Enhanced Solutions to Differential Equations Using Numerical Iterative Approach

Alaa Al-Wadi, Rania Saadeh, Ahmad Qazza and Ghassan Abufoudeh

Abstract—**This paper explores the benefits of integrating numerical iterative methods with the Sawi transform (SWT) to effectively address ordinary and delay differential equations. We focus specifically on the Sawi iterative method (SIM), showcasing how its integration with the SWT simplifies complex differential equations and improves solution accuracy and computational efficiency. Initially, we present fundamental theoretical insights into SWT, emphasizing its key properties, such as linearity, convolution, and scaling, that facilitate converting complicated differential equations into simpler algebraic forms. Through illustrative examples and detailed case studies, we validate the proposed approach by highlighting its effectiveness in producing both analytical and approximate solutions. Our results illustrate the robustness, versatility, and practicality of the combined methods across a variety of complex differential equations. Ultimately, this study emphasizes the significance and effectiveness of the proposed techniques, opening avenues for their broader application in various scientific and engineering contexts.**

Index Terms—**Sawi transform, iterative methods, delay differential equations, Sawi iterative method.**

I. Introduction

IN applied mathematics, efficiently and accurately solving differential equations continues to be a central challenge due to their critical role in modeling phenomena across various scientific and engineering disciplines, including physics, biology, finance, and engineering [1], [2], [3]. Classical approaches, such as Laplace and Fourier transforms [4], [5], have traditionally been employed to address these equations, yet their limitations become evident when handling complex, nonlinear, or more intricate problems [6], [7].

The introduction of the SWT by Mahgoub and Mohand in 2019 has demonstrated significant potential, offering greater flexibility and effectiveness for addressing a wide range of differential equations [8], [9], [10]. The SWT has several advantageous properties, such as linearity, scaling, shifting, and convolution. These properties collectively enable the transformation of complicated differential equations into simpler algebraic representations, thus significantly streamlining the analytical solution process [11], [12], [21].

The unique properties of SWT—including linearity, scaling, shifting, and convolution—facilitate the transformation of complex differential problems into simpler algebraic forms, thereby simplifying the analytical process [15], [16], [17]. Iterative methods, recognized for their effectiveness in achieving convergence and precision, are extensively utilized within computational mathematics and numerical analysis [18], [19], [20]. By integrating these iterative approaches with the SWT, this research aims to address nonlinear and complex differential equations more effectively and accurately [21], [22].

Iterative methods are recognized in numerical analysis and computational mathematics for their ability to efficiently refine solutions and achieve reliable convergence [23], [24], [25].

Recently, the Sawi iterative method (SIM) have shown great promise in solving nonlinear and complex differential equations. The integration of iterative methods with the SWT can further enhance their effectiveness, allowing for the iterative refinement of solutions and the handling of more intricate problems [26], [8], [9]. This paper will demonstrate the application of these techniques through detailed examples and case studies, highlighting their practical utility and advantages [17], [27], [28].

This research contributes to the growing body of knowledge on integral transforms and iterative methods, providing a comprehensive analysis of their combined application in solving differential equations. By presenting detailed theoretical insights and practical implementations, we aim to establish a solid foundation for future research and advancements in this field. The results presented in this study can substantially influence multiple scientific and engineering fields, creating opportunities for broader utilization of the SWT and iterative techniques in addressing practical, real-world challenges.

This paper is organized as follows: In Section 2, we introduce SWT and some basic properties. The application of SWT on DDEs with the iterative method is presented in Section 3. Finally, some illustrative examples are presented in Section 4.

II. Basic Definitions and Properties

Definition 2.1: The Sawi transform (SWT) of a given function $w(t)$, defined on the interval $[0,\infty)$, is denoted by $S[w(t)]$ and defined as follows:

$$S[w(t)] = R(v) = \frac{1}{v^2}\int_0^\infty w(t)e^{-\frac{t}{v}}dt. \quad (1)$$

The inverse of the Sawi transform is expressed by:

$$S^{-1}[R(v)] = \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty} R(v)e^{\frac{t}{v}},dv, \quad c\in\mathbb{R}. \quad (2)$$

It is important to note that if $S[w_1(t)] = R_1(v)$ and $S[w_2(t)] = R_2(v)$, then the SWT is linear, and the following

