# Quasi-uniform DtN-FE Method for the Exterior Scattering Problem

Hongrui Geng, Quanyu Dou, Guizhong Xie, and Hongbo Guan

***Abstract*—In this paper, a quasi-uniform DtN-FE method is presented to deal with the exterior scattering problem in acoustics within an unbounded domain. An artificial boundary is utilized to cause the computational domain to be finite. Then, the original problem is reformulated into an equivalent nonlocal boundary value problem in a bounded domain. A number of a priori error estimates of the DtN (Dirichlet-to-Neumann) finite element method are developed. The error estimates take into account both the influence of the DtN boundary condition truncation and that of the finite element discretization. Numerical experiments are presented which confirm the convergence rates.**

***Index Terms*—DtN mapping, exterior scattering problems, Finite element methods, quasiuniform meshes.**

## I. Introduction

THE propagation of acoustic and electromagnetic waves arises in a wide variety of applications, for example, nonCdestructive testing, ocean science, spectroscopy, remote sensing, and radar imaging. A key aspect of a great number of these problems is that they are most fittingly presented on an unbounded domain, and conditions at spatial infinity must be utilized to specify a unique solution. Such domains and conditions create major challenges for numerical simulations. This paper is focused on the investigation of the numerical solutions of the scattering of two-dimensional time-harmonic acoustic waves by an impenetrable bounded obstacle. Some numerical methods are very appropriate for treating the Helmholtz scattering problem in an infinite domain. The coupling of finite element method (FEM) [1]and some appropriate numerical methods is one of the highly conventional numerical methods. The key technique involves introducing an artificial boundary large enough to enclose the obstacle and imposing a proper artificial boundary condition. Then the exterior problem is reduced to an equivalent nonlocal boundary value problem. Many researchers implemented the coupling procedure to exterior scattering problem through defining a Dirichlet-to-Neumann (DtN) mapping on the artificial boundary [2], [3], [4], [5], [6]. The authors of [2], [3], [7], [4], [8]define it by Fourier expansion series, whereas some researchers represent the DtN mapping by means of basic boundary integral operators [5], [6], [9], [10].There are some extensions of the DtN-FE method on more general artificial boundary in [11], [12], [13]. We also refer to [14], [15], [16] for the boundary integral equation methods and [17], [18] for the perfectly matched layer (PML) method. Corresponding to the exact DtN mapping, there exist several kinds of local boundary conditions [3], [19], [20], [21] as well.

This paper aims to make contributions towards computing exterior scattering problem on quasiuniform triangle elements. With the aim of examining the errors arising from the truncation of the DtN map, Xu and Yin deduce a new and more obvious truncation error indicating exponential decrease between the exact DtN and the truncated one in [8](Theorem 7). Using the new truncation error in [8], the unique solvability of the corresponding truncated variational formula (Theorem 3.2) and the classical finite element analysis, we formulate a priori error estimates which involve the influences of both finite element discretization and the truncations of the DtN map. In addition, a new explicit formula is given to obtain the finite element computed solution of the Galerkin problem in this paper. It is different from the formula (82) in [4], which was derived by equispaced nodes. Finally, we carry out a series of numerical tests to demonstrate the efficiency and accuracy of the new formula.

The layout of the paper is presented as follows: In Section 2, we first describe the conventional Helmholtz exterior problem.Then we convert the exterior problem in acoustics into a nonlocal boundary value problem. The corresponding variational equations and modified formulation are discussed in Section 3. In Section4, using a point estimate of the DtN map, we give a priori error approximations for the Galerkin solution. In the concluding section, we carry out some numerical tests to show efficiency and the accuracy of the proposed method.

## II. Nonlocal boundary value problem

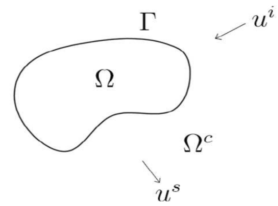


Fig. 1: Boundary value problem (1)-(5).

Let $\Omega \in \mathbb{R}^2$ be a bounded domain having a smooth boundary $\Gamma = \partial\Omega$, and let $\Omega^c = \mathbb{R}^2/\overline{\Omega}$ be the unbounded

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exterior domain beyond $\Gamma$(see Fig. 1). The following exterior Neumann problem in acoustics is considered: Given $\partial u^i/\partial\nu$, find $u \in C^2(\Omega^c) \cap C^1(\overline{\Omega^c})$ satisfying

$$\Delta u + k^2 = 0, \quad \text{in } D^c, \tag{1}$$

$$\frac{\partial u}{\partial \nu} = -\frac{\partial u^i}{\partial \nu}, \text{ on } \Gamma, \tag{2}$$

and the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{\frac{1}{2}}(\frac{\partial u}{\partial r} - ik_2 u) = 0, \ r = |x|, \tag{3}$$

where $k \neq 0$ is the wave numbers with $\mathrm{Im}(k) \geq 0$, $i = \sqrt{-1}$ and $x = (x_1, x_2) \in \mathbb{R}^2$. In this paper, $\nu$ represents the outer unit normal to the boundary and $\partial/\partial\nu$ indicates the outward normal derivative on $\Gamma$. The uniqueness of the problem (1)-(3) are given in [22].

