\varphi_y+\psi_y\alpha)\varphi_y r\\
&+7\psi_{yy}\omega r-4\alpha\Delta)\varphi_y r^2)\varphi_y r)\varphi_y)\varphi_y^3\\
&-((10((2(2\Delta_{xx}+17\Delta_{yy}r^2+11\Delta_{xy}r)\\
&-(29\Delta_x+75\Delta_y r-51r_y\Delta)r_y\\
&-45r_{yy}r\Delta)r-(3\Delta_x+13\Delta_y r\\
&-13r_y\Delta)r_x-r_{xx}\Delta-14r_{xy}r\Delta)\\
&+(6(2((5(\Delta_x+3\Delta_y r)-13r_y\Delta)r\\
&-2r_x\Delta)\nu-35\psi_{yyy}r^3)-5(7\varphi_y\psi_y\omega r\\
&+21\psi_{yy}\nu r-12\omega\Delta)\varphi_y r^2)\varphi_y r)\varphi_y^2\\
&-15((6((5(\Delta_x+3\Delta_y r)-13r_y\Delta)r\\
&-2r_x\Delta)-(7\varphi_y\psi_y\nu r+35\psi_{yy}r\\
&-20\nu\Delta)\varphi_y r^2)\varphi_y+35(\varphi_y\psi_y r\\
&-4\Delta)\varphi_{yy}r^2)\varphi_{yy}r)\varphi_{yy}-5((4((5(\Delta_x\\
&+3\Delta_y r)-13r_y\Delta)r-2r_x\Delta)\\
&-(7\varphi_y\psi_y\nu r+28\psi_{yy}r-16\nu\Delta)\varphi_y r^2)\varphi_y\\
&+10(7\varphi_y\psi_y r-24\Delta)\varphi_{yy}r^2)\varphi_{yyy}\varphi_y r\\
&+5(7\varphi_y\psi_y r-20\Delta)\varphi_{yyyy}\varphi_y^2 r^3)/(\varphi_y^3\Delta), \qquad \text{(B.16)}
\end{aligned}
$$

$$
\begin{aligned}
K_2=&-(((3((5\Delta_{xxy}+7\Delta_{yyy}r^2)r+\Delta_{xxx}\\
&+7\Delta_{xyy}r^2)-(3(13\Delta_{xx}+28\Delta_{yy}r^2\\
&+39\Delta_{xy}r)+(204r_y\Delta-161\Delta_x\\
&-217\Delta_y r)r_y)r_y-(79\Delta_x+116\Delta_y r\\
&-264r_y\Delta)r_{yy}r-54r_{yyy}r^2\Delta)r\\
&-(3(2\Delta_{xx}+7\Delta_{yy}r^2+11\Delta_{xy}r)\\
&+(171r_y\Delta-64\Delta_x-140\Delta_y r)r_y\\
&-72r_{yy}r\Delta-18r_x\Delta_y)r_x-(4\Delta_x\\
&+11\Delta_y r-21r_y\Delta)r_{xx}-12r_{xxy}r\Delta\\
&-r_{xxx}\Delta-((37\Delta_x+53\Delta_y r-150r_y\Delta)r\\
&-33r_x\Delta)r_{xy}-33r_{xyy}r^2\Delta+(3(((2\Delta_{xx}\\
&+7\Delta_{yy}r^2+6\Delta_{xy}r-(13\Delta_x+19\Delta_y r\\
&-20r_y\Delta)r_y-13r_{yy}r\Delta)r^2-((3\Delta_x\\
&+5\Delta_y r-11r_y\Delta)r-r_x\Delta)r_x-r_{xx}r\Delta\\
&-6r_{xy}r^2\Delta)\nu-7\psi_{yyyy}r^5)+(2((5(\Delta_x\\
&+2\Delta_y r)-12r_y\Delta)r-3r_x\Delta)\omega\\
&-21\psi_{yyy}\nu r^3-3(7((\beta\psi+\gamma)\varphi_y\\
&+\psi_y\alpha)\varphi_y r+7\psi_{yy}\omega r\\
&-5\alpha\Delta)\varphi_y r^2)\varphi_y r^2)\varphi_y)\varphi_y^3\\
&-3((10((2\Delta_{xx}+7\Delta_{yy}r^2+6\Delta_{xy}r\\
&-(13\Delta_x+19\Delta_y r-20r_y\Delta)r_y\\
&-13r_{yy}r\Delta)r^2-((3\Delta_x+5\Delta_y r\\
&-11r_y\Delta)r-r_x\Delta)r_x-r_{xx}r\Delta\\
&-6r_{xy}r^2\Delta)+(2(2((5(\Delta_x+2\Delta_y r)\\
&-12r_y\Delta)r-3r_x\Delta)\nu-21\psi_{yyy}r^3)\\
&-(7\varphi_y\psi_y\omega r+21\psi_{yy}\nu r\\
&-15\omega\Delta)\varphi_y r^2)\varphi_y r^2)\varphi_y^2-3((10((5(\Delta_x\\
&+2\Delta_y r)-12r_y\Delta)r-3r_x\Delta)-(7\varphi_y\psi_y\nu r\\
&+35\psi_{yy}r-25\nu\Delta)\varphi_y r^2)\varphi_y\\
&+35(\varphi_y\psi_y r-5\Delta)\varphi_{yy}r^2)\varphi_{yy}r^2)\varphi_{yy}\\
&-((20((5(\Delta_x+2\Delta_y r)-12r_y\Delta)r\\
&-3r_x\Delta)-3(7\varphi_y\psi_y\nu r+28\psi_{yy}r\\
&-20\nu\Delta)\varphi_y r^2)\varphi_y+30(7\varphi_y\psi_y r\\
&-30\Delta)\varphi_{yy}r^2)\varphi_{yyy}\varphi_y r^2+3(7\varphi_y\psi_y r\\
&-25\Delta)\varphi_{yyyy}\varphi_y^2 r^4)/(\varphi_y^3\Delta), \qquad \text{(B.17)}
\end{aligned}
$$