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relation holds for arbitrary constants a and b:

$$\begin{aligned} S[aw_1(t)+bw_2(t)] &= aS[w_1(t)]+bS[w_2(t)] \\ &= aR_1(v)+bR_2(v). \end{aligned}$$

Additionally, the inverse Sawi transform also exhibits linearity. If $S^{-1}[R_1(v)] = w_1(t)$ and $S^{-1}[R_2(v)] = w_2(t)$, then:

$$\begin{aligned} S^{-1}[aR_1(v)+bR_2(v)] &= aS^{-1}[R_1(v)]+bS^{-1}[R_2(v)] \\ &= aw_1(t)+bw_2(t). \end{aligned} \tag{3}$$

Theorem 2.1: Let $w(t)$ be a continuous function defined on $t > 0$ and satisfying the exponential growth condition $|w(t)| \leq \mu e^{\alpha t}$, where $\mu > 0$ and $\alpha \in \mathbb{R}$ are constants. Then the SWT $S[w(t)]$ exists and is well-defined for all v such that:

$$Re\left(\frac{1}{v}\right) > \alpha$$

The Sawi transform satisfies several useful properties, including:

- If $S[w(t)] = R(v)$, then $S[w(at)] = aR(av)$.
- For an exponential multiplier, we have: $S\Big[e^{at}w(t)\Big] = \frac{1}{(1-av)^2}R\Big(\frac{v}{1-av}\Big)$.
- For convolution, the SWT is given by: $S[w_1(t) \times w_2(t)] = v^2R_1(v)R_2(v)$.

Theorem 2.2: If $R(v)$ denotes the SWT of $w(t)$, the transforms of derivatives of $w(t)$ satisfy the following relations:

1) First derivative:
$$S[w'(t)] = \frac{R(v)}{v} - \frac{w(0)}{v^2}. \tag{4}$$
2) Second derivative:
$$S[w''(t)] = \frac{R(v)}{v^2} - \frac{w(0)}{v^3} - \frac{w'(0)}{v^2}. \tag{5}$$
3) Higher-order derivative (general form):
$$S[w^{(n)}(t)] = \frac{R(v)}{v^n} - \sum_{k=0}^{n-1} \frac{w^{(k)}(0)}{v^{n-k+1}}. \tag{6}$$

The following Table I provides examples of the SWT for several fundamental functions.

TABLE I
SWT OF SOME ELEMENTARY FUNCTIONS.

$Sr.No.$	$w(t)$	$S[w(t)]$
1	1	$\frac{1}{v}$
2	t	1
3	$t^n, \quad n \in \mathbb{N}$	$n!v^{n-1}$
4	$t^\alpha, \quad \alpha \in \mathbb{R}^+$	$\Gamma(\alpha+1)\ v^{\alpha-1}$
5	e^{at}	$\frac{1}{v(1-av)}$
6	$\sin at$	$\frac{a}{1+a^2v^2}$
7	$\cos at$	$\frac{1}{v(1+a^2v^2)}$
8	$\sinh at$	$\frac{a}{1-a^2v^2}$
9	$\cosh at$	$\frac{1}{v(1-a^2v^2)}$

III. SWT ITERATIVE METHOD

In this section, we introduce and discuss an iterative technique, specifically focusing on the newly proposed Sawi Iterative Method (SIM), which is particularly suited for solving delay differential equations (DDEs).

A. Iterative Method

We start by considering the general functional equation of the form:

$$w(t) = N(w) + g(t). \tag{7}$$

Here, N represents a nonlinear operator defined on a Banach space S, mapping the space back into itself ($N : S \to S$), and $g(t)$ denotes a given, known function. The solution $w(t)$ to Eq. (7) is assumed to have a series expansion of the form:

$$w(t) = \sum_{i=0}^{\infty} w_i(t). \tag{8}$$

The nonlinear operator N can be represented through a telescopic decomposition as follows:

$$\begin{aligned} N\left[\sum_{i=0}^{\infty} w_i(t)\right] &= N(w_0) \\ &+ \sum_{i=1}^{\infty}\left(N\left[\sum_{k=0}^{i} w_k(t)\right] - N\left[\sum_{k=0}^{i-1} w_k(t)\right]\right). \end{aligned} \tag{9}$$