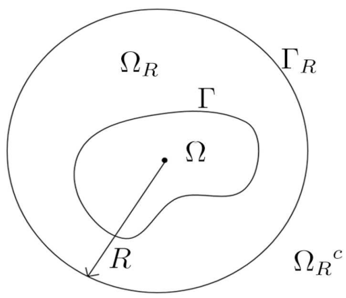


Fig. 2: The nonlocal boundary value problem (7)-(9).

In order to solve numerically problem (1)-(3), we introduce an artificial boundary $\Gamma_R := \{x \in \mathbb{R}^2 : |x| = R\}$ which should be sufficiently large to surround the region $\Omega$ (see Fig. 2). A DtN mapping $T : H^s(\Gamma_R) \to H^{s-1}(\Gamma_R)$, for $\forall\phi \in H^s(\Gamma_R), 1/2 \leq s \in R$ is described as

$$T\phi := \sum_{n=0}^{\infty}{}' \frac{kH_n^{(1)'}(kR)}{\pi H_n^{(1)}(kR)} \int_0^{2\pi} \phi(R,\psi)\cos(n(\theta-\psi))d\psi, \tag{4}$$

or equivalently as

$$T\phi := \sum_{n\in\mathbb{Z}} \frac{kH_n^{(1)'}(kR)}{2\pi H_n^{(1)}(kR)} \int_0^{2\pi} \phi(R,\psi)e^{in(\theta-\psi)}d\psi, \tag{5}$$

here and after, the prime following the summation indicates that the first term in the summation is multiplied by 1/2, and $H_n^{(1)}(\cdot)$ is the Hankel function of the first kind. [4] has confirmed that $T$ is a bounded linear operator mapping $H^s(\Gamma_R)$ to $H^{s-1}(\Gamma_R)$, for any constant $s \geq 1/2$.

The artificial boundary divides the exterior domain $\Omega^c$ into two subdomains. One is the annular region $\Omega_R$ between $\Gamma$ and $\Gamma_R$, and the other is the infinite exterior region $\Omega_R^c = \mathbb{R}^2 \backslash \overline{\Omega \cup \Omega_R}$. On $\Gamma_R$, we utilize the exact radiation boundary condition

$$\frac{\partial u^s}{\partial \nu} = Tu^s \quad \text{on} \quad \Gamma_R. \tag{6}$$

Then the original acoustic scattering problem is reduced to the following equivalent nonlocal boundary value problem: Given $\partial u^i/\partial\nu$, find the scattered field $u \in C^2(\Omega_R) \bigcap C^1(\overline{\Omega_R})$ satisfying

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \Omega_R, \tag{7}$$

$$\frac{\partial u}{\partial \nu} = -\frac{\partial u^i}{\partial \nu} \quad \text{on} \quad \Gamma, \tag{8}$$

$$\frac{\partial u^s}{\partial \nu} = Tu^s \quad \text{on} \quad \Gamma_R. \tag{9}$$

The uniqueness of the solution of (7)- (9) has been given in [4].

## III. Modified nonlocal boundary value problem

The weak formulation of (7)- (9) reads as: Given $\partial u^i/\partial n$, find $u \in H^1(\Omega_R)$ such that

$$a(u,v) + b(u,v) = \ell(v), \quad \forall\ v \in H^1(\Omega_R), \tag{10}$$

where

$$a(u,v) = \int_\Omega \left(\nabla u \cdot \nabla\overline{v} - k^2 u\overline{v}\right) dx \tag{11}$$

and

$$b(u,v) = -\int_{\Gamma_R} (Tu^s)\overline{v}ds \tag{12}$$

are sesquilinear forms defined on $H^1(\Omega_R) \times H^1(\Omega_R)$, and $\ell(v) = \int_\Gamma \frac{\partial u^i}{\partial\nu}\overline{v}ds$ is a linear functional on $H^1(\Omega_R)$ dependent on $\frac{\partial u^i}{\partial\nu} \in H^{-1/2}(\Gamma)$. For the uniqueness of the solution of (10), we refer to [4]. In practical computing, one must to truncate the infinite series of the accurate DtN map at a finite order to acquire an approximate DtN map presented as

$$T^N\phi := \sum_{n=0}^{N}{}' \frac{kH_n^{(1)'}(kR)}{\pi H_n^{(1)}(kR)} \int_0^{2\pi} \phi(R,\psi)\cos(n(\theta-\psi))d\psi, \tag{13}$$

or

$$T^N\phi := \sum_{n=0}^{N}{}' \frac{kH_n^{(1)'}(kR)}{2\pi H_n^{(1)}(kR)} \int_0^{2\pi} \phi(R,\psi)e^{in(\theta-\psi)}d\psi, \tag{14}$$

for all $\varphi \in H^s(\Gamma_R)$, $s \geq 1/2$. Here, the non-negative integer $N$ is named the truncation order of the DtN map. Therefore, we reach a modified nonlocal boundary value problem composed of (7), (8) and

$$\frac{\partial u^s}{\partial \nu} = T^N u \quad \text{on } \Gamma_R. \tag{15}$$