$$
\begin{aligned}
K_1=&-(((7(\Delta_{xxy}+\Delta_{yyy}r^2)r+3\Delta_{xxx}\\
&+7\Delta_{xyy}r^2-(33\Delta_{xx}+28\Delta_{yy}r^2\\
&+49\Delta_{xy}r+2(59r_y\Delta-56\Delta_x\\
&-42\Delta_y r)r_y)r_y-(43\Delta_x+42\Delta_y r\\
&-128r_y\Delta)r_{yy}r-23r_{yyy}r^2\Delta)r^2\\
&-((12\Delta_{xx}+7\Delta_{yy}r^2+21\Delta_{xy}r\\
&+2(86r_y\Delta-49\Delta_x-35\Delta_y r)r_y\\
&-49r_{yy}r\Delta)r+(85r_y\Delta-15\Delta_x\\
&-21\Delta_y r)r_x)r_x-((8\Delta_x+7\Delta_y r\\
&-32r_y\Delta)r-10r_x\Delta)r_{xx}-9r_{xxy}r^2\Delta\\
&-2r_{xxx}r\Delta-((29\Delta_x+21\Delta_y r-95r_y\Delta)r\\
&-46r_x\Delta)r_{xy}r-16r_{xyy}r^3\Delta+(((4\Delta_{xx}\\
&+7\Delta_{yy}r^2+7\Delta_{xy}r-(23\Delta_x+21\Delta_y r\\
&-31r_y\Delta)r_y-17r_{yy}r\Delta)r^2-((9\Delta_x\\
&+7\Delta_y r-27r_y\Delta)r-6r_x\Delta)r_x\\
&-3r_{xx}r\Delta-10r_{xy}r^2\Delta)\nu-7\psi_{yyyy}r^5\\
&+(((5\Delta_x+7\Delta_y r-11r_y\Delta)r-4r_x\Delta)\omega\\
&-7\psi_{yyy}\nu r^3-(7((\beta\psi+\gamma)\varphi_y+\psi_y\alpha)\varphi_y r\\
&+7\psi_{yy}\omega r-6\alpha\Delta)\varphi_y r^2)\varphi_y r^2)\varphi_y r)\varphi_y^3\\
&-((10((4\Delta_{xx}+7\Delta_{yy}r^2+7\Delta_{xy}r\\
&-(23\Delta_x+21\Delta_y r-31r_y\Delta)r_y\\
&-17r_{yy}r\Delta)r^2-((9\Delta_x+7\Delta_y r\\
&-27r_y\Delta)r-6r_x\Delta)r_x-3r_{xx}r\Delta\\
&-10r_{xy}r^2\Delta)+(6(((5\Delta_x+7\Delta_y r\\
&-11r_y\Delta)r-4r_x\Delta)\nu-7\psi_{yyy}r^3)\\
&-(7\varphi_y\psi_y\omega r+21\psi_{yy}\nu r\\
&-18\omega\Delta)\varphi_y r^2)\varphi_y r^2)\varphi_y^2-3((15((5\Delta_x\\
&+7\Delta_y r-11r_y\Delta)r-4r_x\Delta)-(7\varphi_y\psi_y\nu r\\
&+35\psi_{yy}r-30\nu\Delta)\varphi_y r^2)\varphi_y+35(\varphi_y\psi_y r\\
&-6\Delta)\varphi_{yy}r^2)\varphi_{yy}r^2)\varphi_{yy}r-((10((5\Delta_x\\
&+7\Delta_y r-11r_y\Delta)r-4r_x\Delta)-(7\varphi_y\psi_y\nu r\\
&+28\psi_{yy}r-24\nu\Delta)\varphi_y r^2)\varphi_y\\
&+10(7\varphi_y\psi_y r-36\Delta)\varphi_{yy}r^2)\varphi_{yyy}\varphi_y r^3\\
&+(7\varphi_y\psi_y r-30\Delta)\varphi_{yyyy}\varphi_y^2 r^5)/(\varphi_y^3\Delta), \qquad \text{(B.18)}
\end{aligned}
$$

