Integer and Fractional Order q(Alpha)-Delta Integration, Sum and Its Fundamental Theorems

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Abstract—**This research focuses on developing discrete fundamental theorems related to $q(\alpha)$-delta anti-difference operator, applied to a class of $q(\alpha)$-delta integrable functions. The fractional sum of a function f can be expressed in two ways: as a closed form and as a summation form. This dual representation motivates the creation of a new technique to derive several identities for $q(\alpha)$-delta integrable functions, which possess both discrete anti-difference and integer summation referred to as discrete fundamental theorems. Additionally, by introducing the concept of ∞-order $q(\alpha)$-delta integrable functions, we derive the discrete integral related to the fractional sum of f using Newton's method. The theoretical results are illustrated and validated by numerical examples.**

Index Terms—**Discrete integration, Delta integrable function, Closed form, Summation form, Fractional sum.**

I. Introduction

DISCRETE fractional calculus has attracted considerable focus in the literature over the past several decades [1], [11], [12], [17], [19]. Difference equations are meant for discrete process where as the differential equations deals with continuous system. In [2], [13], [14], the authors applied a discrete case approach to a continuous scenario to determine closed form solutions for both continuous and discrete fractional order integration. They derived various theorems and formulas using the Riemann-Liouville fractional integral, incorporating the gamma function. Additionally, they formulated the discrete counterpart of the continuous ν^{th}-fractional order integration. A key innovation of this work lies in the introduction of fractional-order exponential functions and the development of corresponding theorems.

In [4], the authors explored the application of the forward hybrid delta operator, incorporating a shift value, to derive a generalized infinite series for fractional hybrid summation formulas. Additionally, they presented numerical closed form solutions for fractional order hybrid difference equations. In [5], [8], the researchers presented a discrete-time fractional calculus of variations on the time scale $(hZ)_a$, $a \in \mathbb{R}$, $h > 0$. Necessary optimality conditions of first and second order are derived. Also, they provided the examples demonstrating the application of the newly developed Euler-Lagrange and Legendre-type conditions. Addition to this, the authors in [8] proposed a novel approach to quantum calculus by introducing q-symmetric variational calculus. In [6], the authors introduced the q-difference operator Δ_q and presents key findings on the inverse of the q-difference operator of the t^{th}-order, utilizing generalized polynomial factorials and the Stirling numbers of the second kind.

In the articles [7], [15], the authors derived the solutions for a generalized α_i-mixed difference equation in closed form as well as in finite and infinite multi-series representations. By equating the closed-form solutions with the multi-series solutions of the α_i-mixed difference equation, they determined the values of specific finite and infinite multi-series formula. In [9], the researchers explored q-integral representations of the q-gamma and q-beta functions, which reveal a noteworthy q-constant. As an application of these representations, they provided a straightforward conceptual proof of a family of identities related to the Jacobi triple product, including Jacobi's identity, as well as Ramanujan's formula for the bilateral hypergeometric series.

In this study [10], the theory of fractional h-difference equations were introduced and enhanced with valuable tools for explicitly solving discrete equations involving left and right fractional difference operators. In [16], the authors explored various properties of q-exponential functions, both standard and symmetric, for general nonzero complex q. In [18], [20], the authors derived both numerical and closed-form solutions for fractional-order Fibonacci difference equations. Also, they developed generalized infinite series for fractional Fibonacci summation formulas using the forward Fibonacci delta operator with various parameters and its inverse on real-valued functions. In [20], they explored various products of k-Fibonacci and k-Lucas numbers. Also, they presented generalized identities involving these products and establish connection formulas between them using Binet's formula.

The goal our research is to develop discrete fundamental theorems for a class of delta integrable functions using a novel mechanism known as the delta integration method. The ν^{th}-fractional sum of a function f has two forms: closed and summation form, which concept is applied to $q(\alpha)$-delta integration and its sum. Our finding are the extension of the results developed for h-delta operator in [3] and validated with numerical examples.

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II. Preliminaries Related To $q(\alpha)$ - Delta Integration

In this section, we present basic definitions of falling factorials, the $q(\alpha)$-delta operator and summation formula arrived by inverse of $q(\alpha)$-delta operator. It is clear that, whenever f is defined on a set $aq^{\mathbb{Z}} = \{\ldots, aq^{-2}, aq^{-1}, a, aq, aq^2, \ldots\}$, then $\Delta_q f$ is also defined on the set $aq^{\mathbb{Z}}$ for $a \in \mathbb{R} = (-\infty, \infty)$.