The modified variational equation of (10) is as: find $u_N \in H^1(\Omega_R)$ such that

$$a(u_N,v) + b^N(u_N,v) = \ell(v), \quad \forall\ v \in H^1(\Omega_R), \tag{16}$$

where $b^N(u_N,v) = -\int_{\Gamma_R}(T^N u_N)\overline{v}ds$.

*Theorem 3.1:* Suppose that the DtN maps $T$ and $T^N$ are defined as in (5) and (14), correspondingly. Let $u$ be the solution of Helmholtz equation in $\Omega_R$ meeting either (6) or (15) on $\Gamma_R$. Then there exists an $N_0 > 0$, for all $N > N_0$, such that

$$\|(T-T^N)u\|_{H^{s-1}(\Gamma_R)} \leq cq^N\|u\|_{H^{s+t+1/2}(\Omega_R)}, \quad \forall\ t \geq 0, s \geq 1/2, \tag{17}$$

through out this paper, $0 < q < 1$ and $c > 0$ are constants independent of $N$ and $h$, where $h$ is the finite element mesh size, which will be introduced in the following section.

*Proof:* Regarding the proof of the theorem, we look up[8]. ■

Now we show the uniqueness and existence of solutions for the modified variational equation (16) which in the subsequent theorem. The demonstration of the theorem can be found in [4].

*Theorem 3.2:* There exists a constant $N_0 \geq 0$ such that the modified variational formula (16) has a unique solution $u_N \in H^1(\Omega_R)$ for $N \geq N_0$.

## IV. Finite element analysis

We state a prior error estimates for the finite element solution of (16) consisting of error effects of the finite element grid size $h$ and the truncation order $N$ in the relevant Sobolev spaces.

Let $V_h$ represent the conforming finite element space of piecewise polynomial functions. We take into account the Galerkin formulation of (16): Given $\partial u^i/\partial\nu$, find $u_h \in V_h \subset H^1(\Omega_R)$ such that

$$a(u_h, v_h) + b^N(u_h, v_h) = \ell(v_h), \quad \forall\ v_h \in V_h. \tag{18}$$

The discrete sesquilinear form $a(u_h, v_h) + b^N(u_h, v_h)$ was shown to satisfy the BBL-condition [23].

The following two theorems are the primary conclusions of this article, in which some a priori error approximations are provided in $H^1-$norm and $L^2-$norm, respectively.

*Theorem 4.1:* For $2 \leq t \in \mathbb{R}$ and $u_h \in V_h$, we suppose that $u \in H^t(\Omega_R)$. Then there are constants that satisfy $h_0 > 0$ and $N_0 \geq 0$ for any $h \in (0, h_0]$ and $N \geq N_0$, such that

$$\|u - u_h\|_{H^1(\Omega_R)} \leq c\left(h^{t-1} + q^N\right)\|u\|_{H^t(\Omega_R)}. \tag{19}$$

*Proof:* We know from Theorem 5.2 of [4] that

$$\|u - u_h\|_{H^1(\Omega_R)} \leq c\left\{\inf_{v_h \in S_h}\|u - v_h\|_{H^1(\Omega_R)} + \sup_{0\neq\sigma_h \in S_h}\frac{|(b(u,\sigma_h) - b^N(u,\sigma_h)|}{\|\sigma_h\|_{H^1(\Omega_R)}}\right\}, \tag{20}$$

The foremost term on the right hand portion of (20) can be estimated directly by the approximation characteristic of the finite element space $\mathcal{H}_h$ gives as

$$\inf_{v_h \in S_h}\|u - v_h\|_{H^1(\Omega_R)} \leq ch^{t-1}\|u\|_{H^t(\Omega_R)}. \tag{21}$$