$$
\begin{aligned}
K_0=&(((((2(r_{xxy}+2r_{yyy}r^2)r+r_{xxx}\\
&+3r_{xyy}r^2)\Delta+3(3\Delta_x+2\Delta_y r\\
&-8r_y\Delta)r_{yy}r^2)r-((10r_x+11r_y r)\Delta\\
&-(4\Delta_x+\Delta_y r)r)r_{xx}-((13r_x+20r_y r)\Delta\\
&-(7\Delta_x+3\Delta_y r)r)r_{xy}r+(9\Delta_{xx}\\
&+4\Delta_{yy}r^2+7\Delta_{xy}r-2(13\Delta_x+6\Delta_y r\\
&-12r_y\Delta)r_y)r_y r^2-((\Delta_{xxy}+\Delta_{yyy}r^2)r\\
&+\Delta_{xxx}+\Delta_{xyy}r^2)r^2)r-((2((17\Delta_x
\end{aligned}
$$

$$
\begin{aligned}
&+ 5\Delta_y r - 23r_y\Delta)r_y + 6r_{yy}r\Delta) - (6\Delta_{xx}\\
&+ \Delta_{yy}r^2 + 3\Delta_{xy}r))r^2 - (5(3r_x + 8r_y r)\Delta\\
&- 3(5\Delta_x + \Delta_y r)r)r_x)r_x + ((((r_{xx} + 3r_{yy}r^2\\
&+ 2r_{xy}r)\Delta + (5\Delta_x + 3\Delta_y r - 6r_y\Delta)r_y r\\
&- (\Delta_{xx} + \Delta_{yy}r^2 + \Delta_{xy}r)r)r - ((3r_x\\
&+ 7r_y r)\Delta - (3\Delta_x + \Delta_y r)r)r_x)\nu + ((((r_x\\
&+ 2r_y r)\Delta - (\Delta_x + \Delta_y r)r)\omega + \psi_{yyy}\nu r^3\\
&+ (((\beta\psi + \gamma)\varphi_y + \psi_y\alpha)\varphi_y r + \psi_{yy}\omega r\\
&- \alpha\Delta)\varphi_y r^2)\varphi_y + \psi_{yyyy}r^3)r^2)\varphi_y r^2)\varphi_y^3\\
&- (((10(((r_{xx} + 3r_{yy}r^2 + 2r_{xy}r)\Delta + (5\Delta_x\\
&+ 3\Delta_y r - 6r_y\Delta)r_y r - (\Delta_{xx} + \Delta_{yy}r^2\\
&+ \Delta_{xy}r)r)r - ((3r_x + 7r_y r)\Delta - (3\Delta_x\\
&+ \Delta_y r)r)r_x) + (6(((r_x + 2r_y r)\Delta - (\Delta_x\\
&+ \Delta_y r)r)\nu + \psi_{yyy}r^3) + (3(\psi_{yy}\nu r - \omega\Delta)\\
&+ \varphi_y\psi_y\omega r)\varphi_y r^2)\varphi_y r^2)\varphi_y^2 - 3(((5(\psi_{yy}r\\
&- \nu\Delta) + \varphi_y\psi_y\nu r)\varphi_y r^2 + 15((r_x + 2r_y r)\Delta\\
&- (\Delta_x + \Delta_y r)r))\varphi_y - 5(\varphi_y\psi_y r\\
&- 7\Delta)\varphi_{yy}r^2)\varphi_{yy}r^2)\varphi_{yy} + ((((4(\psi_{yy}r - \nu\Delta)\\
&+ \varphi_y\psi_y\nu r)\varphi_y r^2 + 10((r_x + 2r_y r)\Delta - (\Delta_x\\
&+ \Delta_y r)r))\varphi_y - 10(\varphi_y\psi_y r - 6\Delta)\varphi_{yy}r^2)\varphi_{yyy}\\
&+ (\varphi_y\psi_y r - 5\Delta)\varphi_{yyyy}\varphi_y r^2)\varphi_y r^2)r^2)/(\varphi_y^3\Delta). \qquad \text{(B.19)}
\end{aligned}
$$