Definition II.1. For $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, the n^{th}-order falling factorial of t, denoted as $t_q^{(n)}$, is defined by

$$t_q^{(n)} = \prod_{r=0}^{n-1}(t - q^r) \text{ and } t_q^{(0)} = 1, \ t \in \mathbb{R}.$$

Definition II.2. Let $a, q \neq 0$ and $aq^{\mathbb{Z}} = \{\ldots, aq^{-2}, aq^{-1}, a, aq, aq^2, \ldots\}$. Let $f : aq^{\mathbb{Z}} \to \mathbb{R}$. The $q(\alpha)$-delta operator on f is defined as

$$\Delta_{q(\alpha)} f(t) = f(tq) - \alpha f(t), \ t \in aq^{\mathbb{Z}}. \tag{1}$$

The inverse of $q(\alpha)$-delta operator on f is defined by, if there exists $f_1 : aq^{\mathbb{Z}} \to \mathbb{R}$ such that

$$\Delta_{q(\alpha)} f_1(t) = f(t) \Leftrightarrow f_1(t) + c = \Delta_{q(\alpha)}^{-1} f(t), \tag{2}$$

where c is an arbitrary constants. Here, we consider a larger size domain $aq^{\mathbb{Z}}$.

Lemma II.3. *For any positive integer $n > 0$, $q^n \neq \alpha$ and $t \in (-\infty, \infty)$, then $q(\alpha)$-delta operator for the n^{th}-order falling factorial $t_q^{(n)}$ is given by,*

$$\Delta_{q(\alpha)} t_q^{(n)} = (q^n - \alpha) t_q^{(n)}. \tag{3}$$

Proof: From the definition of $q(\alpha)$-delta operator,

$$\begin{aligned}\Delta_{q(\alpha)} t_q^{(n)} &= (tq)_q^{(n)} - \alpha t_q^{(n)} \\ &= tq(t-q)q(t-q^2)q \cdots (t-q^{n-1})q \\ &\quad -\alpha t(t-q)(t-q^2)\cdots(t-q^{n-1}) \\ &= (q^n - \alpha)[t(t-q)(t-q^2)\cdots(t-q^{n-1})]\end{aligned}$$

which gives the equation (3). ■

Lemma II.4. *For any positive integer $n > 0$, $q^n \neq \alpha$ and $t \in (-\infty, \infty)$, then the inverse $q(\alpha)$-delta operator for the n^{th}-order falling factorial $t_q^{(n)}$ is given by,*

$$\Delta_{q(\alpha)}^{-1} t_q^{(n)} = \frac{t_q^{(n)}}{(q^n - \alpha)}. \tag{4}$$

Proof: The proof follows by applying $\Delta_{q(\alpha)}^{-1}$ operator on both sides of the equation (3) . ■

Lemma II.5. *For any positive integer $n > 0$, $q^n \neq \alpha$ and $t \in (-\infty, \infty)$, then the higher order $q(\alpha)$-delta operator for the n^{th}-order falling factorial $t_q^{(n)}$ is given by,*

$$\Delta_{q(\alpha)}^{m} t_q^{(n)} = (q^n - \alpha)^m t_q^{(n)} \tag{5}$$

and its inverse $q(\alpha)$-delta operator for the n^{th}-order falling factorial $t_q^{(n)}$ is given by,

$$\Delta_{q(\alpha)}^{-m} t_q^{(n)} = \frac{t_q^{(n)}}{(q^n - \alpha)^m}. \tag{6}$$

Proof: The proof follows by applying $\Delta_{q(\alpha)}$ and $\Delta_{q(\alpha)}^{-1}$ operator m times on the equations (3) and (4) respectively. ■

Theorem II.6. ***(First Fundamental Theorem)*** *If $\Delta_{q(\alpha)} f_1 = f$ and $m \in \mathbb{N}$, then*

$$f_1(aq^{m+1}) - \alpha^{m+1} f_1(a) = \sum_{s=0}^{m} \alpha^{m-s} f(aq^s). \tag{7}$$

Proof: The given condition $\Delta_q f_1 = f$, and (1) yield for $t \in aq^{\mathbb{Z}}$,

$$f_1(tq) = f(t) + \alpha f_1(t). \tag{8}$$

Replacing t by t/q in (8), we get

$$f_1(t) = f(t/q) + \alpha f_1(t/q). \tag{9}$$

Now, we substitute (9) in (8) to obtain,

$$f_1(tq) = f(t) + \alpha f(t/q) + \alpha^2 f_1(t/q). \tag{10}$$

Simillarly $f_1(t/q), f_1(t/q^2), f_1(t/q^3), \ldots$ and $f_1(t/q^m)$ are obtained by replacing t by $t/q^2, t/q^3, \ldots, t/q^m$ respectively in (9) and then substituting repeatedly all these values again in (10) yields

$$\begin{aligned}f_1(tq) &= f(t) + \alpha f(t/q) + \alpha^2 f(t/q^2) + \alpha^3 f(t/q^3) \\ &\quad + \cdots + \alpha^m f(t/q^m) + \alpha^{m+1} f_1(t/q^m).\end{aligned} \tag{11}$$

Now, (7) follows by taking $t = aq^m$ in (11) and $m \in \mathbb{N}$. ■

The following Corollary II.7 motivates us to develop integer order $q(\alpha)$-delta integration of certain function.