By substituting Eq. (8) and Eq. (9) into the original equation (7), we obtain:

$$\begin{aligned} \sum_{i=0}^{\infty} w_i(t) =& g(t) + N(w_0(t)) \\ &+ \sum_{i=1}^{\infty}\left(N\left[\sum_{k=0}^{i} w_k(t)\right] - N\left[\sum_{k=0}^{i-1} w_k(t)\right]\right). \end{aligned} \tag{10}$$

From this formulation, we derive the following iterative recurrence relations:

$$\begin{aligned} w_0(t) &= g(t), \\ w_1(t) &= N[w_0(t)], \\ w_2(t) &= N[w_0(t)+w_1(t)] - N[w_0(t)], \\ w_3(t) &= N[w_0(t)+w_1(t)+w_2(t)] - N[w_0(t)+w_1(t)], \\ w_4(t) &= N[w_0(t)+w_1(t)+w_2(t)+w_3(t)] \\ &\quad - N[w_0(t)+w_1(t)+w_2(t)], \\ &\vdots \end{aligned}$$

$$w_{m+1}(t) = N\left[\sum_{i=0}^{m} w_i(t)\right] - N\left[\sum_{i=0}^{m-1} w_i(t)\right], \quad m = 1, 2, 3, \ldots$$

Thus, we can succinctly express the above iterative relations as:

$$\begin{aligned} w_1(t) + w_2(t) + \cdots + w_{m+1}(t) &= N[w_0(t)+w_1(t)+\cdots+w_m(t)], \\ m &= 1, 2, 3, \ldots, \end{aligned}$$

with the solution given by the infinite series:

$$w(t) = \sum_{i=0}^{\infty} w_i(t).$$

To ensure the convergence of the proposed iterative method, we introduce and discuss two essential theorems:

Theorem 3.1: Let N be a continuously differentiable nonlinear operator in a neighborhood around w_0. Assume that there exists a constant $L > 0$ satisfying:

$$\|N^{(n)}(w_0)\| = \sup \Big\{ \left|N^{(n)}(w_0)(h_1, h_2, \ldots, h_n)\right| : \|h_i\| \leq 1, \, 1 \leq i \leq n \Big\} \leq L,$$

for each $n \geq 1$. If the terms satisfy $w_i \leq M < \frac{1}{e}$ for all $i = 1, 2, 3, \ldots$, then the series $\sum_{i=0}^{\infty} w_{i+1}$ converges absolutely. Moreover, the following bound holds:

$$\|w_{i+1}\| \leq LM^n e^{n-1}(e-1), \quad n = 1, 2, \ldots$$

Theorem 3.2: Suppose that N is continuously differentiable in a neighborhood of w_0 and satisfies the condition:

$$\|N^{(n)}(w_0)\| \leq M \leq \frac{1}{e},$$

for all $n \geq 1$. Then the infinite series $\sum_{i=0}^{\infty} w_{i+1}$ is absolutely convergent.

B. Sawi iterative method for Solving Ordinary DDEs

To understand the procedure of the presented SIM, let us consider the following DDE in the general form of $n^{th}-$ order DDE

$$\frac{d^n w(t)}{dt^n} + P(w(t)) + N(w(\lambda t)) = f(t), \quad n = 1, 2, \cdots, \quad n \in \mathbb{N}, \quad \lambda \neq 0, \tag{11}$$

subject to the initial conditions

$$w^{(k)}(0) = w_{0\ k}, \quad k = 0, 1, 2, \cdots. \tag{12}$$

In this context, $\frac{d^n w(t)}{dt^n}$ denotes the n^{th} derivative of the unknown function $w(t)$. Furthermore, P represents a bounded linear operator, N is a bounded nonlinear operator, and $f(t)$ is a known continuous function.

To perform the SIM to solve the initial value problem (11) and, (12) we follow the steps.

Step (1): Applying the SWT to both sides of equation (11),

$$S\left[\frac{d^n w}{dt^n}\right] + S[P(w(t))] + S[N(w(\lambda t))] = S[f(t)]. \tag{13}$$

Running SWT on Eq (13), we have

$$\begin{aligned} \frac{R(v)}{v^n} - \sum_{k=0}^{n-1} \frac{w^{(k)}(0)}{v^{n-k+1}} &+ S[P(w(t))] + S[N(w(\lambda t))] \\ &= S[f(t)], \\ \frac{R(v)}{v^n} &= \sum_{k=0}^{n-1} \frac{w^{(k)}(0)}{v^{n-k+1}} + S[f(t)] - S[P(w(t))] \\ &- S[N(w(\lambda t))], \end{aligned}$$

which implies

$$R(v) = \left(\sum_{k=0}^{n-1} \frac{w^{(k)}(0)}{v^{-k+1}}\right) + v^n S[f(t)] - v^n S[P(w(t))] - v^n S[N(w(\lambda t))]. \tag{14}$$