C. Equations for Theorem 2.5 in Section II.

$$
\begin{aligned}
\omega =&(10\varphi_{yyy}\varphi_y\Delta^2 + 15\varphi_{yy}^2\Delta^2 - 24\varphi_{yy}\varphi_y\Delta_y\Delta\\
&+ 6\varphi_{yy}\varphi_y F_2\Delta^2 - 6\varphi_y^2\Delta_{yy}\Delta + 12\varphi_y^2\Delta_y^2\\
&- 3\varphi_y^2\Delta_y F_2\Delta + \varphi_y^2 J_4\Delta^2)/(\varphi_y^4\Delta^2), \qquad \text{(C.1)}\\
F_{1yy} =& - (F_{1y}F_2 - 40F_{2xy} - 16F_{2x}F_2 + 20F_{2yy}r\\
&+ 40F_{2y}r_y + 14F_{2y}F_2 r + 20J_{4x} - 20J_{4y}r\\
&+ 14r_y F_2^2 - 40r_y J_4)/10, \qquad \text{(C.2)}\\
F_{2x} =&(12F_{2y}r - 3F_1F_2 + 6F_2^2 r + 4J_3\\
&- 16J_4 r)/12, \qquad \text{(C.3)}\\
F_{1x} =&(60F_{1y}r - 36F_0F_2 - 15F_1^2 + 66F_1F_2 r\\
&- 36F_2^2 r^2 + 40J_2 - 80J_3 r + 80J_4 r^2)/60, \qquad \text{(C.4)}\\
F_{0x} =&(60F_{0y}r - 51F_0F_1 + 66F_0F_2 r + 36F_1^2 r\\
&- 72F_1F_2 r^2 + 36F_2^2 r^3 + 60J_1 - 80J_2 r\\
&+ 80J_3 r^2 - 80J_4 r^3)/60, \qquad \text{(C.5)}\\
J_0 =&(9F_0^2 - 18F_0F_1 r + 18F_0F_2 r^2 + 9F_1^2 r^2\\
&- 18F_1F_2 r^3 + 9F_2^2 r^4 + 20J_1 r - 20J_2 r^2\\
&+ 20J_3 r^3 - 20J_4 r^4)/20, \qquad \text{(C.6)}\\
\gamma =&(\varphi_{yyyy}\varphi_y\psi_y\Delta^2 + 10\varphi_{yyy}\varphi_{yy}\psi_y\Delta^2\\
&- 6\varphi_{yyy}\varphi_y\psi_{yy}\Delta^2 - 4\varphi_{yyy}\varphi_y\psi_y\Delta_y\Delta\\
&+ \varphi_{yyy}\varphi_y\psi_y F_2\Delta^2 - 12\varphi_{yy}^2\psi_y\Delta_y\Delta\\
&+ 3\varphi_{yy}^2\psi_y F_2\Delta^2 - 4\varphi_{yy}\varphi_y\psi_{yyy}\Delta^2\\
&+ 12\varphi_{yy}\varphi_y\psi_{yy}\Delta_y\Delta - 3\varphi_{yy}\varphi_y\psi_{yy}F_2\Delta^2\\
&- 6\varphi_{yy}\varphi_y\psi_y\Delta_{yy}\Delta + 12\varphi_{yy}\varphi_y\psi_y\Delta_y^2\\
&- 3\varphi_{yy}\varphi_y\psi_y\Delta_y F_2\Delta + \varphi_{yy}\varphi_y\psi_y J_4\Delta^2\\
&- \varphi_y^6\beta\psi\Delta^2 - \varphi_y^5\psi_y\alpha\Delta^2 - \varphi_y^2\psi_{yyyy}\Delta^2\\
&+ 4\varphi_y^2\psi_{yyy}\Delta_y\Delta - \varphi_y^2\psi_{yyy}F_2\Delta^2\\
&+ 6\varphi_y^2\psi_{yy}\Delta_{yy}\Delta - 12\varphi_y^2\psi_{yy}\Delta_y^2\\
&+ 3\varphi_y^2\psi_{yy}\Delta_y F_2\Delta - \varphi_y^2\psi_{yy}J_4\Delta^2\\
&+ \varphi_y K_7\Delta^3)/(\varphi_y^6\Delta^2), \qquad \text{(C.7)}\\
\alpha =&(5\varphi_{yyyy}\varphi_y\Delta^3 + 10\varphi_{yyy}\varphi_{yy}\Delta^3\\
&- 16\varphi_{yyy}\varphi_y\Delta_y\Delta^2 + 4\varphi_{yyy}\varphi_y F_2\Delta^3\\
&- 12\varphi_{yy}^2\Delta_y\Delta^2 + 3\varphi_{yy}^2 F_2\Delta^3\\
&- 18\varphi_{yy}\varphi_y\Delta_{yy}\Delta^2 + 36\varphi_{yy}\varphi_y\Delta_y^2\Delta\\
&- 9\varphi_{yy}\varphi_y\Delta_y F_2\Delta^2 + 3\varphi_{yy}\varphi_y J_4\Delta^3\\
&- 4\varphi_y^2\Delta_{yyy}\Delta^2 + 24\varphi_y^2\Delta_{yy}\Delta_y\Delta\\
&- 3\varphi_y^2\Delta_{yy}F_2\Delta^2 - 24\varphi_y^2\Delta_y^3\\
&+ 6\varphi_y^2\Delta_y^2 F_2\Delta - 2\varphi_y^2\Delta_y J_4\Delta^2\\
&- \varphi_y^2 K_6\Delta^3 + 7\varphi_y^2 K_7 r\Delta^3)/(\varphi_y^5\Delta^3), \qquad \text{(C.8)}\\
J_{3yy} =&(216F_{1y}F_{2y} + 54F_{1y}F_2^2 - 48F_{1y}J_4\\
&+ 360F_{2yy}r_y + 90F_{2yy}F_1 - 180F_{2yy}F_2 r\\
&- 432F_{2y}^2 r + 324F_{2y}r_y F_2 + 189F_{2y}F_1F_2\\
&- 486F_{2y}F_2^2 r - 192F_{2y}J_3 + 864F_{2y}J_4 r\\
&- 60J_{3y}F_2 + 720J_{4xy} + 180J_{4x}F_2\\
&- 240J_{4yy}r - 1200J_{4y}r_y + 60J_{4y}F_2 r\\
&+ 720K_{6x} - 720K_{6y}r - 5040K_{7x}r\\
&+ 5040K_{7y}r^2 + 36r_y F_2^3 - 432r_y F_2 J_4\\
&- 2160r_y K_6 + 15120r_y K_7 r + 504F_0K_7\\
&+ 36F_1F_2^3 - 102F_1F_2J_4 - 504F_1K_7 r\\
&- 72F_2^4 r - 48F_2^2 J_3 + 396F_2^2 J_4 r\\
&+ 504F_2K_7 r^2 + 136J_3J_4 - 544J_4^2 r)/120, \qquad \text{(C.9)}\\
\beta =&(4\varphi_{yyyyy}\Delta^4 - 16\varphi_{yyyy}\Delta_y\Delta^3\\
&+ 4\varphi_{yyyy}F_2\Delta^4 - 24\varphi_{yyy}\Delta_{yy}\Delta^3\\
&+ 48\varphi_{yyy}\Delta_y^2\Delta^2 - 12\varphi_{yyy}\Delta_y F_2\Delta^3\\
&+ 4\varphi_{yyy}J_4\Delta^4 - 16\varphi_{yy}\Delta_{yyy}\Delta^3\\
&+ 96\varphi_{yy}\Delta_{yy}\Delta_y\Delta^2 - 12\varphi_{yy}\Delta_{yy}F_2\Delta^3\\
&- 96\varphi_{yy}\Delta_y^3\Delta + 24\varphi_{yy}\Delta_y^2 F_2\Delta^2\\
&- 8\varphi_{yy}\Delta_y J_4\Delta^3 - 4\varphi_{yy}K_6\Delta^4\\
&+ 28\varphi_{yy}K_7 r\Delta^4 - 4\varphi_y K_{7x}\Delta^4\\
&+ 4\varphi_y K_{7y}r\Delta^4 - 4\varphi_y\Delta_{yyyy}\Delta^3\\
&+ 32\varphi_y\Delta_{yyy}\Delta_y\Delta^2 - 4\varphi_y\Delta_{yyy}F_2\Delta^3\\
&+ 24\varphi_y\Delta_{yy}^2\Delta^2 - 144\varphi_y\Delta_{yy}\Delta_y^2\Delta\\
&+ 24\varphi_y\Delta_{yy}\Delta_y F_2\Delta^2 - 4\varphi_y\Delta_{yy}J_4\Delta^3\\
&+ 96\varphi_y\Delta_y^4 - 24\varphi_y\Delta_y^3 F_2\Delta\\
&+ 8\varphi_y\Delta_y^2 J_4\Delta^2 + 4\varphi_y\Delta_y K_6\Delta^3\\
&- 28\varphi_y\Delta_y K_7 r\Delta^3 - \varphi_y F_1K_7\Delta^4\\
&+ 2\varphi_y F_2K_7 r\Delta^4)/(4\varphi_y^5\Delta^4), \qquad \text{(C.10)}\\
J_{4xyy} =& - (36F_{1y}F_{2yy} + 162F_{1y}F_{2y}F_2\\
&- 72F_{1y}J_{4y} + 36F_{1y}F_2^3\\
&- 168F_{1y}F_2J_4 - 72F_{1y}K_6\\
&- 168F_{1y}K_7 r - 72F_{2yy}F_{2y}r\\
&+ 144F_{2yy}r_y F_2 + 54F_{2yy}F_1F_2\\
&- 108F_{2yy}F_2^2 r - 72F_{2yy}J_3\\
&+ 288F_{2yy}J_4 r + 432F_{2y}^2 r_y
\end{aligned}
$$