Corollary II.7. *Let $t = aq^m$, $m \in \mathbb{N}$ and $\Delta_{q(\alpha)}^{-1} f = f_1$. Then*

$$\Delta_{q(\alpha)}^{-1} f(aq^m) - \alpha^m \Delta_{q(\alpha)}^{-1} f(a) = \sum_{s=0}^{m-1} \alpha^{m-1-s} f(aq^s). \tag{12}$$

Proof: *Since $t = aq^m$, the proof follows by taking $\Delta_{q(\alpha)}^{-1} f = f_1$, (11) and replacing m by $m-1$ in Theorem II.6.*

III. Integer Order $q(\alpha)$ - Delta Integration

The relation (7) is a fundamental theorem of $q(\alpha)$-delta integration. The relations (7) as well as (12) can be considered as first order $q(\alpha)$-delta integration of f. We propose a main theorem for integer order $q(\alpha)$-delta integration in this section, which is an extension of equation (12).

Definition III.1. A function $f : aq^{\mathbb{Z}} \to \mathbb{R}$ is called an n^{th}-order $q(\alpha)$-delta integrable function if there exists a sequence of functions, say $(f_1, f_2, \cdots, f_n)$ such that

$$\Delta_{q(\alpha)}^{r} f_r = f, \ r = 1, 2, 3, \cdots, n. \tag{13}$$

The sequence $(f_1, f_2, \cdots, f_n)$ can be called as $q(\alpha)$-delta integrating sequence of f.

Example III.2. *Let $f(t) = t_q^{(n)}$ with $t \in \mathbb{R}$ and $q^n \neq \alpha$ is an m^{th}-order $q(\alpha)$-delta integrable function having integrating sequence $\left(\frac{t_q^{(n)}}{q^n - \alpha}, \frac{t_q^{(n)}}{(q^n - \alpha)^2}, \ldots, \frac{t_q^{(n)}}{(q^n - \alpha)^m}\right)$, Since*

$$f(t) = t_q^{(n)} = \frac{\Delta_{q(\alpha)} t_q^{(n)}}{q^n - \alpha} = \frac{\Delta_{q(\alpha)}^2 t_q^{(n)}}{(q^n - \alpha)^2} = \ldots = \frac{\Delta_{q(\alpha)}^m t_q^{(n)}}{(q^n - \alpha)^m}, \tag{14}$$

$\forall\ m \in \mathbb{N}$.

Definition III.3. Let $f : aq^{\mathbb{Z}} \to \mathbb{R}$ be an $q(\alpha)$-delta integrable function having $q(\alpha)$-delta integrating sequence $(f_1, f_2, \cdots, f_n)$. Let $t = aq^m$ and $m, n \in \mathbb{N}$. The n^{th}-order $q(\alpha)$-delta integration of f based at a is defined by

$$F_a^n(t) := f_n(tq) - \sum_{r=0}^{n-1} \alpha^{m+1-r} \frac{(m+1)^{(r)}}{r!} f_{n-r}(a). \quad (15)$$

Example III.4. *Let $f(t) = t_q^{(k)}$ with $q^k \neq \alpha$, where $t \in aq^{\mathbb{Z}}$. Then the n^{th}-order $q(\alpha)$-delta integration of $t_q^{(n)}$ based at a is defined as,*

$$F_a^n(t) = f_n(tq) - [\alpha^{m+1} f_n(a) + \frac{m+1}{1!}\alpha^m f_{n-1}(a) + \ldots + \frac{m+1^{(n-1)}}{(n-1)!}\alpha^{m-(n-2)} f_1(a)]$$

$$F_a^n(t) = \frac{(tq)_q^{(k)}}{(q^k-\alpha)^n} - \left[\frac{\alpha^{m+1} a_q^{(k)}}{(q^k-\alpha)^n} + \frac{(m+1)\alpha^m a_q^{(k)}}{(1!)(q^k-\alpha)^{n-1}} + \ldots + \frac{(m+1)^{(n-1)}\alpha^{m-(n-2)} a_q^{(k)}}{((n-1)!)(q^k-\alpha)}\right]$$

$$F_a^n(t) = \frac{(tq)_q^{(k)}}{(q^k-\alpha)^n} - \sum_{r=0}^{n-1} \alpha^{m+1-r} \frac{(m+1)^{(r)}}{r!} \frac{a_q^{(k)}}{(q^k-\alpha)^{n-r}}.$$

Theorem III.5. *Consider the conditions given in Theorem II.6 and let $f : aq^{\mathbb{Z}} \to \mathbb{R}$ be having $q(\alpha)$-delta integrating sequence $(f_1, f_2, \cdots, f_n)$, $t = aq^m$ such that $n, m \in \mathbb{N}$ and ${}_a\Delta_{q(\alpha)}^{-n} f(t)$ be the n^{th}-order $q(\alpha)$-delta integration of f based at a. Then*

$${}_a\Delta_{q(\alpha)}^{-n} f(t) := \frac{1}{(n-1)!} \sum_{s=0}^{m-(n-1)} \alpha^{m-(n-1)-s} (m-s)^{(n-1)} f(aq^s). \quad (16)$$