Step (2): Applying the inverse SWT to both sides of Eq (14), we obtain

$$\begin{aligned} w(t) = &S^{-1}\left[\sum_{k=0}^{n-1} \frac{w^{(k)}(0)}{v^{-k+1}}\right] + S^{-1}[v^n S[f(t)]] \\ &- S^{-1}[v^n S[P(w(t))]] - S^{-1}[v^n S[N(w(\lambda t))]], \end{aligned} \tag{15}$$

which implies,

$$\begin{aligned} w(t) = &S^{-1}\left[\sum_{k=0}^{n-1} \frac{w^{(k)}(0)}{v^{-k+1}} + v^n S[f(t)] - v^n S[P(w(t))]\right] \\ &- S^{-1}[v^n S[N(w(\lambda t))]]. \end{aligned} \tag{16}$$

Step (3): We consider the series solution of Eq (16), of the form

$$w(t) = \sum_{n=0}^{\infty} w_n(t).$$

Substituting it in the Eq (16), to obtain

$$\begin{aligned} \sum_{n=0}^{\infty} w_n(t) = S^{-1}\Bigg[&\sum_{k=0}^{n-1} \frac{w^{(k)}(0)}{v^{-k+1}} + v^n S[f(t)] \\ &- v^n S[P(w(t))]\Bigg] \\ &- S^{-1}\left[v^n S\left[N\left(\sum_{n=0}^{\infty} w_n(\lambda t)\right)\right]\right]. \end{aligned} \tag{17}$$

Step (4): The term of the nonlinear operator N, can be decomposed as follows:

$$\begin{aligned} &\sum_{n=0}^{\infty} N(w_n(\lambda t)) = N(w_0) \\ &+ \sum_{i=1}^{n}\left(N\left[\sum_{n=0}^{k} w_n(\lambda t)\right] - N\left[\sum_{n=0}^{k-1} w_n(\lambda t)\right]\right). \end{aligned} \tag{18}$$

Step (5): By substituting Eq (18), in Eq (17), we get

$$\begin{aligned} \sum_{n=0}^{\infty} w_n(t) = &S^{-1}\Bigg[\sum_{k=0}^{n-1} \frac{w^{(k)}(0)}{v^{-k+1}} + v^n S[f(t)] \\ &- v^n S[P(w(t))]\Bigg] - S^{-1}\Bigg[v^n S\Bigg[N(w_0(\lambda t)) \\ &+ \sum_{i=1}^{\infty}\Bigg\{N\left[\sum_{n=0}^{k} w_n(\lambda t)\right] \\ &- N\left[\sum_{n=0}^{k-1} w_n(\lambda t)\right]\Bigg\}\Bigg]\Bigg]. \end{aligned} \tag{19}$$

Step (6): We can get the recurrence relation from Eq (19), in the previous step

$$
\begin{aligned}
w_0(t) =& S^{-1}\left[\sum_{k=0}^{n-1} \frac{w^{(k)}(0)}{v^{-k+1}} + v^n S[f(t)] \right. \\
&\left. - v^n S[P(w(t))] \right] \\
w_1(t) =& -S^{-1}\left[v^n S[N[w_0(\lambda t)]]\right], \\
&\vdots \\
w(t) =& -S^{-1}\left[v^n S\left[\sum_{i=1}^{\infty}\left\{N\left[\sum_{n=0}^{k} w_n(\lambda t)\right]\right.\right.\right. \\
&\left.\left.\left. - N\left[\sum_{n=0}^{k-1} w_n(\lambda t)\right]\right\}\right]\right],
\end{aligned}
\tag{20}
$$

for $n = 1, 2, \cdots$. Consequently, the analytical solution to the DDE described by Eq. (20) can be represented using the following infinite series form:

$$
w(t) = \lim_{N\to\infty} \sum_{m=0}^{N} w_m(t) = w_0(t) + w_1(t) + w_2(t) + \cdots. \tag{21}
$$

IV. Illustrative Examples

In this part of the study, we introduce some examples, and solve them by SIM, to show the simplicity and efficiency of the method.