$$
\begin{aligned}
&+108F_{2y}^2F_1-540F_{2y}^2F_2r\\
&-144F_{2y}J_{3y}+528F_{2y}J_{4x}\\
&+192F_{2y}J_{4y}r+324F_{2y}r_yF_2^2\\
&-1008F_{2y}r_yJ_4+162F_{2y}F_1F_2^2\\
&-132F_{2y}F_1J_4-396F_{2y}F_2^3r\\
&-180F_{2y}F_2J_3+1320F_{2y}F_2J_4r\\
&+144F_{2y}K_6r-336F_{2y}K_7r^2\\
&-36J_{3y}F_2^2+176J_{3y}J_4\\
&+120J_{4xy}F_2+132J_{4x}F_2^2\\
&-432J_{4x}J_4-240J_{4yyy}r\\
&-960J_{4yy}r_y-120J_{4yy}F_2r\\
&-768J_{4y}r_yF_2-138J_{4y}F_1F_2\\
&+288J_{4y}F_2^2r+184J_{4y}J_3\\
&-1008J_{4y}J_4r+960K_{6xy}\\
&+240K_{6x}F_2-960K_{6yy}r\\
&-3840K_{6y}r_y-240K_{6y}F_2r\\
&-1920K_{7xy}r-2400K_{7xx}\\
&+2880K_{7x}r_y-600K_{7x}F_1\\
&-480K_{7x}F_2r+4320K_{7yy}r^2\\
&+24000K_{7y}r_yr+432K_{7y}F_0\\
&+168K_{7y}F_1r+912K_{7y}F_2r^2\\
&+20160r_y^2K_7+1728r_yF_1K_7\\
&+36r_yF_2^4-264r_yF_2^2J_4\\
&-1248r_yF_2K_6+5280r_yF_2K_7r\\
&+160r_yJ_4^2+408F_0F_2K_7\\
&+150F_1^2K_7+27F_1F_2^4-120F_1F_2^2J_4\\
&-168F_1F_2K_6+168F_1F_2K_7r\\
&-54F_2^5r-36F_2^3J_3+384F_2^3J_4r\\
&+336F_2^2K_6r-1344F_2^2K_7r^2\\
&+160F_2J_3J_4-640F_2J_4^2r-400J_2K_7\\
&+224J_3K_6-368J_3K_7r-896J_4K_6r\\
&+3872J_4K_7r^2+672F_{0y}K_7)/240, \qquad (C.11)
\end{aligned}
$$