Proof: Since $\Delta_{q(\alpha)} f_1 = f$, by putting $\Delta_{q(\alpha)}^{-1} f = f_1$ in Theorem II.6, we get, the first order $q_{\alpha(\ell)}$-delta integration as

$$\Delta_{q(\alpha)}^{-1} f(tq) - \alpha^{m+1} \Delta_q^{-1} f(a) = f_1(tq) - \alpha^{m+1} f_1(a) = \sum_{s=0}^{m} \alpha^{m-s} f(aq^s) \quad (17)$$

$$f_1(tq) - \alpha^{m+1} f_1(a) = f(t) + \alpha f(t/q) + \alpha^2 f(t/q^2) + \cdots + \alpha^m f(t/q^m) \quad (18)$$

Now applying the inverse $q(\alpha)$-delta operator $\Delta_{q(\alpha)}^{-1}$ on both sides of the above equation yields

$$f_2(tq) - \alpha^{m+1} f_2(a) = f_1(t) + \alpha f_1(t/q) + \alpha^2 f_1(t/q^2) + \cdots + \alpha^m f_1(t/q^m) \quad (19)$$

$$= \Big[f(t/q) + \alpha f(t/q^2) + \alpha^2 f(t/q^3) + \cdots + \alpha^m f_1(t/q^m)\Big] + \alpha\Big[f(t/q^2) + \alpha f(t/q^3) + \alpha^2 f(t/q^4) + \cdots + \alpha^{m-1} f_1(t/q^m)\Big] + \cdots + \alpha^{m-1}\Big[f(t/q^m) + f_1(t/q^m)\Big] + \alpha^m f_1(t/q^m)$$

After simplifying this, we get

$$f_2(tq) - \sum_{r=0}^{1} \alpha^{m+1-r} \frac{(m+1)^{(r)}}{r!} f_{2-r}(a) = \sum_{s=0}^{m-1} \alpha^{m-s-1} (m-s) f(aq^s), \quad (20)$$

which is the second order $q(\alpha)$-delta integration formula. Again applying the $\Delta_{q(\alpha)}^{-1}$ operator on both sides of the equation (20) and then proceeding the steps from equation (19) to (20) yields

$$f_3(tq) - \sum_{r=0}^{2} \alpha^{m+1-r} \frac{(m+1)^{(r)}}{r!} f_{3-r}(a) = \sum_{s=0}^{m-2} \alpha^{m-2-s} \frac{(m-s)^{(2)}}{2!} f(aq^s). \quad (21)$$

Similarly applying the $\Delta_{q(\alpha)}^{-1}$ operator repeatedly upto $n-1$ times and proceeding the similar steps, we will get the $(n-1)^{th}$-order $q(\alpha)$-delta integration as

$$f_{n-1}(tq) - \sum_{r=0}^{n-2} \alpha^{m+1-r} \frac{(m+1)^{(r)}}{r!} f_{n-1-r}(a) = \sum_{s=0}^{m-(n-2)} \alpha^{m-(n-2)-s} \frac{(m-s)^{(n-2)}}{(n-2)!} f(aq^s). \quad (22)$$

Hence, again applying the $\Delta_{q(\alpha)}^{-1}$ operator on both sides of above equation and by equation (22), we can easily find the n-th order $q(\alpha)$-delta integration as

$$f_n(tq) - \sum_{r=0}^{n-1} \alpha^{m+1-r} \frac{(m+1)^{(r)}}{r!} f_{n-r}(a) = \sum_{s=0}^{m-(n-1)} \alpha^{m-(n-1)-s} \frac{(m-s)^{(n-1)}}{(n-1)!} f(aq^s) \quad (23)$$

The proof completes by taking

$${}_a\Delta_{q(\alpha)}^{-n} f(t) = f_n(tq) - \sum_{r=0}^{n-1} \alpha^{m+1-r} \frac{(m+1)^{(r)}}{r!} f_{n-r}(a). \quad \blacksquare$$

The following corollary gives fundamental theorem of n^{th}-order $q(\alpha)$-delta integration.

Corollary III.6. *If f is n^{th}-order $q(\alpha)$-delta integrable function based at a, then*

$$F_a^n(tq) := f_n(tq) - \sum_{r=0}^{n-1} \alpha^{m+1-r} \frac{(m+1)^{(r)}}{r!} f_{n-r}(a) = \frac{1}{(n-1)!} \sum_{s=0}^{m-(n-1)} \alpha^{m-n-s+1} (m-s)^{(n-1)} f(aq^s),\ n \in \mathbb{N}. \quad (24)$$

Proof: *The proof follows by* (23) *and*

$${}_a\Delta_q^{-n} f(t) = \frac{1}{(n-1)!} \sum_{s=0}^{m-(n-1)} \alpha^{m-(n-1)-s} (m-s)^{(n-1)} f(aq^s).$$

Corollary III.7. *Let $t = aq^m$ and $n, m \in \mathbb{N}$. If f is n^{th}-order $q(\alpha)$-delta integrable function based at a, then*

$$F_a^n(t) =_a \Delta_{q(\alpha)}^{-n} f(t). \quad (25)$$

Proof: *The proof follows from Corollary III.6 and Definition III.3.*

Remark III.8. If f is n^{th}-order $q(\alpha)$-delta integrable function based at a, then

$$f_n(aq^m) - \sum_{r=0}^{n-1} \alpha^{m-r} \frac{m^{(r)}}{r!} f_{n-r}(a) = \frac{1}{(n-1)!} \sum_{s=0}^{m-n} \alpha^{m-n-s} (m-1-s)^{(n-1)} f(aq^s). \quad (26)$$

Now (26) is obtained from Corollary III.6 by replacing t by t/q.