Example 4.1. Consider the following nonlinear first - order DDE

$$
w'(t) - 2tw^4\left(\frac{t}{2}\right) = 0, \quad t \le 0, \tag{22}
$$

with the initial condition

$$
w(0) = 1. \tag{23}
$$

Solution. Applying the Sawi transform (SWT) to both sides of equation (22), we have:

$$
S[w'(t)] - S\left[2t, w^4\left(\frac{t}{2}\right)\right] = 0. \tag{24}
$$

Evaluating the SWT of equation (24)

$$
\frac{R(v)}{v} - \frac{w(0)}{v^2} - S\left[2t, w^4\left(\frac{t}{2}\right)\right] = 0. \tag{25}
$$

Inserting the initial condition from equation (23) into equation (25), yields:

$$
R(v) = \frac{1}{v} + vS\left[2t, w^4\left(\frac{t}{2}\right)\right]. \tag{26}
$$

Next, taking the inverse Sawi transform of Eq. (26) provides:

$$
S^{-1}[R(v)] = S^{-1}\left[\frac{1}{v} + vS\left[2t, w^4\left(\frac{t}{2}\right)\right]\right].
$$

Employing the inverse Sawi transform properties, the equation becomes:

$$
S^{-1}[R(v)] = S^{-1}\left[\frac{1}{v}\right] + S^{-1}\left[vS\left[2t, w^4\left(\frac{t}{2}\right)\right]\right]. \tag{27}
$$

Performing the inverse SWT operation explicitly, we obtain:

$$
w(t) = 1 + S^{-1}\left[vS\left[2t, w^4\left(\frac{t}{2}\right)\right]\right]. \tag{28}
$$

Now, implementing the iterative procedure to solve Eq. (28), we start with the initial approximation:

$$
w_0(t) = 1, \quad w_0(t/2) = 1.
$$

Then, the first iteration component $w_1(t)$ is calculated by applying the nonlinear operator N to $w_0(t/2)$:

$$
w_1(t) = N\left[w_0\left(\frac{t}{2}\right)\right] = S^{-1}\left[vS\left[2t, \left(w_0\left(\frac{t}{2}\right)\right)^4\right]\right].
$$

Evaluating this expression explicitly, we find:

$$
w_1(t) = S^{-1}[vS[2t]] = t^2.
$$

Hence,

$$
w_1\left(\frac{t}{2}\right) = \frac{t^2}{4}.
$$

To determine the expression for $w_2(t)$, we proceed with the following calculation:

$$
\begin{aligned}
w_2(t) &= N\left[w_0\left(\frac{t}{2}\right) + w_1\left(\frac{t}{2}\right)\right] - N\left[w_0\left(\frac{t}{2}\right)\right] \\
&= S^{-1}\left[12v^3 + 90v^5 + 630v^7 + 2835v^9\right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
w_2\left(\frac{t}{2}\right) &= \frac{t^4}{32} + \frac{t^6}{512} + \frac{t^8}{16384} + \frac{t^{10}}{1310720}, \\
w(t) =& S^{-1}\left[vS\left[\sum_{n=1}^{\infty}\left(N\left(\sum_{n=0}^{k} {w_n}^2\left(\frac{t}{2}\right)\right)\right.\right.\right. \\
&\left.\left.\left. - N\left(\sum_{n=0}^{k-1} {w_n}^2\left(\frac{t}{2}\right)\right)\right)\right]\right].
\end{aligned}
$$

Thus,

$$
w(t) = w_0(t) + w_1(t) + w_2(t) + \cdots.
$$

Now, the approximate solution of this example is,

$$
w(t) = 1 + t^2 + \frac{t^4}{2} + \frac{t^6}{8} + \frac{t^8}{64} + \frac{t^{10}}{1280} + \cdots. \tag{29}
$$

Here, we state that the exact analytical solution of Eq (22), is given by

$$
w(t) = e^{t^2}.
$$

The expressions are simplified using Mathematica version 13.0. In the following Figure 1, we sketch the approximate and exact solutions of Example 4.1.