$$
\begin{aligned}
J_{4x}=&(4J_{4y}r-F_1J_4+2F_2J_4r-4K_5\\
&+24K_6r-84K_7r^2)/4, \qquad (C.12)
\end{aligned}
$$

$$
\begin{aligned}
K_{5yy}=&(672F_{0y}K_7+36F_{1y}F_{2yy}\\
&+162F_{1y}F_{2y}F_2-216F_{1y}J_{4y}\\
&+36F_{1y}F_2^3-144F_{1y}F_2J_4\\
&-216F_{1y}K_6+840F_{1y}K_7r\\
&-72F_{2yy}F_{2y}r+144F_{2yy}r_yF_2\\
&+54F_{2yy}F_1F_2-108F_{2yy}F_2^2r\\
&-72F_{2yy}J_3+288F_{2yy}J_4r\\
&+432F_{2y}^2r_y+108F_{2y}^2F_1\\
&-540F_{2y}^2F_2r-144F_{2y}J_{3y}\\
&+1008F_{2y}J_{4y}r+324F_{2y}r_yF_2^2\\
&-768F_{2y}r_yJ_4+162F_{2y}F_1F_2^2\\
&-204F_{2y}F_1J_4-396F_{2y}F_2^3r\\
&-180F_{2y}F_2J_3+1416F_{2y}F_2J_4r\\
&-528F_{2y}K_5+3600F_{2y}K_6r\\
&-13440F_{2y}K_7r^2-36J_{3y}F_2^2\\
&+96J_{3y}J_4-480J_{4yy}r_y\\
&-60J_{4yy}F_1+120J_{4yy}F_2r\\
&-384J_{4y}r_yF_2-174J_{4y}F_1F_2\\
&+492J_{4y}F_2^2r+192J_{4y}J_3\\
&-1152J_{4y}J_4r-120K_{5y}F_2\\
&+960K_{6xy}+240K_{6x}F_2\\
&+480K_{6yy}r-960K_{6y}r_y\\
&+480K_{6y}F_2r-1920K_{7xy}r\\
&-2400K_{7xx}+2880K_{7x}r_y\\
&-600K_{7x}F_1-480K_{7x}F_2r\\
&-720K_{7yy}r^2+3840K_{7y}r_yr\\
&+432K_{7y}F_0+168K_{7y}F_1r\\
&-1608K_{7y}F_2r^2+10080r_y^2K_7\\
&+1728r_yF_1K_7+36r_yF_2^4\\
&-228r_yF_2^2J_4-384r_yF_2K_6\\
&-768r_yF_2K_7r+240r_yJ_4^2\\
&+408F_0F_2K_7+150F_1^2K_7\\
&+27F_1F_2^4-132F_1F_2^2J_4\\
&-204F_1F_2K_6+420F_1F_2K_7r\\
&+78F_1J_4^2-54F_2^5r-36F_2^3J_3\\
&+408F_2^3J_4r-132F_2^2K_5\\
&+1200F_2^2K_6r-4620F_2^2K_7r^2\\
&+132F_2J_3J_4-684F_2J_4^2r\\
&-400J_2K_7+272J_3K_6\\
&-704J_3K_7r+312J_4K_5\\
&-2960J_4K_6r+11768J_4K_7r^2)/240, \qquad (C.13)
\end{aligned}
$$

$$
\begin{aligned}
F_{0yy}=&-(30F_{0y}F_2+36F_{1y}F_1-36F_{1y}F_2r\\
&-60F_{2yy}r^2+24F_{2y}F_0-36F_{2y}F_1r\\
&-54F_{2y}F_2r^2-40J_{2y}+40J_{3y}r\\
&+80J_{4y}r^2-36r_yF_1F_2+36r_yF_2^2r\\
&+40r_yJ_3-80r_yJ_4r+6F_0F_2^2\\
&-6F_0J_4+9F_1^2F_2-18F_1F_2^2r\\
&-12F_1J_3+24F_1J_4r-6F_2^3r^2\\
&-10F_2J_2+22F_2J_3r+26F_2J_4r^2\\
&-60K_4+180K_5r-180K_6r^2\\
&-420K_7r^3)/60, \qquad (C.14)
\end{aligned}
$$

$$
\begin{aligned}
J_{2x}=&(20J_{2y}r+20J_{3x}r-20J_{3y}r^2\\
&-14F_0J_3+28F_0J_4r-5F_1J_2\\
&+19F_1J_3r-28F_1J_4r^2+10F_2J_2r\\
&-24F_2J_3r^2+28F_2J_4r^3-120K_3\\
&+360K_4r-640K_5r^2+840K_6r^3\\
&-840K_7r^4)/20, \qquad (C.15)
\end{aligned}
$$