The following example illustrates Corollary III.6.

Example III.9. *Consider the function* $f(t) = t$, $t = 40$, $a = 10$, $q = 2$, $m = 2$, $\alpha = 1.5$ *in* (24). *It will becomes*

$$f_2(tq) - \alpha^{m+1} f_2(a) - (m+1)\alpha^m f_1(a) = \sum_{s=0}^{m-1} \alpha^{m-s-1}(m-s)f(aq^s) \quad (27)$$

$$LHS = \frac{tq}{(q-\alpha)^2} - \alpha^{m+1}\frac{a}{(q-\alpha)^2} - (m+1)\alpha^m \frac{a}{q-\alpha}$$
$$= \frac{(40)(2)}{(0.5)^2} - (1.5)^3\frac{10}{(0.5)^2} - (3)(1.5)^2\frac{10}{0.5}$$
$$= 320 - 135 - 135 = 50$$

$$RHS = \sum_{s=0}^{1}(1.5)^{1-s}(2-s)(aq^s)$$
$$= (2)(1.5)(10) + (1)(10)(2) = 30 + 20 = 50$$

Hence, the equation (27) *is verified.*

Corollary III.10. *If* $|q| < 1$, $|\alpha| \le 1$ *and* $\lim_{m\to\infty} f_n(aq^m)$ *converges, then*

$$\lim_{m\to\infty}\Big[\frac{1}{(n-1)!}\sum_{s=0}^{m-n}\alpha^{m-n-s}(m-1-s)^{(n-1)}f(aq^s) + \sum_{r=0}^{n-1}\alpha^{m-r}\frac{m^{(r)}}{r!}f_{n-r}(a)\Big] \quad (28)$$

converges.

Proof: *The proof follows from Remark III.8. As* $m \to \infty$, $f_n(aq^m)$ *converges when* $|q| < 1$, $|\alpha| \le 1$ *and the proof completes from the equation* (26).

The following remark gives $q(\alpha)$-delta integer integration method is coincided with the standard q-delta integer integration when $\alpha = 1$.

Remark III.11. If f is n^{th}-order q-delta integrable function based at a, then

$$f_n(aq^m) - \sum_{r=0}^{n-1}\frac{m^{(r)}}{r!}f_{n-r}(a) = \frac{1}{(n-1)!}\sum_{s=0}^{m-n}(m-1-s)^{(n-1)}f(aq^s), \quad (29)$$

$n \in \mathbb{N}$ and $m - n \in \mathbb{N}$.

The following example illustrates Remark III.11.

Example III.12. *Consider the function* $f(t) = t_q^{(2)}$, $n = 2$, $a = 6$, $q = 2$, $m = 2$ *in* (29), *then it will becomes*

$$f_2(aq^{m+1}) - f_2(a) - (m+1)f_1(a) = \sum_{s=0}^{m-1}(m-s)f(aq^s)$$

$$LHS = \frac{(aq^{m+1})_q^{(2)}}{(q^2-1)^2} - \frac{a_q^{(2)}}{(q^2-1)^2} - (m+1)\frac{a_q^{(2)}}{q^2-1}$$
$$= \frac{(aq^{m+1})\prod_{r=1}^{1}(a-q^r)q^{m+1}}{(q^2-1)^2} - \frac{a\prod_{r=1}^{1}(a-q^r)}{(q^2-1)^2} - (m+1)\frac{a\prod_{r=1}^{1}(a-q^r)}{(q^2-1)}$$
$$= \frac{6\times 8\times 4\times 8}{9} - \frac{6\times 4}{9} - \frac{3\times 6\times 4}{3}$$
$$= 170.67 - 2.67 - 24 = 144$$

$$RHS = \sum_{s=0}^{1}(2-s)(aq^s)_q^{(2)} = \sum_{s=0}^{1}(2-s)aq^s(a-q)q^s$$
$$= \sum_{s=0}^{1}(2-s)(6)2^s(6-2)2^s = 48 + 96 = 144$$

From LHS and RHS, we verified the Remark III.11.

Corollary III.13. *If* $|q| < 1$ *and* $\lim_{m\to\infty} f_n(aq^m)$ *converges, then*

$$\lim_{m\to\infty}\Big[\frac{1}{(n-1)!}\sum_{s=0}^{m-n}(m-1-s)^{(n-1)}f(aq^s) + \sum_{r=0}^{n-1}\frac{m^{(r)}}{r!}f_{n-r}(a)\Big] \quad (30)$$

converges.