We now only need to take into account the second component on the right hand side of (20). The trace theorem implies that there is a bounded linear operator $\gamma_1 : H^1(\Omega_R) \to H^{1/2}(\Gamma_R)$ such that

$$\begin{aligned}\frac{|(b(u,\sigma_h) - b^N(u,\sigma_h)|}{\|\sigma_h\|_{H^1(\Omega_R)}} &= \frac{|\langle(T - T^N)\gamma_1 u, \gamma_1\sigma_h\rangle_{\Gamma_R}|}{\|\sigma_h\|_{H^1(\Omega_R)}}\\ &= \frac{|\langle\gamma_2(T - T^N)\gamma_1 u, \gamma\sigma_h\rangle_{\Omega_R}|}{\|\sigma_h\|_{H^1(\Omega_R)}}\end{aligned} \tag{22}$$

where $\langle\cdot,\cdot\rangle_{\Gamma_R}$ is the standard $L^2$ duality pairing between $H^{-1/2}(\Gamma_R)$ and $H^{1/2}(\Gamma_R)$, and $\gamma_2 : H^{-1/2}(\Gamma_R) \to (H^1(\Omega_R))'$ is the adjoint operator of $\gamma_1$. Consequently, we have

$$\begin{aligned}&\sup_{0\neq\sigma_h \in S_h}\frac{|(b(u,\sigma_h) - b^N(u,\sigma_h)|}{\|\sigma_h\|_{H^1(\Omega_R)}}\\ =&\sup_{0\neq\sigma_h \in S_h}\frac{|\langle\gamma_2(T - T^N)\gamma_1 u, \sigma_h\rangle_{\Omega_R}|}{\|\sigma_h\|_{H^1(\Omega_R)}}\\ =&\|\gamma_2(T - T^N)\gamma_1 u\|_{(H^1(\Omega_R))'}\\ \leq& c\|(T - T^N)\gamma_1 u\|_{H^{-1/2}(\Gamma_R)}\\ \leq& cq^N\|u\|_{H^t(\Omega_R)}.\end{aligned} \tag{23}$$

Noticing the results of Theorem 3.2, the trace theorem and the boundedness of $\gamma_1$ and $\gamma_2$, combining (21) and (23) leads to the desired result. ■

*Theorem 4.2:* With the same assumptions as in Theorem 4.1, there holds the $L^2-$norm error estimate:

$$\begin{aligned}&\|u - u_h\|_{L^2(\Omega_R)}\\ \leq& c\left(h^t + h^{t-1}N^{-1} + q^N\right)\|u\|_{H^t(\Omega_R)}.\end{aligned} \tag{24}$$

*Proof:* Let $d = u - u_h$ denote the finite element error, subtracting (18) from (10), we have the following error equation:

$$a(d, v_h) + b^N(d, v_h) + b(u, v_h) - b^N(u, v_h) = 0, \quad \forall\ v_h \in S_h \tag{25}$$

Now, the auxiliary boundary value problem is considered as: Find $\chi \in C^2(\Omega_R)\bigcap C^1(\overline{\Omega_R})$ such that

$$\Delta\chi + k^2\chi = d \quad \text{in } \Omega_R, \tag{26}$$

$$\frac{\partial\chi}{\partial n} = 0 \quad \text{on } \Gamma, \tag{27}$$

$$\frac{\partial\chi}{\partial n} = T\chi \quad \text{on } \Gamma_R, \tag{28}$$

Let $\chi$ is the weak solution of boundary value problem (26) - (28), $\omega$ should satisfy for any $\forall\ v \in H^1(\Omega_R)$ that,

$$a(v, \chi + b^N(v, \chi) + b(v, \chi) - b^N(v, \chi) = (v, d)_{L^2(\Omega_R)}. \tag{29}$$

Replacing $v$ with $e$ in (29) gives that

$$a(d, \chi) + b^N(d, \chi) + b(d, \chi) - b^N(d, \chi) = \|d\|^2_{L^2(\Omega_R)}. \tag{30}$$

Then subtracting (25) from (30) leads to

$$\begin{aligned}\|d\|^2_{L^2(\Omega_R)} =& a(d, \chi - v_h) + b^N(d, \chi - v_h) + b(d, \chi) -\\ & b^N(d, \chi) + b^N(u, v_h) - b(u, v_h), \forall v_h \in S_h.\end{aligned} \tag{31}$$

Because of the approximation property of $S_h$ and the regularity theory, there yields

$$\begin{aligned}&|a(d, \chi - v_h) + b^N(d, \chi - v_h)|\\ \leq& c\|d\|_{H^1(\Omega_R)}\|\chi - v_h\|_{H^1(\Omega_R)}\\ \leq& ch\|d\|_{H^1(\Omega_R)}\|\chi\|_{H^1(\Omega_R)}\\ \leq& ch\|d\|_{H^1(\Omega_R)}\|d\|_{L^2(\Omega_R)},\end{aligned} \tag{32}$$

where $c > 0$ is a constant. Adhering to the same argument in Theorem 4.1 and selecting $t = 2$, we have

$$\begin{aligned}
&|b(d,\chi) - b^N(d,\chi)| \\
\leq& \|\gamma_2(T - T^N)\gamma_1 d\|_{(H^2(\Omega_R))'} \|\chi\|_{H^2(\Omega_R)} \\
\leq& c\|(T - T^N)\gamma_1 d\|_{H^{-3/2}(\Gamma_R)} \|\chi\|_{H^2(\Omega_R)} \\
\leq& \frac{c}{N}\|d\|_{H^1(\Omega_R)}\|d\|_{L^2(\Omega_R)}.
\end{aligned} \tag{33}$$