$$
\begin{aligned}
J_{1x}=&(60J_{1y}r-40J_{3x}r^2+40J_{3y}r^3\\
&-42F_0J_2+42F_0J_3r-70F_0J_4r^2\\
&-15F_1J_1+42F_1J_2r-52F_1J_3r^2\\
&+70F_1J_4r^3+30F_2J_1r-42F_2J_2r^2\\
&+62F_2J_3r^3-70F_2J_4r^4-600K_2\\
&+1080K_3r-1380K_4r^2+1700K_5r^3\\
&-2100K_6r^4+2100K_7r^5)/60, \qquad (C.16)
\end{aligned}
$$

$$
\begin{aligned}
K_1 = (&3F_0^2F_1 - 6F_0^2F_2r - 6F_0F_1^2r \\
&+ 18F_0F_1F_2r^2 - 12F_0F_2^2r^3 - 8F_0J_1 \\
&+ 16F_0J_2r - 24F_0J_3r^2 + 32F_0J_4r^3 \\
&+ 3F_1^3r^2 - 12F_1^2F_2r^3 + 15F_1F_2^2r^4 \\
&+ 8F_1J_1r - 16F_1J_2r^2 + 24F_1J_3r^3 \\
&- 32F_1J_4r^4 - 6F_2^3r^5 - 8F_2J_1r^2 \\
&+ 16F_2J_2r^3 - 24F_2J_3r^4 + 32F_2J_4r^5 \\
&+ 160K_2r - 240K_3r^2 + 320K_4r^3 \\
&- 400K_5r^4 + 480K_6r^5 - 560K_7r^6)/80, \qquad \text{(C.17)}
\end{aligned}
$$

$$
\begin{aligned}
K_0 = -(&6F_0^3 - 33F_0^2F_1r + 48F_0^2F_2r^2 \\
&+ 48F_0F_1^2r^2 - 126F_0F_1F_2r^3 + 78F_0F_2^2r^4 \\
&+ 40F_0J_1r - 80F_0J_2r^2 + 120F_0J_3r^3 \\
&- 160F_0J_4r^4 - 21F_1^3r^3 + 78F_1^2F_2r^4 \\
&- 93F_1F_2^2r^5 - 40F_1J_1r^2 + 80F_1J_2r^3 \\
&- 120F_1J_3r^4 + 160F_1J_4r^5 + 36F_2^3r^6 \\
&+ 40F_2J_1r^3 - 80F_2J_2r^4 + 120F_2J_3r^5 \\
&- 160F_2J_4r^6 - 400K_2r^2 + 800K_3r^3 \\
&- 1200K_4r^4 + 1600K_5r^5 - 2000K_6r^6 \\
&+ 2400K_7r^7)/400. \qquad \text{(C.18)}
\end{aligned}
$$

References

[1] S. Lie, “Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen x, y, die eine Gruppe von Transformationen gestatten. III,” *Archiv for Matematik og Naturvidenskab*, vol. 8, no. 4, pp. 371-427, 1883.

[2] R. Liouville, “Sur les Invariantes de Certaines Equations,” *J. de lÉcole Polytechnique*, vol. 59, pp. 7-76, 1889.

[3] A. M. Tresse, “Détermination des Invariants Ponctuels de lÉquation Différentielle Ordinaire du Second Ordre $y'' = f(x, y, y')$,” *Preisschriften der fürstlichen Jablonowski schen Geselschaft XXXII, Leipzig, S.Herzel,* 1896.

[4] S. Suksern and C. Sawatdithep, “Reduction of second-order ordinary differential equations into general linear equations via point transformations and its application,” *Appl. Math. Inf. Sci.*, vol. 14, no. 2, pp. 249-259, 2020.

[5] W. Sinkala, “Some Remarks on the Solution of Linearisable Second-Order Ordinary Differential Equations via Point Transformations,” *Journal of Mathematics*, vol. 2020, no. 1, pp. 1-5, 2020.

[6] A. V. Bocharov, V. V. Sokolov and S. I. Svinolupov, “On some equivalence problems for differential equations,” *Preprint ESI*, vol. 54, no. 1, pp. 12 pages, 1993.

[7] G. Grebot, “The characterization of third order ordinary differential equations admitting a transtive fiber-preserving point symmetry group,” *Journal of Mathematical Analysis and Applications*, vol. 206, pp. 364-388, 1997.

[8] N. H. Ibragimov and S. V. Meleshko, “Linearization of third-order ordinary differential equations by point and contact transformations,” *Journal of Mathematical Analysis and Applications*, vol. 308, pp. 266-289, 2005.

[9] A. Y. Al-Dweik, “Third-order ordinary differential equations $y''' = f(x, y, y', y'')$ with maximal symmetry group,” *Quaestiones Mathematicae*, vol. 39, no. 5, pp. 595-601, 2016.