Proof: *The proof follows from Remark III.11. As* $m \to \infty$, $f_n(aq^m)$ *converges when* $|q| < 1$ *and the proof completes from the equation* (29).

IV. Fractional Order $q(\alpha)$ - Delta Integration

The expression (26) in Remark III.8 motivates us to form a conjecture in fractional order $q(\alpha)$-delta integration. In this section, we develop infinite and ν^{th} order $q(\alpha)$-delta integration, which is equal to ν^{th}-order fractional sum of f based at a. For any real $\nu > 0$, the Theorems III.5 and II.6 generate the definition of the ν^{th}-order delta sum and delta integration.

Definition IV.1. If $f : aq^{\nu+\mathbb{Z}} \to \mathbb{R}$ is n^{th}-order $q(\alpha)$-delta integrable function based at a for every $n \in \mathbb{N}$, then f is said to be ∞-order $q(\alpha)$-delta integrable function.

Definition IV.2. Let $f : aq^{\nu+\mathbb{Z}} \to \mathbb{R}$ be a function, $\nu > 0$ and $t = aq^{\nu+m}$. The fractional order (ν^{th}-order) $q(\alpha)$-delta sum of f based at a is defined by

$$_a\Delta_{q(\alpha)}^{-\nu}f(t) = \frac{1}{\Gamma\nu}\sum_{s=0}^{m-\nu}\alpha^{m-\nu-s}\frac{\Gamma(m-s)}{\Gamma(m-s-\nu+1)}f(aq^{\nu+s}). \quad (31)$$

Definition IV.3. Let $f : aq^{\nu+\mathbb{Z}} \to \mathbb{R}$, $\nu > 0$ and $t = aq^{\nu+m}$ such that $m - \nu$. If there exists a function, say $f_a^\nu : a + \nu + \mathbb{N} \to \mathbb{R}$ such that

$$f_a^\nu(t) = \frac{1}{\Gamma\nu}\sum_{s=0}^{m-\nu}\alpha^{m-\nu-s}\frac{\Gamma(m-s)}{\Gamma(m-s-\nu+1)}f(aq^{\nu+s}), \quad (32)$$

then the function f_a^ν is called as ν-order $q(\alpha)$-delta integration of f based at a.

Note that $_a\Delta_{q(\alpha)}^{-\nu}f(t)$ and $f_a^\nu(t)$ are ν^{th}-order $q(\alpha)$-delta sum and ν^{th} order $q(\alpha)$-delta integration of f based at a respectively. The expression of $_a\Delta_{q(\alpha)}^{-\nu}f(t)$ is possible for any given function f by using (31). But finding an exact function for $f_a^\nu(t)$ to a given function f is a challenging task. we obtain $f_a^\nu(t)$ for certain falling factorial function f using the following conjecture.

Conjecture IV.4. Assume that $f : aq^{\nu+\mathbb{Z}} \to \mathbb{R}$ be ∞-order $q(\alpha)$-delta integrable function based at a having integrating sequence $(f_n)_{n=1}^{\infty}$. If $f_n(a) = 0$ for $n = 1, 2, 3, \cdots, n$ then $f_a^{\nu}(t)$ exists and satisfies (32) for $\nu > 0$.

Theorem IV.5. *If $f : aq^{\nu+\mathbb{Z}} \to \mathbb{R}$ be having $q(\alpha)$-delta integrating sequence $(f_1, f_2, \cdots, f_n)$ and also f is in geometric progression, $t = aq^m$ and ${}_a\Delta_{q(\alpha)}^{-n} f(t)$ be the n^{th}-order $q(\alpha)$-delta integration of f based at a, then*

$$f_n(tq) - \frac{\left[\frac{(r+1)^{(n-1)}}{(n-1)!}\alpha^{r+2-n} f\left(\frac{t}{q^{r+1}}\right)\right]^2}{\frac{(r+1)^{(n-1)}}{(n-1)!}\alpha^{r+2-n} f\left(\frac{t}{q^{r+1}}\right) - \frac{(r+2)^{(n-1)}}{(n-1)!}\alpha^{r+3-n} f\left(\frac{t}{q^{r+2}}\right)}$$
$$= \sum_{s=n-1}^{r} \alpha^{s+1-n} \frac{s^{(n-1)}}{(n-1)!} f\left(\frac{t}{q^s}\right) \qquad (33)$$

Proof: Consider the expression (23) in Theorem II.6,

$$\begin{aligned}
f_n(tq) - \alpha^{m+1} f_n(a) - \alpha^m (m+1) f_{n-1}(a) \\
-\alpha^{m-1}\frac{(m+1)^{(2)}}{2!} f_{n-2}(a) - \cdots \\
-\alpha^{m-n+3}\frac{(m+1)^{(n-2)}}{(n-2)!} f_2(a) \\
-\alpha^{m-n+2}\frac{(m+1)^{(n-1)}}{(n-1)!} f_1(a) \\
= \frac{(n-1)^{(n-1)}}{(n-1)!} f\left(\frac{t}{q^{n-1}}\right) + \alpha \frac{(n)^{(n-1)}}{(n-1)!} f\left(\frac{t}{q^n}\right) + \cdots \\
+\alpha^{m-(n-1)}\frac{(m)^{(n-1)}}{(n-1)!} f\left(\frac{t}{q^m}\right).
\end{aligned}$$