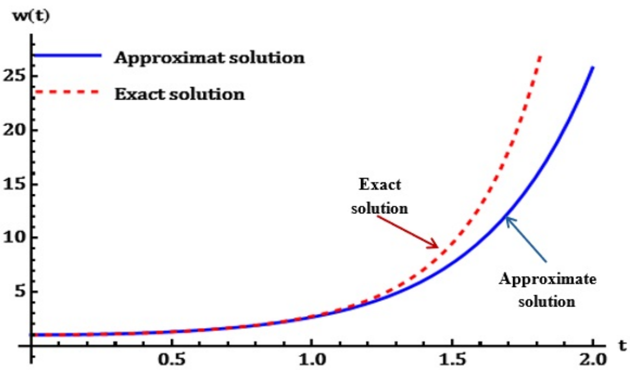

Fig. 1. Exact and approximate solutions of Example 4.1

Example 4.2. Consider the nonlinear proportional DDE

$$w''(t) = 1 - 2w^2\left(\frac{t}{2}\right), \qquad 0 < t < 1, \tag{30}$$

with the initial conditions

$$w(0) = 1, \qquad w'(0) = 0. \tag{31}$$

Solution. Here, we state that the exact solution of Eq (30), is given by

$$w(t) = \cos t. \tag{32}$$

Applying the Sawi transform (SWT) to equation (30), we obtain:

$$S[w''(t)] = S\left[1 - 2w^2\left(\frac{t}{2}\right)\right]. \tag{33}$$

Expanding equation (33) using properties of the SWT, we have:

$$\frac{R(v)}{v^2} - \frac{w'(0)}{v^2} - \frac{w(0)}{v^3} = S[1] - S\left[2w^2\left(\frac{t}{2}\right)\right]. \tag{34}$$

Next, we substitute the initial conditions given by equation (31) into equation (34), simplifying to obtain:

$$\frac{R(v)}{v^2} - \frac{1}{v^3} = \frac{1}{v} - S\left[2w^2\left(\frac{t}{2}\right)\right].$$

Consequently, this leads to:

$$R(v) = \frac{1}{v} + v - v^2 S\left[2w^2\left(\frac{t}{2}\right)\right]. \tag{35}$$

Finally, by applying the inverse Sawi transform to Eq. (35), the solution can be expressed as:

$$S^{-1}[R(v)] = S^{-1}\left[\frac{1}{v} + v - v^2 S\left[2w^2\left(\frac{t}{2}\right)\right]\right].$$

By using the properties of SWT, we have

$$w(t) = 1 + \frac{t^2}{2} - S^{-1}\left[v^2 S\left[2w^2\left(\frac{t}{2}\right)\right]\right]. \tag{36}$$

Applying the iterative technique to equation (36), we obtain the initial approximation components of the solution:

$$w_0(t) = 1 + \frac{t^2}{2},$$
$$w_0\left(\frac{t}{2}\right) = 1 + \frac{t^2}{8}.$$

To determine the expression for $w_1(t)$, we proceed with the following calculation:

$$\begin{aligned} w_1(t) &= N\left[w_0\left(\frac{t}{2}\right)\right] = -S^{-1}\left[v^2 S\left[2w_0\left(\frac{t}{2}\right)^2\right]\right] \\ &= -S^{-1}\left[2\, v^2 S\left[\left(1 + \frac{t^2}{8}\right)^2\right]\right]. \end{aligned}$$

Thus,

$$w_1(t) = -t^2 - \frac{t^4}{24} - \frac{t^6}{960},$$

$$w_1\left(\frac{t}{2}\right) = N\left[w_0\left(\frac{t}{2}\right)\right] = \frac{-t^2}{4} - \frac{t^4}{384} - \frac{t^6}{23040}.$$

Now, we compute $w_2(t)$

$$\begin{aligned} w_2(t) &= N\left[w_0\left(\frac{t}{2}\right) + w_1\left(\frac{t}{2}\right)\right] - N\left[w_0\left(\frac{t}{2}\right)\right] \\ &= -S^{-1}\left[2v^2 S\left[\left(1 + \frac{t^2}{8} - \frac{t^2}{4} - \frac{t^4}{384} - \frac{t^6}{23040}\right)^2 \right.\right. \\ &\quad \left.\left. - \left(1 + \frac{t^2}{8}\right)^2\right]\right] \end{aligned}$$