[10] A. Y. Al-Dweik, M. T. Mustafa, F. M. Mahomed and R. S. Alassar, “Linearization of third-order ordinary differential equations $u''' = f(x, u, u', u'')$ via point transformations,” *Mathematical Methods in the Applied Sciences*, vol. 41, no. 16, pp. 6955-6967, 2018.

[11] S. Suksern and B. Sookcharoenpinyo, “Applications of point transformation on third-order ordinary differential equations,” *IAENG International Journal of Computer Science*, vol. 50, no. 2, pp. 662-672, 2023.

[12] N. H. Ibragimov, S. V. Meleshko and S. Suksern, “Linearization of fourth-order ordinary differential equations by point transformations,” *Journal Of Physics A: Mathematical and Theoretical*, vol. 41, 19 pages, 2008.

[13] D. Lyakhov, V. P. Gerdt and D. L. Michels, “Algorithmic Verification of Linearizability for Ordinary Differential Equations,” *in Proceedings of the 2017 ACM on International Symposium on Symbolic and Algebraic Computation - ISSAC* $'17$, Kaiserslautern, Germany, pp. 285-292, 2017.

[14] H. M. Dutt, and A. Qadir, “Classification of Scalar Fourth Order Ordinary Differential Equations Linearizable via Generalized Lie-Bäcklund Transformations,” *in Springer Proceedings in Mathematics & Statistics*, vol. 266, pp. 67-74, 2018.

[15] M. Euler, N. Euler and P. G. L. Leach, “The Riccati and Ermakov-Pinney Hierarchies,” *Preprint: Institut Mittag-Leffler, The Royal Swedish Academy of Sciences, Auravägen 17, SE-182 60 Djursholm, Sweden*, 2005.

[16] W. Nakpim and S. V. Meleshko, “Linearization of third-order ordinary differential equations by generalized Sundman transformations: The case $X''' + \alpha X = 0$,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, pp. 1717-1723, 2010.

[17] N. Euler and M. Euler, “Sundman symmetries of nonlinear second-order and third-order ordinary differential equations,” *Journal of Nonlinear Mathematical Physics*, vol. 11, no. 3, pp. 399-421, 2004.

[18] P. Guha, A. G. Choudhury and B. Khanra, “On solutions of third and fourth-order time dependent Riccati equations and the generalized Chazy system,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 11, pp. 4053-4063, 2012.

[19] M. J. Ablowitz, S. Chakravarty and R. Halburd, “The generalized Chazy equation and Schwarzian triangle functions,” *Asian J. Math.*, vol. 2, no. 4, pp. 619-624, 1998.

[20] J. Chazy, “Sur les équations différentielles dont líntégrale générale est uniforme et admet des singularities essentielles mobiles,” *C.R. Acad. Sc. Paris*, vol. 149, pp. 563-565, 1909.

[21] J. Chazy, “Sur les équations différentielles dont líntégrale générale posséde une coupure essentielle mobile,” *C.R. Acad. Sc. Paris*, vol. 150, pp. 456-458, 1910.

[22] J. Chazy, “Sur les équations différentielles du troisiéme et dórdre supérieur dont líntégrale générale á ses points critiques fixés,” *Acta Math.*, vol. 34, pp. 317-385, 1911.

[23] P. A. Clarkson and P. J. Olver, “Symmetry and the Chazy equation,” *J. Diff. Eq.*, vol. 124, pp. 225-246, 1996.

[24] A. Volpert, “Traveling wave solutions of parabolic systems,” *Amer. Math. Soc.*, vol. 32, no. 4, 1995.

[25] J. Murray, *Mathematical Biology*, Biomathematics, Springer Berlin Heidelberg, 2013.

[26] P. A. Clarkson and T. J. Priestley, “Symmetries of a class of nonlinear fourth order partial differential equations,” *J. Nonlinear Math. Phys.*, vol. 6, no. 1, pp. 66-98, 1999.

[27] H. S. Bhat and R. C. Fetecau, “A Hamiltonian regularization of the Burgers equation,” *J. Nonlinear Sci*, vol. 16, pp. 615-638, 2006.

[28] R. Camassa, P. H. Chiu, L. Lee and T. W. H. Sheu, “Viscous and inviscid regularizations in a class of evolutionary partial differential equations,” *Journal of Computational Physics*, vol. 229, pp. 6676-6687, 2010.

Supaporn Suksern was born in Chiang Rai, Thailand, in 1978. She received her B.Sc. degree in mathematics from Chiang Mai University, Chiang Mai, Thailand, in 2001, and M.Sc. degree in applied mathematics from Chiang Mai University, Chiang Mai, Thailand, in 2004. She received her Ph.D. degree in applied mathematics from Suranaree University of Technology, Nakhonratchasima, Thailand, in 2009. Currently, she is an Associate Professor in the Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok, Thailand. Her research interests are linearization problems and differential equations.

Supakit Tammakun was born in Phayao, Thailand, in 1991. He received his B.Sc. degree in mathematics from Naresuan University, Phitsanulok, Thailand, in 2014, and M.Sc. degree in mathematics from Naresuan University, Phitsanulok, Thailand, in 2023. Currently, he is a lecturer at the Faculty of Oriental Medicine, Chiang Rai College, Chiang Rai, Thailand. His research interests are linearization problems and differential equations.