Similarly, we can get that

$$\begin{aligned}
&|b^N(u,\chi - v_h) - b(u,\chi - v_h)| \\
\leq& \|\gamma_2(T - T^N)\gamma_1 u\|_{(H^1(\Omega_R))'} \|\chi - v_h\|_{H^1(\Omega_R)} \\
\leq& c\|(T - T^N)\gamma_1 u\|_{H^{-1/2}(\Gamma_R)} \|\chi - v_h\|_{H^1(\Omega_R)} \\
\leq& chq^N\|u\|_{H^t(\Omega_R)}\|d\|_{L^2(\Omega_R)},
\end{aligned}$$

and

$$\begin{aligned}
&|b(u,\chi) - b^N(u,\chi)| \\
\leq& \|\gamma_2(T - T^N)\gamma_1 u\|_{(H^2(\Omega_R))'} \|\chi\|_{H^2(\Omega_R)} \\
\leq& c\|(T - T^N)\gamma_1 u\|_{H^{-3/2}(\Gamma_R)} \|\chi\|_{H^2(\Omega_R)} \\
\leq& cq^N\|u\|_{H^t(\Omega_R)}\|d\|_{L^2(\Omega_R)}.
\end{aligned}$$

Thus, by the triangular inequality, we have

$$\begin{aligned}
&|b^N(u,v_h) - b(u,v_h)| \\
\leq& |b^N(u,\chi - v_h) - b(u,\chi - v_h)| + |b(u,\chi) - b^N(u,\chi)| \\
\leq& c_1hq^N\|u\|_{H^t(\Omega_R)}\|d\|_{L^2(\Omega_R)} \\
&+ c_2hq^N\|u\|_{H^t(\Omega_R)}\|d\|_{L^2(\Omega_R)}.
\end{aligned} \tag{34}$$

Therefore, by the combination of the inequality (31) - (34) and (19), we derive that

$$\begin{aligned}
\|u - u_h\|_{L^2(\Omega_R)} \leq& ch\left(h^{t-1} + q^N\right)\|u\|_{H^t(\Omega_R)} \\
&+ \frac{c}{N}\left(h^{t-1} + q^N\right)\|u\|_{H^t(\Omega_R)} \\
&+ chq^N\|u\|_{H^t(\Omega_R)} + cq^N\|u\|_{H^t(\Omega_R)}.
\end{aligned}$$

Finally, noticing to the fact $h \in (0, h_0]$ and $N \geq N_0$, the desired result is obtained. ■

## V. Numerical experiments

In this part, for the purpose of verifying the effectiveness of the proposed method, we consider a model with a plane wave $u^i = e^{ik_2 x\cdot d}$ with the propagation direction $d = (1, 0)$ around an infinite circular cylinder with a radius $R_0$. The exact solutions $u$ of the exterior problem (1)-(3) can be written as

$$u(r,\theta) = -\sum_{n\in\mathbb{Z}} i^n \frac{J_n'(kR_0)}{H_n^{(1)'}(kR_0)} H_n^{(1)}(kr)e^{in\theta}, \quad \forall\ r \geq R_0, \tag{35}$$

where the prime behind Bessel and Hankel functions denotes the first order derivative.

We select the artificial boundary $\Gamma_R$ as a circle with a radius of $R$. It surrounds the circle with a radius of $R_0$ and has the same center as $\Gamma$. Then, the computational region $\Omega_R$ is the annulus region between $\Gamma$ and $\Gamma_R$ (see Fig. 3). The computational annulus region is further discretized by quasi-uniform triangle elements.

In order to obtain the finite element solution of (18), we have to numerically compute the sesquilinear form

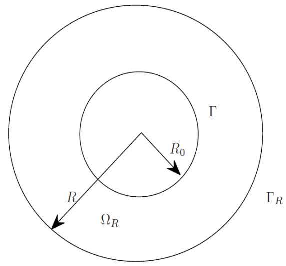


Fig. 3: Computational annulus domain.

$$b^N(u,v) = -\int_{\Gamma_R}(M^N u)\bar{v}ds. \tag{36}$$

The discrete formulation of the integrals reads as

$$\int_{\Gamma_R} M^N\phi_j\overline{\phi_i}ds. \tag{37}$$

The finite element space $V_h$ is composed of piecewise linear functions $\{\phi_j\}_{j=1}^{NP}$, where $NP$ is the total number of freedom. The outer boundary $\Gamma_R$ is discretized by nonequispaced nodes $x_1, x_2, \ldots, x_{N_\theta}$ and the $j^{th}$ point $\{x_j\}_{j=1}^{N_\theta}$ possesses the polar coordinates $(R, \theta_j)$.