Hence,

$$\begin{aligned} w_2(t) =& \frac{t^4}{12} + \frac{t^6}{2880} - \frac{6.5t^8}{645120} - \frac{1.3t^{10}}{6635520} \\ &+ \frac{t^{12}}{583925760} - \frac{t^{14}}{96613171200} + \cdots. \end{aligned} \tag{37}$$

Continuing this iterative process, we can determine subsequent components $w_3(t),\ w_4(t), \cdots$, systematically. The complete solution is then represented as the infinite sum of these iterative terms:

$$w(t) = w_0(t) + w_1(t) + w_2(t) + \cdots.$$

For $n = 2$, we have

$$\begin{aligned} w(t) =& 1 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{960} \\ &- 0.0000100756448412698 41t^8 \\ &- 1.959153163580247 \times 10^{-7} t^{10} + \frac{t^{12}}{583925760} \\ &- \frac{t^{14}}{96613171200} + \cdots. \end{aligned}$$

The expressions are simplified using Mathematica version 13.0. Then, the exact solution is

$$w(t) = \cos t.$$

Now, we present the following Figure 2, in which we sketch the approximate and exact solutions

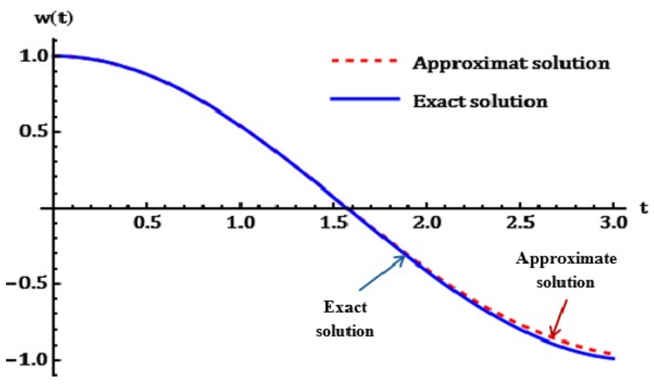

Fig. 2. Exact and approximate solutions of Example 4.2

Example 4.3. Consider the nonlinear 3rd order DDE

$$w'''(t) = 2w^2\left(\frac{t}{2}\right) - 1, \tag{38}$$

with the initial conditions

$$w(0) = 0, \qquad w'(0) = 1, \qquad w''(0) = 0. \tag{39}$$

Here we state that the exact analytical solution of Eq (38) and Eq (39), is given by

$$w(t) = \sin t.$$

Solution. Applying the SWT to each side of Eq. (38) yields the following:

$$S[w''(t)] = S\left[2w^2\left(\frac{t}{2}\right) - 1\right]. \quad (40)$$

Applying the Sawi transform (SWT) to equation (40), we have:

$$\frac{R(v)}{v^3} - \frac{w(0)}{v^4} - \frac{w'(0)}{v^3} - \frac{w''(0)}{v^2} = S\left[2w^2\left(\frac{t}{2}\right)\right] - S[1]. \quad (41)$$

Incorporating the initial conditions given by Eq. (39) into Eq. (41), we obtain:

$$\frac{R(v)}{v^3} - \frac{1}{v^3} = S\left[2w^2\left(\frac{t}{2}\right)\right] - \frac{1}{v}, \quad (42)$$

leading to:

$$R(v) = 1 - v^2 + v^3 S\left[2w^2\left(\frac{t}{2}\right)\right]. \quad (43)$$

Finally, taking the inverse Sawi transform of Eq. (43), we express the solution as:

$$S^{-1}[R(v)] = S^{-1}\left[1 - v^2 + v^3 S\left[2w^2\left(\frac{t}{2}\right)\right]\right]. \quad (44)$$

By using the properties of inverse SWT, we obtain

$$S^{-1}[R(v)] = S^{-1}[1] - S^{-1}[v^2] + S^{-1}\left[v^3 S\left[2w^2\left(\frac{t}{2}\right)\right]\right]. \quad (45)$$

Running inverse SWT on Eq (45), we get

$$w(t) = t - \frac{t^3}{3!} + S^{-1}\left[v^3 S\left[2w^2\left(\frac{t}{2}\right)\right]\right]. \quad (46)$$