As $m \to \infty$, $f_s(a) \to 0$ when $s = 1, 2, \ldots, n$, then appling this into the above equation and it will becomes

$$\begin{aligned}
f_n(tq) = \frac{(n-1)^{(n-1)}}{(n-1)!} f\left(\frac{t}{q^{n-1}}\right) + \alpha \frac{(n)^{(n-1)}}{(n-1)!} f\left(\frac{t}{q^n}\right) \\
+\alpha^2 \frac{(n+1)^{(n-1)}}{(n-1)!} f\left(\frac{t}{q^{n+1}}\right) + \cdots
\end{aligned}$$

Now, we spilt the above infinite series into two series

$$\begin{aligned}
f_n(tq) = \Big[\frac{(n-1)^{(n-1)}}{(n-1)!} f\left(\frac{t}{q^{n-1}}\right) + \alpha \frac{(n)^{(n-1)}}{(n-1)!} f\left(\frac{t}{q^n}\right) \\
+ \cdots + \alpha^{r+1-n}\frac{(r)^{(n-1)}}{(n-1)!} f\left(\frac{t}{q^r}\right)\Big] \\
+ \Big[\alpha^{r+2-n}\frac{(r+1)^{(n-1)}}{(n-1)!} f\left(\frac{t}{q^{r+1}}\right) \\
+\alpha^{r+3-n}\frac{(r+2)^{(n-1)}}{(n-1)!} f\left(\frac{t}{q^{r+2}}\right) + \cdots\Big] \qquad (34)
\end{aligned}$$

Consider the second series of the equation (34)

$$\begin{aligned}
\Big[\alpha^{r+2-n}\frac{(r+1)^{(n-1)}}{(n-1)!} f\left(\frac{t}{q^{r+1}}\right) \\
+\alpha^{r+3-n}\frac{(r+2)^{(n-1)}}{(n-1)!} f\left(\frac{t}{q^{r+2}}\right) + \cdots\Big]
\end{aligned}$$
$$= \frac{\left[\frac{(r+1)^{(n-1)}}{(n-1)!}\alpha^{r+2-n} f\left(\frac{t}{q^{r+1}}\right)\right]^2}{\frac{(r+1)^{(n-1)}}{(n-1)!}\alpha^{r+2-n} f\left(\frac{t}{q^{r+1}}\right) - \frac{(r+2)^{(n-1)}}{(n-1)!}\alpha^{r+3-n} f\left(\frac{t}{q^{r+2}}\right)}$$

Substitute this into the equation (34), we get (33). ■

Corollary IV.6. *If f is ν^{th}-order $q(\alpha)$-delta integrable function based at a, then*

$$f_\nu(tq) - \frac{\left[\frac{1}{\Gamma\nu}\frac{\Gamma(r+2)}{\Gamma(r-\nu+3)}\alpha^{r+2-n} f\left(\frac{t}{q^{r+1}}\right)\right]^2}{\frac{1}{\Gamma\nu}\frac{\Gamma(r+2)}{\Gamma(r-\nu+3)}\alpha^{r+2-n} f\left(\frac{t}{q^{r+1}}\right) - \frac{1}{\Gamma\nu}\frac{\Gamma(r+3)}{\Gamma(r-\nu+4)}\alpha^{r+3-n} f\left(\frac{t}{q^{r+2}}\right)}$$
$$= \frac{1}{\Gamma\nu}\sum_{s=\nu-1}^{r} \frac{\Gamma(s+1)}{\Gamma(s-\nu+2)}\alpha^{s+1-n} f\left(\frac{t}{q^s}\right). \qquad (35)$$

Proof: *The proof follows by Theorem IV.5 and convert* (33) *this into gamma function.*

The following example illustrates Corollary IV.6.

Example IV.7. *Applying $f(t) = t$, $t = 9$, $q = 3$, $r = 4.5$, $\alpha = 0.5$, $\nu = 2.5$ in* (35)*, then it will become as*

$$f_{2.5}(tq) - \frac{\left[\frac{1}{\Gamma 2.5}\frac{\Gamma 6.5}{\Gamma 5}(0.5)^4 f\left(\frac{t}{q^{5.5}}\right)\right]^2}{\frac{1}{\Gamma 2.5}\frac{\Gamma 6.5}{\Gamma 5}(0.5)^4 f\left(\frac{t}{q^{5.5}}\right) - \frac{1}{\Gamma 2.5}\frac{\Gamma 7.5}{\Gamma 6}(0.5)^5 f\left(\frac{t}{q^{6.5}}\right)}$$
$$= \sum_{s=1.5}^{4.5}\frac{1}{\Gamma 2.5}\frac{\Gamma(s+1)}{\Gamma(s-0.5)}(0.5)^{s-1.5} f\left(\frac{t}{q^s}\right)$$