For those $\phi_j$'s which do not vanish on $\Gamma_R$, we have

$$\phi_j(\theta) = \begin{cases} \frac{\theta-\theta_{j-1}}{\Delta\theta_{j-1}}, & \theta_{j-1} \leq \theta \leq \theta_j, \\ \frac{\theta_{j+1}-\theta}{\Delta\theta_j}, & \theta_j \leq \theta \leq \theta_{j+1}, \\ 0, & \text{others}. \end{cases}$$

Here, $\Delta\theta_{j-1} = \theta_j - \theta_{j-1}$, $\Delta\theta_j = \theta_{j+1} - \theta_j$.

In terms of the definition of $M^N$, the computation of integrals (37) is equivalent to calculating the following series

$$\begin{aligned}
&\int_{\Gamma_R} M^N\phi_j\overline{\phi_i}ds \\
=&\sum_{n=0}^{N}{}' \frac{kRH_n^{(1)'}(kR)}{\pi H_n^{(1)}(kR)} \times \\
&\int_0^{2\pi}\int_0^{2\pi}\phi_j(R,\psi)\overline{\phi_i(R,\theta)}\cos(n(\theta-\psi))d\theta d\psi \\
=&\frac{kR}{\pi}\sum_{n=0}^{N}{}' \frac{H_n^{(1)'}(kR)}{n^4 H_n^{(1)}(kR)} \times \\
&\Big\{[\frac{1}{\Delta\theta_{i-1}}(\cos(n\theta_i) - \cos(n\theta_{i-1})) \\
&- \frac{1}{\Delta\theta_i}(\cos(n\theta_{i+1}) - \cos(n\theta_i))] \\
&\cdot [\frac{1}{\Delta\theta_{j-1}}(\cos(n\theta_j) - \cos(n\theta_{j-1})) \\
&- \frac{1}{\Delta\theta_j}(\cos(n\theta_{j+1}) - \cos(n\theta_j))] \\
&+ [\frac{1}{\Delta\theta_{i-1}}(\sin(n\theta_i) - \sin(n\theta_{i-1})) \\
&- \frac{1}{\Delta\theta_i}(\sin(n\theta_{i+1}) - \sin(n\theta_i))] \\
&\cdot [\frac{1}{\Delta\theta_{j-1}}(\sin(n\theta_j) - \sin(n\theta_{j-1})) \\
&- \frac{1}{\Delta\theta_j}(\sin(n\theta_{j+1}) - \sin(n\theta_j))]\Big\}
\end{aligned} \tag{38}$$

Equation (38) plays an important role in the following computations.

In the following part, several numerical examples utilizing linear Lagrange elements will be presented. An original coarse triangular mesh is generated by MATLAB and the mesh is refined uniformly to carry out an investigation of convergence (Fig. 4). Additionally, we invariably fix the radius $R_0 = 1, R = 2$ unless otherwise being stated.

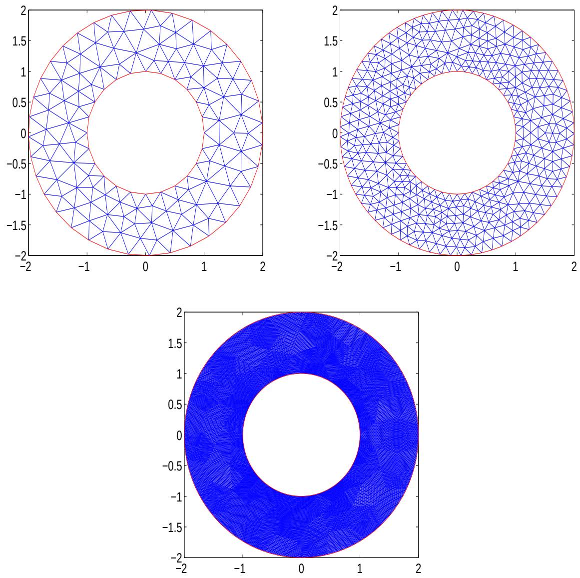

Fig. 4: From left to right: coarse mesh; fine mesh; finest mesh.

**Experiment 1**. In this experiment, we choose the wave number $k = 1$ and the truncation order $N = 15$ of the DtN mapping. Then the solutions for different $h$ are computed and corresponding numerical results and exact solutions are presented in Fig. 5 when $h = 0.4110$ and $h = 0.0262$, respectively. We can find that the numerical solution are in complete accordance with the exact solution if $h$ is small enough. Numerical errors and convergence order are listed in Table I corresponding to the different $h$. The number of degrees of freedom is represented by $dof$. We use the numerical method in [4] to calculate this problem and present the results in Table II. Here $h$ is the diagonal length of the rectangular meshes. It can be seen that our results are consistent with the results in [4]. However, we don't need to project the annular region onto the rectangular region.