Now, to apply the iterative approach to Eq. (28), we first identify the initial component of the solution as:

$$w_0(t) = t - \frac{t^3}{3!}.$$

Evaluating this initial approximation at $t/2$, we obtain:

$$w_0\left(\frac{t}{2}\right) = \frac{t}{2} - \frac{t^3}{48}.$$

To calculate $w_1(t)$, we apply the nonlinear operator N on the initial approximation $w_0(t/2)$. This procedure involves utilizing the inverse Sawi transform combined with the Sawi transform of the squared initial approximation, as shown below:

$$\begin{aligned} w_1(t) &= N\left[w_0\left(\frac{t}{2}\right)\right] = S^{-1}\left[v^3 S\left[2w_0^2\left(\frac{t}{2}\right)\right]\right] \\ &= S^{-1}\left[v^3 S\left[2\left(\frac{t}{2} - \frac{t^3}{48}\right)^2\right]\right] \\ &= \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{t^9}{580608}. \end{aligned}$$

Next, we compute $w_2(t)$ by applying the operator N to the sum of the previous approximations, subtracting the result obtained using only the initial approximation. Explicitly, this calculation is given by:

$$\begin{aligned} w_2(t) &= N\left[w_0\left(\frac{t}{2}\right) + w_1\left(\frac{t}{2}\right)\right] - N\left[w_0\left(\frac{t}{2}\right)\right] \\ &= \frac{31t^5}{3840} - \frac{43t^7}{215040} + \frac{4091t^9}{1486356480} \\ &- \frac{t^{11}}{53222400} + \frac{7t^{13}}{910924185600} \\ &+ \frac{17t^{15}}{34780741632000} + \frac{83t^{17}}{570532583964672000} \\ &+ \frac{t^{19}}{278745919594168320} \\ &+ \frac{t^{21}}{352597191467823267840}. \end{aligned}$$

Thus

$$\begin{aligned} w(t) &= w_0(t) + w_1(t) + w_2(t) + \cdots \\ &= t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{645120} + \frac{4091t^9}{1486356480} \\ &- \frac{t^{11}}{53222400} + \frac{7t^{13}}{910924185600} + \frac{31t^{14}}{2229534720} \\ &+ \frac{17t^{15}}{34780741632000} + \frac{83t^{17}}{570532583964672000} \\ &+ \frac{t^{19}}{278745919594168320} \\ &+ \frac{t^{21}}{352597191467823267840} + \cdots. \end{aligned}$$

The expressions are simplified using Mathematica version 13.0. Now, we present the following Figure 3, in which we sketch the approximate and exact solutions of Example 4.3

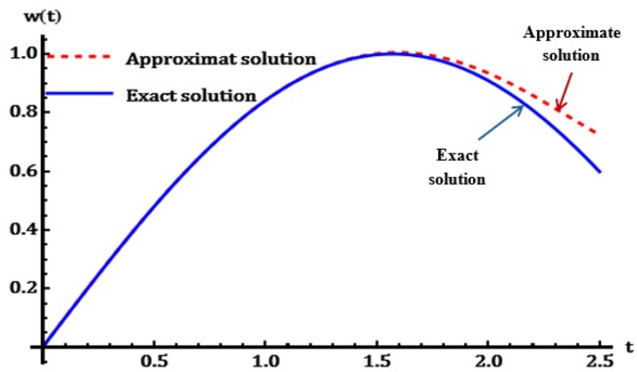

Fig. 3. Exact and approximate solutions of Example 4.3

V. Conclusion

In summary, this study highlights the advantages achieved by combining the SWT with numerical iterative approaches, especially the SIM, when addressing ordinary and delay differential equations. Leveraging the distinctive characteristics of SWT—namely linearity, scaling, shifting, and convolution—allowed us to effectively convert complex differential equations into simpler algebraic forms. Additionally, incorporating iterative numerical methods, specifically SIM, significantly improved solution accuracy, ensured rapid convergence, and increased the practical utility of the solutions.

These approaches not only streamlined the analytical procedures but also extended their applicability to a wider array of differential equations encountered in scientific and engineering contexts. By presenting a robust framework for addressing nonlinear and delay differential equations, this research provides essential methodologies that can be adopted and further developed in various practical and theoretical disciplines.

Overall, this research contributes valuable insights and practical tools to the existing literature on integral transforms and iterative techniques. Future research directions include exploring further applications of the SWT and refining iterative methodologies to optimize their efficiency, accuracy, and convergence for broader real-world scenarios. These advancements promise substantial impacts across various scientific and engineering fields, driving continued innovation and progress in applied mathematics [29], [30], [31].

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