$$\begin{aligned}
LHS &= \frac{tq}{(q-\alpha)^{2.5}} \\
&\quad - \frac{\left[\frac{1}{\Gamma 2.5}\frac{\Gamma 6.5}{\Gamma 5}(0.5)^4 \frac{t}{q^{5.5}}\right]^2}{\frac{1}{\Gamma 2.5}\frac{\Gamma 7.5}{\Gamma 6}(0.5)^4 \frac{t}{q^{5.5}} - \frac{1}{\Gamma 2.5}\frac{\Gamma 7.5}{\Gamma 6}(0.5)^5 \frac{t}{q^{6.5}}} \\
&= 2.7322 - \frac{(0.012059)^2}{0.012059 - 0.002617} \\
&= 2.7322 - 0.015357 = 2.7168 \\
RHS &= \sum_{s=1.5}^{4.5}\frac{1}{\Gamma 2.5}\frac{\Gamma(s+1)}{\Gamma(s-0.5)}(0.5)^{s-1.5}\frac{t}{q^s} \\
&= \sum_{s=1.5}^{4.5}\frac{1}{\Gamma 2.5}\frac{\Gamma(s+1)}{\Gamma(s-0.5)}\frac{9}{3^s} \\
&= \frac{1}{1.3296}[3.612329] = 2.71685
\end{aligned}$$

From LHS and RHS, we verified the Corollary IV.6.

The following Remark gives the $q(\alpha)$-delta fractional integration method is coincided with the standard q-delta fractional integration when $\alpha = 1$.

Remark IV.8. If f is ν^{th}-order q-delta integrable function based at a, then

$$f_\nu(tq) - \frac{\left[\frac{1}{\Gamma\nu}\frac{\Gamma(r+2)}{\Gamma(r-\nu+3)} f\left(\frac{t}{q^{r+1}}\right)\right]^2}{\frac{1}{\Gamma\nu}\frac{\Gamma(r+2)}{\Gamma(r-\nu+3)} f\left(\frac{t}{q^{r+1}}\right) - \frac{1}{\Gamma\nu}\frac{\Gamma(r+3)}{\Gamma(r-\nu+4)} f\left(\frac{t}{q^{r+2}}\right)}$$
$$= \sum_{s=\nu-1}^{r}\frac{1}{\Gamma\nu}\frac{\Gamma(s+1)}{\Gamma(s-\nu+2)} f\left(\frac{t}{q^s}\right). \qquad (36)$$

The following example illustrates Remark IV.8.

Example IV.9. *Applying $f(t) = t$, $t = 10$, $r = 6.5$, $q = 2$, $\nu = 1.5$ in* (36)*, then the equation* (36) *will becomes*

$$f_{1.5}(tq) - \frac{\left[\frac{1}{\Gamma 1.5}\frac{\Gamma 8.5}{\Gamma 8} f\left(\frac{t}{q^{7.5}}\right)\right]^2}{\frac{1}{\Gamma 1.5}\frac{\Gamma 8.5}{\Gamma 8} f\left(\frac{t}{q^{7.5}}\right) - \frac{1}{\Gamma 1.5}\frac{\Gamma 9.5}{\Gamma 9} f\left(\frac{t}{q^{8.5}}\right)}$$
$$= \sum_{s=0.5}^{6.5} \frac{1}{\Gamma 1.5}\frac{\Gamma(s+1)}{\Gamma(s+0.5)} f\left(\frac{t}{q^s}\right)$$

$$LHS = \frac{tq}{(q-1)^{1.5}} - \frac{\left[\frac{1}{\Gamma 1.5}\frac{\Gamma 8.5}{\Gamma 8}\frac{t}{q^{7.5}}\right]^2}{\frac{1}{\Gamma 1.5}\frac{\Gamma 8.5}{\Gamma 8}\frac{t}{q^{7.5}} - \frac{1}{\Gamma 1.5}\frac{\Gamma 9.5}{\Gamma 9}\frac{t}{q^{8.5}}}$$
$$= 20 - \frac{(0.17358)^2}{0.17358 - 0.092214}$$
$$= 20 - 0.370302 = 19.629698$$
$$RHS = \sum_{s=0.5}^{6.5} \frac{1}{\Gamma 1.5}\frac{\Gamma(s+1)}{\Gamma(s+0.5)}\frac{t}{q^s}$$
$$= \sum_{s=0.5}^{6.5} \frac{1}{\Gamma 1.5}\frac{\Gamma(s+1)}{\Gamma(s+0.5)}\frac{10}{2^s}$$
$$= \frac{1}{0.88639}[17.401899] = 19.632328$$

From LHS and RHS, we verified the Remark IV.8.

V. Conclusion

Although the fractional order q-delta sum of a given function f based at a is well-established in the literature, no previous work has explored the fractional order $q(\alpha)$-delta integration of f. In this research, we have developed a discrete fractional integration method for certain classes of functions. Furthermore, we extend these results to the $q(\alpha)$-delta operator, leading to the formulation of several new identities and fundamental theorems. These findings contribute novel insights into the field of discrete fractional calculus.

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