TABLE I: Numerical errors when $k = 1$,$N = 15$.

| $dof$ | $h$ | $L^2$-norm | order | $H^1$-norm | order |
|---|---|---|---|---|---|
| 143 | 0.4110 | 4.4642E-2 | | 2.8550E-1 | |
| 512 | 0.2084 | 1.1637E-2 | 1.98 | 1.4285E-1 | 1.02 |
| 1928 | 0.1042 | 2.9455E-3 | 1.98 | 7.1509E-2 | 1.00 |
| 7472 | 0.0521 | 7.3896E-4 | 1.99 | 3.5775E-2 | 1.00 |
| 29408 | 0.0262 | 1.8492E-4 | 2.02 | 1.7892E-2 | 1.01 |

**Experiment 2**. In this experiment, we calculate the model problem to test the error effect of the numerical discretization. We fix $N = 15$ and take into account three different wave numbers $k = 1, 2, 4$. Fig. 6 shows the log-log plot of errors measured in $L^2$-norm and $H^1$-norm with respect to

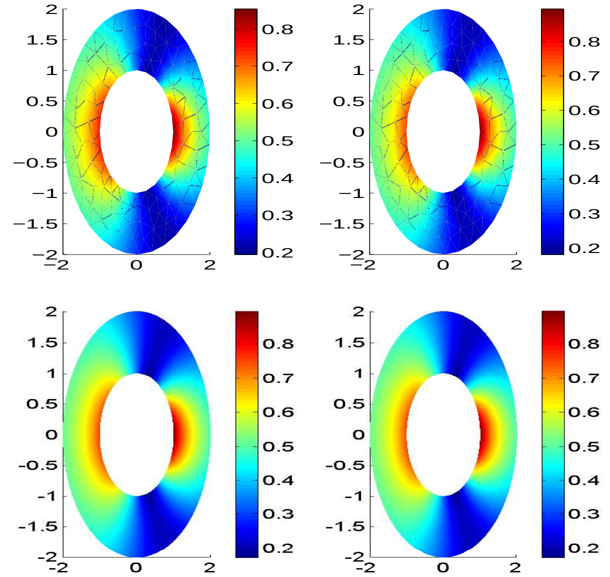

Fig. 5: Absolute values of the numerical solutions (left) and the exact solutions (right) of $u$ on coarse mesh (top) and finest mesh (bottom).

TABLE II: Numerical errors when $k = 1$,$N = 15$.

| $N_r$ | $dof$ | $h$ | $L^2$-norm | order | $H^1$-norm | order |
|---|---|---|---|---|---|---|
| 2 | 24 | 0.9310 | 9.1615E-2 | | 3.8745E-1 | |
| 4 | 80 | 0.4655 | 2.0250E-2 | 2.18 | 1.4723E-1 | 1.40 |
| 8 | 288 | 0.2328 | 4.7179E-3 | 2.10 | 7.1523E-2 | 1.04 |
| 16 | 1088 | 0.1164 | 1.1649E-3 | 2.02 | 3.5745E-2 | 1.00 |
| 32 | 4224 | 0.0582 | 2.8913E-4 | 2.01 | 1.7825E-2 | 1.00 |

$1/h$ ($h$ being the meshsize) and confirms that the optimal order of convergence

$$\|u - u_h\|_{L^2} = O(h^2), \quad \|u - u_h\|_{H^1} = O(h), \tag{39}$$

as depicted in Theorem 4.1 and 4.2 for a truncation order $N$ of the DtN mapping that is large enough. It also indicates that the quality of numerical solutions relies on wave numbers and the accuracy degrades correspondingly as the wave number $k_2$ grows with the identical mesh size $h$.

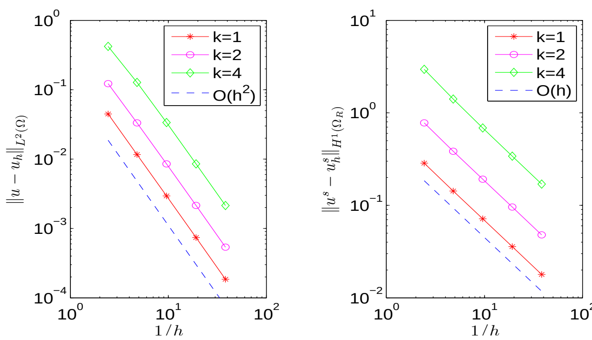


Fig. 6: Log-log plot vs. $1/h$ for errors in $L^2$-norm (left) and $H^1$-norm (right).

**Experiment 3**. It is considered with the effects of truncation order $N$ regarding the overall numerical errors. Let $k = 2$, and compute the numerical errors measured in $L^2$-norm for four distinct meshsizes of $h =$

$0.2084, 0.1042, 0.0521, 0.0262$, in turn. Fig. 7 indicates that the errors diminish very rapidly for all $h$ because of the truncation order $N$. This is agrees with the theory which indicates, based on (24), that the convergence order is $O(\frac{\epsilon(N,u^s)}{N^2})$ provided that $h$ is sufficiently small. In Fig. 7, we recognize that when $N = N_0 = 4$, the accuracy for $h = 0.2084$ and $0.1042$ reaches the optimal level whereas $N = N_0 = 5$ for other magnitudes of $h$. In addition, we note that there are no numerical enhancements in accuracy when $N > N_0$ for every $h$ because error of domain discretization prevails.

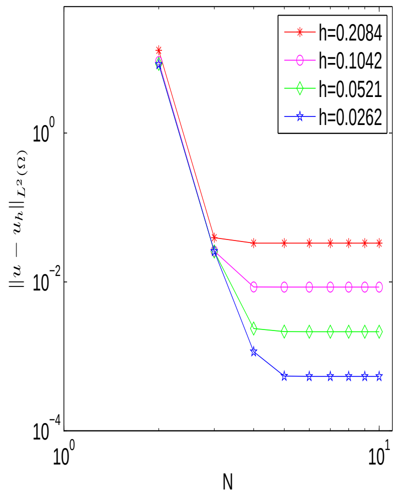


Fig. 7: Log-log plot of errors in $\mathcal{H}^0$-norm vs the truncation order $N$ for various $h$ when $kR = 4$.

**Experiment 4**. This test aims at investigating the numerical rule $N \geq kR$ introduced in [24], [25]. We choose the meshsize $h = 0.0521$. The log-log plots of numerical values evaluated in $L^2$-norm are shown in Fig. 8. It shows that the optimal truncation order $N_0$ rises linearly in proportion to $kR$. For example, we can see from Fig. 8, the optimal truncation order $N_0 = 5$ as $kR = 4$ while $N_0 = 13$ as $kR = 20$. To each $kR$, there is no enhancement of accuracy when $N > N_0$. Our numerical outcomes are in excellent accordance with the numerical rule $N \geq kR$.

## VI. Conclusion

In the present paper, we compute the finite element solution of an exterior Neumann problem through Fourier analysis on quasi-uniform meshes. The initial boundary value problem is reformulated into an equivalent nonlocal boundary value problem in a bounded domain. Uniqueness and existence for the weak solution are demonstrated within suitable function spaces. Priori error estimates of the finite element solution, incorporating the impacts of both the truncation of the DtN operator and the numerical discretisation, are developed in suitable Sobolev spaces. In the end, we carry out a series of numerical experiments to demonstrate the efficiency and accuracy of the DtN-FE method for solving the exterior scattering problem in acoustics.

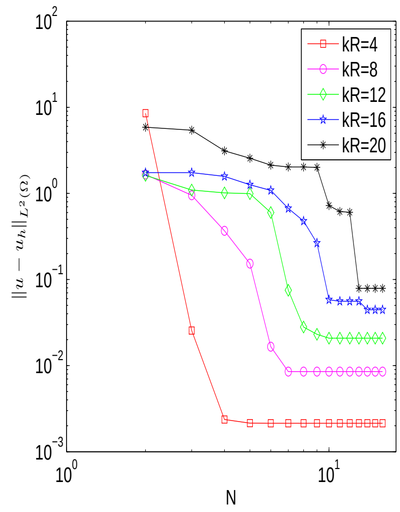


Fig. 8: The relationship between the truncation order $N$ and the variable $kR$ when $h = 0.0521$.

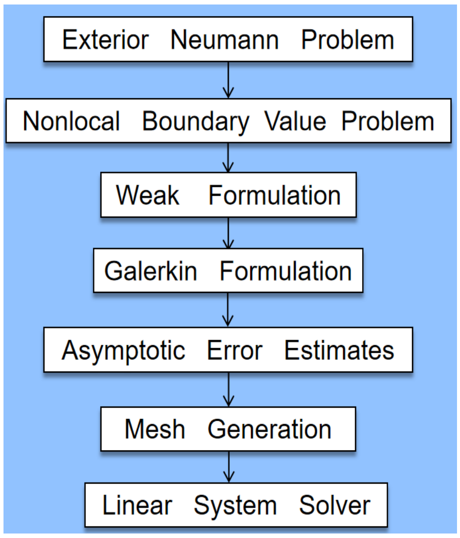


Fig. 9: A schematic procedure for DtN-FE method.

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